# VERONESE SECTIONS AND INTERLACING MATRICES OF POLYNOMIALS AND FORMAL POWER SERIES 

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#### Abstract

The concept of a fully interlacing matrix of formal power series with real coefficients is introduced. This concept extends and strengthens that of an interlacing sequence of realrooted polynomials with nonnegative coefficients, in the special case of row and column matrices. The fully interlacing property is shown to be preserved under matrix products, flips across the reverse diagonal and Veronese sections of the power series involved. These results and their corollaries generalize, unify and simplify several results which have previously appeared in the literature. An application to the theory of uniform triangulations of simplicial complexes is included.


## 1. Introduction

Real-rooted polynomials, especially those with nonnegative coefficients, have been studied intensely within various mathematical disciplines, including algebraic, enumerative and geometric combinatorics [9, 11, 12, 28]. Two important players in this study are the theory of interlacing of polynomials [9, Section 8] [16] and the Veronese construction for formal power series [6, 13, 21, 22, 32. The present work arose from an effort to better understand the connections between these two notions.

To explain the content and motivation for this paper, we begin with a few definitions. A polynomial $P(x) \in \mathbb{R}[x]$ is said to be real-rooted if either it is the zero polynomial, or every complex root of $P(x)$ is real. Given two polynomials $P(x), Q(x) \in \mathbb{R}[x]$, we say that $P(x)$ interlaces $Q(x)$, and write $P(x) \prec Q(x)$, if (a) both $P(x)$ and $Q(x)$ are real-rooted and have nonnegative coefficients; and (b) the roots $\left\{\eta_{i}\right\}$ of $P(x)$ interlace (or alternate to the left of ) the roots $\left\{\theta_{j}\right\}$ of $Q(x)$, in the sense that they can be listed as

$$
\cdots \leq \theta_{3} \leq \eta_{2} \leq \theta_{2} \leq \eta_{1} \leq \theta_{1} \leq 0
$$

By convention, the zero polynomial interlaces and is interlaced by every real-rooted polynomial with nonnegative coefficients.

Given a formal power series $A(x)=\sum_{n \geq 0} a_{n} x^{n} \in \mathbb{R}[[x]]$ and integers $0 \leq k<r$, the $k$ th Veronese $r$-section (or simply $k$ th $r$-section) of $A(x)$ is defined as the formal power series

$$
\begin{equation*}
\mathrm{S}_{k}^{(r)} A(x)=\sum_{n \geq 0} a_{k+r n} x^{n} \tag{1}
\end{equation*}
$$

[^0]We will often suppress the superscript $(r)$ from the notation. Every $r$-section of a real-rooted polynomial with nonnegative coefficients is real-rooted (see, for instance, [1] [16, Theorem 7.65] [25. V 171.4]). The question whether the $r$-section operators preserve interlacing arises naturally: Given that $P(x)$ interlaces $Q(x)$, does the $k$ th $r$-section of $P(x)$ necessarily interlace that of $Q(x)$ for all $r, k$ ? The aim of this paper is to give an affirmative answer to this question (see Corollary 5.6), extend and strengthen the corresponding result in several directions (see Section (5) and show that a natural context to study such questions is provided by the class of AESW series (equivalently, that of Pólya frequency sequences); see Section 2.1. The latter can be viewed as the closure of the set of real-rooted polynomials with nonnegative coefficients and is characterized by the fundamental notion of total positivity.

The main new construction introduced is that of an interlacing matrix of formal power series (see Definition 3.5). The new concept of interlacing, termed as full interlacing, is defined by the total positivity of that matrix. The special cases of row and column matrices strengthen and extend to power series the well known concept of an interlacing sequence of real-rooted polynomials (both concepts reduce to the interlacing relation we have already discussed for rows and columns of length two). Our main results state that the conept of full interlacing behaves well with respect to taking submatrices, matrix products, flips across the reverse diagonal and Veronese sections of the entries of the matrix (see Theorems 4.1, 4.3 and 5.3). They simplify, generalize and unify several results which have previously appeared in the literature.

Our motivation comes from the study of certain operators which are of interest in geometric combinatorics [4 (we describe an application in Section 6 and expect that our results will find more applications there), as well as from the need to develop more tools for proving the interlacing of real-rooted polynomials. An extension of the ideas of this paper to total positivity up to a certain order or level [27] and to tensors of higher rank is left to the future.

## 2. Three old theorems

This section reviews three classical theorems on stability and total positivity, which will serve as the main ingredients for our results, and extends the notions of interlacing and Hurwitz stability from polynomials to formal power series.

Here, $\mathbf{a}=\left(a_{n}\right)_{n \in \mathbb{N}}=\left(a_{0}, a_{1}, a_{2}, \ldots\right)$ will be a sequence of real numbers and $A=A(x)=$ $\sum_{n>0} a_{n} x^{n}$ will be its generating function. We set $a_{n}=0$ for all negative integers $n$. We denote by $\mathcal{A}^{\top}$ the transpose of a matrix $\mathcal{A}$.
2.1. The Aissen-Edrei-Schoenberg-Whitney (AESW) Theorem. The Toeplitz matrix of $A$ is defined as $\operatorname{Toep}[A]=\left(a_{v-u}\right)$; it is indexed by $\mathbb{Z} \times \mathbb{Z}$, with matrix indexing conventions, where $u$ indexes rows and $v$ indexes columns. Here is a small piece:

$$
\operatorname{Toep}[A]=\left(\begin{array}{cccccc}
. & \cdot & . & \cdot & \cdot & \cdot \\
\cdot & a_{0} & a_{1} & a_{2} & a_{3} & \cdot \\
\cdot & 0 & a_{0} & a_{1} & a_{2} & \cdot \\
\cdot & 0 & 0 & a_{0} & a_{1} & \cdot \\
\cdot & 0 & 0 & 0 & a_{0} & \cdot \\
. & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

The sequence a is said to be a Pólya frequency sequence if $\operatorname{Toep}[A]$ is totally positive (TP), meaning that every finite square submatrix has nonnegative determinant. The Aissen-Edrei-Schoenberg-Whitney Theorem [1, 2, 14, 15] classifies the power series $A(x) \in \mathbb{R}[[x]]$ for which
this happens. They are exactly the series of the form

$$
\begin{equation*}
A(x)=c x^{n} \exp (\gamma x) \frac{\prod_{i}\left(1+\alpha_{i} x\right)}{\prod_{j}\left(1-\beta_{j} x\right)} \tag{AESW}
\end{equation*}
$$

for $n \in \mathbb{N}$ and nonnegative real numbers $c, \alpha_{i}, \beta_{j}, \gamma$ such that $\sum_{i} \alpha_{i}$ and $\sum_{j} \beta_{j}$ are finite (we refer to them as AESW series, in this paper). They are also the series in $\mathbb{R}[[x]]$ that are limits (uniformly on compact sets) of polynomials with nonnegative coefficients and only real roots. We note that a polynomial is AESW if and only if it has nonnegative coefficients and only real roots and that a series $A(x) \in \mathbb{R}[[x]]$ with constant term $A(0) \neq 0$ is AESW if and only if $A(-x)^{-1}$ is AESW.
2.2. The Hurwitz stability criterion. The Hurwitz matrix of $A$ is also indexed by $\mathbb{Z} \times \mathbb{Z}$ and defined as $\operatorname{Hur}[A]=\left(a_{2 v-u}\right)$. Here is a small piece:

$$
\boldsymbol{\operatorname { H u r }}[A]=\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & a_{0} & a_{2} & a_{4} & a_{6} & \cdot \\
\cdot & 0 & a_{1} & a_{3} & a_{5} & \cdot \\
\cdot & 0 & a_{0} & a_{2} & a_{4} & \cdot \\
\cdot & 0 & 0 & a_{1} & a_{3} & \cdot \\
\cdot & 0 & 0 & a_{0} & a_{2} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

A polynomial $R(x) \in \mathbb{R}[x]$ is called Hurwitz (or Hurwitz stable) if it has nonnegative coefficients and every root of $R(x)$ has nonpositive real part. The Hurwitz stability criterion [20] (see also [3, 23]) asserts that this happens if and only if $\operatorname{Hur}[R]$ is TP.

More generally, we say that the power series $C(x) \in \mathbb{R}[[x]]$ is Hurwitz precisely when $\operatorname{Hur}[C]$ is TP. Notice that the Hurwitz matrix $\mathbf{H u r}[C]$ consists of two Toeplitz matrices which interleave each other, namely $\operatorname{Toep}[A]$ and $\operatorname{Toep}[B]$, where $C(x)=B\left(x^{2}\right)+x A\left(x^{2}\right)$ (thus, $A(x)$ and $B(x)$ are the odd and even parts of $C(x)$, respectively). As a consequence, if a power series $C(x)=B\left(x^{2}\right)+x A\left(x^{2}\right) \in \mathbb{R}[[x]]$ is Hurwitz, then $A(x)$ and $B(x)$ are both AESW. In particular, if a polynomial $R(x)=Q\left(x^{2}\right)+x P\left(x^{2}\right) \in \mathbb{R}[x]$ is Hurwitz, then both $P(x)$ and $Q(x)$ have nonnegative coefficients and only real roots.
2.3. The Hermite-Biehler Theorem. Recall from the introduction that a polynomial $P(x) \in$ $\mathbb{R}[x]$ is said to interlace a polynomial $Q(x) \in \mathbb{R}[x]$, written $P(x) \prec Q(x)$, if both $P(x)$ and $Q(x)$ are AESW and the roots of $P(x)$ interlace those of $Q(x)$. The Hermite-Biehler Theorem [7, 19] (see also [17]) implies that a polynomial $R(x)=Q\left(x^{2}\right)+x P\left(x^{2}\right) \in \mathbb{R}[x]$ is Hurwitz if and only if $P(x) \prec Q(x)$. The same theorem implies (see [26, Theorem 6.3.4] [31, p. 57]) that for AESW polynomials $P(x)$ and $Q(x)$, this property is equivalent to the complex polynomial $Q(z)+\mathrm{i} P(z)$ being (univariate) stable, meaning that all its roots have nonpositive imaginary part.

We may again generalize the concept of interlacing from polynomials to series as follows. Given power series $A(x), B(x) \in \mathbb{R}[[x]]$ with $A(x)=\sum_{n \geq 0} a_{n} x^{n}$ and $B(x)=\sum_{n \geq 0} b_{n} x^{n}$, we
introduce the interlacing matrix

$$
\text { Lace }\binom{A(x)}{B(x)}:=\operatorname{Hur}\left[B\left(x^{2}\right)+x A\left(x^{2}\right)\right]=\left(\begin{array}{cccccc}
. & . & . & . & . & . \\
. & b_{0} & b_{1} & b_{2} & b_{3} & \cdot \\
. & 0 & a_{0} & a_{1} & a_{2} & \cdot \\
. & 0 & b_{0} & b_{1} & b_{2} & \cdot \\
. & 0 & 0 & a_{0} & a_{1} & \cdot \\
. & 0 & 0 & b_{0} & b_{1} & \cdot \\
. & . & . & . & \cdot & \cdot
\end{array}\right)
$$

and say that $A(x)$ interlaces $B(x)$, denoted $A(x) \prec B(x)$, if Lace $(A(x) B(x))^{\top}$ is TP. This implies that both $A(x)$ and $B(x)$ are AESW. For polynomials, this agrees with the notion of interlacing we have already discussed.

## 3. Fully interlacing matrices

This section introduces the concept of a fully interlacing matrix of formal power series. We begin with the important special case of interlacing sequences. It will be convenient for us to think of sequences of formal power series as column vectors.
3.1. Fully interlacing sequences. Let $\mathcal{A}=\left(A_{0}(x) A_{1}(x) \cdots A_{p-1}(x)\right)^{\top}$ be a sequence of formal power series in $\mathbb{R}[[x]]$ (this indexing convention will be convenient for us later). We call $\mathcal{A}$ pairwise interlacing if $A_{i}(x) \prec A_{j}(x)$ for all $0 \leq i<j \leq p-1$.

Extending the definition of the interlacing matrix of two power series from Section 2.3, we set

$$
\text { Lace }\left(\begin{array}{l}
A(x) \\
B(x) \\
C(x)
\end{array}\right)=\left(\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_{0} & c_{1} & c_{2} & c_{3} & c_{4} & \cdot \\
\cdot & 0 & a_{0} & a_{1} & a_{2} & a_{3} & \cdot \\
. & 0 & b_{0} & b_{1} & b_{2} & b_{3} & \cdot \\
\cdot & 0 & c_{0} & c_{1} & c_{2} & c_{3} & \cdot \\
\cdot & 0 & 0 & a_{0} & a_{1} & a_{2} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

for $A(x)=\sum_{n \geq 0} a_{n} x^{n}, B(x)=\sum_{n \geq 0} b_{n} x^{n}, C(x)=\sum_{n \geq 0} c_{n} x^{n} \in \mathbb{R}[[x]]$ and similarly for $\operatorname{Lace}(\mathcal{A})$ for every sequence $\mathcal{A}=\left(A_{0}(x) A_{1}(x) \cdots A_{p-1}(x)\right)^{\top}$ of power series in $\mathbb{R}[[x]]$. We call $\mathcal{A}$ fully interlacing if the matrix $\operatorname{Lace}(\mathcal{A})$ is TP. Our next two statements follow directly from the definition.

Lemma 3.1 (Shift-invariance). Suppose that $\left(A_{0}(x) A_{1}(x) \cdots A_{p-1}(x)\right)^{\top}$ is a fully interlacing sequence of formal power series in $\mathbb{R}[[x]]$. Then, so is $\left(A_{1}(x) \cdots A_{p-1}(x) x A_{0}(x)\right)^{\top}$.

Proposition 3.2 (Heredity). Every subsequence of a fully interlacing sequence of formal power series in $\mathbb{R}[[x]]$ is fully interlacing. In particular, every fully interlacing sequence is pairwise interlacing.

The next statement is the primary means of producing AESW series, and especially realrooted polynomials, in many applications. The proof is postponed until Example 4.4 in the sequel, since it can be deduced from more general results.

Proposition 3.3 (Convexity). If $\mathcal{A}=\left(A_{0}(x) A_{1}(x) \cdots A_{p-1}(x)\right)^{\top}$ is a fully interlacing sequence of formal power series in $\mathbb{R}[[x]]$, then

$$
A_{0}(x) \prec \sum_{i=0}^{p-1} \lambda_{i} A_{i}(x) \prec A_{p-1}(x)
$$

for all nonnegative real numbers $\lambda_{0}, \lambda_{1}, \ldots, \lambda_{p-1}$. In particular, every series in the cone $\mathbb{R}_{+} \mathcal{A}$ is AESW.

Example 3.4. Pairwise interlacing sequences which are not fully interlacing can be constructed as follows. Let $a, b, c, d, t$ be positive real numbers such that $a \leq c$ and $b \leq d$ and consider the polynomials $P(x)=t+x, Q(x)=(b+x)(d+x)$ and $R(x)=(a+x)(c+x)$. Then, the sequence $(P(x) Q(x) R(x))^{\top}$ is pairwise interlacing if and only if $a \leq b \leq t \leq c \leq d$. The interlacing matrix Lace $(P(x) Q(x) R(x))^{\top}$ has

$$
\operatorname{det}\left(\begin{array}{ccc}
t & 1 & 0 \\
b d & b+d & 1 \\
a c & a+c & 1
\end{array}\right)=t(b+d-a-c)-(b d-a c)
$$

as a minor, hence $(P(x) Q(x) R(x))^{\top}$ is not fully interlacing for $t<(b d-a c) /(b+d-a-c)$. When $(a, b, c, d)=(1,2,3,4)$, any value $2 \leq t<5 / 2$ provides such an example.
3.2. Fully interlacing grids. Let $\mathcal{A}$ be a $p \times q$ matrix of power series. Then, Lace $[\mathcal{A}]$ is obtained by interleaving the Toeplitz matrices of the series in $\mathcal{A}$ in a natural way, for instance:

$$
\text { Lace }\left(\begin{array}{ll}
A(x) & C(x) \\
B(x) & D(x)
\end{array}\right)=\left(\begin{array}{ccccccc}
. & . & . & . & . & . & \cdot \\
\cdot & a_{0} & c_{0} & a_{1} & c_{1} & a_{2} & \cdot \\
\cdot & b_{0} & d_{0} & b_{1} & d_{1} & b_{2} & \cdot \\
. & 0 & 0 & a_{0} & c_{0} & a_{1} & \cdot \\
. & 0 & 0 & b_{0} & d_{0} & b_{1} & \cdot \\
. & 0 & 0 & 0 & 0 & a_{0} & \cdot \\
. & \cdot & . & . & . & . & .
\end{array}\right)
$$

Formally, we have the following definition.
Definition 3.5. Let $\mathcal{A}=\left(A_{i j}(x)\right)$ be a $p \times q$ matrix of power series

$$
\begin{equation*}
A_{i j}(x)=\sum_{n \geq 0} a_{i j}(n) x^{n} \in \mathbb{R}[[x]], \tag{2}
\end{equation*}
$$

with entries indexed by $i \in\{0,1, \ldots, p-1\}$ and $j \in\{0,1, \ldots, q-1\}$. Let Lace $(\mathcal{A})=\left(M_{u v}\right)$ be the $\mathbb{Z} \times \mathbb{Z}$ matrix defined as follows. For $(u, v) \in \mathbb{Z} \times \mathbb{Z}$, let $\left(u^{\prime}, i\right)$ and $\left(v^{\prime}, j\right)$ be the unique pairs of integers such that $u=p u^{\prime}+i$ and $v=q v^{\prime}+j$ with $0 \leq i<p$ and $0 \leq j<q$. Then,

$$
M_{u v}=a_{i j}\left(v^{\prime}-u^{\prime}\right)=\left[x^{v^{\prime}-u^{\prime}}\right] A_{i j}(x)
$$

is the coefficient of $x^{v^{\prime}-u^{\prime}}$ in $A_{i j}(x)$. We call $\mathcal{A}$ fully interlacing if Lace $(\mathcal{A})$ is TP.

Remark 3.6. Let $\boldsymbol{\alpha}_{n}$ be the $p \times q$ matrix $\left(a_{i j}(n)\right)$ for every $n \in \mathbb{N}$, so that $\mathcal{A}=\sum_{n \geq 0} \boldsymbol{\alpha}_{n} x^{n}$. Then, $\operatorname{Lace}(\mathcal{A})$ has the block decomposition

$$
\operatorname{Lace}(\mathcal{A})=\left(\begin{array}{cccccc}
. & . & . & . & . & \cdot \\
. & \boldsymbol{\alpha}_{0} & \boldsymbol{\alpha}_{1} & \boldsymbol{\alpha}_{2} & \boldsymbol{\alpha}_{3} & \cdot \\
. & O & \boldsymbol{\alpha}_{0} & \boldsymbol{\alpha}_{1} & \boldsymbol{\alpha}_{2} & \cdot \\
. & O & O & \boldsymbol{\alpha}_{0} & \boldsymbol{\alpha}_{1} & \cdot \\
. & O & O & O & \boldsymbol{\alpha}_{0} & \cdot \\
. & . & . & . & . & .
\end{array}\right)
$$

in which every block is a $p \times q$ matrix. Clearly, this expression specializes to the definition of the Toeplitz matrix for $p=q=1$.

Example 3.7. Let $A(x), B(x) \in \mathbb{R}[[x]]$.
(a) A $p \times q$ matrix of constant power series is fully interlacing if and only if it is TP.
(b) By the AESW Theorem, the $1 \times 1$ matrix $(A(x))$ is fully interlacing if and only if $A(x)$ is AESW.

Similarly, by definition, the $2 \times 1$ matrix $(A(x) B(x))^{\top}$ is fully interlacing if and only if $A(x) \prec$ $B(x)$. By the Hermite-Biehler Theorem, this statement holds for polynomials in $A(x), B(x) \in$ $\mathbb{R}[x]$ and the classical notion of interlacing.

Definition 3.5 agrees with our earlier notion of fully interlacing sequence when $\mathcal{A}$ is a column matrix and directly implies the following statement.
Proposition 3.8 (Heredity). Every submatrix of a fully interlacing matrix of formal power series in $\mathbb{R}[[x]]$ is fully interlacing.
Remark 3.9. The $2 \times 2$ case of Definition 3.5 corresponds to the "interpolating squares" of [30]. Although it is not clear how these concepts are related, the corresponding "interpolating hypercubes" point towards a concept of fully interlacing tensors of higher rank.

## 4. Flips and products

This section and the following one include the main results of this paper.
Given a $p \times q$ matrix $\mathcal{A}=\left(A_{i j}(x)\right)$, we denote by $\mathcal{A}^{\perp}$ the $q \times p$ matrix obtained by reflecting $\mathcal{A}$ across its reverse diagonal. With indices $0 \leq i<p$ and $0 \leq j<q$, this is the matrix $\mathcal{A}^{\perp}=\left(A_{i j}^{\perp}(x)\right)$ defined by setting $A_{j i}^{\perp}(x)=\left(A_{p-1-i, q-1-j}(x)\right)$; we call it the flip of $\mathcal{A}$. The flip of a $\mathbb{Z} \times \mathbb{Z}$ matrix $\mathbf{M}=\left(M_{u v}\right)$ is defined as the $\mathbb{Z} \times \mathbb{Z}$ matrix $\mathbf{M}^{\perp}=\left(M_{u v}^{\perp}\right)$ such that $M_{v u}^{\perp}=M_{-1-u,-1-v}$ for every $(u, v) \in \mathbb{Z} \times \mathbb{Z}$.

As already discussed, columns of fully interlacing matrices are fully interlacing sequences, in the sense of Section 3.1, when read from top to bottom. The following result implies the same for rows, when read from right to left.
Theorem 4.1. Let $\mathcal{A}$ be a $p \times q$ matrix of formal power series.
(a) $\operatorname{Lace}\left(\mathcal{A}^{\perp}\right)=\operatorname{Lace}(\mathcal{A})^{\perp}$.
(b) $\mathcal{A}^{\perp}$ is fully interlacing if and only if so is $\mathcal{A}$.

Proof. We convert between indices with the following notation: $u, v \in \mathbb{Z}$ and

$$
\begin{aligned}
& u=p u^{\prime}+i \text { with } 0 \leq i<p \text { and } u^{\prime} \in \mathbb{Z}, \\
& v=q v^{\prime}+j \text { with } 0 \leq j<q \text { and } v^{\prime} \in \mathbb{Z} .
\end{aligned}
$$

Note that

$$
\begin{aligned}
& -1-u=p\left(-1-u^{\prime}\right)+(p-1-i) \text { and } \\
& -1-v=q\left(-1-v^{\prime}\right)+(q-1-j)
\end{aligned}
$$

so that $(-1-u)^{\prime}=-1-u^{\prime}$ and $(-1-v)^{\prime}=-1-v^{\prime}$ (modulo $p$ and $q$, respectively).
Let $\mathcal{A}=\left(A_{i j}(x)\right)$, so that $\mathcal{A}^{\perp}=\left(A_{j i}^{\perp}(x)\right)=\left(A_{p-1-i, q-1-j}(x)\right)$. For $u, v \in \mathbb{Z}$ the $(v, u)$-entry of Lace $(\mathcal{A})^{\perp}$ equals

$$
\begin{aligned}
\left(\mathbf{L a c e}(\mathcal{A})^{\perp}\right)_{v u} & =\mathbf{L a c e}(\mathcal{A})_{-1-u,-1-v}=\left[x^{(-1-v)^{\prime}-(-1-u)^{\prime}}\right] A_{p-1-i, q-1-j}(x) \\
& =\left[x^{u^{\prime}-v^{\prime}}\right] \mathcal{A}_{j i}^{\perp}(x)=\left(\mathbf{L a c e}\left(\mathcal{A}^{\perp}\right)\right)_{v u}
\end{aligned}
$$

This proves part (a). Part (b) follows from part (a) and the fact that $\operatorname{det}\left(L^{\perp}\right)=\operatorname{det}(L)$ for every finite square matrix $L$.

The following example shows that the notion of fully interlacing matrix does not behave as well with respect to transposing the matrix.

Example 4.2. For positive real numbers $a, b, c, d$ consider the interlacing matrix of linear forms

$$
\operatorname{Lace}(\mathcal{A})=\operatorname{Lace}\left(\begin{array}{ll}
a+x & c+x \\
b+x & d+x
\end{array}\right)=\left(\begin{array}{ccccccc}
. & \cdot & . & . & . & \cdot & \cdot \\
\cdot & a & c & 1 & 1 & 0 & \cdot \\
\cdot & b & d & 1 & 1 & 0 & \cdot \\
. & 0 & 0 & a & c & 1 & \cdot \\
\cdot & 0 & 0 & b & d & 1 & \cdot \\
\cdot & 0 & 0 & 0 & 0 & a & \cdot \\
. & . & . & . & . & . & .
\end{array}\right)
$$

For this to be TP, its $2 \times 2$ minors must be nonnegative, so $b \leq a \leq c, b \leq d \leq c$ and $b c \leq a d$. The only way for both $\mathcal{A}$ and $\mathcal{A}^{\top}$ to be interlacing is that $a=b=c=d$.

Theorem 4.3. Let $\mathcal{A}$ and $\mathcal{B}$ be $p \times t$ and $t \times q$ matrices of formal power series, respectively.
(a) $\operatorname{Lace}(\mathcal{A B})=\operatorname{Lace}(\mathcal{A}) \operatorname{Lace}(\mathcal{B})$.
(b) If $\mathcal{A}$ and $\mathcal{B}$ are fully interlacing, then so is $\mathcal{A B}$.

Proof. We let Lace $(\mathcal{A})=\mathbf{M}=\left(M_{u w}\right), \operatorname{Lace}(\mathcal{B})=\mathbf{N}=\left(N_{w v}\right), \mathcal{A}=\left(A_{i k}(x)\right), \mathcal{B}=\left(B_{k j}(x)\right)$ and $\mathcal{C}=\mathcal{A B}=\left(C_{i j}(x)\right)$. We convert between indices with the following notation: $u, v, w \in \mathbb{Z}$ and

$$
\begin{array}{r}
u=p u^{\prime}+i \text { with } 0 \leq i<p \text { and } u^{\prime} \in \mathbb{Z} \\
v=q v^{\prime}+j \text { with } 0 \leq j<q \text { and } v^{\prime} \in \mathbb{Z} \\
w=t w^{\prime}+k \text { with } 0 \leq k<t \text { and } w^{\prime} \in \mathbb{Z}
\end{array}
$$

Then, for $u, v \in \mathbb{Z}$, the $(u, v)$-entry of $\operatorname{Lace}(\mathcal{A B})$ equals

$$
\begin{aligned}
\operatorname{Lace}(\mathcal{A B})_{u v} & =\left[x^{v^{\prime}-u^{\prime}}\right] C_{i j}(x)=\left[x^{v^{\prime}-u^{\prime}}\right] \sum_{0 \leq k<t} A_{i k}(x) B_{k j}(x) \\
& =\sum_{0 \leq k<t} \sum_{w^{\prime} \in \mathbb{Z}}\left(\left[x^{w^{\prime}-u^{\prime}}\right] A_{i k}(x)\right)\left(\left[x^{v^{\prime}-w^{\prime}}\right] B_{k j}(x)\right) \\
& =\sum_{w \in \mathbb{Z}} M_{u w} N_{w v}=(\mathbf{M N})_{u v}
\end{aligned}
$$

This proves part (a). Part (b) follows from part (a) and the fact that products of TP matrices are TP.

Example 4.4. (a) The special case $t=q=1$ of Theorem 4.3 asserts that if $A(x) \in \mathbb{R}[[x]]$ is AESW and $\mathcal{A}$ is a fully interlacing sequence of formal power series in $\mathbb{R}[[x]]$, then so is the sequence obtained from $\mathcal{A}$ by multiplying every entry with $A(x)$.
(b) The special case $p=q=1$ of Theorem 4.3 asserts that if

$$
\begin{aligned}
& \left(A_{0}(x) A_{1}(x) \cdots A_{t-1}(x)\right)^{\top} \\
& \left(B_{0}(x) B_{1}(x) \cdots B_{t-1}(x)\right)^{\top}
\end{aligned}
$$

are fully interlacing sequences of power series in $\mathbb{R}[[x]]$, then

$$
\sum_{i=0}^{t-1} A_{t-1-i}(x) B_{i}(x)=A_{0}(x) B_{t-1}(x)+A_{1}(x) B_{t-2}(x)+\cdots+A_{t-1}(x) B_{0}(x)
$$

is AESW. This should be compared with [9, Lemma 7.8.3], which states that in the case that the $A_{i}(x)$ and $B_{i}(x)$ are polynomials with nonnegative coefficients, it suffices to assume that the two sequences are pairwise interlacing. The special case in which the $B_{i}(x)$ are constants is equivalent to the last statement in Proposition 3.3.
(c) Suppose that the $2 \times q$ matrix

$$
\left(\begin{array}{cccc}
A_{q-1}(x) & \cdots & A_{1}(x) & A_{0}(x) \\
B_{q-1}(x) & \cdots & B_{1}(x) & B_{0}(x)
\end{array}\right)
$$

of formal power series in $\mathbb{R}[[x]]$ is fully interlacing. According to Theorem 4.3, multiplying on the right by a column vector of nonnegative real numbers yields a $2 \times 1$ fully interlacing matrix. This means that

$$
\sum_{i=0}^{q-1} \lambda_{i} A_{i}(x) \prec \sum_{i=0}^{q-1} \lambda_{i} B_{i}(x)
$$

for all nonnegative real numbers $\lambda_{0}, \lambda_{1}, \ldots, \lambda_{q-1}$. This result should be compared with [16, Lemma 3.14], where the rows and columns of a $2 \times n$ matrix of polynomials are assumed to be pairwise interlacing and, under additional assumptions, the same conclusion as above is reached. More generally, by the same argument,

$$
\sum_{i=0}^{q-1} C_{i}(x) A_{q-1-i}(x) \prec \sum_{i=0}^{q-1} C_{i}(x) B_{q-1-i}(x)
$$

for every fully interlacing sequence $\left(C_{0}(x) C_{1}(x) \cdots C_{q-1}(x)\right)^{\top}$ of formal power series in $\mathbb{R}[[x]]$.
(d) By Example 3.7 and Theorem 4.3, the fully interlacing property of a matrix $\mathcal{A}$ of power series is preserved under multiplication with totally positive matrices (for instance, with Toeplitz matrices of AESW power series) of appropriate size. In the special case that $\mathcal{A}$ is a column vector with polynomial entries, [16, Theorem 3.7] states that full interlacing can be replaced by pairwise interlacing.

For example, multiplying a fully interlacing column matrix $\left(A_{0}(x) A_{1}(x) \cdots A_{p-1}(x)\right)^{\top}$ on the left with the TP matrix

$$
\left(\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
\lambda_{0} & \lambda_{1} & \cdots & \lambda_{p-1} \\
0 & 0 & \cdots & 1
\end{array}\right)
$$

shows that the triple $\left(A_{0}(x), \sum_{i=0}^{p-1} \lambda_{i} A_{i}(x), A_{p-1}(x)\right)$ is also fully interlacing and proves Proposition 3.3. Multiplying a fully interlacing matrix $\mathcal{A}$ of power series on the left with a row vector (respectively, on the right by a column vector) with all entries equal to one shows that the column sums of $\mathcal{A}$ read from right to left (respectively, the row sums of $\mathcal{A}$ read from top to bottom) are fully interlacing sequences. Similarly, multiplying $\mathcal{A}$ on the left (respectively, on the right) with an upper-triangular square matrix of appropriate size with all entries on or above the main diagonal equal to one yields a fully interlacing matrix whose rows (respectively, columns) are the partial sums of the rows (respectively, columns) of $\mathcal{A}$ read from bottom to top (respectively, from left to right).

Remark 4.5. According to Theorem4.3. multiplication on the left by a $p \times q$ fully interlacing matrix of formal power series in $\mathbb{R}[[x]]$ preserves the fully interlacing property of $q \times 1$ columnmatrices (sequences of length $q$ ) of formal power series. We do not know if the converse holds, namely whether every $p \times q$ matrix of formal power series which preserves the fully interlacing property of sequences of length $q$ must be fully interlacing.

## 5. VEronese sections

We recall that $\mathrm{S}_{k}=\mathrm{S}_{k}^{(r)}$ stands for the $k$ th Veronese $r$-section operator on formal power series. Given a $p \times q$ matrix $\mathcal{A}=\left(A_{i j}(x)\right)$ of formal power series and a positive integer $r$, we denote by $\mathrm{S}_{k}^{(r)} \mathcal{A}$ the $p \times q$ matrix $\left(\mathrm{S}_{k}^{(r)} A_{i j}(x)\right)$ which is obtained by applying $\mathrm{S}_{k}^{(r)}$ to every entry of $\mathcal{A}$ and set

$$
\mathcal{S}^{(r)} \mathcal{A}=\left(\mathrm{S}_{0}^{(r)} \mathcal{A} \mathrm{S}_{1}^{(r)} \mathcal{A} \ldots \mathrm{S}_{r-1}^{(r)} \mathcal{A}\right) \quad \text { and } \quad \mathcal{S}^{(r) \perp} \mathcal{A}=\left(\begin{array}{c}
\mathrm{S}_{r-1}^{(r)} \mathcal{A} \\
\vdots \\
\mathrm{S}_{1}^{(r)} \mathcal{A} \\
\mathrm{S}_{0}^{(r)} \mathcal{A}
\end{array}\right)
$$

Example 5.1. We have

$$
\begin{aligned}
\mathcal{S}^{(r)} A(x) & =\left(\mathrm{S}_{0} A(x) \mathrm{S}_{1} A(x)\right. \\
\mathcal{S}^{(r) \perp} A(x) & =\left(\mathrm{S}_{0} A(x) \mathrm{S}_{r-1} A(x)\right)
\end{aligned}
$$

for every $A(x) \in \mathbb{R}[[x]]$ and

$$
\begin{aligned}
\mathcal{S}^{(2)}(A(x) B(x)) & =\left(\mathrm{S}_{0} A(x) \mathrm{S}_{0} B(x) \mathrm{S}_{1} A(x) \mathrm{S}_{1} B(x)\right) \\
\mathcal{S}^{(2) \perp}(A(x) B(x)) & =\left(\begin{array}{ll}
\mathrm{S}_{1} A(x) & \mathrm{S}_{1} B(x) \\
\mathrm{S}_{0} A(x) & \mathrm{S}_{0} B(x)
\end{array}\right)
\end{aligned}
$$

for all $A(x), B(x) \in \mathbb{R}[[x]]$.
Remark 5.2. Given positive integers $r$ and $s$, let us use the shorthand $\mathrm{R}_{i}=\mathrm{S}_{i}^{(r)}$ and $\mathrm{S}_{j}=\mathrm{S}_{j}^{(s)}$.
(a) We have $\mathrm{R}_{i} \mathrm{~S}_{j} A(x)=\mathrm{S}_{j+s i}^{(r s)} A(x)$ for every $A(x) \in \mathbb{R}[[x]]$ and all $0 \leq i<r, 0 \leq j<s$. Indeed, if $A(x)=\sum_{n \geq 0} a_{n} x^{n}$, then

$$
\begin{aligned}
\mathrm{R}_{i} \mathrm{~S}_{j} A(x) & =\mathrm{R}_{i} \mathrm{~S}_{j} \sum_{n \geq 0} a_{n} x^{n}=\mathrm{R}_{i} \sum_{n \geq 0} a_{j+s n} x^{n}=\sum_{n \geq 0} a_{j+s(i+r n)} x^{n} \\
& =\sum_{n \geq 0} a_{j+s i+r s n} x^{n}=\mathrm{S}_{j+s i}^{(r s)} \sum_{n \geq 0} a_{n} x^{n}=\mathrm{S}_{j+s i}^{(r s)} A(x)
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
\mathcal{S}^{(r) \perp} \mathcal{S}^{(s)} \mathcal{A} & =\left(\begin{array}{cccc}
\mathrm{R}_{r-1} \mathrm{~S}_{0} \mathcal{A} & \mathrm{R}_{r-1} \mathrm{~S}_{1} \mathcal{A} & \cdots & \mathrm{R}_{r-1} \mathrm{~S}_{s-1} \mathcal{A} \\
\vdots & \vdots & \cdots & \vdots \\
\mathrm{R}_{1} \mathrm{~S}_{0} \mathcal{A} & \mathrm{R}_{1} \mathrm{~S}_{1} \mathcal{A} & \cdots & \mathrm{R}_{1} \mathrm{~S}_{s-1} \mathcal{A} \\
\mathrm{R}_{0} \mathrm{~S}_{0} \mathcal{A} & \mathrm{R}_{0} \mathrm{~S}_{1} \mathcal{A} & \cdots & \mathrm{R}_{0} \mathrm{~S}_{s-1} \mathcal{A}
\end{array}\right), \\
\mathcal{S}^{(s)} \mathcal{S}^{(r) \perp} \mathcal{A} & =\left(\begin{array}{cccc}
\mathrm{S}_{0} \mathrm{R}_{r-1} \mathcal{A} & \mathrm{~S}_{1} \mathrm{R}_{r-1} \mathcal{A} & \cdots & \mathrm{~S}_{s-1} \mathrm{R}_{r-1} \mathcal{A} \\
\vdots & \vdots & \cdots & \vdots \\
\mathrm{~S}_{0} \mathrm{R}_{1} \mathcal{A} & \mathrm{~S}_{1} \mathrm{R}_{1} \mathcal{A} & \cdots & \mathrm{~S}_{s-1} \mathrm{R}_{1} \mathcal{A} \\
\mathrm{~S}_{0} \mathrm{R}_{0} \mathcal{A} & \mathrm{~S}_{1} \mathrm{R}_{0} \mathcal{A} & \cdots & \mathrm{~S}_{s-1} \mathrm{R}_{0} \mathcal{A}
\end{array}\right),
\end{aligned}
$$

where $\mathrm{R}_{i} \mathrm{~S}_{j}=\mathrm{S}_{j+s i}^{(r s)}$ and $\mathrm{S}_{j} \mathrm{R}_{i}=\mathrm{S}_{i+r j}^{(r s)}$. Moreover,

$$
\mathcal{S}^{(r)} \mathcal{S}^{(s)} \mathcal{A}=\mathcal{S}^{(r s)} \mathcal{A}=\left(\mathrm{S}_{0}^{(r s)} \mathcal{A} \mathrm{S}_{1}^{(r s)} \mathcal{A} \cdots \mathrm{S}_{r s-1}^{(r s)} \mathcal{A}\right)
$$

is the row vector obtained from the first matrix by reading each row from left to right, starting from the bottom row and continuing to the top, or from the second matrix by reading each column from bottom to top, staring from the leftmost column and continuing to the right. In particular, the operators $\mathcal{S}^{(r)}$ and $\mathcal{S}^{(s)}$ commute.

Theorem 5.3. Let $\mathcal{A}=\left(A_{i j}(x)\right)$ be a $p \times q$ matrix of formal power series. If $\mathcal{A}$ is fully interlacing, then so are the matrices $\mathcal{S}^{(r)} \mathcal{A}$ and $\mathcal{S}^{(r) \perp} \mathcal{A}$ for every positive integer $r$.
Proof. Let us adopt the notation of Equation (21) for the $A_{i j}(x)$. We first consider $\mathcal{S}^{(r)} \mathcal{A}=$ $\left(\mathrm{S}_{0}^{(r)} \mathcal{A} \mathrm{S}_{1}^{(r)} \mathcal{A} \cdots \mathrm{S}_{r-1}^{(r)} \mathcal{A}\right)$, which is a $p \times r q$ matrix with entries

$$
\mathrm{S}_{k}^{(r)} A_{i j}(x)=\sum_{n \geq 0} a_{i j}(k+r n) x^{n}
$$

for $0 \leq k<r$ and $0 \leq i<p, 0 \leq j<q$. Given $(u, v) \in \mathbb{Z} \times \mathbb{Z}$, we set

$$
\begin{aligned}
u=p u^{\prime}+i & \text { with } 0 \leq i<p \text { and } u^{\prime} \in \mathbb{Z}, \\
v=r q v^{\prime}+j^{\prime} & \text { with } 0 \leq j^{\prime}<r q \text { and } v^{\prime} \in \mathbb{Z}, \\
j^{\prime}=k q+j & \text { with } 0 \leq k<r \text { and } 0 \leq j<q .
\end{aligned}
$$

Then, the $(u, v)$-entry of $\operatorname{Lace}\left(\mathcal{S}^{(r)} \mathcal{A}\right)$ is equal to

$$
\begin{aligned}
\left(\operatorname{Lace}\left(\mathcal{S}^{(r)} \mathcal{A}\right)\right)_{u v} & =\left[x^{v^{\prime}-u^{\prime}}\right]\left(\mathcal{S}^{(r)} \mathcal{A}\right)_{i j^{\prime}}(x)=\left[x^{v^{\prime}-u^{\prime}}\right] \mathrm{S}_{k}^{(r)} A_{i j}(x)=a_{i j}\left(k+r\left(v^{\prime}-u^{\prime}\right)\right) \\
& =\left[x^{k+r v^{\prime}-r u^{\prime}}\right] A_{i j}(x)
\end{aligned}
$$

Since $v=\left(k+r v^{\prime}\right) q+j$, we have shown that the the $(u, v)$-entry of $\operatorname{Lace}\left(\mathcal{S}^{(r)} \mathcal{A}\right)$ is equal to the $\left(r u^{\prime}, v\right)$-entry of $\operatorname{Lace}(\mathcal{A})$ for every $(u, v) \in \mathbb{Z} \times \mathbb{Z}$. This means that Lace $\left(\mathcal{S}^{(r)} \mathcal{A}\right)$ is the submatrix of Lace $(\mathcal{A})$ consisting of all rows with indices congruent to $0,1, \ldots, p-1$ modulo $r p$. A similar argument shows that $\mathcal{S}^{(r) \perp} \mathcal{A}$ is the submatrix of $\operatorname{Lace}(\mathcal{A})$ consisting of all columns with indices congruent to $0,1, \ldots, q-1$ modulo $r q$. In particular, $\operatorname{Lace}\left(\mathcal{S}^{(r)} \mathcal{A}\right)$ and Lace $\left(\mathcal{S}^{(r) \perp} \mathcal{A}\right)$ are TP as submatrices of a TP matrix.

Remark 5.4. All $r$-sections $\mathrm{S}_{k}^{(r)} A(x)$ of an AESW power series $A(x) \in \mathbb{R}[[x]]$ are AESW as well; see [1, Theorem 7]. Theorem [5.3 implies the stronger result that $\mathcal{S}^{(r)} A(x)$ is a fully interlacing sequence. The weaker result, stating that $\mathcal{S}^{(r)} A(x)$ is pairwise interlacing for every AESW polynomial $A(x) \in \mathbb{R}[x]$, appears as [16, Theorem 7.65].

Example 5.5. Let

$$
\begin{aligned}
Q_{n}(x) & =\mathrm{S}_{0}^{(2)}(1+x)^{n}
\end{aligned}=\sum_{k \geq 0}\binom{n}{2 k} x^{k}, ~=\sum_{k \geq 0}\binom{n}{2 k+1} x^{k} .
$$

for $n \in \mathbb{N}$. Since $(1+x)^{n} \prec(1+x)^{n+1}$, Theorem 5.3 implies that the matrices

$$
\left(\begin{array}{cc}
Q_{n}(x) & P_{n}(x) \\
Q_{n+1}(x) & P_{n+1}(x)
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{c}
P_{n}(x) \\
P_{n+1}(x) \\
Q_{n}(x) \\
Q_{n+1}(x)
\end{array}\right)
$$

are fully interlacing for every $n \in \mathbb{N}$.
As mentioned in the introduction, the question whether part (b) of the following statement is valid motivated our investigations.

Corollary 5.6. (a) If $A(x), B(x) \in \mathbb{R}[[x]]$ are formal power series and $A(x) \prec B(x)$, then the sequence

$$
\left(\mathrm{S}_{0}^{(r)} B(x) \mathrm{S}_{0}^{(r)} A(x) \mathrm{S}_{1}^{(r)} B(x) \mathrm{S}_{1}^{(r)} A(x) \cdots \mathrm{S}_{r-1}^{(r)} B(x) \mathrm{S}_{r-1}^{(r)} A(x)\right)^{\perp}
$$

is fully interlacing.
(b) If $P(x), Q(x) \in \mathbb{R}[x]$ are polynomials and $P(x) \prec Q(x)$, then the sequence

$$
\left(\mathrm{S}_{0}^{(r)} Q(x) \mathrm{S}_{0}^{(r)} P(x) \mathrm{S}_{1}^{(r)} Q(x) \mathrm{S}_{1}^{(r)} P(x) \cdots \mathrm{S}_{r-1}^{(r)} Q(x) \mathrm{S}_{r-1}^{(r)} P(x)\right)^{\perp}
$$

is pairwise interlacing. In particular, $\mathrm{S}_{k}^{(r)} P(x) \prec \mathrm{S}_{k}^{(r)} Q(x)$ for $0 \leq k<r$.
Proof. This follows by applying Theorem 5.3 to $\mathcal{S}^{(r) \perp} \mathcal{A}$ for $\mathcal{A}=(A(x) B(x))^{\top}$ and $\mathcal{A}=$ $(P(x) Q(x))^{\top}$, respectively.

Various results can be deduced by combining Theorem 5.3 and the results of Section 4. here is an example.

Corollary 5.7. If $A(x), B(x) \in \mathbb{R}[[x]]$ are formal power series and $A(x) \prec B(x)$, then

$$
\sum_{k=0}^{r-1} \lambda_{k} \mathrm{~S}_{k}^{(r)} A(x) \prec \sum_{k=0}^{r-1} \lambda_{k} \mathrm{~S}_{k}^{(r)} B(x)
$$

for all nonnegative real numbers $\lambda_{0}, \lambda_{1}, \ldots, \lambda_{r-1}$.
Proof. This follows by applying Example 4.4 (c) to the matrix $\mathcal{S}^{(r)} \mathcal{A}$ for $\mathcal{A}=(A(x) B(x))^{\top}$, which is fully interlacing by Theorem 5.3.

## 6. An ApPLICATION

This section describes a concrete application to the theory of uniform triangulations of simplicial complexes [4, 5]. We assume familiarity with basic notions about simplicial complexes, their triangulations and their face enumeration; such background can be found in [8, 29].

A triangulation $\Delta^{\prime}$ of an $(n-1)$-dimensional simplicial complex $\Delta$ is said to be uniform [4] if for all $0 \leq i \leq j \leq n$ and for every $(j-1)$-dimensional face $F \in \Delta$, the number of $(i-1)$ dimensional faces of the restriction of $\Delta^{\prime}$ to $F$ depends only on $i$ and $j$. We denote this number by $f_{i j}$ and say that the triangular array $\mathcal{F}=\left(f_{i j}\right)_{0 \leq i \leq j \leq n}$ is the $f$-triangle (of size $n$ ) associated to $\Delta^{\prime}$ and that $\Delta^{\prime}$ is an $\mathcal{F}$-uniform triangulation of $\Delta$. The $h$-polynomial of a simplicial complex $\Delta$, denoted $h(\Delta, x)$, provides a convenient way to encode its face numbers [29, Section II.2]. One of the main results of [4] (see [4, Theorem 4.1]) states that the $h$-polynomial of an $\mathcal{F}$-uniform triangulation $\Delta^{\prime}$ of a simplicial complex $\Delta$ depends only on $h(\Delta, x)$ and $\mathcal{F}$ and describes this dependence explicitly. We may thus denote $h\left(\Delta^{\prime}, x\right)$ by $h_{\mathcal{F}}(\Delta, x)$. The following two families of polynomials play an important role in the theory:

$$
\begin{aligned}
& h_{\mathcal{F}}\left(\sigma_{m}, x\right) \text {, for } m \in\{0,1, \ldots, n\} \text {, } \\
& \theta_{\mathcal{F}}\left(\sigma_{m}, x\right)=h_{\mathcal{F}}\left(\sigma_{m}, x\right)-h_{\mathcal{F}}\left(\partial \sigma_{m}, x\right) \text {, for } m \in\{0,1, \ldots, n\} \text {, }
\end{aligned}
$$

where $\sigma_{m}$ stands for the $(m-1)$-dimensional simplex and $\partial \sigma_{m}$ is its boundary complex. The polynomials $h_{\mathcal{F}}\left(\sigma_{m}, x\right)$ have nonnegative coefficients and, under some mild assumptions, so do the $\theta_{\mathcal{F}}\left(\sigma_{m}, x\right)$. Following [5, Section 3], we say that $\mathcal{F}$ has the strong interlacing property if
(i) $h_{\mathcal{F}}\left(\sigma_{m}, x\right)$ is real-rooted for all $2 \leq m<n$,
(ii) $\theta_{\mathcal{F}}\left(\sigma_{m}, x\right)$ is either identically zero, or a real-rooted polynomial of degree $m-1$ with nonnegative coefficients which is interlaced by $h_{\mathcal{F}}\left(\sigma_{m-1}, x\right)$, for all $2 \leq m \leq n$.
These conditions imply strong real-rootedness properties for the $h$-polynomials of $\mathcal{F}$-uniform triangulations of simplicial complexes and their symmetric decompositions [4, Section 6] [5, Section 4], such as the real-rootedness of $h_{\mathcal{F}}(\Delta, x)$ for every $(n-1)$-dimensional Cohen-Macaulay simplicial complex $\Delta$ [4, Theorem 6.1]. It is thus natural to ask [5, Question 7.1] which $f$-triangles of uniform triangulations have the strong interlacing property. Such $f$-triangles include those of barycentric subdivisions, $r$-fold edgewise subdivisions (for $r \geq n$ ) and $r$-colored barycentric subdivisions [4, Section 7] [5, Section 5]; see [4, Section 5] [24, Section 3] and the references given there for the relevant definitions and background.

Given an $f$-triangle $\mathcal{F}$ of size $n$, we denote by $\operatorname{esd}_{r}(\mathcal{F})$ the $f$-triangle of size $n$ associated to the $r$-fold edgewise subdivision of an $\mathcal{F}$-uniform triangulation of $\sigma_{n}$. The main result of this section is as follows.

Proposition 6.1. Let $\mathcal{F}$ be the $f$-triangle associated to a uniform triangulation of the simplex $\sigma_{n}$. If $\mathcal{F}$ has the strong interlacing property, then so does $\operatorname{esd}_{2}(\mathcal{F})$.
Proof. We assume that $\mathcal{F}$ has the strong interlacing property and write $\mathcal{G}:=\operatorname{esd}_{2}(\mathcal{F})$. We then have (see, for instance, [4, Equation (9)])

$$
h_{\mathcal{G}}(\Delta, x)=\mathrm{S}_{0}^{(2)}\left((1+x)^{n} h_{\mathcal{F}}(\Delta, x)\right)
$$

for every ( $n-1$ )-dimensional simplicial complex $\Delta$.
Since $\mathrm{S}_{k}^{(r)}$ preserves real-rootedness, $h_{\mathcal{G}}\left(\sigma_{m}, x\right)=\mathrm{S}_{0}^{(2)}\left((1+x)^{n} h_{\mathcal{F}}\left(\sigma_{m}, x\right)\right)$ is real-rooted for all $2 \leq m<n$. This verifies condition (i) for $\mathcal{G}$. To verify condition (ii), for $2 \leq m \leq n$ we compute that

$$
\begin{aligned}
\theta_{\mathcal{G}}\left(\sigma_{m}, x\right) & =h_{\mathcal{G}}\left(\sigma_{m}, x\right)-h_{\mathcal{G}}\left(\partial \sigma_{m}, x\right) \\
& =\mathrm{S}_{0}^{(2)}\left((1+x)^{m} h_{\mathcal{F}}\left(\sigma_{m}, x\right)\right)-\mathrm{S}_{0}^{(2)}\left((1+x)^{m-1} h_{\mathcal{F}}\left(\partial \sigma_{m}, x\right)\right) \\
& =\mathrm{S}_{0}^{(2)}\left((1+x)^{m-1}\left(h_{\mathcal{F}}\left(\sigma_{m}, x\right)-h_{\mathcal{F}}\left(\partial \sigma_{m}, x\right)\right)\right)+\mathrm{S}_{0}^{(2)}\left(x(1+x)^{m-1} h_{\mathcal{F}}\left(\sigma_{m}, x\right)\right) \\
& =\mathrm{S}_{0}^{(2)}\left((1+x)^{m-1} \theta_{\mathcal{F}}\left(\sigma_{m}, x\right)\right)+x \mathrm{~S}_{1}^{(2)}\left((1+x)^{m-1} h_{\mathcal{F}}\left(\sigma_{m}, x\right)\right)
\end{aligned}
$$

By assumption, $\theta_{\mathcal{F}}\left(\sigma_{m}, x\right)$ either is identically zero, or has nonnegative coefficients and degree $m-1$. In the former case, $h_{\mathcal{F}}\left(\sigma_{m}, x\right)=h_{\mathcal{F}}\left(\partial \sigma_{m}, x\right)$ and hence $h_{\mathcal{F}}\left(\sigma_{m}, x\right)$ also has nonnegative coefficients and degree $m-1$. Thus, the formula we have reached shows that $\theta_{\mathcal{G}}\left(\sigma_{m}, x\right)$ has nonnegative coefficients and degree $m-1$ as well.

Finally, $h_{\mathcal{F}}\left(\sigma_{m-1}, x\right)$ interlaces $\theta_{\mathcal{F}}\left(\sigma_{m}, x\right)$ and $h_{\mathcal{F}}\left(\sigma_{m}, x\right)$ by assumption and [4, Theorem 6.1]. As a result, $(1+x)^{m-1} h_{\mathcal{F}}\left(\sigma_{m-1}, x\right)$ interlaces both $(1+x)^{m-1} \theta_{\mathcal{F}}\left(\sigma_{m}, x\right)$ and $(1+x)^{m-1} h_{\mathcal{F}}\left(\sigma_{m}, x\right)$ and Corollary 5.6 implies that

$$
\begin{aligned}
\mathrm{S}_{1}^{(2)}\left((1+x)^{m-1} h_{\mathcal{F}}\left(\sigma_{m}, x\right)\right) & \prec \mathrm{S}_{0}^{(2)}\left((1+x)^{m-1} h_{\mathcal{F}}\left(\sigma_{m-1}, x\right)\right) \\
& =h_{\mathcal{G}}\left(\sigma_{m-1}, x\right) \\
& \prec \mathrm{S}_{0}^{(2)}\left((1+x)^{m-1} \theta_{\mathcal{F}}\left(\sigma_{m}, x\right)\right)
\end{aligned}
$$

Equivalently, $h_{\mathcal{G}}\left(\sigma_{m-1}, x\right)$ interlaces both summands of (3). Therefore, $h_{\mathcal{G}}\left(\sigma_{m-1}, x\right)$ interlaces their sum $\theta_{\mathcal{G}}\left(\sigma_{m}, x\right)$ and the proof follows.

Question 6.2. Does Proposition 6.1 continue to hold if $\operatorname{esd}_{2}(\mathcal{F})$ is replaced by $\operatorname{esd}_{r}(\mathcal{F})$ for any $r \geq 2$ ?

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