SOME CLASSIFICATIONS FOR GAUSS MAP OF TUBULAR HYPERSURFACES IN $\mathbb{E}_1^4$ CONCERNING LINEARIZED OPERATORS $\mathcal{L}_k$

AHMET KAZAN[1*], MUSTAFA ALTIN[2] AND NURETTIN CENK TURGAY[3]

Abstract. In this study, we deal with the Gauss map of tubular hypersurfaces in 4-dimensional Lorentz-Minkowski space concerning the linearized operators $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$. We obtain the $\mathcal{L}_1$ (Cheng-Yau) operator of the Gauss map of tubular hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on timelike or spacelike curves with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map, second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1, 2\}$.

1. Introduction

Let (M, g) be a hypersurface of $(n+1)$-dimensional Minkowski space $\mathbb{E}_1^{n+1}$, Δ denote its Laplace operator. A smooth mapping $\phi : M \to \mathbb{E}_1^{n+1}$ is said to be *finite type* if it can be expressed as

$$\phi = \phi_0 + \phi_1 + \cdots + \phi_k,$$

where ϕ_0 is a constant vector and ϕ_i is an eigenvector of Δ corresponding to the eigenvector λ_i for $i = 1, 2, \ldots, k$. More precisely, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, then ψ is said to be *k-type* ([4, 6, 7]). Several results on the study of finite type mappings were summed up in a report by B.-Y. Chen in [5] (See also [8, 22]).

Let N denote the Gauss map of M. From the definition above, one can conclude that N is of *1-type* if and only if it satisfies the equation

$$\Delta N = \lambda(N + C) \tag{1.1}$$

for a constant $\lambda \in \mathbb{R}$ and a constant vector C. However, Gauss map of some important submanifolds such as catenoid and helicoid of the Euclidean 3-space $\mathbb{E}^3$ satisfies

$$\Delta N = f(N + C) \tag{1.2}$$

which is weaker than (1.1), where $f \in C^\infty(M)$ is a smooth function, [10]. These submanifolds whose Gauss map N satisfying (1.2) are said to have *pointwise 1-type Gauss map*. Submanifolds with pointwise 1-type Gauss map have been worked in several papers (cf. [10, 20, 21, 22]).

On the other hand, the Gauss map of some hypersurfaces of semi-Euclidean spaces satisfies the equation

$$\Delta N = f_1 N + f_2 C \tag{1.3}$$

for some smooth functions f_1, f_2 and a constant vector C. A submanifold is said to have *generalized 1-type Gauss map* if its Gauss map satisfies the condition (1.3), [25]. After this definition was given, hypersurfaces of pseudo-Euclidean spaces have been considered in terms of having generalized 1-type Gauss map, [17, 19, 25, 26].

2010 *Mathematics Subject Classification.* 14J70, 53A35.

Key words and phrases. Tubular hypersurface, $\mathcal{L}_1$ (Cheng-Yau) operator, $\mathcal{L}_2$ operator, Generalized $\mathcal{L}_k$ 1-type Gauss map, First and second kind $\mathcal{L}_k$-pointwise 1-type Gauss map, $\mathcal{L}_k$-harmonic Gauss map.

In the recent years, the definition of $\mathcal{L}_k$-finite type maps has been obtained by replacing Δ in the definition above with the sequence of operators $\mathcal{L}_0,\ \mathcal{L}_1,\ \mathcal{L}_2,\ \dots,\ \mathcal{L}_{n-1}$, [1, 2]. Note that, by the definition of these operators, one can obtain $\mathcal{L}_0 = -\Delta$ and $\mathcal{L}_1 = \square$ is called as the Cheng-Yau operator introduced in [9]. By motivating this idea, notion of $\mathcal{L}_k$-pointwise 1-type Gauss map and generalized $\mathcal{L}_k$ 1-type Gauss map was presented in [14] and [18], respectively (see Definition 1). After the case $k = 1$ is studied in these papers, many result obtained on hypersurfaces with certain type of Gauss map, [11, 12, 13, 19, 24, 25, 26].

On the other hand, in [3], the general expression of the canal hypersurfaces that are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones whose centers lie on a non-null curve with non-null Frenet vector fields in $\mathbb{E}^4_1$ has been given and their some geometric invariants such as unit normal vector felds, Gaussian curvatures, mean curvatures and principal curvatures have been obtained. Also, tubular hypersurfaces in $\mathbb{E}^4_1$ by taking constant radius function have been studied in [3].

In this paper, we study the tubular hypersurfaces in Lorentz-Minkowski 4-space $\mathbb{E}^4_1$ with the aid of $\mathcal{L}_k$ operators, $k \in \{1, 2\}$. In Sect. 2, we give basic notation, facts and definitions about hypersurfaces of Minkowski spaces. In Sect. 3 and Sect 4, we consider some classifications of tubular hypersurfaces by considering their Gauss maps in terms of their types with respect to the operators $\mathcal{L}_1$ and $\mathcal{L}_2$.

2. Preliminaries

Let $\mathbb{E}^{n+1}_1$ be the $(n+1)$-dimensional Lorentz-Minkowski space with the canonical pseudo-Euclidean metric $\langle\ ,\rangle$ of index 1 and signature $(-,+,+,...,+)$ given by

$$\langle\ ,\rangle = -dx_1^2 + dx_2^2 + dx_3^2 + ... + dx_{n+1}^2$$

where $(x_1, x_2, ..., x_{n+1})$ is a rectangular coordinate system in $\mathbb{E}^{n+1}_1$.

If $\Gamma : M \longrightarrow \mathbb{E}^{n+1}_1$ is an isometric immersion from an n-dimensional orientable manifold M to $\mathbb{E}^{n+1}_1$, then the induced metric on M by the immersion Γ can be Riemannian or Lorentzian. Let N denotes a unit normal vector field and put $\langle N, N\rangle = \varepsilon = \pm 1$, so that $\varepsilon = 1$ or $\varepsilon = -1$ according to M is endowed with a Lorentzian or Riemannian metric, respectively.

The operator $\mathcal{L}_k$ acting on the coordinate functions of the Gauss map N of the hypersurface M in $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{E}^{n+1}_1$ is

$$\mathcal{L}_k N = -\varepsilon \mathfrak{C}_k \left(\nabla H_{k+1} + \left(nH_1H_{k+1} - (n-k-1)\, H_{k+2}\right) N\right). \tag{2.1}$$

Here,

$$\begin{pmatrix} n \\ k \end{pmatrix} H_k = (-\varepsilon)^k a_k, \quad \left(\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} \right), \tag{2.2}$$

such that

$$\left.\begin{array}{l} a_1 = -\sum_{i=1}^{n} \kappa_i, \\ \\ a_k = (-1)^k \sum_{i_1<i_2<...<i_k}^{n} \kappa_{i_1}\kappa_{i_2}...\kappa_{i_k},\ \ k = 2, 3, ..., n \end{array}\right\} \tag{2.3}$$

and H_k is called the k-th mean curvature of order k of M.

Also, the constant $\mathfrak{C}_k$ is given by

$$\mathfrak{C}_k = \begin{pmatrix} n \\ k+1 \end{pmatrix} (-\varepsilon)^k. \tag{2.4}$$

(For more details about the linearized operator $\mathcal{L}_k$, one can see [16].)

Definition 1. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be non-zero smooth functions on M, $C \in \mathbb{E}^{n+1}_1$ be a non-zero constant vector and $k \in \{0, 1, 2, ..., n\}$.*

If the Gauss map N of an oriented submanifold M in $\mathbb{E}^4_1$ satisfies

- **i:** *$\mathcal{L}_k N = \mathfrak{m}N + \mathfrak{n}C$, then M has generalized $\mathcal{L}_k$ 1-type Gauss map;*
- **ii:** *$\mathcal{L}_k N = \mathfrak{m}N$, then M has first kind $\mathcal{L}_k$-pointwise 1-type Gauss map;*
- **iii:** *$\mathcal{L}_k N = \mathfrak{m}(N + C)$, then M has second kind $\mathcal{L}_k$-pointwise 1-type Gauss map;*
- **iv:** *$\mathcal{L}_k N = 0$, then N is called $\mathcal{L}_k$-harmonic.*

In this study, we will deal with Gauss maps of tubular hypersurfaces in 4-dimensional Lorentz-Minkowski space $\mathbb{E}^4_1$ concerning linearized operators L_1 and L_2. So, let us give some notions in $\mathbb{E}^4_1$.

Let $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ be three vectors in $\mathbb{E}^4_1$. The inner product and vector product are defined by

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = -u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 \tag{2.5}$$

and

$$\overrightarrow{u} \times \overrightarrow{v} \times \overrightarrow{w} = \det \begin{bmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \tag{2.6}$$

respectively. Here $e_i, (i = 1, 2, 3, 4)$ are standard basis vectors.

A vector $\overrightarrow{u} \in E^4_1 - \{0\}$ is called spacelike, timelike or lightlike (null) if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle > 0$ (or $\overrightarrow{u} = 0$), $\langle \overrightarrow{u}, \overrightarrow{u} \rangle < 0$ or $\langle \overrightarrow{u}, \overrightarrow{u} \rangle = 0$, respectively. A curve $\beta(s)$ in $\mathbb{E}^4_1$ is spacelike, timelike or lightlike (null), if all its velocity vectors $\beta'(s)$ are spacelike, timelike or lightlike, respectively and a non-null (i.e. timelike or spacelike) curve has unit speed if $\langle \beta', \beta' \rangle = \mp 1$. Also, the norm of the vector $\overrightarrow{u}$ is $\| \overrightarrow{u} \| = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle|}$ [15].

Let F_1, F_2, F_3, F_4 be unit tangent vector field, principal normal vector field, binormal vector field, trinormal vector field of a timelike or spacelike curve $\beta(s)$, respectively and $\{F_1, F_2, F_3, F_4\}$ be the moving Frenet frame along $\beta(s)$ in $\mathbb{E}^4_1$. The Frenet equations can be given according to the causal characters of non-null Frenet vector fields F_1, F_2, F_3 and F_4 as follows [23]:

If the curve $\beta(s)$ is timelike, i.e. $\langle F_1, F_1 \rangle = -1$, $\langle F_i, F_i \rangle = 1$ $(i = 2, 3, 4)$, then

$$\left.\begin{aligned} F_1' &= k_1F_2, \\ F_2' &= k_1F_1 + k_2F_3, \\ F_3' &= -k_2F_2 + k_3F_4, \\ F_4' &= -k_3F_3; \end{aligned}\right\} \tag{2.7}$$

if the curve $\beta(s)$ is spacelike with timelike principal normal vector field F_2, i.e. $\langle F_2, F_2 \rangle = -1$, $\langle F_i, F_i \rangle = 1$ $(i = 1, 3, 4)$, then

$$\left.\begin{aligned} F_1' &= k_1F_2, \\ F_2' &= k_1F_1 + k_2F_3, \\ F_3' &= k_2F_2 + k_3F_4, \\ F_4' &= -k_3F_3; \end{aligned}\right\} \tag{2.8}$$

if the curve $\beta(s)$ is spacelike with timelike binormal vector field F_3, i.e. $\langle F_3, F_3\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 2, 4)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = -k_1 F_1 + k_2 F_3, \\ F_3' = k_2 F_2 + k_3 F_4, \\ F_4' = k_3 F_3; \end{array}\right\} \tag{2.9}$$

if the curve $\beta(s)$ is spacelike with timelike trinormal vector field F_4, i.e. $\langle F_4, F_4\rangle = -1$, $\langle F_i, F_i\rangle = 1$ $(i = 1, 2, 3)$, then

$$\left.\begin{array}{l} F_1' = k_1 F_2, \\ F_2' = -k_1 F_1 + k_2 F_3, \\ F_3' = -k_2 F_2 + k_3 F_4, \\ F_4' = k_3 F_3. \end{array}\right\} \tag{2.10}$$

Here k_1, k_2, k_3 are the first, second and third curvatures of the non-null curve $\beta(s)$.

Also, if p is a fixed point in $\mathbb{E}_1^4$ and r is a positive constant, then the pseudo-Riemannian hypersphere and the pseudo-Riemannian hyperbolic space are defined by

$$S_1^3(p, r) = \{x \in \mathbb{E}_1^4 : \langle x - p, x - p\rangle = r^2\}$$

and

$$H_0^3(p, r) = \{x \in \mathbb{E}_1^4 : \langle x - p, x - p\rangle = -r^2\},$$

respectively.

If M is an oriented hypersurface in E_1^4, then the gradient of a smooth function $f(s, t, w)$, which is defined in M, can be obtained by

$$\nabla f = \frac{1}{\mathfrak{g}} \left(\begin{array}{c} \left(\left(g_{23}^2 - g_{22}g_{33}\right) f_s + \left(-g_{13}g_{23} + g_{12}g_{33}\right) f_t + \left(g_{13}g_{22} - g_{12}g_{23}\right) f_w\right) \partial s \\ + \left(\left(-g_{13}g_{23} + g_{12}g_{33}\right) f_s + \left(g_{13}^2 - g_{11}g_{33}\right) f_t + \left(-g_{12}g_{13} + g_{11}g_{23}\right) f_w\right) \partial t \\ + \left(\left(g_{13}g_{22} - g_{12}g_{23}\right) f_s + \left(-g_{12}g_{13} + g_{11}g_{23}\right) f_t + \left(g_{12}^2 - g_{11}g_{22}\right) f_w\right) \partial w \end{array} \right), \tag{2.11}$$

where

$$\mathfrak{g} = g_{13}^2 g_{22} - 2g_{12}g_{13}g_{23} + g_{11}g_{23}^2 + g_{12}^2 g_{33} - g_{11}g_{22}g_{33};$$

$\{s, t, w\}$ is a local coordinat system of M; f_s, f_t, f_w are the partial derivatives of f and $g_{11} = \langle \partial s, \partial s\rangle$, $g_{12} = \langle \partial s, \partial t\rangle$, $g_{13} = \langle \partial s, \partial w\rangle$, $g_{22} = \langle \partial t, \partial t\rangle$, $g_{23} = \langle \partial t, \partial w\rangle$, $g_{33} = \langle \partial w, \partial w\rangle$.

3. Some Classifications for Tubular Hypersurfaces Generated by Timelike Curves with L_k Operators in $\mathbb{E}_1^4$

In this section, we obtain the $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$ operators of the Gauss map of the tubular hypersurfaces $\mathcal{T}(s, t, w)$ that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a timelike curve with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map and second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1, 2\}$.

The tubular hypersurfaces $\mathcal{T}(s, t, w)$ that are formed as the envelope of a family of pseudo hyperspheres whose centers lie on a timelike curve with non-null Frenet vectors in $\mathbb{E}_1^4$ can be parametrized by

$$\mathcal{T}(s, t, w) = \beta(s) + r(\cos t \cos w F_2(s) + \sin t \cos w F_3(s) + \sin w F_4(s)). \tag{3.1}$$

The unit normal vector field of (3.1) is

$$N = -\left(\cos t \cos w F_2 + \sin t \cos w F_3 + \sin w F_4\right) \tag{3.2}$$

and so,

$$\langle N, N\rangle = 1. \tag{3.3}$$

The coefficients of the first fundamental form of (3.1) are

$$\left.\begin{array}{l} g_{11} = -(1+rk_1\cos t\cos w)^2 + (rk_2\cos t\cos w - rk_3\sin w)^2 + r^2\left(k_2^2+k_3^2\right)\sin^2 t\cos^2 w, \\ g_{12} = g_{21} = r^2(k_2\cos w - k_3\cos t\sin w)\cos w,\ \ g_{22} = r^2\cos^2 w, \\ g_{13} = g_{31} = r^2 k_3\sin t,\ \ g_{23} = g_{32} = 0,\ \ g_{33} = r^2. \end{array}\right\} \tag{3.4}$$

The principal curvatures of (3.1) are

$$\kappa_1 = \kappa_2 = \frac{1}{r},\ \ \kappa_3 = \frac{k_1\cos t\cos w}{1+rk_1\cos t\cos w}. \tag{3.5}$$

For more details about these hypersurfaces, one can see [3].

3.1. **Some Classifications for Tubular Hypersurfaces Generated by Timelike Curves with $\mathcal{L}_1$ (Cheng-Yau) Operator in $\mathbb{E}_1^4$.**

The functions a_k of the tubular hypersurfaces (3.1) in $\mathbb{E}_1^4$ are obtained from (2.3) and (3.5) by

$$a_1 = \frac{-2-3rk_1\cos t\cos w}{r(1+rk_1\cos t\cos w)},\ \ a_2 = \frac{1+3rk_1\cos t\cos w}{r^2(1+rk_1\cos t\cos w)},\ \ a_3 = -\frac{k_1}{r^2(rk_1+\sec t\sec w)}. \tag{3.6}$$

Also, from (2.11), (3.4) and (3.6), we have

$$\begin{aligned} \nabla a_2 = & -\frac{2\left(k_1'\cos t + k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3}F_1 - \frac{k_1\left(2\cos^2 t\cos(2w)+\cos(2t)-3\right)}{2r^2(1+rk_1\cos t\cos w)^2}F_2 \\ & -\frac{2k_1\sin t\cos t\cos^2 w}{r^2(1+rk_1\cos t\cos w)^2}F_3 - \frac{2k_1\cos t\sin w\cos w}{r^2(1+rk_1\cos t\cos w)^2}F_4. \end{aligned} \tag{3.7}$$

So, from (2.1), (2.2), (3.2), (3.3), (3.6) and (3.7), we reach that

$$\begin{aligned} \mathcal{L}_1 N = & -\frac{2\left(k_1k_2\sin t + k_1'\cos t\right)\cos w}{r(1+rk_1\cos t\cos w)^3}F_1 \\ & -\frac{2\left(rk_1\left(3rk_1\cos^3 t\cos^3 w + 2\cos^2 t\cos(2w)+\cos(2t)\right)+\cos t\cos w\right)}{r^3(1+rk_1\cos t\cos w)^2}F_2 \\ & -\frac{2(1+3rk_1\cos t\cos w)\sin t\cos w}{r^3(1+rk_1\cos t\cos w)}F_3 - \frac{2(3rk_1\cos t\cos w+1)\sin w}{r^3(1+rk_1\cos t\cos w)}F_4. \end{aligned} \tag{3.8}$$

Now, let us give some classifications for the tubular hypersurfaces (3.1) which have generalized $\mathcal{L}_1$ 1-type Gauss map, first kind $\mathcal{L}_1$-pointwise 1-type Gauss map and second kind $\mathcal{L}_1$-pointwise 1-type Gauss map and $\mathcal{L}_1$-harmonic Gauss map.

Let the tubular hypersurfaces $\mathcal{T}(s,t,w)$ have generalized $\mathcal{L}_1$ (Cheng-Yau) 1-type Gauss map, i.e., $\mathcal{L}_1 N = \mathfrak{m}N + \mathfrak{n}C$, where $C = C_1F_1 + C_2F_2 + C_3F_3 + C_4F_4$ is a constant vector. Here, by taking derivatives of the constant vector C with respect to s, from (2.7) we obtain that

$$\left.\begin{array}{l} C_1' + C_2k_1 = 0, \\ C_2' + C_1k_1 - C_3k_2 = 0, \\ C_3' + C_2k_2 - C_4k_3 = 0, \\ C_4' + C_3k_3 = 0. \end{array}\right\} \tag{3.9}$$

Also, by taking derivatives the constant vector C with respect to t and w separately, one can see that the functions C_i depend only on s.

Firstly, let us classificate the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have generalized $\mathcal{L}_1$ (Cheng-Yau) 1-type Gauss map.

From (3.2) and (3.8), we get

$$\left.\begin{aligned} &-\tfrac{2\left(k_1'\cos t+k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3}=\mathfrak{n}C_1,\\ &-\tfrac{2\left(rk_1\left(3rk_1\cos^3 t\cos^3 w+2\cos^2 t\cos(2w)+\cos(2t)\right)+\cos t\cos w\right)}{r^3(1+rk_1\cos t\cos w)^2}=\mathfrak{m}\left(-\cos t\cos w\right)+\mathfrak{n}C_2,\\ &-\tfrac{2(1+3rk_1\cos t\cos w)\sin t\cos w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin t\cos w\right)+\mathfrak{n}C_3,\\ &-\tfrac{2(1+3rk_1\cos t\cos w)\sin w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin w\right)+\mathfrak{n}C_4.\end{aligned}\right\} \tag{3.10}$$

Now let us investigate the non-zero functions $\mathfrak{m}(s,t,w)$ and $\mathfrak{n}(s,t,w)$ from the above four equations.

Firstly, let us assume that $C_1\neq 0$.

In this case, from the first equation of (3.10) it's easy to see that

$$\mathfrak{n}(s,t,w)=-\frac{2\left(k_1'\cos t+k_1k_2\sin t\right)\cos w}{r(1+rk_1\cos t\cos w)^3C_1}. \tag{3.11}$$

Here, when the equation (3.11) is successively substituted into the second, third and fourth equations of (3.10), we obtain

$$\mathfrak{m}(s,t,w)=\frac{2\left(\begin{array}{c}\left(rk_1\left(\begin{array}{c}3rk_1\cos^3 t\cos^3 w\\ +2\cos^2 t\cos(2w)+\cos(2t)\end{array}\right)+\cos t\cos w\right)C_1(1+rk_1\cos t\cos w)\\ -C_2r^2\left(k_1'\cos t+k_1k_2\sin t\right)\cos w\end{array}\right)}{C_1r^3(1+rk_1\cos t\cos w)^3\cos t\cos w},$$

$$\mathfrak{m}(s,t,w)=\frac{2\left(C_1(1+rk_1\cos t\cos w)^2(1+3rk_1\cos t\cos w)-C_3r^2\left(k_1'\cot t+k_1k_2\right)\right)}{C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w)=\frac{2\left(C_1(1+rk_1\cos t\cos w)^2(1+3rk_1\cos t\cos w)-C_4r^2\left(k_1'\cos t+k_1k_2\sin t\right)\cot w\right)}{C_1r^3(1+rk_1\cos t\cos w)^3}.$$

When we equate the functions $\mathfrak{m}(s,t,w)$ found above to each other, we arrive at the following equations:

$$\left(C_3-C_4\sin t\cot w\right)\left(k_1'\cot t+k_1k_2\right)=0, \tag{3.12}$$

$$k_1(C_1\sec t\sec w+k_2r(C_2\tan t-C_4\sin t\cot w))+r(C_1k_1^2-k_1'(C_4\cos t\cot w-C_2))=0, \tag{3.13}$$

$$k_1(rk_2(C_3-C_2\tan t)-C_1\sec t\sec w)-C_1rk_1^2+rk_1'(C_3\cot t-C_2)=0. \tag{3.14}$$

In the equation (3.12), it holds that $k_1'\cot t+k_1k_2\neq 0$. This is because, when $k_1'\cot t+k_1k_2=0$, the function $\mathfrak{n}(s,t,w)$ in the first equation of (3.10) becomes zero. This, in turn, contradicts the definition of the function $\mathfrak{n}(s,t,w)$ in our classification as $\mathcal{L}_1N=\mathfrak{m}N+\mathfrak{n}C$. So, from the equation (3.12) and $k_1'\cot t+k_1k_2\neq 0$, we have $C_3=C_4=0$. When $C_3=C_4=0$, substituting this into the equation (3.14) yields

$$\left(C_1rk_1^2+C_2rk_1'\right)\cos t+C_1k_1\sec w+C_2rk_1k_2\sin t=0.$$

Thus, we have

$$C_1k_1^2+C_2k_1'=C_1k_1=C_2k_1k_2=0$$

and so $C_1 = C_2 = 0$. This is a contradiction.

Secondly, let us assume that $C_1 = 0$.

In this case, from the first equation of the set of equations (3.9) it's easy to see that

$$C_2 k_1 = 0. \tag{3.15}$$

If $k_1 = 0$ in (3.15), then from the second, third and fourth equations of (3.10), it is calculated as

$$\left.\begin{aligned} C_2 r^3 \mathfrak{n}(s,t,w) &= (\mathfrak{m}(s,t,w) r^3 - 2)\cos t \cos w, \\ C_3 r^3 \mathfrak{n}(s,t,w) &= (\mathfrak{m}(s,t,w) r^3 - 2)\sin t \cos w, \\ C_4 r^3 \mathfrak{n}(s,t,w) &= (\mathfrak{m}(s,t,w) r^3 - 2)\sin w, \end{aligned}\right\} \tag{3.16}$$

respectively. Since the functions C_i depend only on s, there is no solution for functions $\mathfrak{n}(s,t,w)$ in (3.16).

Now, let us assume that $C_2 = 0$ in (3.15). In this case, from the second equation of (3.10), it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{2\left(\cos t \cos w + r k_1 \left(\cos(2t) + 3 r k_1 \cos^3 t \cos^3 w + 2\cos^2 t \cos(2w)\right)\right)}{r^3 (1 + r k_1 \cos t \cos w)^2 \cos t \cos w}. \tag{3.17}$$

Here, when the equation (3.17) is successively substituted into the third and fourth equations of (3.10), we obtain

$$\mathfrak{n}(s,t,w) C_3 = \tfrac{-2k_1}{r^2 (1 + r \cos t \cos w k_1)^2} \tan t,$$

$$\mathfrak{n}(s,t,w) C_4 = \tfrac{-2k_1}{r^2 (1 + r k_1 \cos t \cos w)^2} \sec t \tan w.$$

Here, there is no solution for functions $\mathfrak{n}(s,t,w)$.

Hence, we can state the following theorem:

Theorem 1. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with generalized $\mathcal{L}_1$ 1-type Gauss map.*

Now, let us classificate the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have second kind $\mathcal{L}_1$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_1 N = \mathfrak{m}(N + C)$.

From (3.2) and (3.8), we get

$$\left.\begin{aligned} &-\tfrac{2\left(k_1' \cos t + k_1 k_2 \sin t\right)\cos w}{r(1 + r k_1 \cos t \cos w)^3} = \mathfrak{m} C_1, \\ &-\tfrac{2\left(r k_1 \left(3 r k_1 \cos^3 t \cos^3 w + 2\cos^2 t \cos(2w) + \cos(2t)\right) + \cos t \cos w\right)}{r^3 (1 + r k_1 \cos t \cos w)^2} = \mathfrak{m}\left(-\cos t \cos w + C_2\right), \\ &-\tfrac{2(1 + 3 r k_1 \cos t \cos w)\sin t \cos w}{r^3 (1 + r k_1 \cos t \cos w)} = \mathfrak{m}\left(-\sin t \cos w + C_3\right), \\ &-\tfrac{2(1 + 3 r k_1 \cos t \cos w)\sin w}{r^3 (1 + r k_1 \cos t \cos w)} = \mathfrak{m}\left(-\sin w + C_4\right). \end{aligned}\right\} \tag{3.18}$$

Here, from the fourth equation of (3.18) it's easy to see that

$$\mathfrak{m}(s,t,w) = -\frac{2\left(1 + 3 r k_1 \cos t \cos w\right)\sin w}{r^3 (1 + r k_1 \cos t \cos w)\left(-\sin w + C_4\right)}. \tag{3.19}$$

When the equation (3.19) is successively substituted into the second and third equations of (3.18), we obtain

$$
\begin{aligned}
&3C_2r^2k_1^2\cos^2 t\cos^2 w+4C_2rk_1\cos t\cos w+C_2-rk_1=0,\\
&C_4\sin t\cos w-C_3\sin w=0.
\end{aligned}
$$

So, we have

$$k_1=C_2=C_3=C_4=0. \tag{3.20}$$

Now, when the components of the equation (3.20) is substituted into the second or third equations of (3.18), we calculated

$$\mathfrak{m}(s,t,w)=\frac{2}{r^3}. \tag{3.21}$$

Also, from the first equation of (3.18) and (3.21), we have $C_1=0$.

From the calculations made above for classificate the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have second kind $\mathcal{L}_1$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_1N=\mathfrak{m}\,(N+C)$, we can give the following theorem:

Theorem 2. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with second kind $\mathcal{L}_1$-pointwise 1-type Gauss map.*

Moreover, if the function m is constant in Definition 1 (**ii** or **iii**), then we say M has first or second kind $\mathcal{L}_k$-(global) pointwise 1-type Gauss map. Thus, we can state the following theorem:

Theorem 3. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, have first kind $\mathcal{L}_1$-(global) pointwise 1-type Gauss map, i.e., $\mathcal{L}_1N=\mathfrak{m}N$ if and only if $k_1=0$, where $\mathfrak{m}(s,t,w)=\frac{2}{r^3}$.*

Finally, in the equation (3.8), since the coefficients of $F_1,\ F_2,\ F_3$ and F_4 cannot all be zero, we can give the following theorem:

Theorem 4. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, cannot have $\mathcal{L}_1$-harmonic Gauss map.*

3.2. **Some Classifications for Tubular Hypersurfaces Generated by Timelike Curves with $\mathcal{L}_2$ Operator in $\mathbb{E}_1^4$.**

Firstly, it is calculated from (2.11), (3.4) and (3.6) as

$$
\begin{aligned}
\nabla a_3=&\frac{(k_1'\cos t+k_1k_2\sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3}F_1+\frac{k_1\left(2\cos^2 t\cos(2w)+\cos(2t)-3\right)}{4r^3(1+rk_1\cos t\cos w)^2}F_2\\
&+\frac{k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)^2}F_3+\frac{k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)^2}F_4.
\end{aligned}\tag{3.22}
$$

So, from (2.1), (2.2), (3.2), (3.3), (3.6) and (3.22), we have

$$
\begin{aligned}
\mathcal{L}_2N=&\frac{(k_1'\cos t+k_1k_2\sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3}F_1\\
&+\frac{k_1\left(\begin{array}{c}\frac{rk_1}{2}\left(\begin{array}{c}24rk_1\cos^4 t\cos^4 w+12\cos^3 t\cos(3w)\\+19\cos t\cos w+9\cos(3t)\cos w\end{array}\right)\\+6\cos^2 t\cos(2w)+3\cos(2t)-1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3}F_2\\
&+\frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)}F_3+\frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)}F_4.
\end{aligned}\tag{3.23}
$$

Now, let us give some classifications for the tubular hypersurfaces (3.1) which have generalized $\mathcal{L}_2$ 1-type Gauss map, first kind $\mathcal{L}_2$-pointwise 1-type Gauss map and second kind $\mathcal{L}_2$-pointwise 1-type Gauss map and $\mathcal{L}_2$-harmonic Gauss map.

Now, let us classificate the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have generalized $\mathcal{L}_2$ 1-type Gauss map. From (3.2) and (3.23), we get

$$\left.\begin{aligned}
&\frac{\left(k_1'\cos t+k_1k_2\sin t\right)\cos w}{r^2(1+rk_1\cos t\cos w)^3}=\mathfrak{n}C_1,\\
&\frac{k_1\left(\begin{array}{l}\frac{rk_1}{2}\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w+12\cos^3 t\cos(3w)\\+19\cos t\cos w+9\cos(3t)\cos w\end{array}\right)\\+6\cos^2 t\cos(2w)+3\cos(2t)-1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3}=\mathfrak{m}\left(-\cos t\cos w\right)+\mathfrak{n}C_2,\\
&\frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin t\cos w\right)+\mathfrak{n}C_3,\\
&\frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)}=\mathfrak{m}\left(-\sin w\right)+\mathfrak{n}C_4.
\end{aligned}\right\}\tag{3.24}$$

Firstly, let us assume that $C_1\neq 0$.

In this case, from the first equation of (3.24) it's easy to see that

$$\mathfrak{n}(s,t,w)=\frac{(k_1'\cos t+k_1k_2\sin t)\cos w}{r^2(1+rk_1\cos t\cos w)^3C_1}.\tag{3.25}$$

Here, when the equation (3.25) is successively substituted into the second, third and fourth equations of (3.24), we obtain

$$\mathfrak{m}(s,t,w)=\frac{\left(\begin{array}{l}2k_1(-3C_1\cos t\cos w+C_1\sec t\sec w+C_2rk_2\tan t)+2C_2rk_1'\\-6C_1r^2k_1^3\cos^3 t\cos^3 w-C_1rk_1^2\left(6\cos^2 t\cos(2w)+3\cos(2t)+1\right)\end{array}\right)}{2C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w)=\frac{C_3r\left(k_1'\cot t+k_1k_2\right)-3C_1k_1(1+rk_1\cos t\cos w)^2\cos t\cos w}{C_1r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{m}(s,t,w)=\frac{C_4r\left(k_1'\cos t+k_1k_2\sin t\right)\cot w-3C_1k_1(1+rk_1\cos t\cos w)^2\cos t\cos w}{C_1r^3(1+rk_1\cos t\cos w)^3}.$$

When we equate the functions $\mathfrak{m}(s,t,w)$ found above to each other, we arrive at the following equations:

$$(C_4\sin t\cos w-C_3\sin w)\left(k_1'\cos t+k_1k_2\sin t\right)=0,\tag{3.26}$$

$$k_1(C_1\sec t\sec w+rk_2(C_2\tan t-C_4\sin t\cot w))+C_1rk_1^2+rk_1'(C_2-C_4\cos t\cot w)=0,\tag{3.27}$$

$$k_1(C_1\sec t\sec w+rk_2(C_2\tan t-C_3))+C_1rk_1^2+rk_1'(C_2-C_3\cot t)=0.\tag{3.28}$$

In the equation (3.26), it holds that $k_1'\cos t+k_1k_2\sin t\neq 0$. This is because when $k_1'\cos t+k_1k_2\sin t=0$, the function $\mathfrak{n}(s,t,w)$ in the first equation of (3.24) becomes zero. This, in turn, contradicts the definition of the function $\mathfrak{n}(s,t,w)$ in our classification as $\mathcal{L}_2N=\mathfrak{m}N+\mathfrak{n}C$. So, from the equation (3.26) and $k_1'\cos t+k_1k_2\sin t\neq 0$, we have $C_3=C_4=0$. When $C_3=C_4=0$, substituting this into the equation (3.28) yields

$$r\left(C_1k_1^2+C_2k_1'\right)\cos t+C_2rk_1k_2\sin t+C_1k_1\sec w=0.$$

Thus, we have

$$C_1k_1^2+C_2k_1'=C_1k_1=C_2k_1k_2=0$$

and so $C_1 = C_2 = 0$. This is a contradiction.

Secondly, let us assume that $C_1 = 0$.

In this case, from the first equation of the set of equations (3.9) it's easy to see that

$$C_2 k_1 = 0. \tag{3.29}$$

If $k_1 = 0$ in (3.29), then from the second, third and fourth equations of (3.24), it is calculated as

$$\left.\begin{array}{l}\mathfrak{m}(s,t,w)\cos t\cos w = \mathfrak{n}(s,t,w)C_2,\\ \mathfrak{m}(s,t,w)\sin t\cos w = \mathfrak{n}(s,t,w)C_3,\\ \mathfrak{m}(s,t,w)\sin w = \mathfrak{n}(s,t,w)C_4,\end{array}\right\} \tag{3.30}$$

respectively. Since the functions C_i depend only on s, there is no solution for functions $\mathfrak{m}(s,t,w)$ and $\mathfrak{n}(s,t,w)$ in (3.30).

Now, let us assume that $C_2 = 0$ in (3.29). In this case, from the second equation of (3.24), it's easy to see that

$$\mathfrak{m}(s,t,w) = -\frac{k_1\left(\begin{array}{l} rk_1\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w + 12\cos^3 t\cos(3w)\\ +19\cos t\cos w + 9\cos(3t)\cos w\end{array}\right)\\ +12\cos^2 t\cos(2w) + 6\cos(2t) - 2\end{array}\right)}{8r^3(1+rk_1\cos t\cos w)^3\cos t\cos w}. \tag{3.31}$$

Here, when the equation (3.31) is successively substituted into the third and fourth equations of (3.24), we obtain

$$\mathfrak{n}(s,t,w)C_3 = \tfrac{k_1(rk_1\sin t\cos w + \tan t)}{r^3(1+rk_1\cos t\cos w)^3},$$

$$\mathfrak{n}(s,t,w)C_4 = \tfrac{k_1(rk_1\sin w + \sec t\tan w)}{r^3(1+rk_1\cos t\cos w)^3}.$$

Here, there is no solution for functions $\mathfrak{n}(s,t,w)$.

Therefore, we can give the following theorem:

Theorem 5. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with generalized $\mathcal{L}_2$ 1-type Gauss map.*

Now, let us classificate the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have second kind $\mathcal{L}_2$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_2 N = \mathfrak{m}\left(N + C\right)$.

From (3.2) and (3.23), we get

$$\left.\begin{array}{l}\frac{\left(k_1'\cos t + k_1k_2\sin t\right)\cos w}{r^2(1+rk_1\cos t\cos w)^3} = \mathfrak{m}C_1,\\[2ex] \frac{k_1\left(\begin{array}{l}\frac{rk_1}{2}\left(\begin{array}{l}24rk_1\cos^4 t\cos^4 w + 12\cos^3 t\cos(3w)\\ +19\cos t\cos w + 9\cos(3t)\cos w\end{array}\right)\\ +6\cos^2 t\cos(2w) + 3\cos(2t) - 1\end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3} = \mathfrak{m}\left(-\cos t\cos w + C_2\right),\\[2ex] \frac{3k_1\sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin t\cos w + C_3\right),\\[2ex] \frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin w + C_4\right).\end{array}\right\} \tag{3.32}$$

Here, from the last equation of (3.32) it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)\left(-\sin w + C_4\right)}. \tag{3.33}$$

Here, when the equation (3.33) is substituted into the second equation of (3.32), we obtain

$$-1 + 3C_2 \cos t \cos w + 3C_2 r k_1 \cos^2 t \cos^2 w = 0.$$

Since the last equality is never zero, we can give the following theorem:

Theorem 6. *There are no tubular hypersurface (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with second kind $\mathcal{L}_2$-pointwise 1-type Gauss map.*

Now, let us classificate the tubular hypersurfaces $\mathcal{T}(s,t,w)$ which have first kind $\mathcal{L}_2$-pointwise 1-type Gauss map, i.e., $\mathcal{L}_2 N = \mathfrak{m} N$.

From (3.2) and (3.23), we get

$$\left.\begin{array}{l}
\frac{\cos w\left(\cos t k_1' + k_1 k_2 \sin t\right)}{r^2(1+rk_1\cos t\cos w)^3} = 0, \\[2ex]
\dfrac{k_1\left(\begin{array}{l} \frac{1}{2} r k_1 \left(\begin{array}{l} 24 r k_1 \cos^4 t \cos^4 w + 12\cos^3 t \cos(3w) \\ +19\cos t\cos w + 9\cos(3t)\cos w \end{array}\right) \\ +6\cos^2 t\cos(2w) + 3\cos(2t) - 1 \end{array}\right)}{4r^3(1+rk_1\cos t\cos w)^3} = \mathfrak{m}\left(-\cos t\cos w\right), \\[2ex]
\frac{3k_1 \sin t\cos t\cos^2 w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin t\cos w\right), \\[2ex]
\frac{3k_1\cos t\sin w\cos w}{r^3(1+rk_1\cos t\cos w)} = \mathfrak{m}\left(-\sin w\right).
\end{array}\right\} \tag{3.34}$$

Here, from the last equation of (3.34) it's easy to see that

$$\mathfrak{m}(s,t,w) = \frac{-3k_1\cos t\cos w}{r^3(1+rk_1\cos t\cos w)}. \tag{3.35}$$

Here, when the equation (3.35) is substituted into the second equation of (3.32), we obtain

$$k_1(rk_1 + \sec t\sec w) = 0.$$

Since the last equality is never zero, we can give the following theorem:

Theorem 7. *There are no tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, with first kind $\mathcal{L}_2$-pointwise 1-type Gauss map.*

Finally, since the coefficients $F_1,\ F_2,\ F_3$ and F_4 in equation (3.23) are all zero only for $k_1 = 0$, we can give the following theorem:

Theorem 8. *The tubular hypersurfaces (3.1), obtained by pseudo hyperspheres whose centers lie on a timelike curve in $\mathbb{E}_1^4$, have $\mathcal{L}_2$-harmonic Gauss map if and only if $k_1 = 0$.*

4. Some Classifications for Tubular Hypersurfaces Generated by Spacelike Curves with $\mathcal{L}_k$ Operators in $\mathbb{E}_1^4$

In this section, we give the general formulas for $\mathcal{L}_1$ (Cheng-Yau) and $\mathcal{L}_2$ operators of the Gauss maps of the six types of tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on spacelike curves $\beta(s)$ with non-null Frenet vectors in $\mathbb{E}_1^4$ and give some classifications for these hypersurfaces which have generalized $\mathcal{L}_k$ 1-type Gauss map, first kind $\mathcal{L}_k$-pointwise 1-type Gauss map and second kind $\mathcal{L}_k$-pointwise 1-type Gauss map and $\mathcal{L}_k$-harmonic Gauss map, $k \in \{1,2\}$.

The tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vectors F_i in $\mathbb{E}^4_1$ can be parametrized by

$$\left.\begin{array}{l}
\mathcal{T}^{\{2,1\}}(s,t,w)=\beta(s)+r(\cosh t\sinh wF_2(s)+\cosh wF_3(s)+\sinh t\sinh wF_4(s)),\\
\mathcal{T}^{\{2,-1\}}(s,t,w)=\beta(s)+r(\cosh t\cosh wF_2(s)+\sinh wF_3(s)+\sinh t\cosh wF_4(s)),\\
\mathcal{T}^{\{3,1\}}(s,t,w)=\beta(s)+r(\sinh t\sinh wF_2(s)+\cosh t\sinh wF_3(s)+\cosh wF_4(s)),\\
\mathcal{T}^{\{3,-1\}}(s,t,w)=\beta(s)+r(\sinh t\cosh wF_2(s)+\cosh t\cosh wF_3(s)+\sinh wF_4(s)),\\
\mathcal{T}^{\{4,1\}}(s,t,w)=\beta(s)+r(\cosh wF_2(s)+\sinh t\sinh wF_3(s)+\cosh t\sinh wF_4(s)),\\
\mathcal{T}^{\{4,-1\}}(s,t,w)=\beta(s)+r(\sinh wF_2(s)+\sinh t\cosh wF_3(s)+\cosh t\cosh wF_4(s)),
\end{array}\right\} \tag{4.1}$$

respectively. Here, we suppose for $\mathcal{T}^{\{j;\lambda\}}(s,t,w)$ that

i) $\langle F_j,F_j\rangle=-1=\varepsilon_j$ and for $i\neq j$, $\langle F_i,F_i\rangle=1=\varepsilon_i$, $i,j\in\{1,2,3,4\}$,

ii) if the tubular hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda=1$ *or* $\lambda=-1$, respectively (for more details, one can see [3]).

Now, let us write the following lemma which states the general parametric expressions of 6 different types of tubular hypersurfaces given by (4.1) and obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields in $\mathbb{E}^4_1$.

Lemma 1. *The general expression of the tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ that are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve $\beta(s)$ with non-null Frenet vectors $F_i(s)$ in $\mathbb{E}^4_1$ can be given by*

$$\mathcal{T}^{\{j,\lambda\}}(s,t,w)=\beta(s)+r\left(\textstyle\sum_{i=2}^{4}\mu_i^\lambda(s,t,w)F_i(s)\right), \tag{4.2}$$

where

$$\mu_5^\lambda(s,t,w)=\mu_2^\lambda(s,t,w),\ \ \mu_6^\lambda(s,t,w)=\mu_3^\lambda(s,t,w)$$

and for $j=2,3,4$

$$\mu_j^\lambda(s,t,w)=(\sinh w)^{\frac{1+\lambda}{2}}(\cosh w)^{\frac{1-\lambda}{2}}\cosh t,$$

$$\mu_{j+1}^\lambda(s,t,w)=(\sinh w)^{\frac{1-\lambda}{2}}(\cosh w)^{\frac{1+\lambda}{2}},$$

$$\mu_{j+2}^\lambda(s,t,w)=(\sinh w)^{\frac{1+\lambda}{2}}(\cosh w)^{\frac{1-\lambda}{2}}\sinh t.$$

Here, if the canal hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda=1$ or $\lambda=-1$, respectively.

Here, we can give the general parametric expressions of the unit normal vector fields, the coefficients of the first fundamental forms and the principal curvatures of the tubular hypersurfaces $\mathcal{T}^{\{j,\lambda\}}$ parametrized by (4.2).

The unit normal vector fields $N^{\{j,\lambda\}}$ $(j=2,3,4)$ of (4.2) are

$$N^{\{j,\lambda\}}=-(-1)^{(4-j)!}\lambda^j\textstyle\sum_{i=2}^{4}\mu_i^\lambda F_i. \tag{4.3}$$

The coefficients of the first fundamental forms $g_{ik}^{\{j,\lambda\}}$ $(j=2,3,4)$ of (4.2) are

$$\left.\begin{aligned}
g_{11}^{\{j,\lambda\}} &= 1+r^2(k_2)^2\left(-(-1)^{(4-j)!}\left(\mu_3^\lambda\right)^2+(-1)^j\left(\mu_2^\lambda\right)^2\right)\\
&\quad+r^2(k_3)^2\left((-1)^{(5-j)!}\left(\mu_3^\lambda\right)^2+(-1)^j\left(\mu_4^\lambda\right)^2\right)\\
&\quad+2(-1)^{(4-j)!}rk_1\mu_2^\lambda+r^2(k_1)^2\left(\mu_2^\lambda\right)^2-2(-1)^{(5-j)!}r^2k_2k_3\mu_2^\lambda\mu_4^\lambda,\\
g_{12}^{\{j,\lambda\}} &= g_{21}^{\{j,\lambda\}}=r^2(\mu_{j+1}^\lambda)_w\left((-1)^jk_3\left(\mu_2^\lambda\right)_w-(-1)^{(4-j)!}k_2\left(\mu_4^\lambda\right)_w\right),\\
g_{22}^{\{j,\lambda\}} &= r^2\left((\mu_{j+1}^\lambda)_w\right)^2,\\
g_{13}^{\{2,\lambda\}} &= g_{31}^{\{2,\lambda\}}=\lambda r^2\left(-k_2\cosh t+k_3\sinh t\right),\\
g_{13}^{\{3,\lambda\}} &= g_{31}^{\{3,\lambda\}}=-\lambda r^2k_3\cosh t,\\
g_{13}^{\{4,\lambda\}} &= g_{31}^{\{4,\lambda\}}=\lambda r^2k_2\sinh t,\\
g_{23}^{\{j,\lambda\}} &= g_{32}^{\{j,\lambda\}}=0,\\
g_{33}^{\{j,\lambda\}} &= -\lambda r^2.
\end{aligned}\right\}\tag{4.4}$$

The principal curvatures $\kappa_i^{\{j,\lambda\}}$ $(j=2,3,4)$ of (4.2) are

$$\left.\begin{aligned}
\kappa_1^{\{j,\lambda\}} &= \kappa_2^{\{j,\lambda\}}=\tfrac{(-1)^{(4-j)!}\lambda^j}{r},\\
\kappa_3^{\{j,\lambda\}} &= \tfrac{k_1\mu_2^\lambda}{\lambda^j\left(1+(-1)^{(4-j)!}rk_1\mu_2^\lambda\right)}.
\end{aligned}\right\}\tag{4.5}$$

From Lemma 1, (4.3), (4.4) and (4.5), we get

$$\begin{aligned}
\mathcal{L}_1N^{\{j,\lambda\}} &= \frac{2\left(-(-1)^{(5-j)!}k_1k_2\mu_3^\lambda+k_1'\mu_2^\lambda\right)}{r\left((-1)^{(4-j)!}+rk_1\mu_2^\lambda\right)^3}F_1\\
&\quad+\frac{-2\lambda\left(\mu_2^\lambda+3r^2(\mu_2^\lambda)^3\left(k_1\right)^2+k_1\left(\lambda r+4(-1)^{(4-j)!}r(\mu_2^\lambda)^2\right)\right)}{r^3\left((-1)^{(4-j)!}+rk_1\mu_2^\lambda\right)^2}F_2\\
&\quad+\frac{-2\lambda(-1)^{(4-j)!}\mu_3^\lambda\left(1+3(-1)^{(4-j)!}rk_1\mu_2^\lambda\right)}{r^3\left((-1)^{(4-j)!}+rk_1\mu_2^\lambda\right)}F_3+\frac{-2\lambda\mu_4^\lambda\left((-1)^{(4-j)!}+3rk_1\mu_2^\lambda\right)}{r^3\left((-1)^{(4-j)!}+rk_1\mu_2^\lambda\right)}F_4.
\end{aligned}\tag{4.6}$$

Let $\mathcal{T}^{\{j,\lambda\}}(s,t,w)$ have generalized L_1 (Cheng-Yau) 1-type Gauss map, i.e., $\mathcal{L}_1N^{\{j,\lambda\}}=\mathfrak{m}N^{\{j,\lambda\}}+\mathfrak{n}C$, where $C=C_1F_1+C_2F_2+C_3F_3+C_4F_4$ is a constant vector. Here, by taking derivatives of the constant vector C with respect to s, from (2.8)-(2.10) we obtain for $\mathcal{T}^{\{j,\lambda\}}$ that

$$\left.\begin{aligned}
&C_1'+(-1)^{(4-j)!}C_2k_1=0,\\
&C_2'+C_1k_1+(-1)^{(5-j)!}C_3k_2=0,\\
&C_3'+C_2k_2-(-1)^{(4-j)!}C_4k_3=0,\\
&C_4'+C_3k_3=0.
\end{aligned}\right\}\tag{4.7}$$

Also, by taking derivatives the constant vector C with respect to $t,\ w$ separately, one can see that the functions C_i depend only on s.

So, with similar procedure in Subsection 3.1, we can give the following theorems:

Theorem 9. *There are no tubular hypersurfaces (4.2), obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$, with generalized $\mathcal{L}_1$ 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 10. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with second kind $\mathcal{L}_1$-pointwise 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 11. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ have first kind $\mathcal{L}_1$-(global) pointwise 1-type Gauss map, i.e., $\mathcal{L}_1 N^{\{j,\lambda\}} = \mathfrak{m} N^{\{j,\lambda\}}$ in $\mathbb{E}^4_1$ if and only if $k_1 = 0$, where $\mathfrak{m}(s,t,w) = \frac{2\lambda^{j+1}(-1)^{(4-j)!}}{r^3}$.*

Theorem 12. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ cannot have $\mathcal{L}_1$-harmonic Gauss map.*

Also, from Lemma 1, (4.3), (4.4) and (4.5), we get

$$\begin{aligned}
\mathcal{L}_2 N^{\{j,\lambda\}} &= \frac{\lambda^j \left((-1)^j \mu_3^\lambda k_1 k_2 - (-1)^{(4-j)!}\mu_2^\lambda k_1'\right)}{r^2\left((-1)^{(4-j)!} + r k_1 \mu_2^\lambda\right)^3} F_1 \\
&+ \frac{-\lambda^{j+1} k_1 \left(2\lambda(-1)^{(4-j)!} - 3(-1)^j\left(\mu_4^\lambda\right)^2 - 3(-1)^{(5-j)!}\left(\mu_3^\lambda\right)^2 - 3(-1)^{(4-j)!} r k_1 \left(\mu_2^\lambda\right)^3\right)}{r^3\left((-1)^{(4-j)!} + r k_1 \mu_2^\lambda\right)^2} F_2 \\
&+ \frac{\lambda^{j+1}\mu_3^\lambda\left(3(-1)^{(5-j)!} r k_1 \mu_2^\lambda\right)}{r^4\left((-1)^{(5-j)!} + (-1)^j r k_1 \mu_2^\lambda\right)} F_3 + \frac{\lambda^{j+1}\mu_4^\lambda\left(3(-1)^{(5-j)!} r k_1 \mu_2^\lambda\right)}{r^4\left((-1)^{(5-j)!} + (-1)^j r k_1 \mu_2^\lambda\right)} F_4.
\end{aligned} \tag{4.8}$$

Thus, with similar procedure in Subsection 3.2, we can give the following theorems:

Theorem 13. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with generalized $\mathcal{L}_2$ 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 14. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with second kind $\mathcal{L}_2$-pointwise 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 15. *There are no tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ with first kind $\mathcal{L}_2$-pointwise 1-type Gauss map in $\mathbb{E}^4_1$.*

Theorem 16. *The tubular hypersurfaces (4.2) obtained by pseudo hyperspheres and pseudo hyperbolic hyperspheres whose centers lie on a spacelike curve with non-null Frenet vector fields F_i in $\mathbb{E}^4_1$ have $\mathcal{L}_2$-harmonic Gauss map if and only if $k_1 = 0$.*

References

[1] Alias, L., Ferrández, A. and Lucas, P., *Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x = Ax + B$*, Pacific J. Math. **156**(2) (1992), 201-208.

[2] Alias, L.J. and Gürbüz, N., *An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures*, Geom. Dedicata, **121**(1) (2006), 113-127.

[3] Altın, M., Kazan, A. and Yoon, D.W., *Canal Hypersurfaces generated by Non-null Curves in Lorentz-Minkowski 4-Space*, Bull. Korean Math. Soc. **60**(5) (2023), 1299-1320.

[4] Chen, B.-Y., *Total Mean Curvature and Submanifold of Finite Type*, World Scientific Publisher, 1984.
[5] Chen, B.-Y., *A report on submanifolds of finite type*, Soochow J. Math., **22**(2) (1996), 117-337.
[6] Chen, B.-Y., Morvan, J.-M. and Nore, T., *Energy, tension and finite type maps*, Kodai Math. J., **9**(3) (1986), 406–418.
[7] Chen, B.-Y. and Petrovic, M., *On spectral decomposition of immersions of finite type*, Bull. Aust. Math. Soc. **44**(1) (1991), 117-129.
[8] Chen, B.-Y. and Piccinni, P. *Submanifolds with Finite Type Gauss Map*, Bull. Austral. Math. Soc., **35**(2) (1987), 161-186.
[9] Cheng, S.-Y. and Yau, S.-T., *Hypersurfaces with constant scalar curvature*, Math. Ann., **225** (1977), 195-204.
[10] Choi, M. and Kim, Y.H., *Characterization of the helicoid as ruled surface with pointwise 1-type Gauss map*, Bull. Korean Math. Soc., **38**(4) (2001), 753-761.
[11] Kazan, A., Altın, M. and Turgay, N.C., *Rotational hypersurfaces in* $\mathbb{E}_1^4$ with Generalized L_k 1-Type Gauss Map, arXiv:2403.19671v1, (2024).
[12] Kelleci, A. ,*Rotational surfaces with Cheng-Yau operator in Galilean 3-spaces*, Hacet. J. Math. Stat. **50**(2) (2021), 365-376.
[13] Kim, D-S., Kim, J.R. and Kim, Y.H., *Cheng-Yau Operator and Gauss Map of Surfaces of Revolution*, Bull. Malays. Math. Sci. Soc. **39**(4) (2016), 1319-1327.
[14] Kim, Y.H. and Turgay, N.C., *On the surfaces in* $\mathbb{E}^3$ *with* L_1 *pointwise 1-type Gauss map*, (submitted).
[15] Kuhnel, W.: Differential geometry: curves-surfaces-manifolds, American Mathematical Soc., Braunschweig, Wiesbaden, 1999.
[16] Lucas, P. and Ramírez-Ospina, H.F., *Hypersurfaces in the Lorentz-Minkowski space satisfying* $L_k\psi = A\psi + b$, Geom. Dedicata **153**(1) (2011), 151-175.
[17] Qian, J., Fu, X. and Jung, S.D., *Dual associate null scrolls with generalized 1-type Gauss maps*, Mathematics **8**(7) (2020), 1111.
[18] Qian, J., Fu, X., Tian, X. and Kim, Y.H., *Surfaces of Revolution and Canal Surfaces with Generalized Cheng-Yau 1-Type Gauss Maps*, Mathematics **8**(10) (2020), 1728.
[19] Qian, J., Su, M., Kim, Y.H., *Canal surfaces with generalized 1-type Gauss map*, Rev. Union Mat. Argent. **62**(1) (2021), 199-211.
[20] Yeğin Şen, R., Dursun, U., *On submanifolds with 2-type pseudo-hyperbolic Gauss map in pseudo-hyperbolic space*, Mediterr. J. Math. **14**(1) (2017), 1-20.
[21] Takahashi, T., *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18**(4) (1966), 380-385.
[22] Yoon, D.W., *Rotation surfaces with finite type Gauss Map in* E^4, Indian J. Pure. Appl. Math. 32(12) (2001), 1803-1808.
[23] Walrave, J., Curves and surfaces in Minkowski space, Dissertation, K. U. Leuven, Fac. of Science, Leuven, 1995.
[24] Yang, B. and Liu, X., *Hypersurfaces satisfying* $L_r x = Rx$ *in sphere* S^{n+1} *or hyperbolic space* H^{n+1}, Proc. Indian Acad. Sci. (Math. Sci.) **119** (2009), no. 4, 487-499.
[25] Yoon, D.W., Kim, D.S., Kim, Y.H. and Lee, J.W., *Hypersurfaces with generalized 1-type Gauss maps*, Mathematics **6**(8) (2018), 130.
[26] Yoon, D.W., Kim, D.S., Kim, Y.H. and Lee, J.W., *Classifications of flat surfaces with generalized 1-type Gauss map in* $\mathbb{L}^3$, Mediterr. J. Math. **15** (2018), 78.

[1] *Department of Computer Technologies, Doğanşehir Vahap Küçük Vocational School, Malatya Turgut Özal University, Malatya, Turkey*
[2] *Technical Sciences Vocational School, Bingöl University, Bingöl, Turkey*
[3] *Department of Mathematics, Faculty of Science and Letters, İstanbul Technical University, İstanbul, Turkey*
[*] *Corresponding author: ahmet.kazan@ozal.edu.tr*