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Keywords: geometric analysis, Ricci flow, isoperimetric inequalities

Abstract

We revisit the connection between the Ricci flow and isoperimetric inequalities on surfaces which are diffeomorphic to the 2-sphere. We prove that the Cheeger isoperimetric constant is non-decreasing under Ricci flow on topological 2-spheres. A topological 2-sphere with nontrivial curvature is exhibited which is a counterexample to the hypothesis that the Cheeger constant is a strictly increasing function of the Ricci flow.

1 Introduction

There is a rich history of the interaction between the mean curvature flow and geometric inequalities which relate quantities such as area and volume. As an example, the theory of the one-dimensional mean curvature flow (usually known as the curve shortening flow) was developed in part using the isoperimetric inequality. One can also go in the other direction and use the flow theory to derive inequalities (the theory of the curve shortening flow can be used to derive an isoperimetric inequality which generalises inequalities of Alexandrov, Huber and Bol) [1, 2]. Another flow which might be useful in this context is the Ricci flow. Given a smooth closed manifold M equipped with a smooth Riemannian metric g, the Ricci flow is the following PDE:

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}(g),\tag{1}$$

where $\operatorname{Ric}(g)$ is the Ricci curvature of g. The Ricci flow was first used in the context of isoperimetric estimates for surfaces by Hamilton, who showed the following important result [3, 4]:

Proposition 1: Let (M, g(t)) be a solution of the Ricci flow on a topological 2-sphere. At any time t such that $h(M, g(t)) < 16/\sqrt{A}$, one has

$$\frac{d}{dt}h(M,g(t)) \ge 0. \tag{7}$$

If g_t is a metric which evolves on the two-sphere S^2 under Ricci flow, then \overline{C}_S is monotonically nondecreasing for all times t, where \overline{C}_S is defined as

$$\overline{C}_S = \inf_{\gamma} C_S(\gamma). \tag{2}$$

The infimum is taken over all smooth embedded connected curves γ and $C_S(\gamma)$ is the isoperimetric ratio L/\overline{L} , for L the length of the curve γ and $\overline{\gamma}$ the curve of smallest length \overline{L} on the round sphere.

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There are fewer results in higher dimension, but in dimension 3, Agol, Storm and Thurston showed that if one evolves the doubled version of a hyperbolic metric on an acylindrical 3-manifold with minimal

surface boundary under Ricci flow, then the ratio of area to volume with sectional curvature at least -1 and minimal surface boundary is minimized in the case of a totally geodesic hypersurface, which could be viewed as an isoperimetric-type inequality [5]. Another famous example of an isoperimetric-type inequality from mathematical relativity is the Penrose inequality, which relates the total mass of an initial data set to the surface area of its outermost apparent horizon. There is some work which suggests that it might be possible to use the Ricci flow to prove the time-symmetric version of the Penrose inequality, but it is unlikely to be useful for proving the general case [6].

In this article, we revisit the Ricci flow as a tool for establishing isoperimetric inequalities on surfaces. In Section 2, we consider the Cheeger isoperimetric constant and show that it is non-decreasing under Ricci flow on topological 2-spheres. In Section 3, we give an example of a topological 2-sphere whose Cheeger constant need not be a strictly increasing function of the Ricci flow, which establishes that a monotonically non-decreasing Cheeger constant is the optimal possible result. We end in Section 4 with conclusions and suggestions for future work. We note that the results which we establish here likely follow from the sharp estimates obtained by Andrews and Bryan using isoperimetric comparison for normalized Ricci flow on the two-sphere, so that the main benefit of our approach is the straightforward nature of the proofs [7].

2 The Cheeger Constant under Ricci Flow on Topological Spheres

A natural question to ask in the context of isoperimetric estimates is whether the Cheeger isoperimetric constant is always a strictly increasing function of Ricci flow. It is generally not true that the Cheeger constant is a strictly increasing function under Ricci flow in dimension 3. To see this, consider the counterexample of a dumbbell which develops a nondegenerate neckpinch singularity. Neckpinch singularities do not occur in dimension 2, so one can still ask whether the Cheeger constant is a strictly increasing function of Ricci flow if we restrict only to surfaces. This was posed as an open question of Manning during his work on the volume entropy of a compact Riemannian surface with negative curvature under Ricci flow [8]. Papasoglu studied the growth rate of the Cheeger constant of a non-compact surface of bounded genus [9].

As mentioned in Proposition 1, Hamilton had previously shown that the isoperimetric ratio is monotonically non-decreasing on the two-sphere under Ricci flow [4]. We now define the isoperimetric constant which we will use in the article, which is known as the Cheeger constant h(M) for a compact Riemmanian manifold M:

$$h(M) = \inf_{\gamma} \frac{S(\gamma)}{\min(V(A), V(B))},\tag{3}$$

where $S(\gamma)$ is the (n-1)-dimensional volume of a submanifold γ , V(A) is the volume of an *n*-dimensional submanifold, and the infimum is taken over all curves γ which divide the *n*-manifold M into two disjoint submanifolds A and B. We will prove that h is monotonically non-decreasing under Ricci flow on topological 2-spheres. Before proving this, we will state a few more preliminaries. Since we are restricting to closed orientable Riemannian surfaces, if γ separates M into two open manifolds M+ and M_{-} , then the Cheeger constant for the curve γ is equal to

$$h(\gamma) = L_g(\gamma) \left(\frac{1}{A_g(M_+)} + \frac{1}{A_g(M_-)} \right) = L_g(\gamma) \frac{A_g(M)}{A_g(M_+) \cdot A_g(M_-)},\tag{4}$$

where L_g is the length and A_g is the area measured with respect to the metric g. Furthermore, if M is a closed Riemannian surface, there exists a smooth embedded closed curve γ such that [10]

$$h(\gamma) = h(M). \tag{5}$$

Finally, an argument of [4] implies that if (M, g) is a Riemannian surface which is diffeomorphic to S^2 such that $h(M, g) < 16/\sqrt{A}$, then there exists a smooth embedded loop β such that

$$h(\beta) = h(M, g). \tag{6}$$

We will use these facts to prove that the Cheeger constant is non-decreasing on topological spheres under Ricci flow.

Proposition 2: Let (M, g(t)) be a solution of the Ricci flow on a topological 2-sphere. At any time t

such that $h(M, g(t)) < 4\pi/L$, one has

$$\frac{d}{dt}h(M,g(t)) \ge 0. \tag{7}$$

[*Proof*]: We begin by noting a result of Papasoglu which states that the Cheeger constant of a topological 2-sphere M (more specifically, a Riemannian manifold or simplicial complex which is homeomorphic to the 2-sphere) is bounded by its area A [9]:

$$h(M) \le \frac{16}{\sqrt{A}}.$$

In the case of a 2-sphere, this bound is $8/\sqrt{\pi}r$, where r is the radius of the sphere. We next show that if (M, g(t)) is a solution of the Ricci flow on a topological 2-sphere, the Cheeger constants $h(\rho, t)$ for parallel loops γ_{ρ} measured with respect to g(t) satisfy a heat-type equation given by

$$\frac{\partial}{\partial t}(\log h) = \frac{\partial^2}{\partial \rho^2}(\log h) + \frac{\Gamma}{L}\frac{\partial}{\partial \rho}(\log h) + \left(\frac{4\pi - hL}{A}\right)\left(\frac{A_+}{A_-} + \frac{A_-}{A_+}\right),\tag{9}$$

where

$$\Gamma = \int_{\gamma_{\rho}} k \, ds. \tag{10}$$

To prove this, we note that

 $\log h = \log L - \log A_{+} - \log A_{-} + \log A.$ (11)

By known results from the theory of Ricci flow on topological spheres, we have [11]

$$\frac{\partial}{\partial t}\log L = \frac{\partial^2}{\partial \rho^2} (\log L) + \frac{\Gamma}{L} \frac{\partial}{\partial \rho} (\log L), \qquad (12a)$$

$$\frac{\partial}{\partial t}\log A_{\pm} = \frac{\partial^2}{\partial \rho^2} (\log A_{\pm}) + \frac{L^2}{A_{\pm}^2} - \frac{4\pi}{A_{\pm}} + \frac{\Gamma}{L} \frac{\partial}{\partial \rho} (\log A_{\pm}), \tag{12b}$$

$$\frac{d}{dt}\log A = -\frac{8\pi}{A}.$$
(12c)

Algebraic manipulations show that

$$\frac{4\pi}{A_{+}} + \frac{4\pi}{A_{-}} - \frac{8\pi}{A} = \frac{4\pi}{A} \left(\frac{A_{+}}{A_{-}} + \frac{A_{-}}{A_{+}} \right), \tag{13a}$$

$$\frac{L^2}{A_+^2} + \frac{L^2}{A_-^2} = \frac{L^2}{A_+A_-} \left(\frac{A_+}{A_-} + \frac{A_-}{A_+} \right) = \frac{hL}{A} \left(\frac{A_+}{A_-} + \frac{A_-}{A_+} \right).$$
(13b)

If $h < 4\pi/L$, then log h is a supersolution of a heat-type equation.

To prove Proposition 2, it remains to show that

$$\left. \frac{d}{dt} h(M, g(t)) \right|_{t=t_0} \ge -\epsilon, \tag{14}$$

at all times t_0 such that $h(M, g(t)) < 4\pi/L$. Since there exists a loop such that

$$h(\beta) = h(M, g), \tag{15}$$

if this time t_0 exists, there must be a loop γ_0 whose Cheeger constant measured with respect to the metric $g(t_0)$ satisfies

$$h(0, t_0) = h(M, g(t)).$$
(16)

 γ_0 minimises h at time t_0 , which implies that



Figure 1: Pair of pants with spherical caps. The caps have positive curvature, whereas the rest of the surface has negative curvature.

$$\left(\frac{\partial}{\partial\rho}\log h\right)(0,t_0) = 0,\tag{17a}$$

$$\left(\frac{\partial^2}{\partial\rho^2}\log h\right)(0,t_0) \ge 0.$$
(17b)

Since $h \leq 4\pi/L$, the above result for the heat-type equation implies that

$$\left(\frac{\partial}{\partial t}\log h\right)(0,t_0) \ge 0. \tag{18}$$

This can only happen if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all times t between $t_0 - \delta$ and t_0 , one has

$$\log h(0,t) \le \log h(0,t_0) + \epsilon(t_0 - t) = \log h(M,g(t_0)) + \epsilon(t_0 - t).$$
(19)

It follows that for all times t between $t_0 - \delta$ and t_0 , we have

$$\log h(M, g(t)) + \epsilon t \le \log h(M, g(t_0)) + \epsilon t_0.$$
⁽²⁰⁾

Remark 3: It follows that h(M, g(t)) is non-decreasing under Ricci flow on topological 2-spheres by application of the maximum principle, since h(M, g(t)) is a continuous function of t with an upper bound $h(M, g(t)) \leq 4\pi/L$.

3 Counterexamples to Strictly Increasing Cheeger Constant

We have shown with a heat-type equation that the Cheeger constant is non-decreasing under Ricci flow, which does not rule out cases where it momentarily remains the same under Ricci flow. A similar equation can also be used to show that the isoperimetric ratio is non-decreasing under Ricci flow [11]. One might ask if it is possible to prove the stronger result that the constant can only increase under Ricci flow. Some simple counterexamples obviously rule this out in the general case.

Remark 4: The flat 2-torus has vanishing curvature, so the Cheeger constant remains the same under Ricci flow.

Remark 5: Any steady gradient Ricci soliton (these include Witten's black hole and the Bryant soliton).

A much more interesting counterexample would be for the case of a compact surface with variable negative curvature [8]. The methods of this article cannot be used to construct a counterexample in this case, as we have dealt only with topological spheres. We can, however, construct a counterexample amongst topological spheres which have a mixture of both positive and negative curvature. **Remark 6**: Pair of pants with spherical caps. This surface has been studied extensively in the context of ergodic geodesic flows, since it is an example of a sphere with such a flow [12, 13]. Note that the spherical caps have positive curvature, whereas the bulk of the surface has negative curvature -1. Specifically, it is assumed that the caps are rotationally symmetric and that they have exactly opposite curvature to the body of the pair of pants. Shorten the length of each trunk so that the loop γ used in the definition of the Cheeger constant circles across the middle of the surface and splits it into two surfaces A_+ and A_- , the former with mostly positive curvature such that it shrinks under Ricci flow and the latter with mostly negative curvature such that it expands.

If the lengths of the trunks are chosen so that $A_{+} = A_{-}$, we will have

$$\frac{\partial^2}{\partial \rho^2} (\log A_+) = -\frac{\partial^2}{\partial \rho^2} (\log A_-), \tag{21a}$$

$$\frac{\Gamma}{L}\frac{\partial}{\partial\rho}(\log A_{+}) = -\frac{\Gamma}{L}\frac{\partial}{\partial\rho}(\log A_{-}).$$
(21b)

The cuts can also be chosen so that at a certain time t the length of the loop is $L = 2\sqrt{\pi}\sqrt{A_{\pm}}$, which implies that at this time

$$\frac{\partial}{\partial t}(\log A_{+}) = \frac{\partial}{\partial t}(\log A_{-}) = 0.$$
(22)

Since

$$\frac{d}{dt}\log A = -\frac{8\pi}{A},\tag{23}$$

we may at this time re-scale the total area A of the pair of pants so that

$$\frac{\partial^2}{\partial \rho^2} (\log L) + \frac{\Gamma}{L} \frac{\partial}{\partial \rho} (\log L) = \frac{d}{dt} \log A.$$
(24)

Although re-scaling makes the negative curvature piece less negative and the positive curvature piece less positive, we are free to re-scale because the change in both occurs at the same rate. Since there is a time t where

$$\frac{\partial}{\partial t}(\log h) = 0, \tag{25}$$

there is also a time such that

$$\frac{\partial h}{\partial t} = 0, \tag{26}$$

which implies that the Cheeger constant of the surface is non-decreasing, but not strictly increasing as a function of the Ricci flow.

4 Conclusion

In summary, we showed that the Cheeger isoperimetric constant is non-decreasing under Ricci flow on topological 2-spheres. It is still an open question as to whether the Cheeger constant is strictly increasing for Ricci flow on compact surfaces of negative curvature only. It was shown by Katok that in this setting the Cheeger constant is at most $\int_M e^{u_0} dA_{\infty} \sqrt{-2\pi\chi(M)}$, where $\chi(M)$ denotes the Euler characteristic and u_t denotes a function from M to \mathbb{R} . This is lower than $h_{\infty} = \sqrt{-2\pi\chi(M)}$ (the value for constant curvature) and it is argued that h_0 is approached in the limit when L is a closed curve of constant geodesic curvature [14].

Acknowledgments

The author would like to thank Alexander Baumgartner for useful discussions.

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