Quantum Spacetimes from General Relativity?

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Abstract

We introduce a non-commutative product for curved spacetimes, that can be regarded as a generalization of the Rieffel (or Moyal-Weyl) product. This product employs the exponential map and a Poisson tensor, and the deformed product maintains associativity under the condition that the Poisson tensor Θ satisfies $\Theta^{\mu\nu}\nabla_\nu\Theta^{\rho\sigma} = 0$, in relation to a Levi-Cevita connection. We proceed to solve the associativity condition for various physical spacetimes, uncovering non-commutative structures with compelling properties.

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1 Introduction

Why should we quantize spacetime, i.e. what is the necessity to consider a spacetime with quantum features? While, several arguments exist, let us highlight two fundamental issues, for which a solution lies in the concept of a quantum spacetime.

First, there is a **physical limitation** of localization of spacetime points, which stem from the fusion of the quantum-mechanical uncertainty principle with the formation of black holes in general relativity. In particular, localization with extreme precision induces gravitational collapse, rendering spacetime below the Planck scale devoid of operational significance, see Refs.[Ahl94, DFR95].

Second, we mention a **continuity** or transitive argument. If the Einstein equations provide a connection between gravity and spacetime curvature, any theory of quantum gravity must entail the quantization of spacetime or the spacetime curvature.

A direct method to conceptualize quantized spacetime or non-commutative geometry involves replacing the algebra of smooth functions, denoted as $C^\infty(\mathcal{M})$, on a manifold $\mathcal{M}$ with a non-commutative product, see for example the review [Sza03]. This approach parallels the procedure employed in quantum mechanics, where the phase-space algebra is replaced with a non-commutative product, see [BFF$^+$77, BFF$^+$78a, BFF$^+$78b].

In addition to addressing the technical challenges inherent in this approach, it would be desirable to identify equations corresponding to specific spacetimes that yield solutions representing non-commutative structures. This would provide a concrete and systematic framework for understanding the emergence of non-commutative structures in the context of different spacetime geometries analogous to the metric as a dynamical object in general relativity or to the approach of non-commutative geometry via matrix models known as emergent gravity, see [Yan09, Ste07, GSW08].

In a recent work, we presented a deformation quantization framework, following the Rieffel approach, applicable to globally hyperbolic spacetimes possessing a specific Poisson structure, [Muc21]. Crucially, these Poisson structures must adhere to Fedosov type requirements to ensure associativity of the resulting deformed product. By applying this deformation scheme to quantum field theories and their associated states, we established that the deformed state in a non-commutative spacetime exhibits a singularity structure reminiscent of Minkowski spacetime, known as being Hadamard. In particular, we demonstrate that if the undeformed state satisfies the Hadamard condition, then the deformed state also possesses this property.

Within this manuscript, we introduce a definition of a Rieffel product for curved spacetimes, employing the exponential map. We demonstrate that this product satisfies the essential properties of a star product: it is unital, possesses a commutative and flat limit, and exhibits associativity under specific conditions on the Poisson tensor:

$$\Theta^{\mu\nu}\nabla_\nu\Theta^{\rho\sigma} = 0. \tag{1.1}$$

For a non-degenerate Poisson tensor this equation reduces to

$$\nabla_\nu\Theta^{\rho\sigma} = 0, \tag{1.2}$$

which is the Fedosov requirement in disguise. While for a non-degenerate Poisson tensor there are several examples (provided in Section 4), for various significant spacetimes e.g. Schwarzschild or Friedman-Robertson-Walker-Lemaitre, there are no solutions to the covariant constancy of the Poisson tensor given in Equation 1.2, with exception to the trivial solution $\Theta = 0$.

The degenerate case on the other hand offers a bigger variety of solutions and we discuss in this paper the physical implications of various non-commutative geometries that we obtain as solutions to this equation.

Through our analysis, we establish the viability of this Rieffel product as a fundamental tool in the exploration of non-commutative geometries arising in the context of curved spacetimes.

The paper is structured as follows. In Section 2 we define the generalized Rieffel product and prove that it defines an associative star product, if the Poisson tensor fulfills Equation (1.1). Section 3 delves into useful mathematical properties of the Poisson tensor. Sections 4 and 5 study solutions to the Poisson tensor for specific spacetimes (i.e. solutions to the Einstein equations).

2 Mathematical Preliminaries

We begin this section with a result regarding the existence of a unique maximal geodesic, [Lee18, Corollary 4.28].

Corollary 2.1. *Let $\mathcal{M}$ be a smooth manifold and let ∇ be a connection in its respective tangent bundle $T\mathcal{M}$*[1]*. For each $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, there is a unique maximal geodesic $\gamma : I \to \mathcal{M}$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$, defined on some open interval I containing 0.*

This result allows us to define the exponential map, [Lee18, Chapter 5]. The exponential map, 'propagates' the tangent vector v along a geodesic starting from p in the direction specified by v, and $\exp_p(v)$ gives the point reached after moving along this geodesic for a unit parameter length. Hence, the exponential map is the natural generalization of a translation in flat manifolds to the case of curved manifolds. This is our starting point for the generalization of the Rieffel product from flat to curved.

Definition 2.2. *Define a subset $\mathcal{E} \subseteq T\mathcal{M}$,* ***the domain of the exponential map****, by*

$$\mathcal{E} = \{v \in T\mathcal{M} \,|\, \gamma_v \textit{ is defined on an interval containing } [0,1]\},$$

and then define the exponential map $exp : \mathcal{E} \to \mathcal{M}$ by

$$exp(v) = \gamma_v(1),$$

where γ_v is the unique geodesic with initial conditions $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. For each $p \in \mathcal{M}$, the restricted exponential map at p, denoted by $\exp_p$, is the restriction of exp to the set $\mathcal{E}_p = \mathcal{E} \cap T_p\mathcal{M}$.

Definition 2.3. *We denote by $\mathcal{G}_{\mathcal{M}}$ the set of all transformations that are generated by the exponential map of a manifold $\mathcal{M}$. This set forms a group, see [Mü01].*

Equipped with the former definitions we define the generalized Rieffel product in analogy with [Muc21].

Definition 2.4. *Let the smooth action α of the group $\mathcal{G}_{\mathcal{M}}$ denote the geodesic map w.r.t the manifold $(\mathcal{M}, g)$, and let $\Theta \in \Gamma^\infty(\Lambda^2(T_x\mathcal{M}))$ be a Poisson bivector. Then, the formal generalized Rieffel product of two functions $f, g \in C_0^\infty(\mathcal{M})$ is defined as*

$$\begin{aligned}(f \star_\Theta g)\,(x) &\equiv \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, \alpha_{\Theta X}(f(x))\, \alpha_Y(g(x))\, \mathrm{e}^{-\mathrm{i}\,(X,Y)_x}\, \mathrm{d}^N X\, \mathrm{d}^N Y \\ &= \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, f(\exp_{(x)}(\Theta X))\, g(\exp_{(x)}(Y))\; \mathrm{e}^{-\mathrm{i}\,X\cdot Y}\, \mathrm{d}^N X\, \mathrm{d}^N Y\,,\end{aligned}$$

where $X, Y \in T_x\mathcal{M}$ and the integrations are w.r.t. the non-vanishing components (maximally $\dim\mathcal{M}$) and the scalar product $\cdot$ is w.r.t. the metric g_x at the point x. Moreover, the cut-off function $\chi \in C_0^\infty(\mathcal{M} \times \mathcal{M})$ is chosen such that condition $\chi(0,0) = 1$ is fulfilled.

The main difference between the generalized Rieffel product and the one introduced in [Muc21] is the absence of the embedding formalism.

This product satisfies the standard properties of a star product, see [Wal07, Defintion 6.1.1] and [Kon03].

Proposition 2.5. *The generalized Rieffel product fulfills*

- *Unital,*
$$1 \star_\Theta f = f \star_\Theta 1 = f$$

- *The commutative limit,*
$$\lim_{\Theta\to 0} (f \star_\Theta g)(x) = (f \cdot g)(x),$$

[1] A tangent bundle $T\mathcal{M}$ is the collection of all of the tangent spaces $T_p\mathcal{M}$ for all points p on a manifold $\mathcal{M}$.

- *The flat limit, i.e. in case that the manifold is the four-dimensional Minkowski spacetime and a constant Poisson bivector of maximal rank, the generalized Rieffel product turns to the standard Rieffel product.*

Proof. To prove unitality we use the identity element of the exponential map, i.e. $\exp_p(0) = p$, see Corollary 2.1.

$$
\begin{aligned}
(f \star_\Theta 1)(x) &\equiv \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, \alpha_{\Theta X}(f(x))\, \alpha_Y(1)\, \mathrm{e}^{-\mathrm{i}\,(X,Y)_x}\, \mathrm{d}^N X\, \mathrm{d}^N Y \\
&= \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, f(\exp_{(x)}(\Theta X))\;\; \mathrm{e}^{-\mathrm{i}\, X\cdot Y}\, \mathrm{d}^N X\, \mathrm{d}^N Y \\
&= f(\exp_{(x)}(0)) \\
&= f(x),
\end{aligned}
$$

The proof for $(1 \star_\Theta f)(x)$ is analogous.

Assuming we can interchange the limit and the integrals, which can be done under certain assumptions on the functions f, g see [Muc21], we obtain

$$
\begin{aligned}
\lim_{\Theta\to 0} (f \star_\Theta g)(x) &= \lim_{\Theta\to 0} \iint \alpha_{\Theta X}(f(x))\, \alpha_Y(g(x))\;\; \mathrm{e}^{-\mathrm{i}X\cdot Y}\, \mathrm{d}^N X\, \mathrm{d}^N Y \\
&= f(x) \iint g(\exp_{(x)}(Y))\; \mathrm{e}^{-\mathrm{i}X\cdot Y}\, \mathrm{d}^N X\, \mathrm{d}^N Y \\
&= f(x)\cdot g(x)
\end{aligned}
$$

where we used the continuity of f and the property $\exp_{(x)}(0) = x$ of the exponential map.

The exponential map for the Minkowski spacetime is simply a translation, i.e.

$$\exp_{(x)}(v) = x + v,$$

hence we have for the generalized product the Rieffel product,

$$
\begin{aligned}
(f \star_\Theta g)(x) &\equiv \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, \alpha_{\Theta X}(f(x))\, \alpha_Y(g(x))\, \mathrm{e}^{-\mathrm{i}\,(X,Y)_x}\, \mathrm{d}^4 X\, \mathrm{d}^4 Y \\
&= \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, f(\exp_{(x)}(\Theta X))\, g(\exp_{(x)}(Y))\;\; \mathrm{e}^{-\mathrm{i}\, X\cdot Y}\, \mathrm{d}^4 X\, \mathrm{d}^4 Y \\
&= \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, f(x + \Theta X)\, g(x + Y)\;\; \mathrm{e}^{-\mathrm{i}\, X\cdot Y}\, \mathrm{d}^4 X\, \mathrm{d}^4 Y\,.
\end{aligned}
$$

□

Next, we turn to the question of associativity.

Theorem 2.6. *Let the Poisson bivector $\Theta \in \Gamma^\infty(\Lambda^2(T_x\mathcal{M}))$, fulfill the following condition w.r.t. the Levi-Cevita connection*

$$\Theta^{\mu\nu}\nabla_\mu \Theta^{\beta\alpha} = 0. \tag{2.1}$$

Then, the generalized Rieffel product is associative up to second order in Θ, and is explicitly given by

$$(f \star_\Theta g)(x) = f(x)g(x) - i\Theta^{\mu\nu}\, \partial_\mu f(x)\, \partial_\nu g(x) - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu f(x)\, \nabla_\alpha\partial_\beta g(x) + \mathcal{O}(\Theta^3).$$

Proof. See Appendix A.1.

□

The former theorem can be as well obtained by a pull-back from the formula supplied in [Muc21], if one uses the exponential map.

Remark 2.7. *The deformed product obtained by the generalized Rieffel product is, up to second order, equivalent to the star product given by the following Drinfeld twist, [Dri88] in terms of a formal power series*

$$\mu_0\big(\exp(-i\Theta^{\mu\nu}\nabla_\mu \otimes \nabla_\nu)(f\otimes g)\big), \tag{2.2}$$

where $\mu_0 : C^\infty(\mathcal{M})\otimes C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, assuming the associativity condition given in Equation (2.1)*. It is easy to verify that for the flat spacetime and constant Θ, this star product becomes the well-known representation of the Moyal-Weyl product, see [ALV08].*

Proposition 2.8. *The non-commutative structure between the coordinates, expressed by the star commutator is given to all orders in the deformation matrix Θ by*

$$[x^\mu, x^\nu]_\Theta := x^\mu \star_\Theta x^\nu - x^\nu \star_\Theta x^\mu = -2i\Theta^{\mu\nu}. \tag{2.3}$$

Proof. Using the explicit star product we have

$$\begin{aligned} x^\mu \star_\Theta x^\nu &= x^\mu\, x^\nu - i\Theta^{\mu\nu} - \frac{1}{2}\Theta^{\kappa\alpha}\Theta^{\lambda\beta}\nabla_\kappa\partial_\lambda x^\mu\, \nabla_\alpha\partial_\beta x^\nu + \mathcal{O}(\Theta^3) \\ &= x^\mu\, x^\nu - i\Theta^{\mu\nu} - \frac{1}{2}\Theta^{\kappa\alpha}\Theta^{\lambda\beta}\, \underbrace{\nabla_\kappa\delta^\mu_\lambda}_{=0}\, \underbrace{\nabla_\alpha\delta^\nu_\beta}_{=0} \end{aligned}$$

and due to the vanishing of the second order term it can be easily seen that all the higher order terms vanish as well. □

3 The Associativity Condition

In this section we investigate the mathematical aspects of the condition of associativity of the generalized star product, i.e.

$$\Theta^{\mu\nu}\nabla_\mu\Theta^{\beta\alpha} = 0. \tag{3.1}$$

First, we state a result that is essential in regards to this work, [Vai94, Proposition 1.5]

Proposition 3.1. *Let $(\mathcal{M}, \nabla)$ be a manifold endowed with a torsion-less linear connection. Then, the bivector Θ defines a Poisson structure on $\mathcal{M}$ if and only if*

$$\Theta^{\alpha\beta}\nabla_\alpha\Theta^{\mu\nu} + \Theta^{\alpha\mu}\nabla_\alpha\Theta^{\nu\beta} + \Theta^{\alpha\nu}\nabla_\alpha\Theta^{\beta\mu} = 0. \tag{3.2}$$

Next, we give a result that follows from the former proposition.

Proposition 3.2. *A bivector $\Theta \in \Gamma^\infty(\Lambda^2(T\mathcal{M}))$ fulfilling the associativity condition* (3.1) *, w.r.t. the Levi-Cevita connection, fulfills Equation* (3.2) *and the Jacobi identity [Vai94, Equation 1.5]*

$$\Theta^{\alpha\beta}\partial_\alpha\Theta^{\mu\nu} + \Theta^{\alpha\mu}\partial_\alpha\Theta^{\nu\beta} + \Theta^{\alpha\nu}\partial_\alpha\Theta^{\beta\mu} = 0,$$

and therefore defines a Poisson structure (or Poisson tensor, see [Wal07, Definition 4.1.7.]) on $\mathcal{M}$.

Proof. We begin the proof by stating that Equation (3.2) trivially follows from Condition (3.1). Inserting the Christoffel symbols we obtain

$$\begin{aligned} &\Theta^{\alpha\beta}\partial_\alpha\Theta^{\mu\nu} + \Theta^{\alpha\beta}\Theta^{\lambda\nu}\,\Gamma^\mu_{\alpha\lambda} + \Theta^{\alpha\beta}\Theta^{\mu\lambda}\,\Gamma^\nu_{\alpha\lambda} \\ &+\Theta^{\alpha\mu}\partial_\alpha\Theta^{\nu\beta} + \Theta^{\alpha\mu}\Theta^{\lambda\beta}\,\Gamma^\nu_{\alpha\lambda} + \Theta^{\alpha\mu}\Theta^{\nu\lambda}\,\Gamma^\beta_{\alpha\lambda} \\ &+\Theta^{\alpha\nu}\partial_\alpha\Theta^{\beta\mu} + \Theta^{\alpha\nu}\Theta^{\lambda\mu}\,\Gamma^\beta_{\alpha\lambda} + \Theta^{\alpha\nu}\Theta^{\beta\lambda}\,\Gamma^\mu_{\alpha\lambda} \\ &\Theta^{\alpha\beta}\partial_\alpha\Theta^{\mu\nu} + \Theta^{\alpha\mu}\partial_\alpha\Theta^{\nu\beta} + \Theta^{\alpha\nu}\partial_\alpha\Theta^{\beta\mu} = 0 \end{aligned}$$

where we used the skew-symmetry of the object Θ and the symmetry of the Christoffel symbols to cancel various terms, i.e.

$$\begin{aligned}
&\Theta^{\alpha\beta}\Theta^{\lambda\nu}\,\Gamma^{\mu}_{\alpha\lambda}+\Theta^{\alpha\nu}\Theta^{\beta\lambda}\,\Gamma^{\mu}_{\alpha\lambda}\\
=&\Theta^{\alpha\beta}\Theta^{\lambda\nu}\,\Gamma^{\mu}_{\alpha\lambda}+\Theta^{\lambda\nu}\Theta^{\beta\alpha}\,\Gamma^{\mu}_{\lambda\alpha}\\
=&\Theta^{\alpha\beta}\Theta^{\lambda\nu}\,\Gamma^{\mu}_{\alpha\lambda}-\Theta^{\lambda\nu}\Theta^{\alpha\beta}\,\Gamma^{\mu}_{\alpha\lambda}\\
=&0.
\end{aligned}$$

□

The former proposition serves to assure the tensor structure of Θ in case that the associativity condition is fulfilled.

Assuming non-degeneracy of the Poisson tensor the associativity condition reads

$$\nabla_{\mu}\Theta^{\beta\alpha}=0.$$

Such a connection is called a Poisson connection, see [BFF+78b] and [Vai94, Chapter 1.4]. It is further proven in [Vai94] that any Poisson manifold with constant rank Poisson bivector possesses Poisson connections. If the covariant derivative is constant, then the tensor has constant rank and is therefore very close to being "symplectic"[2]. In case of maximal rank of the Poisson tensor, the inverse of Θ is given by the symplectic structure ω. For the symplectic case, Fedosov, [Fed94] has proven that a deformation quantization exists if the symplectic structure satisfies [Fed94, Definition 2.3].

$$\nabla\omega=0.$$

Turning to other literature, let us mention that in [Bou03, Defintion 1.1.] a triple $(\mathcal{M},\Theta,g)$ with a Poisson tensor being covariant constant is called a (Pseudo-)Riemannian Poisson manifold, see [Bou03, Defintion 1.1.]. It is a generalization of Kähler manifolds. The covariant constancy of the Poisson tensor appears also in the definition of the Poisson-Riemannian manifold, [BM17, Equation 3.1]. See in connection of the covariant constancy of the Poisson tensor [Haw04, Haw07].

In the following sections we assume the case of non-degenerate and degenerate Θ hence where Equation (3.1) holds. We systematically derive the resulting Poisson structures on a case-by-case basis.

4 NC spacetimes - The Non-degenerate Cases

In the section we prove the existence of various Poisson structures fulfilling the condition of associativity of the generalized star product, i.e.

$$\Theta^{\mu\nu}\nabla_{\nu}\Theta^{\beta\alpha}=0, \tag{4.1}$$

w.r.t. the Levi-Cevita connection, assuming that Θ is non-degenerate.

4.1 The Flat Case

The metric of Minkowski spacetime $(\mathcal{M},\eta)$ in Cartesian coordinates $\{t,x,y,z\in\mathbb{R}\}$ is given by

$$\boxed{\eta=-dt^2+dx^2+dy^2+dz^2.} \tag{4.2}$$

For this case we evaluate in the following the Poisson tensor. This example is of importance due to the following reason: It solidifies the fact that the flat limit reduces to the Rieffel product. In particular, for the flat case, the non-degenerate Poisson structure is unique.

Result 4.1. *The non-degenerate Poisson structure, fulfilling Equation* (4.1)*, w.r.t. the Minkowski spacetime* $(\mathcal{M},\eta)$ *is unique and given by a constant Poisson structure.*

[2] I am indebted to Stefan Waldmann for this comment.

Proof. The proof is straightforward, since for a flat Levi-Cevita connection, Equation (4.1) turns to

$$\Theta^{\mu\nu}\partial_\nu\Theta^{\beta\alpha} = 0$$

together with the non-degeneracy we have

$$\partial_\mu\Theta^{\beta\alpha} = 0.$$

□

Note that in the non-degenerate case, the **only** solution is a constant Poisson tensor, and thus the flat limit from result 2.5 is satisfied by default. We choose a standard representation of Θ, with the only non-vanishing and constant components $\Theta^{01} = \kappa_e$ and $\Theta^{23} = \kappa_m$, see for example [GL07] or [BLS11].

4.1.1 The Flat Case in Spherical Coordinates

In spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0,\pi), \varphi \in [0, 2\pi)\}$, the Minkowski metric reads

$$\boxed{\eta = -dt^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).} \tag{4.3}$$

with the following Christoffel symbols, [MG09, Section 2.1.3.]

$$\Gamma^r_{\vartheta\vartheta} = -r, \qquad \Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{4.4a}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta. \tag{4.4b}$$

Either one solves the following differential equations assuming a non-degenerate $\tilde{\Theta}$ that is time-independent,

$$\partial_0\tilde{\Theta}^{\alpha\beta} = 0$$

and the remaining differential equations split in spatial and tempo-spatial part, i.e.

$$\begin{aligned}\partial_k\tilde{\Theta}^{0i} &= -\tilde{\Theta}^{0j}\Gamma^i_{kj}\\ \partial_k\tilde{\Theta}^{ij} &= -\tilde{\Theta}^{rj}\Gamma^i_{kr} + \tilde{\Theta}^{ri}\Gamma^j_{kr}.\end{aligned}$$

or one uses the tensor property of $\tilde{\Theta}$ as follows. Denote by J the Jacobian matrix of partial derivatives, i.e.

$$J^\rho{}_\sigma = \frac{\partial y^\rho}{\partial x^\sigma}$$

where x are the Cartesian coordinates and y represent the spherical coordinates.

Result 4.2. *The transformed Poisson tensor $\tilde{\Theta}$ is given in relation to the matrix Θ obtained in Result 4.1 by*

$$\tilde{\Theta}^{\rho\sigma} = J^\rho{}_\mu\Theta^{\mu\nu}J^\lambda{}_\nu, \tag{4.5}$$

which reads explicitly

$$\tilde{\Theta}^{\rho\sigma} = \begin{pmatrix} 0 & \kappa_e\cos(\varphi)\sin(\vartheta) & \kappa_e\frac{\cos(\vartheta)\cos(\varphi)}{r} & -\kappa_e\frac{\csc(\vartheta)\sin(\varphi)}{r} \\ -\kappa_e\cos(\varphi)\sin(\vartheta) & 0 & -\kappa_m\frac{\sin(\varphi)}{r} & -\kappa_m\frac{\cos(\varphi)\cot(\vartheta)}{r} \\ -\kappa_e\frac{\cos(\vartheta)\cos(\varphi)}{r} & \kappa_m\frac{\sin(\varphi)}{r} & 0 & \kappa_m\frac{\cos(\varphi)}{r^2} \\ \kappa_e\frac{\csc(\vartheta)\sin(\varphi)}{r} & \kappa_m\frac{\cos(\varphi)\cot(\vartheta)}{r} & -\kappa_m\frac{\cos(\varphi)}{r^2} & 0 \end{pmatrix}. \tag{4.6}$$

Proof. The proof is done via matrix multiplication. □

This example, though seemingly trivial in its use of tensor properties, is crucial for solidifying the understanding that the representation of the Poisson tensor, and consequently the resulting deformation of the product, responds appropriately to coordinate changes.

4.1.2 Rindler Wedges

A sub-region of the Minkowski spacetime defined by $|t| < x$, called the right Rindler wedge, can be considered as a static globally hyperbolic spacetime, such as Minkowksi, on its own right. It describes the spacetime geometry experienced by an accelerating observer in flat spacetime and is often used in quantum field theory as a simplified model to study the Hawking effect, see Fulling–Davies–Unruh effect [Unr76, Ful73, Dav75]. A convenient coordinate system for the right Rindler wedge is given by, [CHM08]

$$t = a^{-1}e^{a\xi}\sinh a\tau, \quad x = a^{-1}e^{a\xi}\cosh a\tau, \tag{4.7}$$

where a is a positive constant. Then, the metric takes the form

$$\boxed{g = -e^{2a\xi}(d\tau^2 - d\xi^2) + dy^2 + dz^2.} \tag{4.8}$$

Result 4.3. *Using the tensor property of Θ we obtain by Equation* (4.5) *for the components of the Poisson tensor $\tilde{\Theta}$ for the right Rindler wedge*

$$\tilde{\Theta}^{01} = \kappa_e\, e^{-2a\xi}, \qquad \tilde{\Theta}^{23} = \Theta^{23} = \kappa_m. \tag{4.9}$$

Analog considerations can be done for the left Rindler wedge ($|t| < -x$).

4.2 The Case $\mathbb{R}^2 \times \mathbb{S}^2$

Next, we search for a solution of Equation (4.1) for a spacetime that is not entirely flat. We choose the spacetime manifold ($\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^2, g = \eta \oplus g_{\mathbb{S}^2}$), where $g_{\mathbb{S}^2} = d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$ is the Euclidean metric of a unit two-sphere. Thus, the full metric reads

$$\boxed{g = -dt^2 + dx^2 + d\vartheta^2 + \sin^2(\vartheta)d\varphi^2} \tag{4.10}$$

with coordinates $\{t, x \in \mathbb{R}\}$ and where the angles ϑ and φ are restricted to the ranges $0 < \vartheta < \pi$, $0 \leq \varphi < 2\pi$.

Result 4.4. *The Poisson tensor w.r.t. the manifold ($\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^2, g = \eta \oplus g_{\mathbb{S}^2}$), fulfilling the associativity condition* (4.1)*, is given by the components $\Theta^{01} = \kappa_e$, and $\Theta^{23} = C_1 \csc(\vartheta)$, with $C_1 \in \mathbb{R}$, with all other components vanishing.*

Proof. The proof for the induced metric on $\mathbb{R}^2$ is equivalent to the proof in the former section. For the sphere we have the following Christoffel symbols

$$\Gamma^2_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -\sin(\vartheta)\cos(\vartheta) \end{pmatrix}, \qquad \Gamma^3_{ij} = \begin{pmatrix} 0 & \frac{\cos(\vartheta)}{\sin(\vartheta)} \\ \frac{\cos(\vartheta)}{\sin(\vartheta)} & 0 \end{pmatrix}$$

where $i, j = 2, 3$. The covariant derivatives read,

$$\begin{aligned} \nabla_i h &= \partial_i h + \Gamma^2_{ik}\Theta^{k3} + \Gamma^3_{ik}\Theta^{1k} \\ &= \partial_i h + \Gamma^2_{i1}h + \Gamma^3_{i3}h, \end{aligned}$$

where we defined the function $h := \Theta^{23}$. Setting the covariant derivative equal to zero, we have for the derivatives the following differential equations

$$\begin{aligned} -\partial_2 h &= \frac{\cos(\vartheta)}{\sin(\vartheta)}h \\ -\partial_3 h &= \Gamma^2_{32}h + \Gamma^3_{33}h = 0. \end{aligned}$$

The solution of these equations is given by $h = C_1 \csc(\vartheta)$. □

If we embed the two-sphere into a higher-dimensional Euclidean space, the derived Poisson tensor takes the following well-known form

$$\Theta^{AB} = \varepsilon^{ABC}X_C,$$

where the indices $A, B, C = 1, 2, 3$ and X is the coordinate of the embedding-point on the sphere satisfying

$$X_A X^A = 1.$$

This is known as the fuzzy sphere; see [GP93, GP95, GKP96], [Mad99, Chapter 7.3] and references therein.

4.3 A simplified Bianchi Universe Type 1

In this Section we consider the case of a manifold product of a two-dimensional de Sitter spacetime $(d\mathbb{S}^2, g_{d\mathbb{S}^2})$ and a two-dimensional Euclidean space $(\mathbb{R}^2, \delta)$, which is a special case of the Bianchi Universe Type 1, where the scale factors in y and z direction are set equal to one. The metric describing this spacetime is thus given by

$$\boxed{g = -dt^2 + a(t)^2 dx^2 + dy^2 + dz^2}$$

where $\{t, x, y, z \in \mathbb{R}\}$ and $a(t)^2$ is the scale factor.

Result 4.5. *The Poisson tensor w.r.t. the simplified Bianchi Universe Type* 1*, given by* $(\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^2, g = \eta \oplus g_{\mathbb{S}^2})$*, fulfilling the associativity condition* (4.1)*, is given by the components* $\Theta^{01} = C_1\, a(t)^{-1}$ *and* $\Theta^{23} = C_2$*, where* $C_1, C_2 \in \mathbb{R}$*, with all other components vanishing.*

Proof. Due to the simplicity of the metric, the only non-vanishing Christoffel symbols are

$$\Gamma^0_{11} = a\,\dot{a}, \qquad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}}{a}. \tag{4.11}$$

Setting all components of Θ equal to zero, except for Θ^{01} and Θ^{23}, the covariant constant condition on the Poisson tensor, i.e. $\nabla_\sigma \Theta^{\mu\nu} = 0$, renders the following differential equations

$$\partial_i \Theta^{\mu\nu} = 0, \tag{4.12}$$

$$\partial_0 \Theta^{01} = -\Gamma^1_{01} \Theta^{01} \tag{4.13}$$

$$\partial_0 \Theta^{23} = 0. \tag{4.14}$$

To which the solutions are $\Theta^{01} = C_1/a(t)$ and $\Theta^{23} = C_2$, with $C_1, C_2 \in \mathbb{R}$. □

5 NC spacetimes - The Degenerate Cases

Next, we turn to the case of a degenerate Poisson tensor. The necessity for this requirement stems from the fact that in the presence of a non-vanishing Riemann tensor, the condition of a covariant constant Poisson tensor becomes too restrictive. Specifically, in most four-dimensional spacetimes, this condition results in a vanishing Poisson tensor, which follows from[3]

$$[\nabla_\gamma, \nabla_\delta]\Theta^{\alpha\beta} = -R^{\alpha}{}_{\lambda\gamma\delta}\Theta^{\lambda\beta} - R^{\beta}{}_{\lambda\gamma\delta}\Theta^{\alpha\lambda} = 0. \tag{5.1}$$

In particular, it leads for most physically relevant spacetimes (e.g. Schwarzschild, cosmological spacetimes) in four dimensions to a vanishing Poisson tensor and hence denies the possibility of an associative deformed product using the geodesic map as given in Definition 2.4. Hence, we consider in this section the possibility of having a degenerate Poisson tensor. This refined approach will remove the previous restrictive limitation of covariant constancy and enable a wider array of solutions within the associativity condition, see Equation (2.1).

[3] I am thankful to M. Fröb for this comment.

5.1 Morris-Thorne Wormhole

In [MT88] the authors introduced a traversable wormhole spacetime $(\mathcal{M}, g_{WH})$, with the goal to be used for rapid interstellar travel, by the following metric,

$$\boxed{g_{WH} = -dt^2 + dl^2 + (b_0^2 + l^2)\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)} \tag{5.2}$$

where b_0 is the throat radius and l is the proper radial coordinate; and $\{t \in \mathbb{R}, l \in \mathbb{R}, \vartheta \in (0,\pi), \varphi \in [0, 2\pi)\}$, see Figure 1.

The Christoffel symbols for this spacetime are given by [MG09, Section 2.16]

$$\Gamma^2_{12} = \frac{l}{b_0^2 + l^2}, \qquad \Gamma^3_{13} = \frac{l}{b_0^2 + l^2}, \qquad \Gamma^1_{22} = -l, \tag{5.3a}$$

$$\Gamma^3_{23} = \cot\vartheta, \qquad \Gamma^1_{33} = -l\sin^2\vartheta, \qquad \Gamma^2_{33} = -\sin\vartheta\,\cos\vartheta. \tag{5.3b}$$

and the non-vanishing components of the Riemann and Ricci tensor are given by

$$R_{1212} = -\frac{b_0^2}{b_0^2 + l^2}, \quad R_{1313} = -\frac{b_0^2 \sin^2\vartheta}{b_0^2 + l^2}, \quad R_{2323} = b_0^2 \sin^2\vartheta \tag{5.4}$$

$$R_{11} = -2\frac{b_0^2}{\left(b_0^2 + l^2\right)^2}. \tag{5.5}$$

Result 5.1. *The associativity condition* (2.1) *has no solutions for the case of the simple wormhole spacetime* $(\mathcal{M}, g_{WH})$ *and non-degenerate Poisson tensor.*

Proof. Using Equation (5.1) we have for $\alpha = \gamma$,

$$R_{\lambda\delta}\Theta^{\lambda\beta} + R^{\beta}{}_{\lambda\alpha\delta}\Theta^{\alpha\lambda} = 0.$$

Choosing the indices $\beta = i$ and $\delta = k$ to be spatial we have

$$R_{lk}\Theta^{li} + R^{i}{}_{ljk}\Theta^{jl} = 0,$$

which for $k = 2$ gives us

$$R^{i}{}_{lj2}\Theta^{jl} = 0, \tag{5.6}$$

and setting $i = 1$ or $i = 3$ renders

$$R^{1}{}_{lj2}\Theta^{jl} = R^{1}{}_{212}\Theta^{21} = 0,$$
$$R^{1}{}_{lj3}\Theta^{jl} = R^{1}{}_{313}\Theta^{31} = 0$$

from which we have $\Theta^{21} = \Theta^{31} = 0$. Setting in Equation (5.6) the index $k = 3$ and choosing $i = 2$ renders $\Theta^{23} = 0$. Hence, $\Theta^{jk} = 0$ eliminates the possibility of a non-degenerate solution. □

We conclude from the former result that we have to search for solutions assuming a degenerate Poison tensor.

Result 5.2. *The Poisson tensor w.r.t. the wormhole spacetime* $(\mathcal{M}, g_{WH})$*, fulfilling the associativity condition* (2.1)*, is given by two-sets of solutions. The first, is given by*

$$\Theta^{02} = \frac{C_1}{b_0^2 + l^2}, \tag{5.7}$$

$$\Theta^{01} = \pm\frac{\sqrt{-C_1^2 + 2b_0^2 C_2 + 2l^2 C_2}}{\sqrt{b_0^2 + l^2}}, \tag{5.8}$$

with C_1, $C_2 \in \mathbb{R}$ *and all other components are equal to zero. The second solution is given by* $\Theta^{\mu\nu} = 0$ *except for the component* Θ^{12} *that has the following explicit form*

$$\Theta^{12}(l) = \frac{C_3}{\sqrt{b_0^2 + l^2}},$$

with $C_3 \in \mathbb{R}$.

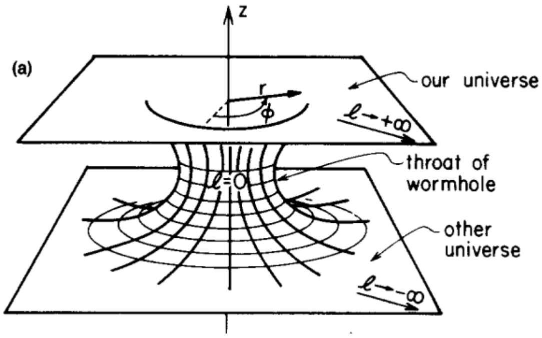

Figure 1: Embedding Diagram for a wormhole that connects two different figures, [MT88, Fig. 1.].

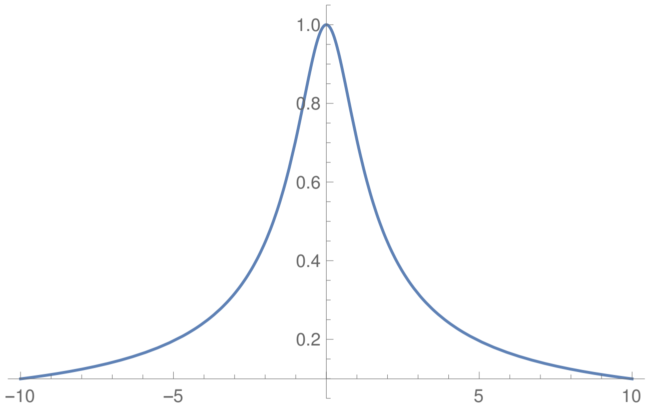

Figure 2: Plot of Function $\Theta^{12}(l)$

Proof. See Appendix A.2. □

For the plot of function $\Theta^{12}(l)$ (with scale by C_1/b_0 and $b_0 = 1$) see Figure 2. The solution states the following: the non-commutative scale vanishes for large l which is far away from the throat radius but becomes maximal at the throat of the wormhole (i.e. $l = 0$, see Figure 1). Assuming a quantum structure of the spacetime, this is a natural picture much in analogy with the Planck cell picture of the phase-space. In particular, the tighter the spatial part (namely the circle, where each point is a sphere) of the space-time is squeezed, the larger the non-commutative effect becomes. The quantum push-back from the commutator relations are larger, the smaller the radius b_0 of the worm hole is. This is consistent with the physical picture of such a quantum spacetime, i.e. the tighter the spacetime is confined the larger non-commutative effects will become. Since these effects act as a repulsive potential pushing the spacetime walls of the wormhole apart (the closer they are together). These quantum effects might prove fruitful when combating negative energies, of which this spacetime suffers, i.e. making rapid interstellar travel possible without negative energy densities. This, of course, has to be proven in the context of deformed Einstein Equations[4].

5.2 Spherically Symmetric, Static Spacetimes

For a large class of static, spherically symmetric spacetimes $(\mathcal{M}, g)$, the general form of the metric is given by

$$\boxed{g = -\exp\left(2\alpha(r)\right)dt^2 + \exp\left(-2\alpha(r)\right)dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right)} \tag{5.9}$$

with $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ and $\alpha(r)$ is a function of the radial component r. The non-vanishing Christoffel symbols for this class of spacetimes are given by

[4]See the last paragraph in Section 6.

$$\begin{aligned} &\Gamma^1_{00} = \exp\left(4\alpha(r)\right)\partial_1\alpha(r)\,, && \Gamma^0_{10} = \partial_1\alpha(r)\,, && \Gamma^1_{11} = -\partial_1\alpha(r)\,, \\ &\Gamma^1_{22} = -r\exp\left(2\alpha(r)\right)\,, && \Gamma^1_{33} = -r\exp\left(2\alpha(r)\right)\sin^2\vartheta\,, && \Gamma^2_{12} = \frac{1}{r}\,, \\ &\Gamma^2_{33} = -\sin\vartheta\cos\vartheta\,, && \Gamma^3_{13} = \frac{1}{r}\,, && \Gamma^3_{23} = \cot\vartheta\,. \end{aligned} \tag{5.10}$$

Result 5.3. *Excluding the case of constant α for the spherically symmetric, static spacetimes the associativity condition* (2.1) *has no solutions for the case of non-degenerate Poisson tensor.*

Proof. See Appendix A.3. □

Result 5.4. *The Poisson tensor w.r.t. class of static, spherically symmetric spacetimes $(\mathcal{M}, g)$ with the metric tensor of the form given in Equation* (5.9)*, fulfilling the associativity condition* (2.1)*, is given for general α by two-sets of solutions. The first, is given by*

$$\Theta^{01} = C_1, \tag{5.11}$$

with C_1 a real constant and all other components equal to zero. The second solution is given by $\Theta^{\mu\nu} = 0$ except for the component $f := \Theta^{12}$ that is the solution of the following differential equation,

$$\partial_1 f = \left(\partial_1\alpha(r) - \frac{1}{r}\right) f. \tag{5.12}$$

namely

$$f(r) = \frac{C_2}{r}\exp(\alpha(r)) \tag{5.13}$$

$$= \frac{C_2}{r}\sqrt{|g_{00}|}, \tag{5.14}$$

with $C_2 \in \mathbb{R}$.

Corollary 5.5. *In case the function $\alpha(r)$ has the following explicit form*

$$\exp(-2\alpha(r)) = 1 - C_3 e^{z_1} r^2,$$

where $\{z_1 \in \mathbb{C} : e^{z_1} \in \mathbb{R}\}$ and C_3 is a constant of spatial dimension -2, as for (Anti-)de Sitter in static coordinates, we have the following set of solutions

$$\Theta^{13}(r) = \exp(\alpha(r))\frac{1}{r}, \qquad \Theta^{23}(r,\vartheta) = \exp(-\alpha(r))\frac{\cot\vartheta}{r^2}$$

with all other components vanishing.

Proof. See Appendix A.4. □

In the following subsections, we investigate the explicit non-commutative structures for spacetimes that are static and spherical symmetric and where the metric takes the form given in Equation (5.9).

5.2.1 Schwarzschild

The Schwarzschild spacetime $(\mathcal{M}_{BH}, g_{BH})$ is a solution to the Einstein field equations of general relativity that describes the spacetime geometry outside a spherically symmetric, non-rotating mass M. The metric has the following form,

$$\boxed{g_{BH} = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right).} \tag{5.15}$$

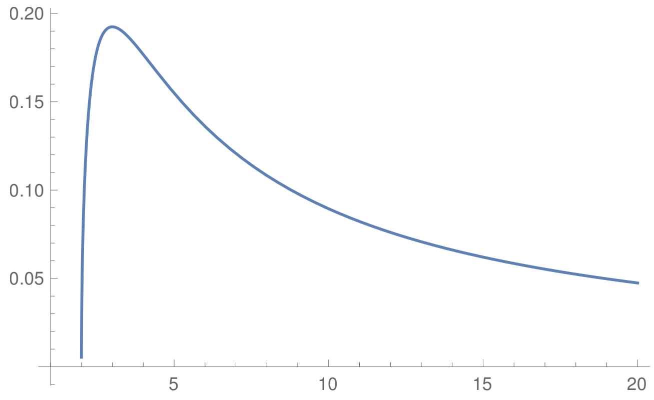

Figure 3: Plot of Function $f(r)$

Result 5.6. *The Poisson tensor w.r.t. Schwarzschild spacetime $(\mathcal{M}_{BH}, g_{BH})$, fulfilling the associativity condition* (2.1)*, is given by two sets of solutions. The first, is given by*

$$\Theta^{01} = C_1, \tag{5.16}$$

with C_1 being a real constant and all other components equal to zero. The second solution is given by $\Theta^{\mu\nu} = 0$ except for the component $f := \Theta^{12}$ that is given by

$$f(r) = \frac{C_2}{r}\sqrt{1 - \frac{2M}{r}}, \tag{5.17}$$

where $C_2 \in \mathbb{R}$.

Proof. See result 5.4. □

If we plot the function $f(r)$ (scaled by C_2 and setting $M = 1$), see Figure 3, we see that at the event horizon, the non-commutative scale vanishes. The non-commutative strength then inclines and reaches a maximum at $r = 3M$ and declines with r inverse. The radius $r = 3M$ is the inner most stable circular orbit (ISCO) for massive particles. This is the closest radius at which a stable circular orbit is possible around a Schwarzschild black hole. The further we move from the non-rotating mass the more the non-commutative scale becomes negligible, thus demonstrating a deep interconnection between spacetime curvature and non-commutative effects of spacetime.

5.2.2 Reissner–Nordstrøm

The Reissner-Nordstrøm spacetime $(\mathcal{M}_{RN}, g_{RN})$ is a solution to the Einstein field equations that describes the spacetime around a spherically symmetric, electrically charged with charge Q, non-rotating mass M, with space-time metric

$$\boxed{g_{RN} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2(\vartheta) d\varphi^2\right).}$$

Result 5.7. *The Poisson tensor w.r.t. Reissner-Nordstrøm $(\mathcal{M}_{RN}, g_{RN})$, fulfilling the associativity condition* (2.1)*, is given by two-sets of solutions. The first, is given by*

$$\Theta^{01} = C_1, \tag{5.18}$$

with C_1 being a real constant and all other components equal to zero. The second solution is given by $\Theta^{\mu\nu} = 0$ except for the component $f := \Theta^{12}$ that is given by

$$f(r) = \frac{C_2}{r}\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}, \tag{5.19}$$

where $C_2 \in \mathbb{R}$.

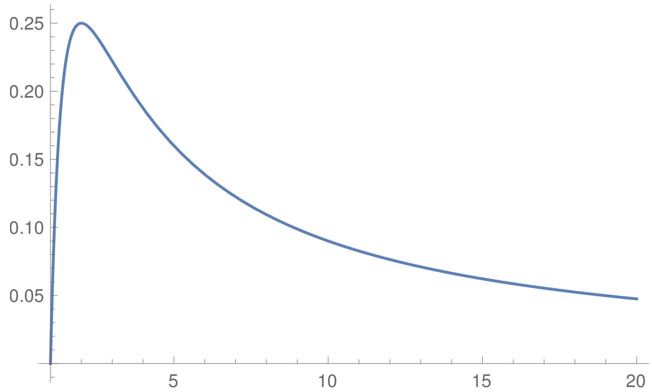

Figure 4: Plot of Function $f(r)$

Proof. See result 5.4. □

The interpretation is similar to the uncharged black hole, see Figure 4, where we set $M = Q = 1$.

5.2.3 Kottler or Schwarzschild-(anti-)deSitter Spacetime

The Kottler spacetime $(\mathcal{M}_K, g_K)$ is a solution to the Einstein equation describing the spacetime geometry outside a spherically symmetric mass distribution, with mass M, in the presence of a non-zero cosmological constant Λ. It is represented by the metric [MG09, Section 2.15]

$$\boxed{g_K = -\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2}. \tag{5.20}$$

If $\Lambda > 0$ the metric is also known as Schwarzschild-de Sitter metric, whereas if $\Lambda < 0$ it is called Schwarzschild-anti-de Sitter.

Result 5.8. *The Poisson tensor w.r.t. Kottler spacetime $(\mathcal{M}_K, g_K)$, fulfilling the associativity condition* (2.1)*, is given by two-sets of solutions. The first, is given by*

$$\Theta^{01} = C_1, \tag{5.21}$$

with C_1 being real constant and all other components equal to zero. The second solution is given by $\Theta^{\mu\nu} = 0$ except for the component $f := \Theta^{12}$ that is given by

$$f(r) = \frac{C_1}{r}\sqrt{\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)}, \tag{5.22}$$

where $C_1 \in \mathbb{R}$.

Proof. See result 5.4. □

5.3 Friedmann-Robertson-Walker-Lemaitre Spacetimes

In this section we turn our attention to a non-static spacetime, namely Friedmann-Robertson-Walker-Lemaitre (**FRWL**) spacetimes. These spacetimes are a class of cosmological solutions to Einstein's field equations that describe homogeneous and isotropic expanding or contracting universes on large scales. For the following context we consider the FRWL spacetime $(\mathcal{M}_{FRW}, g_{FRW})$ for the case of flat spatial geometry, i.e.,

$$\boxed{g_{FRW} = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),} \tag{5.23}$$

where $\{t, x, y, z \in \mathbb{R}\}$ and $a(t)^2$ is the scale factor. Assuming a homogeneous and isotropic spatial part of the spacetime non-commutativity, renders for the Poisson tensor the following conditions,

$$\partial_i \Theta^{\alpha\beta} = 0, \qquad\qquad \Theta^{ij} = 0.$$

This is easily seen, by taking the Lie derivatives of the tensor Θ along the Killing vector fields of FRWL. Hence, the associativity condition has to be considered for the degenerate case.

Result 5.9. *The Poisson tensor for the FRWL manifold $(\mathcal{M}_{FRW}, g_{FRW})$, with vanishing spatial derivatives and spatial non-commutativity, i.e. $\partial_i \Theta^{\alpha\beta} = 0$ and $\Theta^{ij} = 0$, is given by*

$$\Theta^{0j} = \frac{\Theta}{a(t)}\, e^j,$$

where e^j is the unit-vector in j-direction and Θ is the Planck-length λ_p^2 squared.

Proof. See Appendix A.5. □

The resulting non-commutative structure according to result 2.8 is therefore

$$[t, x^i]_\Theta = -2i \frac{\Theta}{a(t)}\, e^i. \tag{5.24}$$

A similar commutation relation was obtained in [FM21, Theorem II.6]. Let us apply these commutation relations heuristically to the big bang singularity. To make the considerations more clear, we write the commutator relations as the product of uncertainties.

$$\Delta T \cdot \Delta X^i \geq \lambda_P^2 \, |\langle a(T)^{-1} \rangle| \tag{5.25}$$

for all i, where (T, X^i) are the operator representations (assuming they exist) of the commutator relation given in Equation (5.24). If one approaches the big bang ($t \to 0$), then $a \to 0$ (see [Wal10, Page 107]), assuming the expectation value thereof behaves analogously, the time becomes definite. This induces the uncertainty in space to go to infinity. Let us assume that the spacetime fabric is interwoven, i.e. no time without space and vice versa. Then, the uncertainty relations (5.24) suggest that the occurrence of a big bang singularity can be circumvented by embracing features of a quantum spacetime. The underlying physical concept is: attempts to confine the time dimension will induce quantum effects counteracting by exerting pressure in the spatial dimensions, thereby averting the occurrence of a Big Bang singularity (i.e. averting a single point). In particular, there will be a minimum scale factor, where the product of the uncertainties in the Inequality (5.25) is equal to the right hand side, implying a minimal size of the universe. The removal of the Big Bang singularity by a non-commutative structure has as well been considered in [MMMZ04, TV14, GNV14, Ste18].

The inflationary phase, characterized by the scale factor $a(t) = e^{Ht}$, can be viewed through this noncommutative lens, as well. The smaller t, the bigger quantum effects and the quantum push-back will be in spatial directions. In particular, very early on $t \approx 10^{-43}s - 10^{-36}s$, the non-commutative scale will be the largest contributing to inflation, as a repulsive potential, e.g. acting as an inflaton field. Directly after inflation the non-commutativity strength decreases exponentially.

These statements concluded from the non-vanishing commutator relations have to be strengthened by studying deformed Einstein equations or/and QFT in these regimes. This is work in progress.

6 Concluding Remarks and Outlook

While the associativity condition has supplied us with a plethora of physically meaningful Poisson tensors corresponding to the specific spacetimes, the condition is still unclear from a physical point of view. In the case of non-degenerate Poisson tensor we can draw, however, some parallels to the metric compatibility condition. In particular $\nabla\pi = 0$, means that that the non-commutative commutation relations remain consistent for observers moving along geodesics.

In addition to the physical interpretation, another question remains open: How can we establish a connection between the Poisson tensor and the underlying geometry through deformation quantization? In our investigation, we adopted an approach centered on taking a classical spacetime metric. By imposing the associativity condition (stemming from requiring a star product), we derived the non-commutative structure that aligns with this metric. Despite this alignment, these objects (metric and Poisson tensor) remain distinct entities, lacking a unified framework. Furthermore, our Poisson tensors, although not explicitly stated, carry a deformation parameter of first order, usually identified with the Planck length. Ideally, we want a method that generates corrections to the metric tensor in terms of this deformation parameter and enforcing the associativity condition w.r.t. this perturbed metric, should give us a Poisson tensor of next order in the deformation parameter. This approach is analogous to solving the semi-classical Einstein equations, see [Hac10, Pin11, Sie15, JA21] and references therein.

There are (at least) two possibilities of unifying the different geometries given by the metric and Poisson tensor.

The first possibility involves expressing each metric in terms of vielbeins and then employing a generalized deformed product between them, utilizing the evaluated Poisson tensor. This approach, akin to methodologies found in literature such as [Cha01, CTSZ08, CTZ08, BF14, TS24], typically yields the classical metric alongside perturbations introduced by the deformation parameter.

Another avenue for connecting the metric and the Poisson tensor arises from quantum field theory (QFT) in curved spacetimes, as demonstrated in [Muc21]. Here, we utilize deformation quantization on the two-point function, enabling the definition of a deformed product for pairs of points through geodesic transport. Subsequently, we evaluate the semi-classical Einstein equations with respect to the deformed states. By carefully rearranging the resulting expressions, we can draw conclusions regarding the alteration of geometric quantities, such as the Ricci tensor and scalar. Alternatively, we can interpret the new equations as quantum-corrected semi-classical equations or deformed Einstein equations. This approach offers potential insights into the relationship between the non-commutative structure and the metric and is work in progress.

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A Proofs

A.1 Proof of Theorem 2.6

Proof. First, we express the generalized Rieffel product in orders of the deformation matrix

$$
\begin{aligned}
(f \star_\Theta g)\,(x) &= \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, f(\exp_{(x)}(\Theta X))\, g(\exp_{(x)}(Y))\ \mathrm{e}^{-\mathrm{i}\,X\cdot Y}\,, \\
&= \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, (f(x) + (\Theta X)^\mu \nabla_\mu f + \frac{1}{2}(\Theta X)^\mu (\Theta X)^\nu \nabla_\mu \nabla_\nu f)\, g(\exp_{(x)}(Y))\ \mathrm{e}^{-\mathrm{i}\,X\cdot Y} \\
&= \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, f(x)\, g(\exp_{(x)}(Y))\ \mathrm{e}^{-\mathrm{i}\,X\cdot Y} \\
&+ \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, (\Theta X)^\mu (\nabla_\mu f(x))\, g(\exp_{(x)}(Y))\, \mathrm{e}^{-\mathrm{i}\,X\cdot Y} \\
&+ \frac{1}{2} \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, ((\Theta X)^\mu (\Theta X)^\nu \nabla_\mu \partial_\nu f(x))\, g(\exp_{(x)}(Y))\, \mathrm{e}^{-\mathrm{i}\,X\cdot Y} \\
&= f(x) g(x) - i\Theta^{\mu\nu} \partial_\mu f(x)\, \partial_\nu g(x) \\
&+ \frac{1}{4} \nabla_\mu \partial_\nu f\, \nabla_\rho \partial_\sigma g\ \lim_{\epsilon\to 0} \iint \chi(\epsilon X, \epsilon Y)\, ((\Theta X)^\mu (\Theta X)^\nu\, Y^\rho Y^\sigma\ \mathrm{e}^{-\mathrm{i}\,X\cdot Y}
\end{aligned}
$$

$$
\begin{aligned}
&= f(x)g(x) - i\Theta^{\mu\nu}\partial_\mu f(x)\,\partial_\nu g(x) \\
&- \frac{1}{4}\left(\Theta^{\mu\rho}\Theta^{\nu\sigma} + \Theta^{\mu\sigma}\Theta^{\nu\rho}\right)\nabla_\mu\partial_\nu f(x)\,\nabla_\rho\partial_\sigma g(x)
\end{aligned}
$$

where in the last lines we used the Taylor expansion of a smooth function of the exponential map,

$$
f(\exp_{(x)}(\Theta X)) = f(x) + (\Theta X)^\mu \nabla_\mu f + \frac{1}{2}(\Theta X)^\mu (\Theta X)^\nu \nabla_\mu\nabla_\nu f + \mathcal{O}(\Theta^3)
$$

rewrote terms as differential operators

$$
\begin{aligned}
\iint \chi(\epsilon X, \epsilon Y)\,(\Theta X)^\mu\, \mathrm{e}^{-\mathrm{i}\,X\cdot Y} &= \iint \chi(\epsilon X, \epsilon Y)\,\Theta^\mu_{\ \nu} X^\nu\, \mathrm{e}^{-\mathrm{i}\,X\cdot Y} \\
&= i\iint \chi(\epsilon X, \epsilon Y)\,\Theta^\mu_{\ \nu} g^{\nu\sigma}\partial_\sigma\, \mathrm{e}^{-\mathrm{i}\,X\cdot Y} \\
&= i\iint \chi(\epsilon X, \epsilon Y)\,\Theta^{\mu\sigma}\partial_\sigma\, \mathrm{e}^{-\mathrm{i}\,X\cdot Y}
\end{aligned}
$$

and used partial integration. Next, note that we have the relation

$$
\nabla_\rho\partial_\sigma g = \nabla_\sigma\partial_\rho g
$$

since the function g is smooth and the Christoffel symbols are symmetric in the lower indices. Using this relation we have

$$
\begin{aligned}
\Theta^{\mu\sigma}\Theta^{\nu\rho}\nabla_\rho\partial_\sigma g &= \Theta^{\mu\sigma}\Theta^{\nu\rho}\nabla_\sigma\partial_\rho g \\
&= \Theta^{\mu\rho}\Theta^{\nu\sigma}\nabla_\rho\partial_\sigma g,
\end{aligned}
$$

which turns the star product to

$$
(f \star_\Theta g)(x) = f(x)g(x) - i\Theta^{\mu\nu}\partial_\mu f(x)\,\partial_\nu g(x) - \frac{1}{2}\Theta^{\mu\rho}\Theta^{\nu\sigma}\,\nabla_\mu\partial_\nu f(x)\,\nabla_\rho\partial_\sigma g(x).
$$

Next, we prove associativity where we first consider

$$
\begin{aligned}
&((f \star_\Theta g) \star_\Theta h) = (F \star_\Theta h) \\
&= Fh - i\Theta^{\mu\nu}\,\partial_\mu F\,\partial_\nu h - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu F\,\nabla_\alpha\partial_\beta h \\
&= (fg - i\Theta^{\mu\nu}\,\partial_\mu f\,\partial_\nu g - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu f\,\nabla_\alpha\partial_\beta g)h \\
&- i\Theta^{\mu\nu}\,\partial_\mu(fg - i\Theta^{\mu\nu}\,\partial_\mu f\,\partial_\nu g)\,\partial_\nu h - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu(fg)\,\nabla_\alpha\partial_\beta h \\
&= (fg - i\Theta^{\mu\nu}\,\partial_\mu f\,\partial_\nu g - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu f\,\nabla_\alpha\partial_\beta g)h \\
&- i\Theta^{\mu\nu}(\,\partial_\mu f\,g + f\,\partial_\mu g - i\,\partial_\mu\Theta^{\alpha\beta}\,\partial_\alpha f\,\partial_\beta g - i\Theta^{\alpha\beta}\,\partial_\mu\partial_\alpha f\,\partial_\beta g - i\Theta^{\alpha\beta}\,\partial_\alpha f\,\partial_\mu\partial_\beta g)\,\partial_\nu h \\
&- \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}(\nabla_\mu\partial_\nu f\,g + 2\partial_\mu f\partial_\nu g + f\,\nabla_\mu\partial_\nu g)\,\nabla_\alpha\partial_\beta h
\end{aligned}
$$

Next, we consider

$$
(f \star (g \star_\Theta h)) = (f \star_\Theta G)
$$

$$= fG - i\Theta^{\mu\nu}\,\partial_\mu f\,\partial_\nu G - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu f\,\nabla_\alpha\partial_\beta G$$

$$= f(gh - i\Theta^{\mu\nu}\,\partial_\mu g\,\partial_\nu h - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu g\,\nabla_\alpha\partial_\beta h)$$

$$- i\Theta^{\mu\nu}\,\partial_\mu f\,\partial_\nu(gh - i\Theta^{\alpha\beta}\,\partial_\alpha g\,\partial_\beta h) - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu f\,\nabla_\alpha\partial_\beta(gh)$$

$$= f(gh - i\Theta^{\mu\nu}\,\partial_\mu g\,\partial_\nu h - \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu g\,\nabla_\alpha\partial_\beta h)$$

$$- i\Theta^{\mu\nu}\,\partial_\mu f(\,\partial_\nu g\,h + g\,\partial_\nu h - i\,\partial_\nu\Theta^{\alpha\beta}\,\partial_\alpha g\,\partial_\beta h - i\Theta^{\alpha\beta}\;\partial_\nu\partial_\alpha g\,\partial_\beta h - i\Theta^{\alpha\beta}\,\partial_\alpha g\;\partial_\nu\partial_\beta h)$$

$$- \frac{1}{2}\Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu f(\,\nabla_\alpha\partial_\beta g\,h + 2\partial_\alpha g\partial_\beta h + g\,\nabla_\alpha\partial_\beta h)$$

comparing the two expressions $f \star (g \star_\Theta h)$ and $(f \star g) \star_\Theta h$ we have,

$$- i\Theta^{\mu\nu}(-i\,\partial_\mu\Theta^{\alpha\beta}\,\partial_\alpha f\,\partial_\beta g - i\Theta^{\alpha\beta}\;\partial_\mu\partial_\alpha f\,\partial_\beta g)\,\partial_\nu h - \Theta^{\mu\alpha}\Theta^{\nu\beta}\,\partial_\mu f\,\partial_\nu g\,\nabla_\alpha\partial_\beta h$$

$$= -i\Theta^{\mu\nu}\,\partial_\mu f(-i\,\partial_\nu\Theta^{\alpha\beta}\,\partial_\alpha g\,\partial_\beta h - i\Theta^{\alpha\beta}\,\partial_\alpha g\;\partial_\nu\partial_\beta h) - \Theta^{\mu\alpha}\Theta^{\nu\beta}\nabla_\mu\partial_\nu f\,\partial_\alpha g\,\partial_\beta h$$

which summarizes to

$$- \Theta^{\mu\nu}(\,\partial_\mu\Theta^{\alpha\beta}\,\partial_\alpha f\,\partial_\beta g)\,\partial_\nu h - \Theta^{\mu\nu}\Theta^{\alpha\beta}\,\Gamma^\gamma_{\nu\beta}\,\partial_\mu f\,\partial_\alpha g\,\partial_\gamma h$$

$$= -\Theta^{\mu\nu}\,\partial_\mu f(\,\partial_\nu\Theta^{\alpha\beta}\,\partial_\alpha g\,\partial_\beta h) - \Theta^{\mu\nu}\Theta^{\alpha\beta}\,\Gamma^\gamma_{\mu\alpha}\,\partial_\gamma f\,\partial_\nu g\,\partial_\beta h$$

using the Jacobi identity we obtain

$$\Theta^{\mu\nu}\,\partial_\mu\Theta^{\beta\alpha} + \Theta^{\mu\nu}\Theta^{\alpha\delta}\,\Gamma^\beta_{\mu\delta} + \Theta^{\mu\nu}\Theta^{\delta\beta}\,\Gamma^\alpha_{\mu\delta} = 0$$

which can be written as the vanishing of the covariant derivative

$$\Theta^{\mu\nu}\,\nabla_\mu\Theta^{\beta\alpha} = 0.$$

□

A.2 Proof of Result 5.2

Proof. The associativity condition for $\alpha = 0$ and $\beta = i$ reads

$$\Theta^{\mu\nu}\partial_\nu\Theta^{0i} = -\Theta^{\mu j}\Theta^{0k}\,\Gamma^i_{jk}.$$

Assuming $\partial_{0,3}\Theta = 0$ and using the notation $a, b = 1, 2$ we have

$$\Theta^{\mu a}\partial_a\Theta^{0i} = -\Theta^{\mu j}\Theta^{0k}\,\Gamma^i_{jk}.$$

We have for $i = 1$

$$\begin{aligned}\Theta^{\mu a}\partial_a\Theta^{01} &= -\Theta^{\mu j}\Theta^{0k}\,\Gamma^1_{jk}\\ &= -\Theta^{\mu 2}\Theta^{02}\,\Gamma^1_{22} - \Theta^{\mu 3}\Theta^{03}\,\Gamma^1_{33}\end{aligned}$$

while for $i = 2$

$$\Theta^{\mu a}\partial_a\Theta^{02} = -\Theta^{\mu j}\Theta^{0k}\,\Gamma^2_{jk}$$

$$= -\Theta^{\mu 1}\Theta^{02}\,\Gamma^2_{12} - \Theta^{\mu 2}\Theta^{01}\,\Gamma^2_{21} - \Theta^{\mu 3}\Theta^{03}\,\Gamma^2_{33}$$

while for $i = 3$

$$\begin{aligned}\Theta^{\mu a}\partial_a\Theta^{03} &= -\Theta^{\mu j}\Theta^{0k}\,\Gamma^3_{jk}\\ &= -\Theta^{\mu 1}\Theta^{03}\,\Gamma^3_{13} - \Theta^{\mu 3}\Theta^{01}\,\Gamma^3_{31} - \Theta^{\mu 2}\Theta^{03}\,\Gamma^3_{23} - \Theta^{\mu 3}\Theta^{02}\,\Gamma^3_{32}\end{aligned}$$

For $\alpha = i$ and $\beta = j$ we have

$$\Theta^{\mu a}\partial_a\Theta^{ij} = -\Theta^{\mu k}\Theta^{sj}\,\Gamma^i_{ks} - \Theta^{\mu k}\Theta^{is}\,\Gamma^j_{ks}$$

One set of solutions is given by setting $\Theta^{ik} = 0$ and $\Theta^{03} = 0$ rendering the following set of differential equations

$$\begin{aligned}\Theta^{0a}\partial_a\Theta^{01} &= -(\Theta^{02})^2\,\Gamma^1_{22}\\ \Theta^{0a}\partial_a\Theta^{02} &= -2\Theta^{01}\Theta^{02}\,\Gamma^2_{12}\end{aligned}$$

Further assuming that the functions do not depend on the angle ϑ we have

$$\Theta^{01}\partial_1\Theta^{01} = -(\Theta^{02})^2\,\Gamma^1_{22} \tag{A.1}$$

$$\partial_1\Theta^{02} = -2\Theta^{02}\,\Gamma^2_{12} \tag{A.2}$$

to which the solutions are

$$\Theta^{02} = \frac{C_1}{b_0^2 + l^2} \tag{A.3}$$

$$\Theta^{01} = \pm\frac{\sqrt{-C_1^2 + 2a^2C_2 + 2l^2C_2}}{\sqrt{a^2+l^2}} \tag{A.4}$$

Setting the dimension-full constant $C_1 = 0$ we have a constant non-commutativity between the temporal and radial component.

The next set of solutions we consider is $\Theta^{0k} = 0$ and $\Theta^{13} = \Theta^{23} = 0$, rendering

$$\Theta^{\mu a}\partial_a\Theta^{12} = -\Theta^{\mu 1}\Theta^{12}\,\Gamma^2_{12}$$

which is a set of two differential equations

$$\begin{aligned}\partial_1\Theta^{12} &= -\Theta^{12}\,\Gamma^2_{12}\\ \partial_2\Theta^{12} &= 0,\end{aligned}$$

with solution

$$\Theta^{12}(l) = \frac{C_1}{\sqrt{b_0^2 + l^2}}$$

□

A.3 Proof of Result 5.3

Proof. The components of the Riemann and Ricci tensor are given by

$$R^0{}_{101} = \left(-\partial_1^2\alpha - 2\left(\partial_1\alpha\right)^2\right), \quad R^0{}_{202} = -re^{2\alpha}\partial_1\alpha, \quad R^0{}_{303} = -re^{2\alpha}\partial_1\alpha\,\sin^2\vartheta$$

$$R^1{}_{212} = -re^{2\alpha}\partial_1\alpha, \quad R^1{}_{313} = -re^{2\alpha}\partial_1\alpha\sin^2\vartheta, \quad R^2{}_{323} = \left(1 - e^{2\alpha}\right)\sin^2\vartheta$$

$$R_{00} = e^{4\alpha}\left(\partial_1^2\alpha + 2\left(\partial_1\alpha\right)^2 + \frac{2}{r}\partial_1\alpha\right), \quad R_{11} = -\left(\partial_1^2\alpha + 2\left(\partial_1\alpha\right)^2 + \frac{2}{r}\partial_1\alpha\right),$$

$$R_{22} = -e^{2\alpha}\left(r\left(2\partial_1\alpha\right)+1\right)+1, \quad R_{33} = R_{22}\sin^2\vartheta$$

Using Equation (5.1) we have for $\alpha = \gamma$

$$R_{\lambda\delta}\Theta^{\lambda\beta} + R^{\beta}{}_{\lambda\alpha\delta}\Theta^{\alpha\lambda} = 0.$$

Using the explicit expressions for the components we have

β	δ	Solution
0	1	$\Theta^{01} = 0$
0	2	$\Theta^{02} = 0$ or Solution (A.6)
0	3	$\Theta^{03} = 0$ or Solution (A.6)
1	2	$\Theta^{12} = 0$ or Solution (A.6)
1	3	$\Theta^{13} = 0$ or Solution (A.6)
2	0	$\Theta^{02} = 0$ or Schwarzschild Solution $\to \Theta^{02} = 0$
2	1	$\Theta^{12} = 0$ or Schwarzschild Solution $\to \Theta^{12} = 0$
2	3	$\Theta^{23} = 0$
3	1	$\Theta^{13} = 0$ or Schwarzschild Solution $\to \Theta^{13} = 0$

where we have the differential equation

$$r\partial_1\alpha + 1 - e^{-2\alpha} = 0 \tag{A.5}$$

with solution

$$\alpha(r) = \frac{1}{2}\log(1 - C_1/r^2) \tag{A.6}$$

The only non-vanishing component is Θ^{03} for the Solution (A.6), rendering a degenerate Θ. □

A.4 Proof of Result 5.4

Proof. Assuming $\partial_{0,3}\Theta = 0$ and using the indices $a, b = 1, 2$ we have for the associativity condition (4.1),

$$\Theta^{\mu a}\partial_a\Theta^{0k} = -\Theta^{\mu 0}\Theta^{1k}\,\Gamma^0_{01} - \Theta^{\mu 1}\Theta^{0k}\,\Gamma^0_{10} - \Theta^{\mu j}\Theta^{0i}\,\Gamma^k_{ji}$$

This gives us

$$\begin{aligned}
\Theta^{\mu a}\partial_a\Theta^{01} &= -\Theta^{\mu 1}\Theta^{01}\,\Gamma^0_{10} - \Theta^{\mu j}\Theta^{0i}\,\Gamma^1_{ji} \\
&= -\Theta^{\mu 1}\Theta^{01}\,\Gamma^0_{10} - \Theta^{\mu 1}\Theta^{01}\,\Gamma^1_{11} - \Theta^{\mu 2}\Theta^{02}\,\Gamma^1_{22} - \Theta^{\mu 3}\Theta^{03}\,\Gamma^1_{33} \\
&= -\Theta^{\mu 2}\Theta^{02}\,\Gamma^1_{22} - \Theta^{\mu 3}\Theta^{03}\,\Gamma^1_{33}
\end{aligned}$$

since $\Gamma^0_{10} + \Gamma^1_{11} = 0$,

$$\begin{aligned}
\Theta^{\mu a}\partial_a\Theta^{02} &= -\Theta^{\mu 0}\Theta^{12}\,\Gamma^0_{01} - \Theta^{\mu 1}\Theta^{02}\,\Gamma^0_{10} - \Theta^{\mu j}\Theta^{0i}\,\Gamma^2_{ji} \\
&= -\Theta^{\mu 0}\Theta^{12}\,\Gamma^0_{01} - \Theta^{\mu 1}\Theta^{02}\,\Gamma^0_{10} - \Theta^{\mu 3}\Theta^{03}\,\Gamma^2_{33} - \Theta^{\mu 1}\Theta^{02}\,\Gamma^2_{12} - \Theta^{\mu 2}\Theta^{01}\,\Gamma^2_{21}
\end{aligned}$$

$$\begin{aligned}
\Theta^{\mu a}\partial_a\Theta^{03} &= -\Theta^{\mu 0}\Theta^{13}\,\Gamma^0_{01} - \Theta^{\mu 1}\Theta^{03}\,\Gamma^0_{10} - \Theta^{\mu j}\Theta^{0i}\,\Gamma^3_{ji} \\
&= -\Theta^{\mu 0}\Theta^{13}\,\Gamma^0_{01} - \Theta^{\mu 1}\Theta^{03}\,\Gamma^0_{10} - \Theta^{\mu 1}\Theta^{03}\,\Gamma^3_{13} - \Theta^{\mu 3}\Theta^{01}\,\Gamma^3_{31} - \Theta^{\mu 2}\Theta^{03}\,\Gamma^3_{23} - \Theta^{\mu 3}\Theta^{02}\,\Gamma^3_{32}
\end{aligned}$$

Second, we consider the terms where $\alpha = i$, $\beta = k$

$$\Theta^{\mu a}\partial_a\Theta^{ik} = -\Theta^{\mu\nu}\Theta^{\lambda k}\,\Gamma^i_{\nu\lambda} - \Theta^{\mu\nu}\Theta^{i\lambda}\,\Gamma^k_{\nu\lambda}$$

rendering

$$\begin{aligned}\Theta^{\mu a}\partial_a\Theta^{12} &= -\Theta^{\mu\nu}\Theta^{\lambda 2}\,\Gamma^1_{\nu\lambda} - \Theta^{\mu\nu}\Theta^{1\lambda}\,\Gamma^2_{\nu\lambda}\\ &= -\Theta^{\mu 0}\Theta^{02}\,\Gamma^1_{00} - \Theta^{\mu 1}\Theta^{12}\,\Gamma^1_{11} - \Theta^{\mu 3}\Theta^{32}\,\Gamma^1_{33} - \Theta^{\mu 3}\Theta^{13}\,\Gamma^2_{33} - \Theta^{\mu 1}\Theta^{12}\,\Gamma^2_{12}\end{aligned}$$

$$\begin{aligned}\Theta^{\mu a}\partial_a\Theta^{13} &= -\Theta^{\mu\nu}\Theta^{\lambda 3}\,\Gamma^1_{\nu\lambda} - \Theta^{\mu\nu}\Theta^{1\lambda}\,\Gamma^3_{\nu\lambda}\\ &= -\Theta^{\mu 0}\Theta^{03}\,\Gamma^1_{00} - \Theta^{\mu 1}\Theta^{13}\,\Gamma^1_{11} - \Theta^{\mu 2}\Theta^{23}\,\Gamma^1_{22} - \Theta^{\mu 1}\Theta^{13}\,\Gamma^3_{13} - \Theta^{\mu 2}\Theta^{13}\,\Gamma^3_{23} - \Theta^{\mu 3}\Theta^{12}\,\Gamma^3_{32}\end{aligned}$$

$$\begin{aligned}\Theta^{\mu a}\partial_a\Theta^{23} &= -\Theta^{\mu\nu}\Theta^{\lambda 3}\,\Gamma^2_{\nu\lambda} - \Theta^{\mu\nu}\Theta^{2\lambda}\,\Gamma^3_{\nu\lambda}\\ &= -\Theta^{\mu 1}\Theta^{23}\,\Gamma^2_{12} - \Theta^{\mu 2}\Theta^{13}\,\Gamma^2_{21} - \Theta^{\mu 1}\Theta^{23}\,\Gamma^3_{13} - \Theta^{\mu 3}\Theta^{21}\,\Gamma^3_{31} - \Theta^{\mu 2}\Theta^{23}\,\Gamma^3_{23}\end{aligned}$$

One solution is given by setting all components equal to zero except Θ^{01} rendering the only remaining differential equation

$$\partial_1\Theta^{01} = 0,$$

with solution $\Theta^{01} = C_{01}$, with $C_{01} \in \mathbb{R}$. A different set of solutions is given if we set Θ^{0i} equal to zero. Then, the differential equations reduce to

$$f\partial_2 f = gh\,\Gamma^1_{33} - g^2\,\Gamma^2_{33} \tag{A.7}$$

$$-f\partial_1 f = f^2\,\Gamma^1_{11} + h^2\,\Gamma^1_{33} - hg\,\Gamma^2_{33} + f^2\,\Gamma^2_{12} \tag{A.8}$$

$$-g\partial_1 f - h\partial_2 f = gf\,\Gamma^1_{11} + gf\,\Gamma^2_{12} \tag{A.9}$$

$$f\partial_2 g = -fh\,\Gamma^1_{22} - 2fg\,\Gamma^3_{23} \tag{A.10}$$

$$-f\partial_1 g = fg\,\Gamma^1_{11} + fg\,\Gamma^3_{13} - hf\,\Gamma^3_{32} \tag{A.11}$$

$$-g\partial_1 g - h\partial_2 g = g^2\,\Gamma^1_{11} + h^2\,\Gamma^1_{22} + g^2\,\Gamma^3_{13} + hg\,\Gamma^3_{23} \tag{A.12}$$

$$f\partial_2 h = -fh\,\Gamma^3_{23} \tag{A.13}$$

$$-f\partial_1 h = 3fh\,\Gamma^2_{12} \tag{A.14}$$

$$-g\partial_1 h - h\partial_2 h = 3gh\,\Gamma^2_{12} + h^2\,\Gamma^3_{23} \tag{A.15}$$

where we defined the functions $f := \Theta^{12}$, $g := \Theta^{13}$, $h := \Theta^{23}$. First, let us consider the solutions of the choices given in the following table,

f	g	h	Solution
0	0	h	Eq. (A.8) $\Rightarrow h = 0$
f	0	0	I
0	g	0	Eq. (A.7) $\Rightarrow g = 0$
f	0	h	Eq. (A.12) $\Rightarrow h = 0 \Rightarrow I$

where I is the solution of the following differential equations,

$$\partial_2 f = 0 \tag{A.16}$$

$$\partial_1 f = -\left(\Gamma^1_{11} + \Gamma^2_{12}\right) f. \tag{A.17}$$

We are left with three cases

1. $f, g \neq 0$ while $h = 0$,
2. $g, h \neq 0$ while $f = 0$,
3. $f, g, h \neq 0$.

For Case 1 we have the following non-vanishing differential equations

$$f \partial_2 f = -g^2\, \Gamma^2_{33} \tag{A.18}$$

$$-\partial_1 f = f\, \Gamma^1_{11} + f\, \Gamma^2_{12} \tag{A.19}$$

$$\partial_2 g = -2g\, \Gamma^3_{23} \tag{A.20}$$

$$-\partial_1 g = g\, \Gamma^1_{11} + g\, \Gamma^2_{12} \tag{A.21}$$

These differential equations are solved by a separation ansatz, that is,

$$g(r, \vartheta) = f_1(r) g_2(\vartheta)$$

where the radial part of both functions is equal, since they satisfy the same differential equation. The solution of Equation (A.20) is given by

$$g(r, \vartheta) = C_2^2\, f_1^2(r)\, \csc(\vartheta)^2$$

Inserting this solution in to Equation (A.18) after simply rewriting it as

$$\frac{1}{2} \partial_2 (f^2) = -g^2\, \Gamma^2_{33}$$

gives us for the function f the following solution

$$f(r, \vartheta) = \sqrt{C_1(r) - C_2^2 f_1^2(r) \csc(\vartheta)^2}$$

Inserting this solution into Equation (A.19) gives us for the function $C_1(r) = C_3^2 f_1^2(r)$ rendering

$$f(r, \vartheta) = C_4 f_1(r) \sqrt{1 - C_5^2\, \csc(\vartheta)^2}, \tag{A.22}$$

where $C_5 = C_2^2 / C_3^2$, with $C_2, \cdots, C_5 \in \mathbb{R}$. To ensure that the solution f is real valued, the constant $C_2 = 0$ renders $g = 0$ and we return to the Solution I.

Taking into account the case 2, that is, $f = 0$ and $g, h \neq 0$ we have the following equations.

$$0 = h\,\Gamma^1_{33} - g\,\Gamma^2_{33} \tag{A.23}$$

$$-g\partial_1 g - h\partial_2 g = g^2\,\Gamma^1_{11} + h^2\,\Gamma^1_{22} + g^2\,\Gamma^3_{13} + hg\,\Gamma^3_{23} \tag{A.24}$$

$$-g\partial_1 h - h\partial_2 h = 3gh\,\Gamma^2_{12} + h^2\,\Gamma^3_{23} \tag{A.25}$$

From Equation (A.23) we obtain

$$h = g\,\Gamma^2_{33}/\Gamma^1_{33} =: b(r,\vartheta) g$$

where $b(r,\vartheta) := \exp(-2\alpha(r))\frac{\cot\vartheta}{r}$. Plugging this into the other two differential equations gives us

$$-g\partial_1 g - bg\partial_2 g = g^2\,\Gamma^1_{11} + g^2\,\Gamma^3_{13} \tag{A.26}$$
$$= \left(\frac{1}{r} - \partial_1\alpha(r)\right) g^2 \tag{A.27}$$

where we used the fact that $b^2\,\Gamma^1_{22} + b\,\Gamma^3_{23} = 0$.

$$-g\partial_1 g - bg\partial_2 g = 3g^2\,\Gamma^2_{12} - g^2\exp(-2\alpha(r))\frac{1}{r} + g^2\left(-2\partial_1\alpha(r) - \frac{1}{r}\right) \tag{A.28}$$
$$= \left(\frac{2}{r} - \exp(-2\alpha(r))\frac{1}{r} - 2\partial_1\alpha(r)\right) g^2 \tag{A.29}$$

where in the last lines we used the relation $b\,\Gamma^3_{23} + \partial_2 b = -\exp(-2\alpha(r))\frac{1}{r}$ which follows from the derivatives of the function b that are given by

$$\begin{aligned}
\partial_1 b(r,\vartheta) &= -2\partial_1\alpha(r)\exp(-2\alpha(r))\frac{\cot\vartheta}{r} - \exp(-2\alpha(r))\frac{\cot\vartheta}{r^2} \\
&= \left(-2\partial_1\alpha(r) - \frac{1}{r}\right) b(r,\vartheta), \\
\partial_2 b(r,\vartheta) &= -\csc(\vartheta)^2\exp(-2\alpha(r))\frac{1}{r}.
\end{aligned}$$

Subtracting the equations (A.26) and (A.28) renders

$$0 = \partial_1\alpha(r) - \frac{1}{r} + \exp(-2\alpha(r))\frac{1}{r}$$

The solution is given by

$$\exp(-2\alpha(r)) = 1 - C_1 e^{z_1} r^2,$$

where $z_1 \in \mathbb{C} : e^{z_1} \in \mathbb{R}$ and C_1 is a constant of spatial dimension -2. Therefore, there is only a solution for a certain class of $\alpha(r)$. Since the differential equations for the function g do not depend on the angel ϑ we can assume $\partial_2 g = 0$ which renders the differential equation (A.26)

$$-\partial_1 g = \left(\frac{1}{r} - \partial_1\alpha(r)\right) g, \tag{A.30}$$

which has for the certain class of solutions w.r.t. α the following form

$$g(r) = \exp(\alpha(r))\frac{1}{r}.$$

This solution gives us thus the following for the function h,

$$h(r,\vartheta) = \exp(-\alpha(r))\frac{\cot\vartheta}{r^2}.$$

The last set of solutions is given by Case 3, namely assuming that no function vanishes. For this case, we have,

$$f\partial_2 f = gh\,\Gamma^1_{33} - g^2\,\Gamma^2_{33} \tag{A.31}$$

$$-f\partial_1 f = f^2\,\Gamma^1_{11} + h^2\,\Gamma^1_{33} - hg\,\Gamma^2_{33} + f^2\,\Gamma^2_{12} \tag{A.32}$$

$$-g\partial_1 f - h\partial_2 f = gf\,\Gamma^1_{11} + gf\,\Gamma^2_{12} \tag{A.33}$$

$$\partial_2 g = -h\,\Gamma^1_{22} - 2g\,\Gamma^3_{23} \tag{A.34}$$

$$-\partial_1 g = g\,\Gamma^1_{11} + g\,\Gamma^3_{13} - h\,\Gamma^3_{32} \tag{A.35}$$

$$\partial_2 h = -h\,\Gamma^3_{23} \tag{A.36}$$

$$-\partial_1 h = 3h\,\Gamma^2_{12} \tag{A.37}$$

The strategy here is to first solve h by a separation of variables ansatz and then use the solution to solve for g. Finally, plugging the functions g and h into the differential equations for f renders the last differential equations. Therefore, we first solve for h

$$h(r,\vartheta) = \frac{C_1}{r^3}\csc(\vartheta)$$

Using this to solve Equation (A.34) we obtain

$$g(r,\vartheta) = C_1\frac{e^{2\alpha(r)}}{r^2}\cot(\vartheta)\csc(\vartheta) + C_3(r)\csc^2(\vartheta)$$

Plugging this into differential equation (A.35) gives us a differential equation for $C_3(r)$

$$\partial_1 C_3(r) = -C_1\cos(\vartheta)\frac{e^{2\alpha(r)}}{r^2}\partial_1\alpha + C_1\cos(\vartheta)\frac{e^{2\alpha(r)}}{r^3} + C_3(r)\partial_1\alpha - \frac{C_3(r)}{r} + C_1\frac{\cos(\vartheta)}{r^3} \tag{A.38}$$

Since the function $C_3(r)$ has to be independent of the angle ϑ we take the derivative w.r.t. this angle which results in

$$-re^{2\alpha(r)}\partial_1\alpha(r) + e^{2\alpha(r)} + 1 = 0$$

This therefore results as with Case 2 into a differential equation of $\alpha(r)$. The solution in case $C_1 \neq 0$ is given by

$$\alpha(r) = 1/2\log(C_4 r^2 - 1).$$

However, since r is unbounded the solution $C_1 = 0$ reduces Case 3 to Case 1, i.e. Solution I, see Equation (A.16).

□

A.5 Proof of Result 5.9

Proof. The only non-vanishing Christoffel symbols for the FRWL spacetime in case of flat spatial geometry are given by

$$\Gamma^0_{ij} = \delta_{ij}\, a\, \dot{a}, \qquad \Gamma^i_{0j} = \delta^i{}_j \dot{a}/a \tag{A.39}$$

We have the following equation that follows from Condition (4.1)

$$\Theta^{\mu\nu}\partial_\nu\Theta^{\alpha\beta} = -\Theta^{\mu\nu}\Theta^{\lambda\beta}\,\Gamma^\alpha_{\nu\lambda} - \Theta^{\mu\nu}\Theta^{\alpha\lambda}\,\Gamma^\beta_{\nu\lambda}$$

Assuming that the Poisson tensor does not depend on the spatial coordinates reduces the differential equations to

$$\Theta^{\mu 0}\partial_0\Theta^{\alpha\beta} = -\Theta^{\mu\nu}\Theta^{\lambda\beta}\,\Gamma^\alpha_{\nu\lambda} - \Theta^{\mu\nu}\Theta^{\alpha\lambda}\,\Gamma^\beta_{\nu\lambda}$$

For $\mu = 0$ we have

$$0 = -\Theta^{0i}\Theta^{\lambda\beta}\,\Gamma^\alpha_{i\lambda} - \Theta^{0i}\Theta^{\alpha\lambda}\,\Gamma^\beta_{i\lambda}$$

choosing $\alpha = 0$, $\beta = j$ renders

$$\begin{aligned} 0 &= -\Theta^{0i}\Theta^{\lambda j}\,\Gamma^0_{i\lambda} - \Theta^{0i}\Theta^{0\lambda}\,\Gamma^j_{i\lambda} \\ &= -\Theta^{0i}\Theta^{kj}\,\Gamma^0_{ik} \end{aligned}$$

Next, we choose $\alpha = k$, $\beta = j$ renders

$$\begin{aligned} 0 &= -\Theta^{0i}\Theta^{\lambda j}\,\Gamma^k_{i\lambda} - \Theta^{0i}\Theta^{k\lambda}\,\Gamma^j_{i\lambda} \\ &= -\Theta^{0i}\Theta^{0j}\,\Gamma^k_{i0} - \Theta^{0i}\Theta^{k0}\,\Gamma^j_{i0} \\ &= -\Theta^{0k}\Theta^{0j}\,\dot{a}/a - \Theta^{0j}\Theta^{k0}\,\dot{a}/a \end{aligned}$$

which is automatically fulfilled since $\Theta^{0j} = -\Theta^{j0}$. For $\mu = l$ we have

$$\Theta^{l0}\partial_0\Theta^{\alpha\beta} = -\Theta^{l\nu}\Theta^{\lambda\beta}\,\Gamma^\alpha_{\nu\lambda} - \Theta^{l\nu}\Theta^{\alpha\lambda}\,\Gamma^\beta_{\nu\lambda}$$

choosing $\alpha = 0$, $\beta = j$ reduces to

$$\begin{aligned} \Theta^{l0}\partial_0\Theta^{0j} &= -\Theta^{l\nu}\Theta^{\lambda j}\,\Gamma^0_{\nu\lambda} - \Theta^{l\nu}\Theta^{0\lambda}\,\Gamma^j_{\nu\lambda} \\ &= -\Theta^{li}\Theta^{kj}\,\Gamma^0_{ik} - \Theta^{l0}\Theta^{0k}\,\Gamma^j_{0k} \\ &= -\Theta^{li}\Theta^{kj}\,\Gamma^0_{ik} - \Theta^{l0}\Theta^{0j}\,\dot{a}/a \end{aligned}$$

and as last equations for $\alpha = k$, $\beta = j$ we have

$$\begin{aligned} \Theta^{l0}\partial_0\Theta^{kj} &= -\Theta^{l\nu}\Theta^{\lambda j}\,\Gamma^k_{\nu\lambda} - \Theta^{l\nu}\Theta^{k\lambda}\,\Gamma^j_{\nu\lambda} \\ &= -\Theta^{l0}\Theta^{ij}\,\Gamma^k_{0i} - \Theta^{li}\Theta^{0j}\,\Gamma^k_{i0} - \Theta^{l0}\Theta^{ik}\,\Gamma^j_{0i} - \Theta^{li}\Theta^{0k}\,\Gamma^j_{i0} \end{aligned}$$

Assuming that $\Theta^{ik} = 0$ the equations reduce to

$$\partial_0\Theta^{0j} = -\Theta^{0j}\,\dot{a}/a.$$

□

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