A NOTE ON THE ERDŐS MATCHING CONJECTURE

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ABSTRACT. The Erdős Matching Conjecture states that the maximum size f(n,k,s) of a family $\mathcal{F}\subseteq \binom{[n]}{k}$ that does not contain s pairwise disjoint sets is $\max\{|\mathcal{A}_{k,s}|,|\mathcal{B}_{n,k,s}|\}$, where $\mathcal{A}_{k,s}=\binom{[sk-1]}{k}$ and $\mathcal{B}_{n,k,s}=\{B\in \binom{[n]}{k}:B\cap [s-1]\neq\emptyset\}$. The case s=2 is simply the Erdős-Ko-Rado theorem on intersecting families and is well understood. The case n=sk was settled by Kleitman and the uniqueness of the extremal construction was obtained by Frankl. Most results in this area show that if k,s are fixed and n is large enough, then the conjecture holds true. Exceptions are due to Frankl who proved the conjecture and considered variants for $n\in[sk,sk+c_{s,k}]$ if s is large enough compared to k. A recent manuscript by Guo and Lu considers non-trivial families with matching number at most s in a similar range of parameters.

In this short note, we are concerned with the case $s \geq 3$ fixed, k tending to infinity and $n \in \{sk, sk+1\}$. For n = sk, we show the stability of the unique extremal construction of size $\binom{sk-1}{k} = \frac{s-1}{s} \binom{sk}{k}$ with respect to minimal degree. As a consequence we derive $\lim_{k \to \infty} \frac{f(sk+1,k,s)}{\binom{sk+1}{k}} < \frac{s-1}{s} - \varepsilon_s$ for some positive constant ε_s which depends only on s.

1. Introduction

We use standard notation. Let [n] denote the set of the first n positive integers. For a set X, $\binom{X}{k}$ stands for the family of all k-element subsets of X and a k-graph (or k-uniform hypergraph) on X is a subset of $\binom{X}{k}$. The members of X are called vertices and the k-sets are called k-edges or hyperedges or just edges. A k-graph is intersecting if every pair of edges has a nonempty intersection. A k-graph on X for which there is an $x \in X$ that is in every edge is called a star.

One of the major unsolved problems in extremal set theory is due to Erdős: The matching number $\nu(\mathcal{F})$ of a family \mathcal{F} of sets is the size of the largest set of pairwise disjoint sets that \mathcal{F} contains. Let f(n, k, s) denote the maximum size of a k-graph on n vertices with matching number strictly smaller than s. Because $\nu(\mathcal{F}) = 1$ if and only if \mathcal{F} is intersecting, the well-known Erdős-Ko-Rado theorem [5] gives

$$f(n, k, 2) = \begin{cases} \binom{n-1}{k-1}, & \text{if } n \ge 2k; \\ \binom{n}{k}, & \text{if } n \le 2k-1. \end{cases}$$

The special case of f(sk, k, s) was proved by Kleitman.

Theorem 1 (Kleitman [22]). If $k \geq 2$ and $s \geq 2$ are positive integers, then $f(sk, k, s) = \binom{sk}{k} - \binom{sk-1}{k-1} = \frac{s-1}{s} \binom{sk}{k}$ with equality if and only if there is an element $x \in [n]$ such that every k-edge fails to contain x.

There are two natural families of k-sets with matching number s-1:

$$\mathcal{A}_{k,s} = {[sk-1] \choose k}$$
 $\mathcal{B}_{n,k,s} = \left\{ B \in {[n] \choose k} : B \cap [s-1] \neq \emptyset \right\}.$

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Erdős conjectured that for any values of n, k, s, one of these families achieves the maximum value.

Conjecture 2 (Erdős Matching Conjecture (EMC) [4]). For all n, k, s with $n \geq sk$, we have $f(n, k, s) = \max\{|\mathcal{A}_{k,s}|, |\mathcal{B}_{n,k,s}|\}$.

It can be computed that $|\mathcal{A}_{k,s}| < |\mathcal{B}_{n,k,s}|$ for all $n \ge s(k+1)$. Most results concerning Conjecture 2 determine that $f(n,k,s) = |\mathcal{B}_{n,k,s}|$ if n is large enough compared to k and s [15, 20], with the current best bounds due to Frankl [8] for $n \ge (2s+1)k-s$ and Frankl and Kupavskii [14] for $n \ge \frac{5}{3}sk - \frac{2}{3}s$ if s is large enough. The case k = 3 is also completely settled by Frankl [9]. Minimum degree versions of the problem are also studied [1, 18, 21] and general inequalities on f(n,k,s) are known [14, 19] for n = cks with c > 1 and k and/or s is large enough.

Not too many results are known in the region when n is very close to sk: Frankl [10] proved that $f(n,k,s) = |\mathcal{A}_{k,s}|$, if $sk \leq n \leq s(k+\varepsilon)$ where ε depends only on k, but s has to be large enough with respect to k ($s \geq k^2 + k + 1$ suffices). Frankl [12] and recently Guo and Lu [17] considered non-trivial families (i.e. those with no isolated vertices) in a similar range of parameters.

Our aim is to obtain bounds on f(n,k,s) when n=sk+1, but with s fixed and k being large. Observe that the EMC states that there should be a huge difference between the cases s=2 and $s\geq 3$: on the one hand $\frac{f(2k,k,2)}{\binom{2k}{k}}=\frac{1}{2}$ and $\frac{f(2k+1,k,2)}{\binom{2k+1}{k}}\to \frac{1}{2}$ as k tends to infinity. On the other hand, $\frac{f(sk,k,s)}{\binom{sk}{k}}=\frac{s-1}{s}$, while $\frac{|\mathcal{A}_{k,s}|}{\binom{sk+1}{k}}\to (\frac{s-1}{s})^2$ and $\frac{|\mathcal{B}_{sk+1,k,s}|}{\binom{sk+1}{k}}\to 1-(1-\frac{1}{s})^{s-1}$ as k tends to infinity, thus the EMC would yield an immediate drop in the limiting constant. Our main result shows that there is indeed a gap between $\frac{f(sk,k,s)}{\binom{sk}{k}}=\frac{s-1}{s}$ and $\frac{f(sk+1,k,s)}{\binom{sk+1}{k}}$.

Theorem 3. For any $s \ge 3$ there exists a positive real ε_s such that $f(sk+1,k,s) \le \left(\frac{s-1}{s} - \varepsilon_s\right) {sk+1 \choose k}$ holds for all k.

To prove Theorem 3, as an intermediate step, we will show the following stability version of Kleitman's bound on f(sk, k, s) with respect to the minimum degree.

Theorem 4. There exist absolute constants C and δ_0 such that, if $s \geq 3$ and $\delta \leq \delta_0$, then any family $\mathcal{F} \subseteq {[sk] \choose k}$ with $\nu(\mathcal{F}) \leq s-1$ and minimum degree at least $\delta {sk-1 \choose k-1}$ satisfies $|\mathcal{F}| \leq \left(\frac{s-1}{s} - \frac{4(s-2)\delta}{s^2C}\right){sk \choose k}$.

2. Tools of the proofs: removal Lemmas and shifting

In this section, we introduce some terminology and results from the literature that we will use in our proofs of Theorem 3 and Theorem 4. Removal lemmas are widely used in combinatorics, they state that if some combinatorial structure \mathcal{S} contains only a few copies of some pattern P, then \mathcal{S} can be made P-free while almost keeping the structure \mathcal{S} . We will need a removal lemma for intersecting families. A general result is the following.

Theorem 5 (Friedgut, Regev [16]). Let $\gamma > 0$ and n, k positive integers with $\gamma n < k < (q/2 - \gamma)n$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that any $\mathcal{F} \subseteq \binom{[n]}{k}$ with at most $\delta |\mathcal{F}| \binom{n-k}{k}$ disjoint pairs can be made intersecting by removing at most $\varepsilon \binom{n-1}{k-1}$ sets from \mathcal{F} .

As we will point out in the last section, Theorem 5 could also be used to derive our results, but the following statement is better suited for our purposes.

Theorem 6 (Das, Tran [3]). There exists an absolute constant C such that if n, k, ℓ satisfy $n > 2k\ell^2$, then for any $\mathcal{F} \subseteq \binom{[n]}{k}$ with at most $\binom{\ell}{2} + \beta \binom{n-1}{k-1} \binom{n-k-1}{k-1}$ disjoint pairs where $\max\{2\ell|\alpha|, |\beta|\} \leq \frac{n-2k}{(20C)^2n}$ there exists a family \mathcal{S} that is a union of ℓ stars such that $|\mathcal{F}\triangle\mathcal{S}| \leq C((2\ell-1)\alpha+2\beta)\frac{n}{n-2k}\binom{n-1}{k-1}$.

We state the case where $\ell = 1, \alpha = 0, n = sk$ as a corollary, since this is what we will use in our proofs.

Corollary 1. There exists an absolute constant C such that for any $\mathcal{F} \subseteq \binom{[sk]}{k}$ with at most $s\beta\binom{sk-1}{k-1}\binom{(s-1)k}{k}$ disjoint pairs where $\beta \leq \frac{s-2}{s(20C)^2}$ there exists a star \mathcal{S} such that $|\mathcal{F}\triangle\mathcal{S}| \leq s$ $\frac{2s}{s-2}C\beta\binom{sk-1}{k-1}$.

Or equivalently, if the maximum degree $\Delta(\mathcal{F})$ of a family $\mathcal{F} \subseteq \binom{[sk]}{k}$ is at most $(1-\beta)\binom{sk-1}{k-1}$ where $\beta \leq \frac{s-2}{s(20C)^2}$, then \mathcal{F} contains at least $\frac{2\beta(s-2)}{C}\binom{sk-1}{k-1}\binom{(s-1)k}{k}$ disjoint pairs.

Let us now turn to our other tool in proving our results. The shifting operation was introduced by Erdős, Ko, and Rado [5] and is a very frequently used powerful tool in extremal set theory. For a set F and family \mathcal{F} of sets we define

$$S_{i,j}(F) = \begin{cases} F \setminus \{j\} \cup \{i\}, & \text{if } i \notin F, j \in F, F \setminus \{j\} \cup \{i\} \notin F; \\ F, & \text{otherwise.} \end{cases}$$

 $S_{i,j}(\mathcal{F}) = \{S_{i,j}(F) : F \in \mathcal{F}\}.$ A family $\mathcal{F} \subseteq {[n] \choose k}$ is left-compressed if $S_{i,j}(\mathcal{F}) = \mathcal{F}$ for all $1 \le i < j \le n$. We will use the following lemma [6, Lemma 4.2(iv)].

Lemma 7 (Frankl [6]). If $\mathcal{F}' \subseteq {[n] \choose k}$ has matching number $\nu(\mathcal{F}) \leq s$, then there is a leftcompressed $\mathcal{F} \subseteq \binom{[n]}{k}$ with $\nu(\mathcal{F}) \leq s$ and $|\mathcal{F}| = |\mathcal{F}'|$.

We will need some more properties of left-compressed families. To state them, we need some definitions: for a family $\mathcal{F} \subseteq \binom{[n]}{k}$ and $x, y \in [n]$, we write $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}$, $\mathcal{F}_{\overline{x}} = \{F \in \mathcal{F} : x \notin F\}$, $\mathcal{F}_{x,y} = \{F \in \mathcal{F} : x, y \in F\}$ and $\mathcal{F}_{x,\overline{y}} = \{F \in \mathcal{F} : x \in F, y \notin F\}$.

Lemma 8. For any left-compressed family $\mathcal{F} \subseteq \binom{[n]}{k}$, we have

- (a) $\frac{n-k}{k} |\mathcal{F}_n| \le |\mathcal{F}_{\overline{n}}|,$ (b) $\frac{|\mathcal{F}_n|}{\binom{n-1}{k-1}} \le \frac{|\mathcal{F}_{n-1,\overline{n}}|}{\binom{n-2}{k-1}}.$

Proof. Inequality (a) follows from counting the pairs (F,j) where $F \in \mathcal{F}_n$ and $j \notin F$. The number of such pairs is $(n-k)|\mathcal{F}_n|$. Moreover, the set $S_{j,n}(F)$ is in \mathcal{F} and any set $F \in \mathcal{F}_{\overline{n}}$ can be the $S_{j,n}$ -image of at most k sets in \mathcal{F}_n . Hence the number of such pairs is at most $k|\mathcal{F}_{\overline{n}}|$

To see (b), a similar double counting argument establishes $\frac{n-k}{k-1}|\mathcal{F}_{n,n-1}| \leq |\mathcal{F}_{n,\overline{n-1}}|$ and thus $|\mathcal{F}_n| = |\mathcal{F}_{n,n-1}| + |\mathcal{F}_{n,\overline{n-1}}| \le \frac{n-1}{n-k}|\mathcal{F}_{n,\overline{n-1}}|$. Also, applying $S_{n-1,n}$, we obtain $|\mathcal{F}_{n,\overline{n-1}}| \le |\mathcal{F}_{n-1,\overline{n}}|$ Therefore

$$\frac{|\mathcal{F}_n|}{\binom{n-1}{k-1}} \le \frac{\frac{n-1}{n-k}|\mathcal{F}_{n,\overline{n-1}}|}{\frac{n-1}{n-k}\binom{n-2}{k-1}} \le \frac{|\mathcal{F}_{n-1,\overline{n}}|}{\binom{n-2}{k-1}}$$

as claimed.

For any r-graph H, the sequence $\frac{\exp_r(n,H)}{\binom{n}{r}}$ is monotone decreasing.

3. Proofs

Proof of Theorem 4. Let $\mathcal{F} \subseteq {[sk] \choose k}$ with $\nu(\mathcal{F}) \leq s-1$ and $\delta(\mathcal{F}) \geq \delta {sk-1 \choose k-1}$. Let us write $\mathcal{G} = {[sk] \choose k} \setminus \mathcal{F}$ and consider an arbitrary subfamily $\mathcal{G}' \subseteq \mathcal{G}$ with $|\mathcal{G}'| = \frac{1}{s} {sk \choose k} = {sk-1 \choose k-1}$. (By Theorem 1, we know $|\mathcal{G}| \geq \frac{1}{s} {sk \choose k}$, and so such a \mathcal{G}' exists.)

The minimum degree condition on \mathcal{F} implies $\Delta(\mathcal{G}') \leq (1-\delta)\binom{sk-1}{k-1}$. Corollary 1 yields that if δ is sufficiently small, then \mathcal{G}' (and thus \mathcal{G}) contains at least $\frac{2\delta(s-2)}{C}\binom{sk-1}{k-1}\binom{(s-1)k}{k}$ disjoint pairs.

Every set G is contained in $M = \frac{1}{(s-1)!} \prod_{j=1}^{s-1} {jk \choose k}$ s-partitions of [sk]. Also, every disjoint pair G, G' is contained in $M' = \frac{1}{(s-2)!} \prod_{j=1}^{s-2} {jk \choose k}$ s-partitions, and clearly one fixed s-partition contains ${s \choose 2}$ disjoint pairs. Let us count the pairs (G, π) with π being an s-partition of [sk] into k-sets, and $G \in \mathcal{G}$ being one of those k-sets. On the one hand the number of such pairs is $|\mathcal{G}| \cdot M$. On the other hand, by definition of \mathcal{F} and \mathcal{G} , every π contains at least one set of \mathcal{G} . Also, by the above observations, the number of π 's containing at least two sets from \mathcal{G} is at least $\frac{2\delta(s-2)}{C} {sk-1 \choose k-1} {s-1 \choose k} \cdot \frac{M'}{{s \choose 2}}$. Putting these together,

$$|\mathcal{G}| \cdot M \ge \frac{1}{s!} \prod_{j=1}^{s} {jk \choose k} + \frac{2\delta(s-2)}{C} {sk-1 \choose k-1} {(s-1)k \choose k} \cdot \frac{M'}{{s \choose 2}}.$$

Dividing by M, we get $|\mathcal{G}| \ge {sk-1 \choose k-1} + \frac{4(s-2)\delta}{sC} {sk-1 \choose k-1}$ and thus

$$|\mathcal{F}| = \binom{sk}{k} - |\mathcal{G}| \le \binom{sk-1}{k} - \frac{4(s-2)\delta}{sC} \binom{sk-1}{k-1} = \left(\frac{s-1}{s} - \frac{4(s-2)\delta}{s^2C}\right) \binom{sk}{k}$$

as claimed. \Box

Proof of Theorem 3. Let $\mathcal{F} \subseteq {[sk+1] \choose k}$ be a family of sets with $\nu(\mathcal{F}) < s$, $|\mathcal{F}| = f(sk+1,k,s)$ and let us write n = sk+1.

By Lemma 7, we can assume that \mathcal{F} is left-compressed. By Theorem 1, we have $|\mathcal{F}_{\overline{n}}| \leq \frac{s-1}{s} {sk \choose k}$. Let δ_0 , C be as in Theorem 4, and let $\varepsilon_s^* = \min\left\{\frac{4(s-2)\delta_0}{s^2C}, \frac{s-1}{s} - \delta_0\right\}$.

Suppose first $|\mathcal{F}_n| \leq \left(\frac{s-1}{s} - \varepsilon_s^*\right) {sk \choose k-1}$, then

$$|\mathcal{F}| = |\mathcal{F}_{\overline{n}}| + |\mathcal{F}_n| \le \frac{s-1}{s} \binom{sk}{k} + \left(\frac{s-1}{s} - \varepsilon_s^*\right) \binom{sk}{k-1} \le \left(\frac{s-1}{s} - \frac{\varepsilon_s^*}{s+1}\right) \binom{sk+1}{k}.$$

On the other hand, if $|\mathcal{F}_n| \geq \left(\frac{s-1}{s} - \varepsilon_s^*\right) {sk \choose k-1}$, then we use the fact that left-compression implies $\delta(\mathcal{F}_{\overline{n}}) = |\mathcal{F}_{n-1,\overline{n}}|$ and Lemma 8(b) to conclude that

$$\frac{\delta(\mathcal{F}_{\overline{n}})}{\binom{sk-1}{k-1}} = \frac{|\mathcal{F}_{n-1,\overline{n}}|}{\binom{sk-1}{k-1}} \ge \frac{|\mathcal{F}_n|}{\binom{sk}{k-1}} \ge \frac{s-1}{s} - \varepsilon_s^*.$$

So one can apply Theorem 4 with $\delta = \delta_0$ and since $\frac{s-1}{s} - \epsilon_s^* \ge \delta_0$, it gives $|\mathcal{F}_{\overline{n}}| \le \left(\frac{s-1}{s} - \epsilon_s^*\right) \binom{sk}{k}$. According to Lemma 8(a), we have $|\mathcal{F}_n| \le \frac{k}{(s-1)k+1} |\mathcal{F}_{\overline{n}}|$ and thus

$$|\mathcal{F}| = |\mathcal{F}_n| + |\mathcal{F}_{\overline{n}}| \le \left(\frac{s-1}{s} - \varepsilon_s^*\right) \binom{sk+1}{k}.$$

So $\varepsilon_s = \varepsilon_s^*/(s+1)$ works in both cases.

4. Concluding remarks

As we mentioned in Section 2, Theorem 3 and Theorem 4 can be proved using the " ε - δ removal lemma" of Theorem 5 with the help of Frankl's result [7] on the maximum size of intersecting families with bounded maximum degree. Also, there is a large literature on the size of intersecting families with constraints on some other parameters: minimum degree [21] and diversity [11, 13, 23, 24]. Supersaturation results on the minimum number of disjoint pairs in a family are also known [2]. Although, exact results are usually known for values of n large enough compared to k, weaker bounds could be turned into bounds on f(n, k, s) with the methods used in our note.

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