The Method of Invariant Sets of Descending Flow for Locally Lipschitz Functionals *

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Abstract In this paper, we extend the method of invariant sets of descending flow that proposed by Sun Jingxian for smooth functionals to the locally Lipschitz functionals. By this way, we obtain the existence results for the positive, negative and sign-changing critical points of the locally Lipschitz functionals, and apply these theoretical results to the study of differential inclusion problems with p-Laplacian. In order to obtain the above results, we develop some new techniques: 1) We establish the method of how to extend the pseudogradient field to the whole space on the premise of preserving the useful information of the local pseudo-gradient field; 2) In the case of set-valued mapping, a pseudo-gradient field is established to make both the cone and the negative cone being invariant sets of descending flow. To obtain our main results, a new class of (PS) condition is also proposed.

Key words Locally Lipschitz Functionals, Critical Points, Differential Inclusion Problems

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1 Introduction

The method of invariant sets of descending flow was proposed by Sun Jingxian in [2]. So far, it has been widely used to study the existence of solutions of elliptic equation boundary value problems. Now let us recall some theories of the method of invariant sets of descending flow. Let E be a real Banach space, $J: E \to \mathbb{R}$ a C^1 functional. Let $K = \{u \in E: J'(u) = 0\}, E_0 = E \setminus K, W: E_0 \to E$ be a pseudo-gradient vector field of J. Now we consider the following initial value problem in E_0

$$\begin{cases} \frac{du(t)}{dt} = -W(u(t)) & t \geqslant 0, \\ u(0) = u_0. \end{cases}$$
 (1.1)

By the theory of ordinary differential equations in Banach spaces, (1.1) has a unique solution, denoted by $u(t, u_0)$, with its right maximal existence interval $[0, T(u_0))$.

The following Definition 2.1,2.2, and Theorem 1.1,1.2 can be fund in [1,2].

Definition 1.1.(Invariant sets of descending flow) Let M be a nonempty subset of E, we call M is a invariant set of descending flow generated by W of J, if $\{u(t, u_0) : 0 \le t < T(u_0)\} \subset M$ hold for all $u_0 \in M \setminus K$.

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Theorem 1.1. If $J \in C^1(E, \mathbb{R})$, $M \subset E$ is a closed invariant set of descending flow of (1.1), let $\alpha = \inf_{u \in M} J(u) > -\infty$, J satisfies the (PS) condition on M, then α is a critical value of J, and there exist $u_0 \in M$ such that $J'(u_0) = 0$, $J(u_0) = \alpha$.

Theorem 1.1 states that finding different critical points may be attributed to finding different invariant sets of descending flow.

Definition 1.2. Let M and D be invariant sets of descending flow of J, $D \subset M$, let

$$C_M(D) = \{u_0 \in M : there \ exists \ 0 \leqslant t' < T(u_0) \ such \ that \ u(t', u_0) \in D\}.$$

Then $C_M(D)$ is a invariant set of descending flow of J expanded by D. If $D = C_M(D)$, then D is called a complete invariant set of descending flow of J with respect to M.

Theorem 1.2. Let M be a closed and connected invariant set of descending flow of J, $D \subset M$ is an open invariant sets of descending flow of J. If $C_M(D) \neq M$, $\inf_{u \in \partial_M D} J(u) > -\infty$ and J satisfies the (PS) condition on $M \setminus D$, then

$$c = \inf_{u \in \partial_M C_M(D)} J(u) \geqslant \inf_{u \in \partial_M D} J(u) > -\infty,$$

and c is a critical value of J, there exist critical points $u^* \in \partial_M C_M(D)$ corresponding to c of J.

Theorem 1.2 states that there would exist a new critical point if $\partial_M C_M(D) \cap \partial_M(D) = \emptyset$. In order to determine whether a closed convex set is an invariant set of descending flow, Sun [3] also introduced the Schauder invariance condition. For more detailed theories on Sun's method, one can refer to the excellent paper [1] by Liu and Sun. This method has been extensively applied by many authors in the past 20 years or so to study the existence of solutions for various of boundary value problems; see [1,4-8,31] and the references therein. For instance, Liu and Sun [1] applied the method to semilinear elliptic boundary value problems and to second-order Hamiltonian systems. A typical result in [1] is as follows. Consider the semilinear Dirichlet problem

$$-\Delta u = f(x, u) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f \in C(\bar{\Omega} \times \mathbb{R})$ is a superlinear function with subcritical growth. Assume that f(x,u) + mu is increasing for some m > 0. If (1.2) has a supersolution ϕ and a subsolution ψ such that $\phi \leqslant \psi$ and both ϕ and ψ are not solutions of (1.2), then (1.2) has at least four solutions.

Chang [9], in ordered to study equations with discontinuities, developed an extension of the classical smooth critical point theory, to non-smooth locally Lipschitz functionals. The theory of Chang was based on the sub-differential of locally Lipschitz functionals due to Clarke [10]. Using this sub-differential, Chang proposed a generalization of the deformation lemma and obtained a Mountain Pass Theorem for Locally Lipschitz functionals. Subsequently, many other saddle points theorems for locally Lipschitz functionals have been obtained and been applied to the studying of the existence of solutions for various of elliptic equations boundary value with discontinuities; see [11-24] and the references therein.

Now, a natural question arises is whether the method of invariant sets of descending flow proposed by Sun can be applied to locally Lipschitz functionals. This paper will serve to fulfill this purpose. We will try to use this method to obtain the existence results for critical points of locally Lipschitz functionals. A direct extension of this method to the locally Lipschitz functional is faced with serious difficulties. As is well known, to use the method of invariant sets of descending flow for C^1 functionals we need first construct a pseudogradient vector field over the Banach space. However, for the Lipschitz functional case, one usually only can construct locally a pseudogradient vector field over a subset of the Banach space rather than the whole Banach space. This is the

first difficulty need to overcome. In this paper, we will analyze the altitude of the critical point of the locally Lipschitz functionals in advance, construct locally a pseudogradient vector field on a neighborhood of the altitude, and then extend this pseudogradient vector field to the entire Banach space. We can then use the method of invariant sets of descending flow that Sun have proposed to obtain the existence results for the critical points for locally Lipschitz functionals. In the course of the study, we need to determine whether a closed convex set is a invariant set of the descending flow generated by the pseudogradient vector field. In the cases of the functionals being of C^1 , we use the Schauder invariance condition presented in the literature [1-3]. However, in the cases of the functionals being of locally Lipschitz, no one has yet used the Schauder invariance condition to establish invariant flows on convex closed sets. This is the second difficulty we face. To this end, based on the conclusion in [20] about the relationship between the critical points on the whole space and the critical points on the closed convex sets under the Schauder invariance condition, and using the method in [9,14] to establish the descending flow on closed convex sets, we get the result that the closed convex set is a descending flow invariant set. Using the above method, in this paper we get the existence results of at least one sign-changing, at least one positive and at least one negative critical point of the locally Lipschitz functionals. The theoretical results can be applied to the study of the existence of sign-changing solutions for differential inclusion problems with p-Laplacian and so extend the relevant results in literatures.

2 Main Results

In what follows of this section we will let X and E be two real Banach spaces with the norms $\|\cdot\|$ and $\|\cdot\|_1$, respectively. Assume that X is reflexive, E is densely imbedded in X. Let X^* be the topological dual of X and $\langle\cdot,\cdot\rangle$ denote the duality pairing between X^* and X. Let P be a closed convex cone of X, that is, P is closed convex set in X, $\lambda x \in P$ for all $x \in P$ and $\lambda \geqslant 0$, and $P \cap (-P) = \{0\}$. Let $P_1 = P \cap E$. We assume that P_1 has a nonempty interior in the E topology, and denote its interior in the E topology by $\inf P_1$. For $x \in X$ and $A \subset X$, let $\operatorname{dist}_X(x,A) = \inf_{y \in A} \|x - y\|$. For any R > 0, let $B(0,R) = \{x \in X : \|x\| < R\}$, $B_1(0,R) = \{x \in E : \|x\|_1 < R\}$ and $S_R = \{x \in X : \|x\| = R\}$. Given a subset $A \subset E$, we write $\partial_E A$ for the boundary of A in E.

Let us recall some theories concerning the sub-differential theory of locally Lipschitz functionals due to Clarke [10]. A functional $\varphi: X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, there exists a neighbourhood U of x and a constant k > 0 depending on U such that $|\varphi(z) - \varphi(y)| \le k||z - y||$ for all $z, y \in U$. For such a functional we define generalized directional derivative $\varphi^0(x; h)$ at $x \in X$ in the direction $h \in X$ by

$$\varphi^{0}(x;h) = \lim_{x' \to x} \sup_{\lambda \downarrow 0^{+}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \mapsto \varphi^0(x; h)$ is sublinear, continuous. So by the Hahn-Banach theorem we know that $\varphi^0(x; \cdot)$ is the support function of a nonempty, convex and w^* -compact set

$$\partial \varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \leqslant \varphi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial \varphi(x)$ is called the generalized or Clarke sub-differential of φ at x. A point $x \in X$ is a critical point of φ if $0 \in \partial \varphi(x)$. Let $\mathbb{K} = \{x \in X : 0 \in \partial \varphi(x)\}.$

Proposition 2.1 ([12,13]). 1) If $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz functionals, then $\partial(\varphi + \psi)(x) \subset \partial\varphi(x) + \partial\psi(x)$, while for any $\lambda \in \mathbb{R}$ we have $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$; 2) If $\varphi : X \to \mathbb{R}$ is also convex, then this sudifferential coincides with the sub-differential in the sense of convex

analysis. If φ is strictly differentiable, then $\partial \varphi(x) = \{\varphi'(x)\}$; 3) If $\varphi : X \to \mathbb{R}$ is locally Lipschitz functional, $\partial \varphi(u)$ is a weakly*-compact subset of X^* which is bounded by the Lipschitz constant k > 0 of φ near u.

S.T. Kyritsi and N. S. Papageorgiou [14] developed a critical point theory for nonsmooth locally Lipschitz functionals defined on closed, convex set extending this way the work of Struwe. Let $C \subset X$ be a nonempty, non-singleton, closed and convex set. For $x \in C$ we define

$$m_C(x) = \inf_{x^*} \sup_{y} \left\{ \langle x^*, x - y \rangle : y \in C, ||x - y|| < 1, x^* \in \partial \varphi(x) \right\}.$$

Evidently, $m_C(x) \ge 0$ for all $x \in C$. This quantity can be viewed as a measure of the generalized slope of φ at $x \in C$. If φ admits an extension $\hat{\varphi} \in C^1(X)$, then $\partial \varphi(x) = \{\varphi'(x)\}$ and so we have

$$m_C(x) = \sup \{ \langle \varphi'(x), x - y \rangle : y \in C, ||x - y|| < 1 \},$$

which is the quantity used by Struwe [29,p.147]. Also if C = X, then we have

$$m_C(x) = m(x) = \inf\{\|x^*\|_* : x^* \in \partial \varphi(x)\},\$$

which is the quantity used by Chang [9].

Now let us introduce the outwardly directed condition and the Schauder invariance condition for set value mappings in a manner as [20]. Let X be reflexive. As usual, we will identify X^{**} with X while $F: X^* \mapsto 2^X$ will denote the duality map, given by

$$\mathsf{F}(x^*) := \{ x \in X : \langle x^*, x \rangle = \|x^*\|_*^2 = \|x\|^2 \}, \ \forall x^* \in X^*.$$

The set $F(x^*)$ turns out to be nonempty, convex, and closed; see,e.g. [13, pp. 311-319]. As in [20], we define

$$\nabla \varphi(x) := \mathsf{F}(\partial \varphi(x)), \quad x \in X. \tag{2.1}$$

Clearly, $\nabla \varphi(x)$ depends on the choice of the duality pairing between X and X^* whenever it is compatible with the topology of X. If X is a Hilbert space, the duality paring becomes the scalar product and (2.1) gives the usual gradient. Write I for the identity operator on X.

Suppose C is a convex and closed set of X. Let $\delta_C: X \to \mathbb{R} \cup \{+\infty\}$ be the indicator function of C, namely

$$\delta_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then we have

$$\partial \delta_C(x) = \{ x^* \in X^* : \langle x^*, z - x \rangle \leqslant 0, \forall z \in C \}.$$

The set $\partial \delta_C(x)$ is usually called normal cone to C at x.

Definition 2.1 ([20]). Suppose X is reflexive, C is a convex and closed set of $X, \varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous. If for $x \in \partial C$,

$$(-\partial \delta_C(x)) \cap \partial \varphi(x) \subset \{0\},\$$

then we say that $\partial \varphi$ turns out to be outwardly directed on ∂C . This clearly rewrites as

$$\forall z^* \in \partial \varphi(x) \setminus \{0\} \text{ there exists } z \in C \text{ fulfilling } \langle z^*, z - x \rangle < 0.$$

Definition 2.2 ([20], Schauder invariance condition). Suppose X is reflexive, C is a convex and closed set of X, $\varphi: X \mapsto \mathbb{R}$ is locally Lipschitz continuous. Then we call φ satisfies the Schauder invariance condition on ∂C if $(I - \nabla \varphi)(\partial C) \subset C$.

Remark 2.1. The well known Schauder invariance condition for a C^1 -functional φ on a Hilbert space X reads as $(I - \varphi')(C) \subset C$; see [1-3,25]. It has been extended to Banach spaces in [26]. The notation of Schauder invariance condition was firstly put forward by Sun Jingxian in [3]. It follows from [20, Theorem 4.4 and 4.5] we have the following Lemma 2.1.

Lemma 2.1 ([20]). Suppose X is reflexive, C is a convex and closed set of X, $\varphi : X \mapsto \mathbb{R}$ is locally Lipschitz continuous, φ is outwardly directed at ∂C or φ satisfies the Schauder invariance condition on ∂C . Then $m_C(x) = 0$ if and only if m(x) = 0.

Lemma 2.2 ([36], Von Neumann). Let X, Y be two Hausdorff topological linear spaces, $C \subset X$, $D \subset Y$ be two convex and compact sets. Let $\psi : X \times Y \mapsto \mathbb{R}$ satisfy: 1) $x \mapsto \psi(x,y)$ is upper semi-continuous and concave; 2) $y \mapsto \psi(x,y)$ is lower semi-continuous and convex. Then ψ has at least one saddle point $(\bar{x}, \bar{y}) \in C \times D$.

Definition 2.3. Let D be a nonempty closed subset of X. We say that φ satisfies the non-smooth CPS-condition on D, denoted by $(CPS)_D$, if every sequence $\{x_n\} \subset D$ such that $\{\varphi(x_n)\}$ is bounded and $(1 + ||x_n||)m_D(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence. If D = X, denoted simply by (CPS).

Let us introduce the following conditions:

- $(H_1) \varphi : X \mapsto \mathbb{R}$ satisfies the conditions (CPS), $(CPS)_P$ and $(CPS)_{-P}$;
- $(H_2) \varphi(0) = 0$ and 0 is a strictly local minimum point;
- (H₃) There exist sub-space E_1 of E and $R_0 > 0$ such that dim $E_1 = 2$, $E_1 \cap \operatorname{int}(P_1) \neq \emptyset$ and

$$\sup_{u \in S_{R_0} \cap E_1} \varphi(u) < 0; \tag{2.2}$$

- (H₄) either φ is outwardly directed at $\partial(\pm P)$, or $(I \nabla \varphi)(\pm P) \subset \pm P$;
- $(\mathbf{H}_5) \left(\mathbb{K} \setminus \{0\} \right) \cap \left(\partial_E P_1 \cup \partial_E (-P_1) \right) = \emptyset;$
- (H₆) for each a < b, $\varphi^{-1}([a,b]) \cap K$ is compact in E.

Remark 2.2. It follows from [20, Theorem 4.5] that φ is outwardly directed at $\partial(\pm P)$ if $(I - \nabla \varphi)(\pm P) \subset \pm P$. Here, in condition (H₄), we list the above two conditions at the same time for the purpose of application convenience.

We have the following main result.

Theorem 2.1. Suppose that $(H_1) \sim (H_6)$ hold. Then φ has at least one positive, one negative and one sign-changing critical point.

It follows from $E_1 \cap \text{int } P_1 \neq \emptyset$ that $E_1 \cap \text{int}(-P_1) \neq \emptyset$. Let $d_1 = \sup_{x \in B(0,R_2) \cap E_1} \varphi(x) + 1$ and

 $D_0 = \varphi^{-1}([-1, d_1])$. Let $K = \mathbb{K} \cap D_0$. Because 0 is an isolate critical point, we may take $\bar{R}_1 > 0$ small enough such that $B(0, \bar{R}_1) \cap (K \setminus \{0\}) = \emptyset$. Since $E \hookrightarrow X$, we take $R_1 > 0$ small such that $B_1(0, R_1) \subset B(0, \bar{R}_1)$. By using the condition $(H_1), (H_5)$ and (H_6) we may take $\delta > 0$ small enough such that

$$K_{3\delta} \cap (\partial_E P_1 \cup \partial_E (-P_1) \cup B_1(0, R_1)) = \emptyset, \tag{2.3}$$

where $K_{3\delta} = \{x \in D_0 : \operatorname{dist}_E(x, K) < 3\delta\}.$

To show Theorem 2.1 we need to give some lemmas.

Lemma 2.3. There exists a locally Lipschitz mapping $v: D_0 \setminus K_\delta \mapsto X$ such that $||v(x)|| \le 2(1+||x||)$ for any $x \in D_0 \setminus K_\delta$, $\langle x^*, v(x) \rangle \geqslant \frac{\gamma}{16}$ for some $\gamma > 0$ and all $x^* \in \partial \varphi(x)$. Moreover, $v: D_0 \setminus K_\delta \mapsto E$ is locally Lipschitz and

$$x - \frac{1}{1 + ||x||} v(x) \in P_1 \text{ for any } x \in P \cap (D_0 \backslash K_\delta), \tag{2.4}$$

$$x - \frac{1}{1 + ||x||} v(x) \in -P_1 \text{ for any } x \in -P \cap (D_0 \backslash K_\delta).$$
 (2.5)

Proof. Let $S = D_0 \setminus (P \cup (-P))$. First we claim that

$$(1 + ||x||)m(x) \geqslant \gamma, \quad \forall x \in S \backslash K_{\delta}, \tag{2.6}$$

$$(1 + ||x||)m_P(x) \geqslant \gamma, \quad \forall x \in (D_0 \cap P) \setminus K_\delta, \tag{2.7}$$

$$(1+||x||)m_{-P}(x) \geqslant \gamma, \quad \forall x \in (D_0 \cap (-P)) \setminus K_\delta. \tag{2.8}$$

We only show that (2.7) holds. In a similar way we can show that (2.6) and (2.8) hold. Arguing by make contradiction that (2.7) does'n hold. Then there exists an sequence $\{x_n\} \subset (D_0 \cap P) \setminus K_\delta$ such that $(1 + ||x_n||) m_P(x_n) \to 0$ as $n \to \infty$. Obviously, $\{\varphi(x_n)\}$ is bounded. Since φ satisfies the $(CPS)_P$ condition, up to a subsequence if necessary, we may assume that $x_n \to x_0$ as $n \to \infty$ for some $x_0 \in (D_0 \cap P) \setminus K_\delta$. Since $m_P(\cdot) : P \to \mathbb{R}$ is lower semi-continuous, we have $m_P(x_0) = 0$. It follows from (H_4) and Lemma 2.1 that $m(x_0) = 0$, which is a contradiction. Thus, (2.7) holds.

For each $x_0 \in S \setminus K_\delta$, take $w_0^* \in \partial \varphi(x_0)$ such that $m(x_0) = ||w_0^*||_* > 0$. Then we have $B(0, ||w_0^*||_*) \cap \partial \varphi(x_0) = \emptyset$, where

$$B(0, ||w_0^*||_*) = \{z^* \in X^* : ||z^*||_* < ||w_0^*||_*\}.$$

So, by using the separation theorem in the weak*-topology, we can find $u_1(x_0) \in X$ with $||u_1(x_0)|| = 1$ such that for all $z^* \in B(0, ||w_0^*||_*)$ and $y^* \in \partial \varphi(x_0)$,

$$\langle z^*, u_1(x_0) \rangle \leqslant \langle w_0^*, u_1(x_0) \rangle \leqslant \langle y^*, u_1(x_0) \rangle.$$

Recall that

$$||w_0^*||_* = \sup \{\langle z^*, u_1(x_0) \rangle : z^* \in B(0, ||w_0^*||_*) \},$$

then we have by (2.6),

$$\langle y^*, u_1(x_0) \rangle \geqslant ||w_0^*||_* > \frac{\gamma}{2(1+||x_0||)}.$$

Since the map $x \mapsto \partial \varphi(x)$ is usc. from X into X_w^* , we may take an open neighborhood $B_1(x_0, r_1(x_0))$ of x_0 , such that

$$\langle y^*, u_1(x_0) \rangle > \frac{\gamma}{4(1+||y||)}, \quad \forall y^* \in \partial \varphi(y), \ y \in U_1(x_0),$$
 (2.9)

where $U_1(x_0) = B_1(x_0, r_1(x_0)) \cap (S \setminus K_{\delta})$. Since $x_0 \in S \setminus K_{\delta}$ and $S \setminus K_{\delta}$ is an open subset of $D_0 \setminus K_{\delta}$, we may take $r_1(x_0) > 0$ small enough such that $U_1(x_0) \subset S \setminus K_{\delta}$.

Pick $x_0 \in (P \cap D_0) \setminus K_\delta$. Let $C = (\{x_0\} - P) \cap \overline{B}(0,1)$ and $D = \partial \varphi(x_0)$, where $\overline{B}(0,1) = \{x \in X : ||x|| \leq 1\}$. Let X_w and X_w^* be the spaces X and X^* furnished their weak topology respectively. Since X is reflexive, D is compact in X_w^* and C is compact in X_w . Obviously, both C and D are convex. Let $\psi : C \times D \mapsto \mathbb{R}$ be defined by $\psi(x, y^*) = \langle y^*, x \rangle$ for any $(x, y^*) \in C \times D$. It follows from Lemma 2.2 that ψ has at least one saddle point $(u_2(x_0), x_0^*) \in C \times D$, that is

$$\min_{x^* \in D} \max_{x \in C} \langle x^*, x \rangle = \langle x_0^*, u_2(x_0) \rangle = \max_{x \in C} \min_{x^* \in D} \langle x^*, x \rangle, \ \forall x^* \in \partial \varphi(x_0), x \in (\{x_0\} - P) \cap \bar{B}(0, 1).$$

Thus, we have

$$\langle x_0^*, x \rangle \leqslant \langle x_0^*, u_2(x_0) \rangle \leqslant \langle x^*, u_2(x_0) \rangle, \ \forall x^* \in \partial \varphi(x_0), x \in (\{x_0\} - P) \cap \bar{B}(0, 1).$$

It follows from (2.7) that for any $x^* \in \partial \varphi(x_0)$,

$$\langle x^*, u_2(x_0) \rangle \geqslant \langle x_0^*, u_2(x_0) \rangle = m_P(x_0) > \frac{\gamma}{2(1 + ||x_0||)}.$$

By using the fact that $x \mapsto \partial \varphi(x)$ is upper semi-continuous, we know that there exists an open neighborhood $B_2(x_0, r_2(x_0))$ of x_0 such that for any $y \in U_2(x_0)$, $y^* \in \partial \varphi(y)$,

$$\langle y^*, u_2(x_0) \rangle > \frac{\gamma}{4(1+||y||)}.$$
 (2.10)

where $U_2(x_0) = B_2(x_0, r_2(x_0)) \cap (D_0 \backslash K_\delta)$. Since $x_0 \in (P \cap D_0) \backslash K_\delta \subset D_0 \backslash (K_\delta \cup (-P))$ and $D_0 \backslash (K_\delta \cup (-P))$ is an open set of $D_0 \backslash K_\delta$, we may take $r_2(x_0) > 0$ small such that $U_2(x_0) \cap (-P) = \emptyset$.

Similarly, by (2.8) we can show that for each $x_0 \in (D_0 \cap (-P)) \setminus K_\delta$, there exist $r_3(x_0) > 0$, $u_3(x_0) \in X$ with $||u_3(x_0)|| \leq 1$, such that

$$x_0 - u_3(x_0) \in -P, \quad U_3(x_0) \cap P = \emptyset$$

and for any $y \in U_3(x_0), y^* \in \partial \varphi(y)$,

$$\langle y^*, u_3(x_0) \rangle > \frac{\gamma}{4(1+\|y\|)},$$
 (2.11)

where $U_3(x_0) := B_3(x_0, r_3(x_0)) \cap (D_0 \backslash K_\delta)$.

By 3) in Proposition 2.1, we may assume that $||x^*||_* \leq L_{\alpha,i}$ for some $L_{\alpha,i} > 0$ and any $x \in U_i(x_\alpha)$ with i = 1, 2, 3 and $x^* \in \partial \varphi(x)$. Also, we assume that for i = 1, 2, 3, $B_i(x_\alpha, r_i(x_\alpha))$ has a small radium $r_i(x_\alpha) > 0$ such that $(1 + ||x||)(1 + ||x_\alpha||)^{-1} \leq 2$ for each $x \in U_i(x_\alpha)$, and

$$0 < r_i(x_\alpha) \le \min\left\{\frac{1}{2}, \frac{\gamma}{16(1 + ||x_\alpha||)L_{\alpha,i}}\right\}.$$

Let

$$\mathscr{A}_1 = \{ U_1(x_0) : x_0 \in S \setminus K_\delta \},$$

$$\mathscr{A}_2 = \{ U_2(x_0) : x_0 \in (D_0 \cap P) \setminus K_\delta \},$$

$$\mathscr{A}_3 = \{ U_3(x_0) : x_0 \in (D_0 \cap (-P)) \setminus K_\delta \},$$

and $\mathscr{A}=\mathscr{A}_1\cup\mathscr{A}_2\cup\mathscr{A}_3$. Then \mathscr{A} is an open cover of $D_0\backslash K_\delta$. By paracompactness we can find a locally finite refinement $\mathscr{B}=\{V_\alpha:\alpha\in\Lambda\}$ and a locally Lipschitz partition of unit $\{\gamma_\alpha:\alpha\in\Lambda\}$ sub-ordinate to it. For each $\alpha\in\Lambda$ we can find $x_\alpha\in D_0\backslash K_\delta$ such that $V_\alpha\subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha)\in\{1,2,3\}$, and $U_{i(\alpha)}(x_\alpha)\in\mathscr{A}$. To this $x_\alpha\in D_0\backslash K_\delta$ corresponds the element $w_\alpha^{i(\alpha)}$ such that $\|w_\alpha^{i(\alpha)}\|\leqslant 1$, and $w_\alpha^{i(\alpha)}=u_{i(\alpha)}(x_\alpha)$ if $V_\alpha\subset U_{i(\alpha)}(x_\alpha)$ for some $i(\alpha)\in\{1,2,3\}$. Since E is densely imbedded in X, we may take $\bar{x}_\alpha, \bar{w}_\alpha^{i(\alpha)}\in E$, such that $\|\bar{w}_\alpha^{i(\alpha)}\|\leqslant 1$, $\bar{x}_\alpha-\bar{w}_\alpha^{i(\alpha)}\in P_1$ when $x_\alpha\in P$, $\bar{x}_\alpha-\bar{w}_\alpha^{i(\alpha)}\in -P_1$ when $x_\alpha\in -P$ and

$$\max \left\{ \|w_{\alpha}^{i(\alpha)} - \bar{w}_{\alpha}^{i(\alpha)}\|, \|x_{\alpha} - \bar{x}_{\alpha}\| \right\} < \min \left\{ \frac{1}{2}, \frac{\gamma}{32(1 + \|x_{\alpha}\|)L_{\alpha, i(\alpha)}} \right\}. \tag{2.12}$$

Now, let $v: D_0 \backslash K_\delta \mapsto X$ be defined by

$$v(x) = (1 + ||x||) \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (\bar{w}_{\alpha}^{i(\alpha)} - \bar{x}_{\alpha} + x).$$
 (2.13)

It is easy to see that $v: D_0 \setminus K_\delta \mapsto X$ is locally Lipschitz. Since $E \hookrightarrow X$, by [32, Lemma 2.3] we see that $\gamma_\alpha: E \to \mathbb{R}$ is also locally Lipschitz. So, we can prove that $v: D_0 \setminus K_\delta \mapsto E$ is locally Lipschitz.

By (2.12) and (2.13), we have

$$||v(x)|| \le (1 + ||x||) \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (||\bar{w}_{\alpha}^{i(\alpha)}|| + ||\bar{x}_{\alpha} - x_{\alpha}|| + ||x_{\alpha} - x||) \le 2(1 + ||x||),$$

and

$$\begin{aligned} (1 + \|x\|) & & \left| \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) \langle x^{*}, x - \bar{x}_{\alpha} \rangle \right| \\ & \leqslant \sum_{\alpha \in \Lambda} \frac{1 + \|x\|}{1 + \|x_{\alpha}\|} (1 + \|x_{\alpha}\|) \gamma_{\alpha}(x) \|x^{*}\|_{*} \left(\|x - x_{\alpha}\| + \|x_{\alpha} - \bar{x}_{\alpha}\| \right) \\ & \leqslant 2 \sum_{\alpha \in \Lambda} L_{\alpha, i(\alpha)} \left(\|x - x_{\alpha}\| + \|x_{\alpha} - \bar{x}_{\alpha}\| \right) (1 + \|x_{\alpha}\|) \\ & < \frac{\gamma}{16} \end{aligned}$$

for each $x \in D_0 \backslash K_\delta$. On the other hand,

$$\left| \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (1 + ||x||) \langle x^*, \bar{w}_{\alpha}^{i(\alpha)} - w_{\alpha}^{i(\alpha)} \rangle \right| \leq \sum_{\alpha \in \Lambda} \frac{1 + ||x||}{1 + ||x_{\alpha}||} (1 + ||x_{\alpha}||) \gamma_{\alpha}(x) ||x^*||_* ||\bar{w}_{\alpha}^{i(\alpha)} - w_{\alpha}^{i(\alpha)}|| \\
\leq 2 \sum_{\alpha \in \Lambda} (1 + ||x_{\alpha}||) L_{\alpha, i(\alpha)} ||\bar{w}_{\alpha}^{i(\alpha)} - w_{\alpha}^{i(\alpha)}|| < \frac{\gamma}{16},$$

and so, by $(2.9)\sim(2.11)$ we have

$$\begin{split} \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (1 + \|x\|) \langle x^*, \bar{w}_{\alpha}^{i(\alpha)} \rangle &= \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (1 + \|x\|) \langle x^*, \bar{w}_{\alpha}^{i(\alpha)} - w_{\alpha}^{i(\alpha)} + w_{\alpha}^{i(\alpha)} \rangle \\ &= \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (1 + \|x\|) \langle x^*, w_{\alpha}^{i(\alpha)} \rangle \\ &+ \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (1 + \|x\|) \langle x^*, \bar{w}_{\alpha}^{i(\alpha)} - w_{\alpha}^{i(\alpha)} \rangle \\ &\geqslant \frac{\gamma}{4} - \frac{\gamma}{16} \geqslant \frac{\gamma}{8}. \end{split}$$

Then, we have

$$\langle x^*, v(x) \rangle = \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (1 + ||x||) \langle x^*, \bar{w}_{\alpha}^{i(\alpha)} \rangle + \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (1 + ||x||) \langle x^*, x - \bar{x}_{\alpha} \rangle \geqslant \frac{\gamma}{16}.$$

For any $x \in P \cap (D_0 \backslash K_\delta)$, we have

$$x - \frac{1}{1 + ||x||} v(x) = \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (\bar{x}_{\alpha} - \bar{w}_{\alpha}^{i(\alpha)}) \in P_1.$$

This implies that (2.4) holds. Similarly, (2.5) also holds. The proof of is complete.

Let $l_1, l_2 : D_0 \mapsto \mathbb{R}$ be defined by for any $x \in D_0$,

$$l_1(x) = \frac{\operatorname{dist}_X(x, K_{\delta})}{\operatorname{dist}_X(x, K_{\delta}) + \operatorname{dist}_X(x, D_0 \setminus K_{2\delta})},$$

$$l_2(x) = \frac{\operatorname{dist}_X(x, (\varphi^{-1}([d_1 - \frac{1}{4}, d_1]) \cup \varphi^{-1}([-1, -\frac{1}{2}])))}{\operatorname{dist}_X(x, \varphi^{-1}([-\frac{1}{4}, d_1 - \frac{1}{2}])) + \operatorname{dist}_X(x, (\varphi^{-1}([d_1 - \frac{1}{4}, d_1]) \cup \varphi^{-1}([-1, -\frac{1}{4}])))},$$

$$v_1(x) = \begin{cases} v(x), & \text{if } x \in D_0 \setminus K_{\delta}; \\ 0, & K_{\delta}, \end{cases}$$

and

$$V(x) = \begin{cases} l_1(x)l_2(x)v_1(x), & \text{if } x \in D_0; \\ 0, & \text{othewise.} \end{cases}$$

Then $l_1, l_2 : X \mapsto \mathbb{R}$ are locally Lipschitz. Since $E \hookrightarrow X$, $l_1, l_2 : E \mapsto \mathbb{R}$ are also locally Lipschitz. Obviously, $V: X \mapsto X$ is locally Lipschitz.

Consider the following initial value problem

$$\begin{cases}
\frac{du}{dt} = -V(u), \\
u(0) = v_0 \in X.
\end{cases}$$
(2.14)

By the theories for initial value problems of ordinary equations in Banach space, we see that (2.14) has a unique solution $\sigma(t, v_0)$, with its right maximal existence interval $[0, T(v_0))$ in the X topology, and its right maximal existence interval $[0, T_1(v_0))$ in the E topology. Since $E \hookrightarrow X$, we have $T_1(v_0) \leq T(v_0)$.

As the Definition 1.1 and 1.2, we give the following Definition 2.3 and 2.4.

Definition 2.4. A nonempty subset D of E is called an invariant set of descending flow of (2.14) if $o(v_0) \subset D$ for all $v_0 \in D$, where $o(v_0) = \{\sigma(t, v_0) \subset E : t \in [0, T_1(v_0))\}.$

Definition 2.5. Let $M \subset E$ be a connected invariant set of descending flow of (2.14), D be an open subset of M and be an invariant set of descending flow of (2.14). Denote

 $C_M(D) = \{v_0 : v_0 \in D \text{ or } v_0 \in M \setminus D \text{ and there exists } t' \in (0, T_1(v_0)) \text{ such that } \sigma(t', v_0) \in D\}.$

If $D = C_M(D)$, then D is called a complete invariant set of descending flow of (2.14) in M.

Lemma 2.4. For each $v_0 \in E$, the solution $\sigma(t, v_0)$ of (2.14) has the following properties:

- 1) $T(v_0) = +\infty$;
- 2) $T_1(v_0) = +\infty$ if $o(v_0) \subset D_0 \backslash K_\delta$ and $v_0 \in (D_0 \cap E) \backslash K_\delta$;
- 3) $\varphi(\sigma(t, v_0))$ is non-increasing in $t \in [0, +\infty)$;
- 4) P_1 , $-P_1$, int P_1 and int $(-P_1)$ are all invariant sets of descending flow of (2.14); 5) For each $v_0 \in \varphi^{-1}\left((-\infty, d_1 \frac{1}{2}]\right) \cap E$ with $\inf_{u \in o(v_0)} \varphi(u) \geqslant -\frac{1}{4}$, there exists $\tau(v_0) \geqslant 0$ such that $\sigma(\tau(v_0), v_0) \in K_{2\delta}$.

Proof. 1) First we prove that $T(v_0) = +\infty$. Arguing by make contradiction that $T(v_0) < \infty$ $+\infty$. By (2.14) we have

$$\|\sigma(t, v_0) - v_0\| \le \int_0^t \|V(\sigma(s, v_0))\| ds \le 2 \int_0^t (1 + \|\sigma(s, v_0)\|) ds.$$

So, we have

$$\frac{1}{2} \|\sigma(t, v_0) - v_0\| \leq \int_0^t (1 + \|\sigma(s, v_0)\|) ds
\leq \int_0^t \|\sigma(s, v_0) - v_0\| ds + (1 + \|v_0\|)t,$$

By the well known Gronwall's inequality, we have

$$\frac{1}{2} \| \sigma(t, v_0) - v_0 \| \leq \int_0^t (1 + \|v_0\|) e^{t-s} ds + (1 + \|v_0\|) t
\leq (1 + \|v_0\|) (e^t - 1) + (1 + \|v_0\|) t
\leq (1 + \|v_0\|) (t + e^t - 1)
\leq (1 + \|v_0\|) (T(v_0) + e^{T(v_0)}).$$

So, we have

$$\|\sigma(t, v_0)\| \le 2(1 + \|v_0\|)(T(v_0) + e^{T(v_0)}) + 2\|v_0\| =: M_0(v_0).$$

Take $\{t_n\} \subset [0, T(v_0))$ such that $t_n \to T^-(v_0)$ and for $n = 1, 2, \dots$,

$$|t_n - t_{n-1}| < \frac{1}{2 \cdot 2^n (1 + M_0(v_0))}.$$

Then we have

$$\|\sigma(t_n, v_0) - \sigma(t_{n-1}, v_0)\| \le \int_{t_{n-1}}^{t_n} \|V(\sigma(s, v_0))\| ds$$

$$\le 2 \int_{t_{n-1}}^{t_n} (1 + \|\sigma(s, v_0)\|) ds$$

$$\le 2(1 + M_0(v_0))(t_n - t_{n-1}) < \frac{1}{2^n}.$$

This implies that $\{\sigma(t_n, v_0)\}$ is a Cauchy sequence. Thus, there exists $\bar{u} \in X$ such that $\sigma(t_n, v_0) \to \bar{u}$ as $t_n \to T^-(v_0)$. Then, we can show that $\sigma(t, v_0) \to \bar{u}$ as $t \to T^-(v_0)$.

Now we consider the initial value problem

$$\begin{cases}
\frac{du}{dt} = -V(u), \\
u(0) = \bar{u},
\end{cases}$$
(2.15)

Then (2.15) has a unique solution on $[0, \bar{\delta})$ for some $\bar{\delta} > 0$, and so (2.14) has a unique solution on $[0, T(v_0) + \bar{\delta})$, which is a contradiction. Thus, we have $T(v_0) = +\infty$.

2) Let the operator $A: E \to X$ be defined by

$$Ax = x - \frac{1}{1 + ||x||}V(x), \quad \forall x \in E.$$

Then we have for any $x \in E$,

$$Ax = \left\{ \begin{array}{ll} x, & \text{if } x \in \left(E \backslash \varphi^{-1}(-\frac{1}{2}, d_1 - \frac{1}{4}) \right) \cup K_{\delta}; \\ x - \frac{1}{1 + \|x\|} v(x), & \text{if } x \in \varphi^{-1}([-\frac{1}{4}, d_1 - \frac{1}{2}]) \backslash K_{2\delta}; \\ l(x) \left(x - \frac{1}{1 + \|x\|} v(x) \right) + \left(1 - l(x) \right) x, & \text{otherwise.} \end{array} \right\} \in E,$$

where $l(x) := l_1(x)l_2(x) \in (0,1)$, and

$$Ax = \sum_{\alpha \in \Lambda} \gamma_{\alpha}(x) (\bar{w}_{\alpha}^{i(\alpha)} - \bar{x}_{\alpha}), \ \forall x \in D_0 \backslash K_{2\delta}.$$
 (2.16)

Let

$$\mu(t) = \int_0^t (1 + \|\sigma(s, v_0)\|) ds \text{ for } t \in [0, +\infty).$$

Obviously, $\mu:[0,T_1(v_0))\to[0,+\infty)$ is increasing. And so, μ^{-1} , the inverse function of μ , exists. Then we have

$$\begin{cases} \frac{d}{dt} (e^{\mu(t)} \sigma(t, v_0)) = e^{\mu(t)} (1 + ||\sigma(t, v_0)||) A \sigma(t, v_0), \\ \sigma(0, v_0) = v_0. \end{cases}$$

By direct computation, we have

$$\sigma(t, v_0) = e^{-\mu(t)} v_0 + e^{-\mu(t)} \int_0^t e^{\mu(s)} (1 + \|\sigma(s, v_0)\|) A\sigma(s, v_0) ds.$$
 (2.17)

where the integral is in the sense of X topology.

By (2.16), one can easily show that for each T > 0, $\{A(\sigma(t, v_0)) : t \in [0, T]\}$ is contained in a finite-dimensional subspace of X. Now we show that $T_1(v_0) = +\infty$ when $o(v_0) \subset D_0 \setminus K_{2\delta}$. Arguing by make contradiction that $T_1(v_0) < +\infty$. Take $T > T_1(v_0)$. Note $\{\sigma(t, v_0) : t \in [0, T]\}$ is bounded in X. Then, there exists $M_1(T) > 0$ such that for all $s \in [0, T]$,

$$||e^{\mu(s)}(1+||\sigma(s,v_0)||)A\sigma(s,v_0)||_1 \leqslant M_1(T).$$

Let $\{t_n\} \subset [0, T_1(v_0))$ such that $t_n \to T_1^-(v_0)$ as $n \to \infty$. Note (2.17) also holds in which the integral is in the sense of E topology for any $t \in [0, T_1(v_0))$. For each $n = 2, 3, \dots$, let $\tau_n = \max\{t_n, t_{n-1}\}$ and $\tau_{n-1} = \min\{t_n, t_{n-1}\}$. Then we have

$$\begin{split} \|\sigma(t_{n},v_{0}) - \sigma(t_{n-1},v_{0})\|_{1} & \leq |e^{-\mu(t_{n})} - e^{-\mu(t_{n-1})}| \Big(\|v_{0}\|_{1} \\ & + \int_{0}^{t_{n}} e^{\mu(s)} \Big(1 + \|\sigma(s,v_{0})\| \Big) \|A\sigma(s,v_{0})\|_{1} ds \Big) \\ & + e^{-\mu(t_{n-1})} \int_{\tau_{n-1}}^{\tau_{n}} e^{\mu(s)} \Big(1 + \|\sigma(s,v_{0})\| \Big) \|A\sigma(s,v_{0})\|_{1} ds \\ & \leq |e^{-\mu(t_{n})} - e^{-\mu(t_{n-1})}| \Big(\|v_{0}\|_{1} + TM_{1}(T) \Big) + M_{1}(T)|t_{n} - t_{n-1}|. \end{split}$$

So, $\{\sigma(t_n, v_0)\}$ is a Cauchy sequence in E. Assume that $\sigma(t_n, v_0) \to \bar{u}$ in E as $t_n \to T_1^-(v_0)$. Since $V : E \mapsto E$ is also locally Lipschitz, one can easily get a contradiction as the proof of $T(v_0) = +\infty$. Thus, we have $T_1(v_0) = +\infty$.

3) Let $h(t, v_0) = \varphi(\sigma(t, v_0))$ for all $t \in [0, T_1(v_0))$. It is easy to see that $h(t, v_0)$ is locally Lipschitz in $t \in [0, +\infty)$, hence differentiable almost everywhere. According to Leburng's Mean Theorem we have

$$\frac{\partial}{\partial s}h(s, v_0) \leqslant \max\left\{\left\langle w^*, \frac{\partial}{\partial s}\sigma(s, v_0)\right\rangle : w^* \in \partial\varphi(\sigma(s, v_0))\right\} \text{ a.e.}$$

$$= -\min\left\{\left\langle w^*, V(\sigma(s, v_0))\right\rangle : w^* \in \partial\varphi(\sigma(s, v_0))\right\} \text{ a.e.}$$

$$\leqslant \begin{cases} -\frac{\gamma}{16}, & \text{if } \sigma(s, v_0) \in \varphi^{-1}([-\frac{1}{4}, d_1 - \frac{1}{2}]) \setminus K_{2\delta}; \\ 0, & \text{otherwise} \end{cases} \tag{2.18}$$

for almost every $t \in [0, T_1(v_0))$. Consequently, $\varphi(\sigma(t, v_0))$ is non-increasing in $t \in [0, T_1(v_0))$.

4) For $u \in P_1$, it follows from Lemma 2.3 that for $\lambda > 0$ small enough,

$$u + \lambda(-V(u)) = \lambda l_1(u)l_2(u)(1 + ||u||)\left(u - \frac{v_1(u)}{1 + ||u||}\right) + \left(1 - \lambda l_1(u)l_2(u)(1 + ||u||)\right)u \in P_1.$$

It follows from the theorem due to Brezis-Martin (see [27]) that P_1 is an invariant sets of descending flow of (2.14). In a similar way we can show that $-P_1$ is also an invariant set of descending flow of (2.14).

Next we show that $\operatorname{int} P_1$ is an invariant set of (2.14). Take $u \in \operatorname{int} P_1$. Note (2.17) also holds where the integral is in the sense of E topology. Make a variable change $\tau = e^{\mu(s)} - 1$ in (2.17). Then we have

$$s = \mu^{-1} \left(\ln(1+\tau) \right), \ ds = \frac{e^{-\mu(s)}}{\mu'(s)} d\tau$$

and

$$e^{-\mu(t)} \int_0^t e^{\mu(s)} (1 + \|\sigma(s, v_0)\|) A\sigma(s, v_0) ds$$

$$= e^{-\mu(t)} \int_0^{e^{\mu(t)} - 1} A\sigma(\mu^{-1}(\ln(1 + \tau)), u) d\tau \text{ (the integral is in the sense of } E \text{ topology)}$$

$$= \lim_{n \to \infty} \frac{1}{n} (1 - e^{-\mu(t)}) \sum_{k=0}^{n-1} A\sigma(\mu^{-1}(\ln(1 + \frac{k(e^{\mu(t)} - 1)}{n}), u).$$

It follows from (2.16) and Lemma 2.3 that $A(P_1) \subset P_1$. Since P_1 is an invariant set of descending flow of (2.14), we have

$$\bar{v} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} A\sigma \Big(\mu^{-1} \Big(\ln(1 + \frac{k(e^{\mu(t)} - 1)}{n} \Big), u \Big) \in P_1.$$

It follows from (2.17) that

$$\sigma(t, u) = e^{-\mu(t)}u + (1 - e^{-\mu(t)})\bar{v}. \tag{2.19}$$

Since $u \in \text{int} P_1$, there exists $r_0 > 0$ such that $B_1(u, r_0) \subset \text{int} P_1$, where $B_1(u, r_0) = \{y \in E : \|y - u\|_1 < r_0\}$. For each $\bar{x} \in B_1(\sigma(t, u), e^{-\mu(t)}r_0)$, let

$$\bar{y} = u + e^{\mu(t)}(\bar{x} - \sigma(t, u)).$$

Then we have

$$\|\bar{y} - u\|_1 = e^{\mu(t)} \|\bar{x} - \sigma(t, u)\|_1 < r_0.$$

This implies that $\bar{y} \in B_1(u, r_0)$. On the other hand, by (2.19) we have

$$\bar{x} = e^{-\mu(t)}\bar{y} - e^{-\mu(t)}u + \sigma(t, u) = e^{-\mu(t)}\bar{y} + (1 - e^{-\mu(t)})\bar{v} \in P_1.$$

So, $B_1(\sigma(t,u), e^{-\mu(t)}r_0) \subset P_1$ and $\sigma(t,u) \in \text{int}P_1$ for all $t \in [0, T_1(v_0))$. This implies that int P_1 is an invariant set for the descending flow of (2.14).

Similarly, int $(-P_1)$ is an invariant set for the descending flow of (2.14).

- 5) For each $v_0 \in \varphi^{-1}((-\infty, d_1 \frac{1}{2}]) \cap E$ with $\inf_{u \in o(v_0)} \varphi(u) \geqslant -\frac{1}{4}$, we assume the conclusion
- 5) doesn't hold. It follows from (2.14) and (2.18) that

$$\varphi(v_0) - \varphi(\sigma(t, v_0)) = -\int_0^t \frac{\partial}{\partial s} h(s, v_0) ds \geqslant \frac{\gamma}{16} t \text{ for } t \in [0, T_1(v_0)).$$

In this case we have $T_1(v_0) = +\infty$ if $o(v_0) \in D_0 \setminus K_\delta$. So, for $t_0 = 16\gamma^{-1}(d_1 + 1)$, we have

$$\varphi(\sigma(t_0, v_0)) \leqslant \varphi(v_0) - \frac{\gamma}{16}t_0 \leqslant d_1 - \frac{1}{2} - \frac{\gamma}{16}t_0 < -1,$$

which contradicts to $\inf_{u \in o(v_0)} \varphi(u) \geqslant -\frac{1}{4}$.

The proof is complete.

Similar to the proof of Theorem 1.2 we have the following Lemma 2.5.

Lemma 2.5. Let $G \subset E$ be a connected and invariant set of (2.14), and D be an open invariant subset of G. Then the following assertions hold:

- 1) $C_G(D)$ is an open subset of G;
- 2) $\partial_G C_G(D)$ is an invariant set of descending flow of (2.14);
- 3) $\inf_{u \in \partial_G C_G(D)} \varphi(u) \geqslant \inf_{u \in \partial_G(D)} \varphi(u).$

Lemma 2.6 ([35]). Assume U is bounded connected open set of \mathbb{R}^2 and $(0,0) \in U$, then there exists a connected component Γ' of the boundary of U, such that each one side ray l emitting from the origin satisfies $l \cap \Gamma' \neq \emptyset$.

The Proof of Theorem 2.1.

Take $d_0 \in (0, d_1 - \frac{1}{2})$. Let O be the connected component of $\varphi^{-1}(-\infty, d_0) \cap E$ containing 0. It follows from Lemma 2.5 that $C_E(O)$ is an open invariant set of descending flow of (2.14). By

Lemma 2.5 (3) and (2.2), we see that $C_E(O) \neq E$, and so $\partial_E C_E(O) \neq \emptyset$. By Lemma 2.4 and 2.5 we have

$$\inf_{u \in \partial_E C_E(O)} \varphi(u) \geqslant \inf_{u \in \partial_E(O)} \varphi(u) = d_0. \tag{2.20}$$

Since $C_E(O)$ is open in E, $C_E(O) \cap E_1 \subset B(0, R_0)$ is an open and bounded subset of E_1 containing 0. It follows from Lemma 2.6 that there exists a connected component Γ' of the boundary of $C_E(O) \cap E_1$, such that each one side ray l emitting from the origin satisfies $l \cap \Gamma' \neq \emptyset$. Let Γ be the connected component of $\partial_E C_E(O)$ containing Γ' . It follows from Lemma 2.5 that Γ is an invariant set of descending flow of (2.14).

It follows from Lemma 2.4 that $\operatorname{int}(P_1)$ is an invariant set of descending flow of (2.14). Thus, $\Gamma \cap \operatorname{int}(P_1)$ is an invariant set of descending flow of (2.14). Take $\widetilde{v}_1 \in S_{R_0} \cap E_1 \cap \operatorname{int}(P_1)$ and let \widehat{l} be the ray emitting from the origin and passing through \widetilde{v}_1 . Then, we have $\widetilde{l} \cap (\Gamma \cap P_1) \neq \emptyset$, and so $\Gamma \cap P_1 \cap \varphi^{-1}((-\infty, d_1 - 1]) \neq \emptyset$. Take $v_0 \in \Gamma \cap P_1 \cap \varphi^{-1}((-\infty, d_1 - 1])$. It follows from (2.20) that $\inf_{u \in o(v_0)} \varphi(u) \geqslant -\frac{1}{4}$, that is $o(v_0) \subset \varphi^{-1}([d_0, d_1 - 1])$. Then we have the following two cases:

- 1) If $v_0 \in K_{2\delta}$, by (2.3) there must exists a u_1 with $u_1 \in P_1 \cap \varphi^{-1}([-\frac{1}{4}, d_1]) \cap (K \setminus \{0\})$.
- 2) If $v_0 \in (\Gamma \cap P_1 \cap \varphi^{-1}([d_0, d_1 1])) \setminus K_{2\delta}$, by (5) in Lemma 2.4 we see that $\sigma(\tau(v_0), v_0) \in K_{2\delta} \cap P_1$ for some $\tau(v_0) > 0$, by the same reason as above we see that there must exists a u_1 with

$$u_1 \in P_1 \cap \varphi^{-1}([-\frac{1}{4}, d_1 - \frac{1}{2}]) \cap (K \setminus \{0\}).$$

Hence, φ has at least one positive critical point u_1 .

Similarly, we can show that φ has at least one negative critical point u_2 .

Now we show that φ has at least one sign-changing critical point u_3 . Obviously, $\Gamma \cap \operatorname{int}(P_1)$ and $\Gamma \cap \operatorname{int}(-P_1)$ are two open invariant sets of descending flow of (2.14) in Γ . It follows from Lemma 2.5 that $C_{\Gamma}(\Gamma \cap \operatorname{int}(P_1))$ and $C_{\Gamma}(\Gamma \cap \operatorname{int}(-P_1))$ are two open invariant sets of descending flow of (2.14) in Γ . By the connectedness of Γ , we see that

$$D_1 := \Gamma \setminus (C_{\Gamma}(\Gamma \cap \operatorname{int}(P_1)) \cup C_{\Gamma}(\Gamma \cap \operatorname{int}(-P_1)) \neq \emptyset.$$

Let

$$D_2 := \Gamma' \setminus (C_{\Gamma}(\Gamma \cap \operatorname{int}(P_1)) \cup C_{\Gamma}(\Gamma \cap \operatorname{int}(-P_1)).$$

Obviously, $D_2 \subset D_1$. Also by the connectedness of Γ' , we have $D_2 \neq \emptyset$. Take $v_0 \in D_2$, by (2.20) we have $\inf_{u \in o(v_0)} \varphi(u) \geqslant -\frac{1}{4}$. In a similar way we can show that φ has a sign-changing critical point

 u_3 . Indeed, if $v_0 \in K_{2\delta}$, by (2.3) there must exist a $u_3 \in (E \setminus (P \cup (-P))) \cap \varphi^{-1}([-\frac{1}{4}, d_1 - \frac{1}{2}]) \cap K$. If $v_0 \in (D_2 \cap \varphi^{-1}([d_0, d_1 - \frac{1}{2}])) \setminus K_{2\delta}$, by (5) in Lemma 2.4 we see that $\sigma(\tau(v_0), v_0) \in K_{2\delta} \cap D_1$ for some $\tau(v_0) > 0$, by the same reason as above we see that there must exists a u_3 with

$$u_3 \in (E \setminus (P \cup (-P))) \cap \varphi^{-1}([-\frac{1}{4}, d_1 - \frac{1}{2}]) \cap (K \setminus \{0\}).$$

Hence, φ has at least one sign-changing critical point u_3 . The proof is complete.

Remark 2.3. It is important in the proof that P_1 and $-P_1$ have nonempty interiors. Instead of these two sets, the authors of [28] used the neighborhoods of cones and negative cones. Therefore, we can establish a result similar to Theorem 2.1 based on that condition of [28]. However, for the sake of brevity, we will not discuss the relevant results in this article.

Assume that $(I - \nabla \varphi)(\pm P) \subset \pm P$. As in the proof of Lemma 2.3, for each $x_0 \in P$, there exists $(u_2(x_0), x_0^*) \in (\{x_0\} - P) \cap \bar{B}(0, 1)) \times \partial \varphi(x_0)$ satisfying $m_P(x_0) = \langle x_0^*, u_2(x_0) \rangle$. Take $y_0 \in \{x_0\} - \mathsf{F}(x_0^*)$. It follows from the condition $(I - \nabla \varphi)(P) \subset P$ that $y_0 \in P$. So, if $||x_0 - y_0|| < 1$,

$$m_P(x_0) = \langle x_0^*, u_2(x_0) \rangle \geqslant \langle x_0^*, x_0 - y_0 \rangle = \langle x_0^*, \mathsf{F}(x_0^*) \rangle = ||x_0^*||_*^2 \geqslant m^2(x_0);$$

if $||x_0 - y_0|| \ge 1$, let $z_0 = x_0 + \frac{y_0 - x_0}{2||x_0 - y_0||}$, we have $z_0 \in P$ and $||x_0 - z_0|| = \frac{1}{2}$, and so

$$m_P(x_0) = \langle x_0^*, u_2(x_0) \rangle \geqslant \langle x_0^*, x_0 - z_0 \rangle$$

$$= \frac{1}{2||x_0 - y_0||} \langle x_0^*, x_0 - y_0 \rangle$$

$$= \frac{1}{2||x_0||_*} ||x_0^*||_*^2 = \frac{1}{2} ||x_0^*||_* \geqslant \frac{1}{2} m(x_0).$$

Thus, we have

$$m_P(x) \geqslant \min\{\frac{1}{2}, m(x)\}m(x) \text{ for all } x \in P.$$
 (2.21)

Similarly, we have

$$m_{-P}(x) \ge \min\{\frac{1}{2}, m(x)\}m(x) \text{ for all } x \in -P.$$
 (2.22)

Definition 2.6. We say that φ satisfies the (PS) condition, for every sequence $\{x_n\} \subset X$ such that $\{\varphi(x_n)\}$ is bounded and $m(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

By using (2.21) and (2.22), in a similar way to show Theorem 2.1 we can prove the following Theorem 2.2.

Theorem 2.2. Suppose that (H_2) , (H_3) , (H_5) and (H_6) hold, $(I - \nabla \varphi)(\pm P) \subset \pm P$, φ satisfies the condition (PS). Then φ has at least one positive, one negative and one sign-changing critical point.

3 Applications to Differential Inclusion Problems

Consider the following differential inclusions problems

$$\begin{cases} -\operatorname{div}(\|Du(x)\|^{p-2}Du(x)) - \lambda |u(x)|^{p-2}u(x) \in \partial j(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
(3.1_{\lambda})

where Ω is a bounded open domain in \mathbb{R}^N with a $C^{1,\alpha}$ -boundary $\partial\Omega$ (0 < α < 1), 1 < p < $+\infty$, the reaction term $\partial j(x,s)$ is the generalized gradient of a non-smooth potential $s\mapsto j(x,s)$, which is subject to the following conditions.

- (\mathbf{H}_j) $j: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function and there exist constants $a_1 > 0$, $p < q < p^*$ such that
 - (i) $j(x,\cdot)$ is locally Lipschitz for almost every $x\in\Omega$ and j(x,0)=0 a.e. on Ω ;
 - (ii) $|\xi| \leq a_1(1+|s|^{q-1})$ a.e. in Ω and for all $s \in \mathbb{R}$, $\xi \in \partial j(x,s)$;
 - (iii) there exist constants $\mu > p$ and M > 0 such that

$$\inf_{x\in\Omega}j(x,M)>0 \text{ and } \mu j(x,z)\leqslant -j^o(x,z;-z) \text{ a.e. on } x\in\Omega \text{ all } z\geqslant M;$$

(iv)

$$\lim_{z \to 0} \sup \frac{pj(x,z)}{z^p} = 0$$

uniformly with respect to $x \in \Omega$;

(v) $zw(x) \ge 0$ for each $w(x) \in \partial j(x,z)$ a.e. Ω and $z \in \mathbb{R}$.

Remark 3.1. The main purpose of this part is to show that our theoretical results can be applied to the study of differential inclusion problems. It should be pointed out that some of the conditions we listed above are similar to those in [18], and the proofs of some of the following lemmas are also similar to those in [18,23].

Let

$$||u|| = \left(\int_{\Omega} ||Du||^p dx\right)^{\frac{1}{p}}, |u|_r = \left(\int_{\Omega} |u|^r dx\right)^{\frac{1}{r}}$$

be the standard norms of $W_0^{1,p}(\Omega)$, respectively $L^r(\Omega)$ for $1 < r < p^*$. Let $X = W_0^{1,p}(\Omega)$ and $E = C_0^1(\Omega)$. For $\lambda > 0$, we introduce the energy functional $\varphi_{\lambda} : X \to \mathbb{R}$ by

$$\varphi_{\lambda}(u) = \frac{1}{p} ||u||_p^p - \frac{\lambda}{p} |u|_p^p - \int_{\Omega} j(x, u(x)) dx,$$

Let $P = \{u \in X : u(x) \ge 0 \text{ a.e. } \Omega\}$ and $P_1 = P \cap E$. Given $\lambda > 0$, we say that $u \in X$ is a (weak) solution of (3.1_{λ}) if $\Delta_p u \in L^{q'}(\Omega)$, where $\frac{1}{q} + \frac{1}{q'} = 1$, and

$$-\Delta_p u(x) \in \lambda |u(x)|^{p-2} u(x) + \partial j(x, u(x))$$
 for almost every $x \in \Omega$.

Let $K_{\lambda} = \{x \in X : 0 \in \partial \varphi_{\lambda}(x)\}$ for $\lambda > 0$.

Recall some facts about the spectrum of the p-Laplacian with Dirichlet boundary condition. Consider the nonlinear eigenvalue problem

$$\begin{cases} -\operatorname{div}(\|Du(x)\|^{p-2}Du(x)) = \lambda |u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
(3.2_{\lambda})

Let λ_1 be the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Then λ_1 is positive, isolated and simple. There is the following variational characterization of λ_1 using Rayleigh quotient:

$$\lambda_1 = \inf \left\{ \frac{\|Du\|_p^p}{|u|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\}.$$

This minimum is actually realized at normalized eigenfunction u_1 . The Ljusternik-Schnirelmann theory gives, in addition to λ_1 , a whole strictly increasing sequence of positive numbers $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$ for which there exist nontrivial solutions for problem (3.2_{λ}) . In what follows we let $u_2 \in W_0^{1,p}(\Omega)$ be a nontrivial solutions for problem (3.2_{λ}) corresponding to λ_2 , and $E_1 = \text{span } \{u_1, u_2\}$.

Theorem 3.1. Assume (H_j) holds. Then for $\lambda \in (0, \lambda_1)$, (3.1_{λ}) has at least one positive solution, one negative solution and one sign-changing solution.

Lemma 3.1. If $u \in K_{\lambda}$, then $u \in C_0^1(\bar{\Omega})$ and u solves (3.1_{λ}) . Moreover, if $u \in \pm P \cap K_{\lambda}$ and $u \neq 0$, then $u \in int(\pm P_1) \cap K_{\lambda}$.

Proof. The proof is similar to Proposition 3.1 and 3.2 in [23]. Let $A: W_0^{1,p}(\Omega) \mapsto W^{-1,p'}(\Omega)$ be defined by

$$\langle A(u), v \rangle = \int_{\Omega} \|Du(z)\|^{p-2} (Du(z), Dv(z))_{\mathbb{R}^N} dz \text{ for } u, v \in W_0^{1,p}(\Omega).$$

It is known that A is monotone and demi-continuous, hence maximal monotone, and so generalized pseudomonotone.

Obviously, $u \mapsto \frac{1}{p} ||u||^p$ is a C^1 -functional whose derivative is the operator A. Aubin-Clarke's Theorem ensures that the functional

$$u \mapsto \int_{\Omega} j(x, u) dx$$

is Lipschitz continuous on any bounded subset of $L^q(\Omega)$ and its gradient is included in the set

$$N(u) = \{ w \in L^{q'}(\Omega) : w(x) \in \partial j(x, u(x)) \text{ for almost every } x \in \Omega \}.$$

Since X continuously embedded in $L^q(\Omega)$, the function φ_{λ} turns out to be locally Lipschitz on X. So, we have

$$\partial \varphi_{\lambda}(u) \subset A(u) - \lambda |u|^{p-2} u - N(u).$$
 (3.3)

Now, if $u \in X$ complies with $0 \in \partial \varphi_{\lambda}(u)$ then

$$A(u) = \lambda |u|^{p-2}u + w \text{ in } X^*$$

for some $w \in N(u)$. Hence, $\Delta_p u \in L^{q'}(\Omega)$ and u solves (3.1_{λ}) . By the condition (H_i) (ii) and (3.3)we get the estimate

$$-u\Delta_p u \leqslant a_1(|u|+|u|^q)$$
 a.e. in Ω .

Hence, by [12, Theorem 1.5.5], we have $u \in L^{\infty}(\Omega)$. From (H_j) (ii) it follows $\Delta_p u \in L^{\infty}(\Omega)$. So, by [12, Theorem 1.5.6], we have $u \in C_0^1(\bar{\Omega})$.

Let $u \in P \cap K$ and $u \neq 0$. By (H_i) (v), for each $c_0 > 0$ we have

$$\Delta_p u = -\lambda u^{p-1} - w \leqslant c_0 u^{p-1}$$

for some $w \in \partial j(x, u)$. The Vázquez maximum principle yields $u \in \text{int}(P_1)$.

Similarly, if $u \in -P \cap K_{\lambda}$ and $u \neq 0$, then $u \in \text{int}(-P_1) \cap K_{\lambda}$. The proof is complete.

Lemma 3.2. The functional $\varphi_{\lambda}: X \mapsto \mathbb{R}$ satisfies the conditions (CPS), (CPS)_P and $(CPS)_{-P}$ for $\lambda > 0$.

Proof. The proof is similar to claim 1 of Theorem 3.1 in [18]. For reader's convenience we give the details of the process. We divide the proof into three steps.

Step 1. In this step we prove the assertion: if $\{x_n\} \subset W_0^{1,p}(\Omega)$ is bounded, and either $(1 + \|x_n\|)m(x_n) \to 0$, or $(1 + \|x_n\|)m_{\pm P}(x_n) \to 0$ as $n \to +\infty$, then $\{x_n\}$ has a convergent subsequence.

We only consider the case of $(1 + ||x_n||)m(x_n) \to 0$ as $n \to +\infty$. In a similar way we can prove the case of $(1 + ||x_n||)m_{\pm P}(x_n) \to 0$. Since $\{x_n\} \subset W_0^{1,p}(\Omega)$ is bounded, by passing to a subsequence if necessary, we may assume

$$x_n \rightharpoonup x$$
 in $W_0^{1,p}(\Omega), x_n \rightarrow x$ in $L^r(\Omega)$ for $1 < r < p^*, x_n(z) \rightarrow x(z)$ a.e. on Ω

and $|x_n(z)| \leq k(z)$ a.e. on Ω , for all $n \geq 1$, with $k \in L^{q'}(\Omega)$. Take $x_n^* \in \partial \varphi_\lambda(x_n)$ such that $m(x_n) = ||x_n^*||_*$ for $n \ge 1$. Then we have

$$x_n^* = A(x_n) - \lambda |x_n|^{p-2} x_n - u_n \tag{3.4}$$

with $u_n \in L^{q'}(\Omega)$, satisfying $u_n(x) \in \partial \varphi(x, x_n(x))$ a.e. on Ω .

Now, we can deduce from $(1+||x_n||)m(x_n)\to 0$ that $|\langle x_n^*,x_n-x\rangle|\leqslant \frac{1}{n}||x_n-x||$. This reads

$$\left| \langle A(x_n), x_n - x \rangle - \lambda \int_{\Omega} |x_n|^{p-2} x_n(x_n - x) dz - \int_{\Omega} u_n(x_n - x) dz \right| \leqslant \frac{1}{n} ||x_n - x||.$$

Consequently, we have

$$\lambda \int_{\Omega} |x_n|^{p-2} x_n(x_n-x) dz \to 0 \text{ and } \int_{\Omega} u_n(x_n-x) dz \to 0 \text{ as } n \to \infty,$$

and so,

$$\lim_{n \to \infty} \langle A(x_n), x_n - x \rangle = 0.$$

Since A is generalized pseudomonotone, we have $\langle A(x_n), x_n \rangle \to \langle A(x), x \rangle$, or equivalently, $||Dx_n||_p \to ||Dx||_p$. Recalling that $Dx_n \rightharpoonup Dx$ in $L^p(\Omega, \mathbb{R}^N)$ and $L^p(\Omega, \mathbb{R}^N)$ being uniformly convex we have $Dx_n \to Dx$ in $L^p(\Omega, \mathbb{R}^N)$ which means $x_n \to x$ in $W_0^{1,p}(\Omega)$.

Step 2. φ_{λ} satisfies the condition (CPS).

Let $\{x_n\} \subset X$ be such that $|\varphi_{\lambda}(x_n)| \leq M_1$ for some $M_1 > 0$ and $(1 + ||x_n||)m(x_n) \to 0$ as $n \to \infty$, and so $m(x_n) \to 0$ as $n \to \infty$. Take $x_n^* \in \partial \varphi(x_n)$ such that $m(x_n) = ||x_n^*||_*$ for $n \geq 1$. Then (3.4) holds. Since $m(x_n) \to 0$ as $n \to \infty$, we can say that $|\langle x_n^*, x_n \rangle| \leq \frac{1}{n} ||x_n||$. So,

$$-\|Dx_n\|_p^p + \lambda |x_n|_p^p - \int_{\Omega} j^o(z, x_n(z); -x_n(z)) dz \leqslant \frac{1}{n} \|x_n\|.$$
 (3.5)

Similarly, since $|\varphi_{\lambda}(x_n)| \leq M_1$ for all $n \geq 1$, we have

$$\frac{1}{p} \|Dx_n\|_p^p - \frac{\lambda}{p} |x_n|_p^p - \int_{\Omega} j(z, x_n(z)) dz \leqslant M_1.$$
 (3.6)

By (3.5) and (3.6) we obtain

$$\left(\frac{\mu}{p} - 1\right) \|Dx_n\|_p^p - \lambda \left(\frac{\mu}{p} - 1\right) |x_n|_p^p \\
- \int_{\Omega} \left(\mu j(z, x_n(z)) + j^o(z, x_n(z); -x_n(z))\right) dz \leqslant \mu M_1 + \frac{1}{n} \|x_n\|.$$

By (H_i) (ii)(iii) we have for some $\beta_1 > 0$,

$$\int_{\Omega} \left(\mu j(z, x_n(z)) + j^o(z, x_n(z); -x_n(z)) \right) dz \geqslant -\beta_1.$$

Then, we have for $n \ge 1$,

$$\left(\frac{\mu}{p} - 1\right)\left(1 - \frac{\lambda}{\lambda_1}\right) \|Dx_n\|_p^p \leqslant \mu M_1 + \frac{1}{n} \|x_n\| + \beta_1.$$

Because $\lambda < \lambda_1$ and $\mu > p$, we infer that $\{x_n\} \subset W_0^{1,p}(\Omega)$ is bounded. So, by the assertion in Step 1 we see that $\{x_n\}$ has a convergent subsequence.

Step 3. φ_{λ} satisfies the condition $(CPS)_P$ and $(CPS)_{-P}$.

Let $\{x_n\} \subset P$ be such that $|\varphi_{\lambda}(x_n)| \leq M_2$ for some $M_2 > 0$, and $(1 + ||x_n||)m_P(x_n) \to 0$ as $n \to \infty$. Let $h_p: X_w^* \times X \mapsto \mathbb{R}$ be defined by

$$h_P(x^*, x) = \sup \{ \langle x^*, x - y \rangle : y \in P, ||y - x|| < 1 \}.$$

Then, $h_p: X_w^* \times X \mapsto \mathbb{R}$ is lower semi-continuous; See [Lemma 1, 14] or [Lemma 2.2.1, 12]. Now we can find $x_n^* \in \partial \varphi_\lambda(x_n)$ such that $m_P(x_n) = h_P(x_n^*, C_1(x_n))$, where $C_1(x_n) = (x_n - P) \cap B(0, 1)$. So, we have

$$\langle x_n^*, x_n - y \rangle \le h_P(x_n^*, C_1(x_n)) \text{ for all } y \in P, ||x_n - y|| < 1.$$
 (3.7)

Let $y_n = x_n + \frac{1}{2||x_n||} x_n$ for $n \ge 1$. Then, $y_n \in P$, $||x_n - y_n|| < 1$. And so, by (3.7) we have

$$(1 + ||x_n||)\langle x_n^*, x_n - y_n \rangle = -\frac{(1 + ||x_n||)}{2||x_n||} \langle x_n^*, x_n \rangle$$

$$= -\frac{(1 + ||x_n||)}{2||x_n||} \Big(\langle A(x_n), x_n \rangle - \lambda ||x_n|||_p^p + \int_{\Omega} j^o(z, x_n(z); -x_n(z)) dz \Big)$$

$$\leqslant (1 + ||x_n||) h_P(x_n^*, C_1(x_n)) =: \varepsilon_n \text{ with } \varepsilon_n \downarrow 0.$$

So we obtain

$$-\left(\langle A(x_n), x_n \rangle - \lambda |x_n|_p^p + \int_{\Omega} j^o(z, x_n(z); -x_n(z)) dz\right) \leqslant \frac{2||x_n||}{(1+||x_n||)} \varepsilon_n.$$

which reads

$$-\|Dx_n\|_p^p + \lambda |x_n|_p^p + \int_{\Omega} j^o(z, x_n(z); -x_n(z)) dz \leqslant \frac{2\|x_n\|}{(1+\|x_n\|)} \varepsilon_n = \varepsilon_n', \varepsilon_n' \downarrow 0.$$

Then, by using the arguments from (3.5) to the end of the step 2 we can show that $\{\|x_n\|\}$ is bounded in $W_0^{1,p}(\Omega)$. By the assertion in Step 1 we see that $\{x_n\}$ has a convergent subsequence. Hence, φ_{λ} satisfies the condition $(CPS)_P$.

Similarly, we can show that φ_{λ} satisfies the condition $(CPS)_{-P}$. The proof is complete.

Lemma 3.3. There exists $R_0 > 0$ such that for $\lambda \in (0, \lambda_1)$,

$$\sup_{u \in S_{R_0} \cap E_1} \varphi_{\lambda}(u) < 0. \tag{3.8}$$

Proof. The proof is similar to claim 3 of Theorem 3.1 in [18]. For almost all $x \in \Omega$ and all $z \in \mathbb{R}$, the function $s \mapsto \frac{1}{s^{\mu}} j(x, sz)$ is locally Lipschitz on $(0, +\infty)$. Using the mean value theorem for locally Lipschitz functions, for s > 1 we can find $\theta \in (1, s)$ such that

$$\frac{1}{s^{\mu}}j(x,sz) - j(x,z) \in \left(-\frac{\mu}{\theta^{\mu+1}}j(x,\theta z) + \frac{1}{\theta^{\mu}}\partial_z(x,\theta z)z\right)(s-1)
= \frac{s-1}{\theta^{\mu+1}}\left(-\mu j(x,\theta z) + \partial_z j(x,\theta z)\theta z\right).$$

By (H_i) (iv), for almost all $x \in \Omega$ and all $z \ge M$, we have

$$\frac{1}{s^{\mu}}j(x,sz) - j(x,z) \geqslant \frac{s-1}{\theta^{\mu+1}} \left(-\mu j(x,\theta z) - j^o(x,\theta z; -\theta z) \right) \geqslant 0.$$

Then for almost all $x \in \Omega$ and all $z \ge M$, we have

$$j(x,z) = j\left(x, \frac{z}{M}M\right) \geqslant \left(\frac{z}{M}\right)^{\mu} j(x,M) \geqslant \left(\frac{z}{M}\right)^{\mu} \inf_{x \in \Omega} j(x,M).$$

Let $p_1 \in (p, \mu)$. So, it is seen that for a given $\eta > 0$ we can find a constant $c_{\eta} > 0$ such that

$$j(x,z) \geqslant \frac{\eta}{p} z^{p_1} - c_{\eta} \text{ for a.e. } x \in \Omega.$$
 (3.9)

Let $u_0 \in S_1 := \{u \in W_0^{1,p}(\Omega) : ||Du||_p = 1\}$. It follows from (3.9) that

$$\varphi_{\lambda}(tu_{0}) = \frac{1}{p} \|D(tu_{0})\|_{p}^{p} - \frac{\lambda t^{p}}{p} |u_{0}|_{p}^{p} - \int_{\Omega} j(x, tu_{0}(x)) dx$$

$$\leq \frac{t^{p}}{p} - \frac{\lambda t^{p}}{p} |u_{0}|_{p}^{p} - \frac{\eta t^{p_{1}}}{p} |u_{0}|_{p}^{p} + c_{\eta} |\Omega|.$$

This implies that

$$\lim_{t \to +\infty} \varphi_{\lambda}(tu_0) = -\infty.$$

Since $E_1 \cap S_1$ is compact, there exists $R_0 > 0$ such that (3.8) holds. The proof is complete.

Lemma 3.4. Let $\lambda \in (0, \lambda_1)$. There exists $R_1 \in (0, R_0)$ such that $\beta_d := \inf_{u \in S_d} \varphi_{\lambda}(u) > 0$ for any $d \in (0, R_1)$.

Proof. By the condition (H_j) (ii) (iv), for $\varepsilon > 0$ with $\lambda + \varepsilon < \lambda_1$, we can find a $c_{\varepsilon} > 0$ such that

$$0 \leqslant j(x,z) \leqslant \frac{\varepsilon}{p}|z|^p + c_{\varepsilon}|x|^q \text{ for a.e. } x \in \Omega, z \in \mathbb{R}.$$

So, we have

$$\varphi_{\lambda}(u) \geqslant \frac{1}{p} \left(1 - \frac{\lambda + \varepsilon}{\lambda_1}\right) \|Du\|_p^p - c_1 \|Du\|_p^q, \quad \forall u \in W_0^{1,p}(\Omega)$$

for some $c_1 > 0$. Thus, there exists $R_1 \in (0, R_0)$ small enough such that $\beta_d := \inf_{u \in S_d} \varphi_{\lambda}(u) > 0$ for any $d \in (0, R_1)$, and 0 is a strictly local minimum point. The proof is complete.

Lemma 3.5. φ_{λ} is outwardly directed on $\pm P$ for $\lambda \in (0, \lambda_1)$. **Proof.** The following elementary inequality is well known:

$$(|y|^{p-2}y - |h|^{p-2}h, y - h)_{\mathbb{R}^N} \geqslant \begin{cases} c_1(p)(|y| + |h|)^{p-2}|y - h|^2 & \text{if } 1$$

for all $y, h \in \mathbb{R}^N$, where $c_1(p), c_2(p) > 0$ are constants. For each $u \in P$, $u^* = Au - \lambda |u|^{p-2}u - w \neq 0$, let $v = (-\Delta_p)^{-1}(\lambda |u|^{p-2}u + w)$, where $w \in L^{q'}(\Omega)$ and $w \in \partial j(x,u)$. Then, we have

$$\langle u^*, u - v \rangle = \langle Au - \lambda | u |^{p-2} u - w, u - v \rangle$$

$$= \langle Au + \Delta_p v, u - v \rangle = \langle Au, u - v \rangle + \langle \Delta_p v, u - v \rangle$$

$$= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (u - v) + \int_{\Omega} (u - v) \Delta_p v$$

$$= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (u - v) - \int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla (u - v)$$

$$\geq \begin{cases} c_1(p) \int_{\Omega} (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u - v)|^2 & \text{if } 1$$

This implies that φ_{λ} is outwardly directed on P. Similarly, φ_{λ} is outwardly directed on -P. The proof is complete.

Lemma 3.6. For any $a, b \in \mathbb{R}$ with a < b, $K_{\lambda} \cap \varphi_{\lambda}^{-1}([a, b])$ is compact in E. **Proof.** For each $u \in K_{\lambda} \cap \varphi_{\lambda}^{-1}([a, b])$, it follows from the proof of Lemma 3.1 that u = 0. $(-\Delta_p)^{-1}(\lambda|u|^{p-2}u+w)$ for some $w\in N(u)$. By a similar way as the proof of Lemma 3.2 we can prove that $K_{\lambda} \cap \varphi_{\lambda}^{-1}([a,b])$ is bounded in X. Let

$$B(\lambda):=\big\{\lambda|u|^{p-2}u+w:u\in K_\lambda\cap\varphi_\lambda^{-1}([a,b]),w\in N(u)\big\}.$$

If p > N then $X \hookrightarrow L^{\infty}(\Omega)$. So, $K_{\lambda} \cap \varphi_{\lambda}^{-1}([a,b])$ is bounded in $L^{\infty}(\Omega)$ if p > N. It follows the condition (H_j) (ii) that the set $B(\lambda)$ is bounded in $L^{\infty}(\Omega)$. According to [33], there exists $0 < \alpha < 1$ and $c_1 > 0$ such that

$$\|(-\Delta_p)^{-1}u\|_{C^{1,\alpha}} \le \|u\|_{\infty}^{\frac{1}{p-1}} \text{ for all } u \in L^{\infty}(\Omega).$$
 (3.10)

Hence, $K_{\lambda} \cap \varphi_{\lambda}^{-1}([a,b])$ is bounded in $C^{1,\alpha}(\bar{\Omega})$, and is compact in E.

If $1 , take <math>p^* > r > \frac{qN}{p}$. It follows the condition (H_j) (ii) that the set $B(\lambda)$ is bounded in $L^r(\Omega)$. According to [34] we have

$$\|(-\Delta_p)^{-1}u\|_{\infty} \leqslant c_2 \|u\|_r^{\frac{1}{p-1}} \text{ for all } u \in L^r(\Omega).$$
 (3.11)

It follows from (3.11) that $K_{\lambda} \cap \varphi_{\lambda}^{-1}([a,b])$ is bounded in $L^{\infty}(\Omega)$. Then, by (3.10), we see that $K_{\lambda} \cap \varphi_{\lambda}^{-1}([a,b])$ is compact in E. The proof is complete.

The Proof of Theorem 3.1. The conclusion can be easily proved by Lemma 3.1 \sim 3.6, and Theorem 2.1. The proof is complete.

Remark 3.2. There have been some papers studied the existence for sign-changing solutions of differential inclusion problems; see [23,24] and the references therein. For example, by combing

variational methods with truncation techniques the paper [23] obtained the existence of positive, negative and nodal solutions to differential inclusion problems with a parameter. Here, our method is different to that in [23,24].

Remark 3.3. Zhang, Chen and Li [7], and Bartsch T. and Liu [8] study p-Laplacian equations boundary value problems with smooth nonlinearities. They proved some results of at least three solutions: one positive, one negative, and one sign-changing. To show their main results, they construct of special continuously differentiable pseudo-gradient vector fields which ensuring that both the cone and negative cone are invariant set of descending flow generated by that pseudo-gradient vector fields. Theorem 3.1 can be thought as an extension of some of the main results in [7,8]. Here, we allow the nonlinearity to have discontinuity. Moreover, our method to construct the pseudo-gradient vector fields is different to that in [7,8].

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