Asymptotic behavior of solutions of the nonlinear Beltrami equation with the Jacobian

Igor Petkov, Ruslan Salimov, Mariia Stefanchuk

Abstract. We investigate the asymptotic behavior at infinity of regular homeomorphic solutions of the nonlinear Beltrami equation with the Jacobian on the right-hand side. The sharpness of the above bounds is illustrated by several examples.

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1 Introduction

Nowadays various kinds of nonlinear counterparts of the classical Beltrami equation attarct attention of many mathematicians. There are several tools for studying the main features of such equations and asymptotic behaviers of their solutions. The so-called directional dilatations and isoperimetric inequality allow us to establish some crutial properties of the regular solutions; see, e.g. [1]–[14].

Let $\mathbb{C}$ be the complex plane. In the complex notation $f = u + iv$ and $z = x + iy$, the *Beltrami equation* in a domain $G \subset \mathbb{C}$ has the form

$$f_{\overline{z}} = \mu(z) f_z, \tag{1.1}$$

where $\mu\colon G \to \mathbb{C}$ is a measurable function and

$$f_{\overline{z}} = \frac{1}{2}(f_x + i f_y) \qquad \text{and} \qquad f_z = \frac{1}{2}(f_x - i f_y)$$

are formal derivatives of f in $\overline{z}$ and z, while f_x and f_y are partial derivatives of f in the variables x and y, respectively.

Various existence theorems for solutions of the Sobolev class have been recently established applying the modulus approach for a quite wide class of linear and quasilinear degenerate Beltrami equations; see, e.g. [2], [5]–[8], [15], [16].

Let $z_0 \in \mathbb{C}$ and $\mathcal{K}_{z_0}\colon G \to \mathbb{C}$ be a measurable function. We consider the following equation

$$f_{\overline{z}} - \frac{z-z_0}{\overline{z-z_0}}\, f_z = \mathcal{K}_{z_0}(z)|J_f(z)|^{1/2}\,, \tag{1.2}$$

where $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$ is a Jacobian of f.

Under $\mathcal{K}_{z_0}(z) \equiv 0$ equation (1.2) reduces to the standard linear Beltrami equation (1.1) with the complex coefficient $\mu(z) = \frac{z-z_0}{\overline{z-z_0}}$. In other cases, the equation (1.2) provides a partial case of the general nonlinear system of equations (7.33) given in [16, Sect. 7.7].

Applying the formal derivatives

$$rf_r \;=\; (z-z_0)f_z + (\overline{z-z_0})\, f_{\overline{z}}\,, \quad f_\theta \;=\; i((z-z_0)f_z - (\overline{z-z_0})\, f_{\overline{z}})\,, \tag{1.3}$$

see, e.g. [16, (21.51)], one can rewrite the equation (1.2) in the polar coordinates (r,θ) $(z = z_0 + re^{i\theta})$:

$$f_\theta = \sigma_{z_0}|J_f|^{1/2} \tag{1.4}$$

with

$$\sigma_{z_0} = \sigma_{z_0}(z) = -i\mathcal{K}_{z_0}(z)(\overline{z-z_0}) \tag{1.5}$$

and

$$J_f = J_f(z_0 + re^{i\theta}) = \frac{1}{r}\,\mathrm{Im}\,\left(\overline{f_r}\, f_\theta\right), \tag{1.6}$$

where f_θ and f_r are the partial derivatives of f by θ and r, respectively, see, e.g. [16, (21.52)].

Next, in the case $z_0 = 0$ we put $\mathcal{K}_{z_0}(z) = \mathcal{K}(z)$ and $\sigma_{z_0}(z) = \sigma(z)$.

Example 1.1. Let A, B, C are complex numbers and $|A| \neq |B|$. Note that the linear mapping

$$f(z) = A\overline{z} + Bz + C$$

is the solution of the equation (1.2) with

$$\mathcal{K}_{z_0}(z) = \frac{A(\overline{z-z_0}) - B(z-z_0)}{|\Delta|^{\frac{1}{2}}(\overline{z-z_0})},$$

where $\Delta = |B|^2 - |A|^2 \neq 0$.

Example 1.2. Let us consider the following area preserving quasiconformal mapping

$$f(z) = ze^{2i\ln|z|}$$

in $\mathbb{B}$. We see that

$$f_z = (1+i)e^{2i\ln|z|}, \quad f_{\overline{z}} = \frac{iz}{\overline{z}}\, e^{2i\ln|z|},$$

and therefore the straight forward computation shows that

$$J_f = |f_z|^2 - |f_{\bar z}|^2 = |1+i|^2 - 1 = 1$$

for $z \in \mathbb{B}$.

Thus, it is obvious that the mapping f is a solution of the equation

$$f_{\overline{z}} - \frac{z}{\overline{z}}\, f_z = \mathcal{K}(z)|J_f(z)|^{1/2}$$

with $\mathcal{K}(z) = -\frac{z}{\overline{z}}\, e^{2i \ln |z|}$.

Note that the equation (1.2) can be written in the form of a system of two real partial differential equations

$$\begin{cases} (y-y_0)u_x - (x-x_0)u_y = k_1|J_f|^{1/2}, \\ (y-y_0)v_x - (x-x_0)v_y = k_2|J_f|^{1/2}, \end{cases}$$

where $k_1 = -\mathrm{Im}((\overline{z-z_0})\,\mathcal{K}_{z_0}(z))$, $k_2 = \mathrm{Re}((\overline{z-z_0})\,\mathcal{K}_{z_0}(z))$, $z = x+iy$ and $z_0 = x_0 + iy_0$.

The nonlinear equation (1.2) provides partial cases of the nonlinear system of two real partial differential equations; see (1) in [17], [18], cf. [19]. Note that various nonlinear systems of partial differential equations studied in a quite large specter of aspects can be found in [9]–[13], [16]–[36].

2 Auxiliary Results

A mapping $f\colon G \to \mathbb{C}$ is called *regular at a point* $z_0 \in G$, if f has the total differential at this point and its Jacobian $J_f = |f_z|^2 - |f_{\bar{z}}|^2$ does not vanish, cf. [37, I. 1.6]. A homeomorphism f of Sobolev class $W^{1,1}_{\rm loc}$ is called *regular*, if $J_f > 0$ a.e. (almost everywhere). By a *regular homeomorphic solution of equation* (1.2) we call a regular homeomorphism $f\colon G \to \mathbb{C}$, which satisfies (1.2) a.e. in G.

Later on we use the following notations

$$B(z_0, r) = \{z \in \mathbb{C} : |z-z_0| < r\} \quad \text{and} \quad \gamma(z_0, r) = \{z \in \mathbb{C} : |z-z_0| = r\}.$$

Given a set $E \in \mathbb{C}$, $|E|$ denotes the two dimensional Lebesgue measure of a set E. We denote by $S_f(z_0, r) = |f(B(z_0, r))|$.

The mapping $f : G \to \mathbb{C}$ has the *N-property* (by Luzin), if the condition $|E| = 0$ implies that $|f(E)| = 0$.

Let G be a domain in $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be a regular homeomorphism of the Sobolev class $W^{1,1}_{\rm loc}$. Given a point $z_0 \in G$, the *angular dilatation* of f with respect to z_0 is the function

$$D_f(z, z_0) = \frac{|f_\theta(z_0 + re^{i\theta})|^2}{r^2 J_f(z_0 + re^{i\theta})}\,,$$

where $z = z_0 + re^{i\theta}$ and J_f is the Jacobian of f, see the equation (3.10) in [14].

For $D_f(z, z_0)$ denote

$$d_f(z_0, r) = \frac{1}{2\pi r} \int\limits_{\gamma(z_0, r)} D_f(z, z_0)\, |dz|\,. \tag{2.1}$$

Proposition 2.1. *Let $f\colon \mathbb{C}\to\mathbb{C}$ be a regular homeomorphism of the Sobolev class $W^{1,1}_{\rm loc}$ that possesses the N-property, $z_0\in\mathbb{C}$. Then*

$$S_f'(z_0,r)\geqslant 2\,\frac{S_f(z_0,r)}{r\,d_f(z_0,r)} \tag{2.2}$$

for a.a. (almost all) $r\in(0,+\infty)$.

Proof. Denote by $L_f(z_0,r)$ the length of curve $f(z_0+re^{i\theta})$, $0\leqslant\theta\leqslant 2\pi$. For a.a. $r\in(0,+\infty)$,

$$L_f(z_0,r)=\int\limits_0^{2\pi}|f_\theta(z_0+re^{i\theta})|\,d\theta=\int\limits_0^{2\pi}D_f^{\frac12}(z_0+re^{i\theta},z_0)J_f^{\frac12}(z_0+re^{i\theta})\,r\,d\theta\,,$$

and by Hölder's inequality,

$$L_f^2(z_0,r)\;\leqslant\;\int\limits_0^{2\pi}D_f(z_0+re^{i\theta},z_0)\,r\,d\theta\int\limits_0^{2\pi}J_f(z_0+re^{i\theta})\,r\,d\theta\,. \tag{2.3}$$

Due to Lusin's N-property and the Fubini theorem,

$$S_f(z_0,r)=\int\limits_{B(z_0,r)}J_f(z)\,dxdy=\int\limits_0^r\int\limits_0^{2\pi}J_f(z_0+\rho e^{i\theta})\rho\,d\theta\,d\rho\,,$$

hence, for a.a. $r\in(0,+\infty)$

$$S_f'(z_0,r)\;=\;\int\limits_0^{2\pi}J_f(z_0+re^{i\theta})\,r\,d\theta\,.$$

Estimating the last integral by (2.3) and using (2.1), one obtains

$$S_f'(z_0,r)\;\geqslant\;\frac{L_f^2(z_0,r)}{2\pi r\,d_f(z_0,r)} \tag{2.4}$$

for a.a. $r\in(0,+\infty)$. Combining (2.4) with the planar isoperimetric inequality

$$L_f^2(z_0,r)\geqslant 4\pi S_f(z_0,r)$$

implies the desired relation (2.2). $\square$

Proposition 2.2. *Let $f\colon\mathbb{C}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (1.2) which belongs to Sobolev class $W^{1,2}_{\rm loc}$. Then*

$$S_f'(z_0,r)\geqslant 2\,\frac{S_f(z_0,r)}{r\,\kappa(z_0,r)} \tag{2.5}$$

for a.a. $r\in(0,+\infty)$ and here

$$\kappa(z_0,r)=\frac{1}{2\pi r}\int\limits_{\gamma(z_0,r)}|\mathcal{K}_{z_0}(z)|^2\,|dz|.$$

Proof. We note that, according to Corollary B in [38], the homeomorphism $W^{1,2}_{\rm loc}$ possesses the N-property. Since f is a regular homeomorphic solution of equation (1.2), we get

$$D_f(z,z_0)=\frac{|f_\theta(z_0+re^{i\theta})|^2}{r^2J_f(z_0+re^{i\theta})}=\frac{|\sigma_{z_0}(z_0+re^{i\theta})|^2}{r^2}\,,$$

where $z=z_0+re^{i\theta}$.

Next, due to the relation (1.5), we obtain

$$D_f(z,z_0)=|\mathcal{K}_{z_0}(z)|^2$$

and

$$d_f(z_0,r)=\frac{1}{2\pi r}\int\limits_{\gamma(z_0,r)}D_f(z,z_0)\,|dz|=\frac{1}{2\pi r}\int\limits_{\gamma(z_0,r)}|\mathcal{K}_{z_0}(z)|^2\,|dz|\,.$$

Thus, applying Proposition 2.1, we obtain (2.5).

□

Lemma 2.1. *Let $f\colon\mathbb{C}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (1.2) which belongs to Sobolev class $W^{1,2}_{\rm loc}$. Then*

$$S_f(z_0,r_0)\leqslant S_f(z_0,R)\,\exp\left(-2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}\right)\tag{2.6}$$

for all $R>r_0>0$.

Proof. Let $0<r_0<R<\infty$. By Proposition 2.2, for a.a. $r\in(0,+\infty)$

$$\frac{S'_f(z_0,r)}{S_f(z_0,r)}\,dr\geqslant 2\,\frac{dr}{r\,\kappa(z_0,r)}$$

and integrating over the segment $[r_0,R]$, we obtain

$$\int\limits_{r_0}^{R}\frac{S'_f(z_0,r)}{S_f(z_0,r)}\,dr\geqslant 2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}.$$

Hence,

$$\int\limits_{r_0}^{R}(\ln S_f(z_0,r))'\,dr\geqslant 2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}.$$

Note that the function $\psi(r)=\ln S_f(z_0,r)$ is nondecreasing on $(0,+\infty)$, and

$$\int\limits_{r_0}^{R}(\ln S_f(z_0,r))'\,dr=\int\limits_{r_0}^{R}\psi'(r)dr\leqslant\psi(R)-\psi(r_0)=\ln\frac{S_f(z_0,R)}{S_f(z_0,r_0)},$$

see Theorem IV.7.4 in [39]. Combining the last two inequalities, we have

$$\ln\frac{S_f(z_0,R)}{S_f(z_0,r_0)}\geqslant 2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}.$$

Thus, we obtain that

$$S_f(z_0,r_0)\leqslant S_f(z_0,R)\,\exp\left(-2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}\right).$$

The lemma is proved. □

3 Asymptotic behavior at infinity of regular homeomorphic solutions

Here we study an asymptotic behavior at infinity of regular homeomorphic solutions of the equation (1.2).

Theorem 3.1. *Let $f\colon\mathbb{C}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (1.2) which belongs to Sobolev class $W^{1,2}_{\rm loc}$, $r_0>0$. Then*

$$\liminf_{R\to\infty} M_f(z_0,R)\exp\left(-\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}\right)\geqslant m_f(z_0,r_0)>0, \tag{3.1}$$

where

$$M_f(z_0,R)=\max_{|z-z_0|=R}|f(z)-f(z_0)|$$

and

$$m_f(z_0,r_0)=\min_{|z-z_0|=r_0}|f(z)-f(z_0)|.$$

Proof. By Lemma 2.1, we have

$$S_f(z_0,r_0)\leqslant S_f(z_0,R)\,\exp\left(-2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}\right)$$

for all $R>r_0>0$. Since f is homeomorphism, we obtain

$$\pi\, m_f^2(z_0,r_0)\leqslant S_f(z_0,r_0)\leqslant S_f(z_0,R)\,\exp\left(-2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}\right)\leqslant$$

$$\leqslant \pi\, M_f^2(z_0,R)\,\exp\left(-2\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}\right).$$

Hence,

$$m_f(z_0, r_0) \leqslant M_f(z_0, R)\, \exp\left(-\int\limits_{r_0}^{R} \frac{dr}{r\,\kappa(z_0, r)}\right).$$

Finally, passing to the lower limit as $R \to \infty$ in the last inequality, we obtain the relation (3.1). $\square$

Corollary 3.1. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a regular homeomorphic solution of the equation (1.2) which belongs to Sobolev class $W^{1,2}_{\rm loc}$ and $\alpha > 0$. If $\kappa(z_0, r) \leqslant \alpha$ for a.a. $r \geqslant 1$, then*

$$\liminf_{R\to\infty} \frac{M_f(z_0, R)}{R^{1/\alpha}} \geqslant m_f(z_0, 1) > 0,$$

where $m_f(z_0, 1) = \min\limits_{|z-z_0|=1} |f(z) - f(z_0)|$.

Example 3.1. Assume that $\alpha > 0$. Consider the equation

$$f_{\overline{z}} - \frac{z}{\overline{z}}\, f_z = -\alpha^{\frac{1}{2}}\, \frac{z}{\overline{z}}\, |J_f(z)|^{\frac{1}{2}} \tag{3.2}$$

in the complex plane $\mathbb{C}$. In the polar coordinates system, this equation takes the form

$$f_\theta = i\alpha^{\frac{1}{2}} r e^{i\theta} |J_f|^{\frac{1}{2}}.$$

It's easy to check that the mapping

$$f = \begin{cases} |z|^{\frac{1}{\alpha}-1} z, \ z \neq 0, \\ 0, \ z = 0 \end{cases}$$

is regular homeomorphism and belongs to Sobolev class $W^{1,2}_{\rm loc}(\mathbb{C})$. We show that f is a solution of the equation (3.2). We write this mapping in the polar coordinates $f(re^{i\theta}) = r^{\frac{1}{\alpha}} e^{i\theta}$. The partial derivatives of f by r and θ are

$$f_r = \frac{1}{\alpha}\, r^{\frac{1-\alpha}{\alpha}} e^{i\theta}, \quad f_\theta = i r^{\frac{1}{\alpha}} e^{i\theta}$$

and by (1.6) we have

$$J_f(re^{i\theta}) = \frac{1}{\alpha}\, r^{\frac{2(1-\alpha)}{\alpha}} > 0.$$

Next we find

$$\sigma = \frac{f_\theta}{J_f^{\frac{1}{2}}} = i\alpha^{\frac{1}{2}} r e^{i\theta}.$$

Consequently, $\sigma = i\alpha^{\frac{1}{2}} z$ and by the relation (1.5) we obtain

$$\mathcal{K}(z) = -\frac{\sigma(z)}{i\,\overline{z}} = -\alpha^{\frac{1}{2}}\, \frac{z}{\overline{z}}$$

and
$$\kappa(z_0,r)=\frac{1}{2\pi r}\int\limits_{\gamma(z_0,r)}|\mathcal{K}(z)|^2\,|dz|=\alpha,\quad z_0=0.$$
Obviously, $\kappa(z_0,r)$ satisfies condition of Corollary 3.1.

On the other hand, we have
$$M_f(z_0,R)=\max_{|z-z_0|=R}|f(z)-f(z_0)|=R^{\frac{1}{\alpha}}$$
and
$$m_f(z_0,1)=\min_{|z-z_0|=1}|f(z)-f(z_0)|=1.$$
It follows that
$$\lim_{R\to\infty}\frac{M_f(z_0,R)}{R^{\frac{1}{\alpha}}}=1.$$

Corollary 3.2. *If for some $\alpha>0$ the condition $|\mathcal{K}_{z_0}(z)|\leqslant\alpha$ holds for a.a. $z\in\{z\in\mathbb{C}\colon |z-z_0|\geqslant 1\}$, then*
$$\liminf_{R\to\infty}\frac{M_f(z_0,R)}{R^{\beta}}\geqslant m_f(z_0,1)>0,$$
where $\beta=1/\alpha^2$ and $m_f(z_0,1)=\min\limits_{|z-z_0|=1}|f(z)-f(z_0)|$.

Later on we denote
$$e_1=e,\ e_2=e^e,\ ...,\ e_{k+1}=e^{e_k}$$
and
$$\ln_1 t=\ln t,\ \ln_2 t=\ln\ln t,\ ...,\ \ln_{k+1}t=\ln\ln_k t,$$
where $k\geqslant 1$ are integer.

Theorem 3.2. *Let $f\colon\mathbb{C}\to\mathbb{C}$ be a regular homeomorphic solution of the equation (1.2) which belongs to Sobolev class $W^{1,2}_{\mathrm{loc}}$ and $\alpha>0$. If*
$$\kappa(z_0,r)\leqslant\alpha\prod_{k=1}^{N}\ln_k r$$
for a.a. $r\geqslant e_N$, then
$$\liminf_{R\to\infty}\frac{M_f(z_0,R)}{(\ln_N R)^{1/\alpha}}\geqslant m_f(z_0,e_N)>0,$$
where $m_f(z_0,e_N)=\min\limits_{|z-z_0|=e_N}|f(z)-f(z_0)|$.

Proof. Note that

$$\int\limits_{e_N}^{R} \frac{dr}{r\,\kappa(z_0,r)} \geqslant \frac{1}{\alpha}\int\limits_{e_N}^{R} \frac{dr}{r\prod\limits_{k=1}^{N}\ln_k r} = \frac{1}{\alpha}\int\limits_{1}^{\ln_N R} \frac{dt}{t} = \ln(\ln_N R)^{1/\alpha}.$$

Hence,

$$\exp\left(-\int\limits_{e_N}^{R} \frac{dr}{r\,\kappa(z_0,r)}\right) \leqslant \exp\Big(-\ln(\ln_N R)^{1/\alpha}\Big) = \frac{1}{(\ln_N R)^{1/\alpha}}.$$

Thus, by Theorem 3.1, we obtain

$$\liminf_{R\to\infty} \frac{M_f(z_0,R)}{(\ln_N R)^{1/\alpha}} \geqslant \liminf_{R\to\infty} \frac{M_f(z_0,R)}{\exp\left(\int\limits_{e_N}^{R} \frac{dr}{r\,\kappa(z_0,r)}\right)} \geqslant m_f(z_0,e_N).$$

□

Corollary 3.3. *If for some* $\alpha > 0$

$$|\mathcal{K}_{z_0}(z)| \leqslant \alpha\left(\prod_{k=1}^{N} \ln_k |z-z_0|\right)^{1/2}$$

for a.a. $z \in \{z \in \mathbb{C}\colon |z-z_0| \geqslant e_N\}$*, then*

$$\liminf_{R\to\infty} \frac{M_f(z_0,R)}{(\ln_N R)^{\beta}} \geqslant m_f(z_0,e_N) > 0,$$

where $\beta = 1/\alpha^2$ *and* $m_f(z_0,e_N) = \min\limits_{|z-z_0|=e_N} |f(z)-f(z_0)|$.

Letting $n = 1$ in Theorem 3.2 gives

Corollary 3.4. *If for some* $\alpha > 0$ *the condition*

$$\kappa(z_0,r) \leqslant \alpha \ln r$$

holds for a.a. $r \geqslant e$*, then*

$$\liminf_{R\to\infty} \frac{M_f(z_0,R)}{(\ln R)^{1/\alpha}} \geqslant m_f(z_0,e) > 0,$$

where $m_f(z_0,e) = \min\limits_{|z-z_0|=e} |f(z)-f(z_0)|$.

Corollary 3.5. *If for some $\alpha > 0$ the condition*

$$|\mathcal{K}_{z_0}(z)| \leqslant \alpha(\ln|z-z_0|)^{1/2}$$

holds for a.a. $z \in \{z \in \mathbb{C} \colon |z-z_0| \geqslant e\}$, *then*

$$\liminf_{R\to\infty} \frac{M_f(z_0,R)}{(\ln R)^{\beta}} \geqslant m_f(z_0,e) > 0,$$

where $\beta = 1/\alpha^2$ *and* $m_f(z_0,e) = \min\limits_{|z-z_0|=e} |f(z)-f(z_0)|$.

Example 3.2. Assume that $\alpha > 0$. Consider the equation

$$f_{\overline{z}} - \frac{z}{\overline{z}}\, f_z = \mathcal{K}(z)\, |J_f(z)|^{\frac{1}{2}}, \tag{3.3}$$

where

$$\mathcal{K}(z) = \begin{cases} -\alpha^{\frac{1}{2}}(\ln|z|)^{\frac{1}{2}}(\ln\ln|z|)^{\frac{1}{2}}\frac{z}{\overline{z}},\ |z| \geqslant e^e, \\ -\frac{z}{\overline{z}},\ |z| < e^e, \end{cases}$$

in the complex plane $\mathbb{C}$. In the polar coordinates system, this equation takes the form

$$f_\theta = \sigma |J_f|^{\frac{1}{2}},$$

where

$$\sigma(re^{i\theta}) = \begin{cases} i\alpha^{\frac{1}{2}}(\ln r)^{\frac{1}{2}}(\ln\ln r)^{\frac{1}{2}} re^{i\theta},\ r \geqslant e^e, \\ ire^{i\theta},\ 0 \leqslant r < e^e. \end{cases}$$

It's easy to check that the mapping

$$f = \begin{cases} (\ln\ln|z|)^{\frac{1}{\alpha}}\frac{z}{|z|},\ |z| \geqslant e^e, \\ e^{-e}z,\ |z| < e^e \end{cases}$$

is regular homeomorphism and belongs to Sobolev class $W^{1,2}_{\rm loc}(\mathbb{C})$. We show that f is a solution of the equation (3.3). We write this mapping in the polar coordinates

$$f(re^{i\theta}) = \begin{cases} (\ln\ln r)^{\frac{1}{\alpha}} e^{i\theta},\ r \geqslant e^e, \\ e^{-e}re^{i\theta},\ 0 \leqslant r < e^e. \end{cases}$$

The partial derivatives of f by r and θ are

$$f_r = \begin{cases} \alpha^{-1}(\ln\ln r)^{\frac{1-\alpha}{\alpha}}(\ln r)^{-1} r^{-1} e^{i\theta},\ r \geqslant e^e, \\ e^{-e}e^{i\theta},\ 0 \leqslant r < e^e \end{cases}$$

and

$$f_\theta = \begin{cases} i(\ln\ln r)^{\frac{1}{\alpha}} e^{i\theta},\ r \geqslant e^e, \\ ie^{-e}re^{i\theta},\ 0 \leqslant r < e^e. \end{cases}$$

Consequently by (1.6) we find

$$J_f(re^{i\theta}) = \begin{cases} \alpha^{-1}(\ln\ln r)^{\frac{2-\alpha}{\alpha}}(\ln r)^{-1}r^{-2}, \ r \geqslant e^e, \\ e^{-2e}, \ 0 \leqslant r < e^e \end{cases}$$

and

$$\sigma(re^{i\theta}) = \frac{f_\theta}{J_f^{\frac{1}{2}}} = \begin{cases} i\alpha^{\frac{1}{2}}(\ln\ln r)^{\frac{1}{2}}(\ln r)^{\frac{1}{2}}re^{i\theta}, \ r \geqslant e^e, \\ ire^{i\theta}, \ 0 \leqslant r < e^e. \end{cases}$$

Hence,

$$\sigma = \begin{cases} i\alpha^{\frac{1}{2}}(\ln\ln|z|)^{\frac{1}{2}}(\ln|z|)^{\frac{1}{2}}z, \ |z| \geqslant e^e, \\ iz, \ |z| < e^e. \end{cases}$$

Next due to the relation (1.5)

$$\mathcal{K}(z) = -\frac{\sigma(z)}{i\,\overline{z}} = \begin{cases} -\alpha^{\frac{1}{2}}(\ln\ln|z|)^{\frac{1}{2}}(\ln|z|)^{\frac{1}{2}}\frac{z}{\overline{z}}, \ |z| \geqslant e^e, \\ -\frac{z}{\overline{z}}, \ |z| < e^e \end{cases}$$

and for $z_0 = 0$ we have

$$\kappa(z_0, r) = \frac{1}{2\pi r}\int\limits_{\gamma(z_0,r)}|\mathcal{K}(z)|^2\,|dz| = \begin{cases} \alpha\ln r\cdot\ln\ln r, \ r \geqslant e^e, \\ 1, \ 0 < r < e^e. \end{cases}$$

Note that $\kappa(z_0, r)$ does not satisfy condition of Corollary 3.4, because

$$\lim_{r\to\infty}\frac{\kappa(z_0,r)}{\ln r} = \infty.$$

On the other hand, we have $M_f(z_0, R) = (\ln\ln R)^{\frac{1}{\alpha}}$. It follows that

$$\lim_{R\to\infty}\frac{M_f(z_0,R)}{(\ln R)^{\frac{1}{\alpha}}} = 0.$$

4 Non-existence theorems

In this section we find sufficient conditions under which the equation (1.2) has no homeomorphic regular solutions belonging to the Sobolev class $W^{1,2}_{\rm loc}$.

Theorem 4.1. *Let $\mathcal{K}_{z_0}\colon \mathbb{C}\to\mathbb{C}$ be a measurable function. Then there are no regular homeomorphic solutions $f\colon\mathbb{C}\to\mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\rm loc}$ with asymptotic condition*

$$\liminf_{R\to\infty} M_f(z_0,R)\exp\left(-\int\limits_{r_0}^{R}\frac{dr}{r\,\kappa(z_0,r)}\right) = 0 \qquad (4.1)$$

for some $r_0 > 0$, where $M_f(z_0,R) = \max\limits_{|z-z_0|=R}|f(z)-f(z_0)|$.

Proof. Suppose that there is a regular homeomorphic solution $f\colon \mathbb{C} \to \mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\mathrm{loc}}$ for which the condition (4.1) is satisfied. Then, by the Theorem 3.1, we get

$$m_f(z_0, r_0) = \min_{|z-z_0|=r_0} |f(z) - f(z_0)| = 0.$$

This contradicts the fact that the mapping f is homeomorphic. $\square$

Corollary 4.1. *Let $\mathcal{K}_{z_0}\colon \mathbb{C} \to \mathbb{C}$ be a measurable function and $\alpha > 0$. If $\kappa(z_0, r) \leqslant \alpha$ for a.a. $r \geqslant 1$, then there are no regular homeomorphic solutions $f\colon \mathbb{C} \to \mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\mathrm{loc}}$ with asymptotic condition*

$$\liminf_{R\to\infty} \frac{M_f(z_0, R)}{R^{1/\alpha}} = 0,$$

where $M_f(z_0, R) = \max\limits_{|z-z_0|=R} |f(z) - f(z_0)|$.

Corollary 4.2. *Let $\mathcal{K}_{z_0}\colon \mathbb{C} \to \mathbb{C}$ be a measurable function and $\alpha > 0$. If $|\mathcal{K}_{z_0}(z)| \leqslant \alpha$ for a.a. $z \in \{z \in \mathbb{C}\colon |z - z_0| \geqslant 1\}$, then there are no regular homeomorphic solutions $f\colon \mathbb{C} \to \mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\mathrm{loc}}$ with asymptotic condition*

$$\liminf_{R\to\infty} \frac{M_f(z_0, R)}{R^{\beta}} = 0,$$

where $\beta = 1/\alpha^2$.

Corollary 4.3. *Let $\mathcal{K}_{z_0}\colon \mathbb{C} \to \mathbb{C}$ be a measurable function and $\alpha > 0$. If*

$$\kappa(z_0, r) \leqslant \alpha \prod_{k=1}^{N} \ln_k r$$

for a.a. $r \geqslant e_N$, then there are no regular homeomorphic solutions $f\colon \mathbb{C} \to \mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\mathrm{loc}}$ with asymptotic condition

$$\liminf_{R\to\infty} \frac{M_f(z_0, R)}{(\ln_N R)^{1/\alpha}} = 0,$$

where $M_f(z_0, R) = \max\limits_{|z-z_0|=R} |f(z) - f(z_0)|$.

Corollary 4.4. *Let $\mathcal{K}_{z_0}\colon \mathbb{C} \to \mathbb{C}$ be a measurable function and $\alpha > 0$. If*

$$|\mathcal{K}_{z_0}(z)| \leqslant \alpha \left(\prod_{k=1}^{N} \ln_k |z - z_0| \right)^{1/2}$$

for a.a. $z \in \{z \in \mathbb{C}\colon |z - z_0| \geqslant e_N\}$, then there are no regular homeomorphic solutions $f\colon \mathbb{C} \to \mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\mathrm{loc}}$ with asymptotic condition

$$\liminf_{R\to\infty} \frac{M_f(z_0, R)}{(\ln_N R)^{\beta}} = 0,$$

where $\beta = 1/\alpha^2$.

Corollary 4.5. *Let $\mathcal{K}_{z_0} \colon \mathbb{C} \to \mathbb{C}$ be a measurable function and $\alpha > 0$. If*

$$\kappa(z_0, r) \leqslant \alpha \ln r$$

for a.a. $r \geqslant e$, then there are no regular homeomorphic solutions $f \colon \mathbb{C} \to \mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\rm loc}$ with asymptotic condition

$$\liminf_{R\to\infty} \frac{M_f(z_0, R)}{(\ln R)^{1/\alpha}} = 0,$$

where $M_f(z_0, R) = \max\limits_{|z-z_0|=R} |f(z) - f(z_0)|$.

Corollary 4.6. *Let $\mathcal{K}_{z_0} \colon \mathbb{C} \to \mathbb{C}$ be a measurable function and $\alpha > 0$. If*

$$|\mathcal{K}_{z_0}(z)| \leqslant \alpha(\ln |z - z_0|)^{1/2}$$

for a.a. $z \in \{z \in \mathbb{C} \colon |z - z_0| \geqslant e\}$, then there are no regular homeomorphic solutions $f \colon \mathbb{C} \to \mathbb{C}$ of the equation (1.2) from Sobolev class $W^{1,2}_{\rm loc}$ with asymptotic condition

$$\liminf_{R\to\infty} \frac{M_f(z_0, R)}{(\ln R)^{\beta}} = 0,$$

where $\beta = 1/\alpha^2$.

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CONTACT INFORMATION

I. Petkov
Admiral Makarov National University of Shipbuilding,
Ukraine, Mykolaiv, 9 Heroes of Ukraine Avenue, 54007,
igorpetkov83@gmail.com

R. Salimov
Institute of Mathematics of NAS of Ukraine,
3 Tereschenkivska St., Kiev-4, 01024, Ukraine,
ruslan.salimov1@gmail.com

M. Stefanchuk
Institute of Mathematics of NAS of Ukraine,
3 Tereschenkivska St., Kiev-4, 01024, Ukraine,
stefanmv43@gmail.com