Asymptotically flat galactic rotation curves in gravity theory

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Abstract

We present a new set of four-geometries exhibiting asymptotically flat galactic rotation curves. These are found as explicit solutions to 5D vacuum Hilbert-Palatini theory, where the fifth dimension has vanishing proper length. In the emergent 4D dynamics, governed by the condition that the Ricci scalar must vanish (upto a cosmological constant), these correspond to anisotropic effective pressure. The enhancement in the deflection angle of a light ray penetrating the halo is obtained, which could provide a realistic testing ground for the model as a purely geometric alternative to 'dark matter'. For very large halo radii, the leading nonbaryonic contribution to the bending angle is predicted to be $\frac{3\pi v^2}{2c^2}$ (v being the asymptotic rotational velocity), a constant that is different from the result for an isothermal CDM halo.

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I. INTRODUCTION

Perhaps, a dynamical explanation of asymptotically flat rotation curves in spiral galaxies solely through spacetime geometry is far from being out of question. Despite the astonishing success enjoyed by Einstein's gravity theory within solar system scales, its apparent failure in this context stems from the fact that the predicted Newtonian fall-off for the circular velocity $v(r)$ of a test particle at a radial distance r far away from the luminous mass distribution happens to be contrary to what is in fact observed [1, 2].

While one could either continue exploring new forms of ('dark') matter in the energy-momentum tensor of Einstein gravity or invoke arbitrary (geometric) modications of gravity Lagrangian itself, we shall adopt a rather conservative outlook here. We shall be concerned with the general question as to whether it is possible to construct a purely geometric theory of gravity, where it must:
a) obey 4D general covariance, b) follow from a gravity action linear in curvature tensor so that the field equations contain no higher than second order derivatives, c) involve no matter-coupling, d) admit galactic spacetimes with asymptotically flat rotation curves as explicit solutions and e) can reproduce Einstein gravity in vacuum or with some restriction on the energy-momentum tensor in a suitable limit.

The quest for a modified theory of gravity towards an explanation of the asymptotic behaviour of rotation curves has a long history though. For instance, MOND [3, 4] and conformal gravity [5] represent two of the most extensively studied approaches in four dimensions. However, finding a generally covariant action principle with metric as the only gravitational degree of freedom is not possible in the first case [6, 7]. In the latter, the issues of defining the test mass through a conformally invariant geodesic equation and of obtaining frame-independent results turn out to be quite subtle [8, 9]. Other approaches, such as Braneworld gravity [10, 11] and $f(R)$ or more generalized extensions of the action principle [12] etc. have also been explored in this context. In any case, none of these proposals do follow from the principles laid out in the previous paragraph.

Here, in this context, we focus on the recent formulation of five dimensional gravity in vacuum based on an extra dimension of vanishing proper length [13, 14]. The four dimensional effective theory is characterized by the vanishing of the Ricci scalar with torsion (upto an effective cosmological constant). Restricting to the case of vanishing four-torsion, the most general static spherically symmetric solutions are obtained. Remarkably, we find that the class where the metric exhibits a small deviation from the Newtonian form implies asymptotically flat rotation curves. Interpreted in terms of an effective energy-momentum tensor in the 4D emergent Einstein equation, these solutions are associated with anisotropic effective pressure. The other (Newtonian) class reproduces a generalized version of Einstein-Maxwell-(A)dS geometry, with a 'charge' that is purely geometric though.

Next, we study the deflection of light passing through the galactic spacetime solution so obtained. Both cases, involving a finite and infinite halo radius respectively, are presented along with testable results. We conclude with a summary and a few relevant remarks.

II. FIELD EQUATIONS AND GALACTIC SPACETIME SOLUTIONS

The fundamental theory is defined by the following action principle:

$$\mathcal{L}(\hat{e},\hat{w}) = \frac{1}{L^3}\epsilon^{\mu\nu\alpha\beta\rho}\epsilon_{IJKLM}\hat{e}^I_\mu \hat{e}^J_\nu \hat{e}^K_\alpha \hat{R}_{\beta\rho}{}^{LM}(\hat{w}),$$

where L is the (fundamental) five-dimensional Planck-length and $\hat{R}_{\beta\rho}{}^{LM}(\hat{w}) = \partial_{[\beta}\hat{w}_{\rho]}{}^{LM} + \hat{w}_{[\beta}{}^{LK}\hat{w}_{\rho]K}{}^{M}$ is the field-strength. The internal metric is defined as $\eta_{IJ} = [-1,1,1,1,\sigma]$, where $\sigma = \pm 1$ can take either of these values. The resulting field equations are given by:

$$\epsilon^{\mu\nu\alpha\beta\rho}\epsilon_{IJKLM}\hat{e}^I_\mu \hat{e}^J_\nu \hat{D}_\alpha(\hat{w})\hat{e}^K_\beta = 0, \tag{1a}$$

$$\epsilon^{\mu\nu\alpha\beta\rho}\epsilon_{IJKLM}\hat{e}^I_\mu \hat{e}^J_\nu \hat{R}^{\ KL}_{\alpha\beta}(\hat{w}) = 0, \tag{1b}$$

where $\hat{D}_\mu$ is the gauge-covariant derivative with respect to the five dimensional connection $\hat{w}_\mu{}^{IJ}$. For a vielbein of the form:

$$\hat{e}^I_\mu = \begin{bmatrix} \hat{e}^i_a \equiv e^i_a & \hat{e}^i_v = 0 \\ \hat{e}^5_a = 0 & \hat{e}^5_5 = 0, \end{bmatrix}$$

the full set of general solutions, already presented in [13], could be summarized as follows. The connection equations are solved as:

$$\hat{w}_v{}^{IJ} \ = \ 0, \ \hat{w}_a{}^{4i} = M^{ij}e_{aj} \ [M^{kl} = M^{lk}], \tag{2}$$

$$\hat{w}_a{}^{ij} \ = \ \bar{w}_a{}^{ij}(e) + K_a{}^{ij} \tag{3}$$

where M^{kl} are arbitrary spacetime fields and $K_a^{\ ij}$ denote the four dimensional contortion satisfying $e_j^a K_a^{\ ij} = 0$. The vielbein equations of motion imply:

$$\begin{aligned} \hat{D}_v M^{kl} &= 0, \ \ [\delta_b^a \delta_{kl} - e_k^a e_{bl}] \, D_a M^{kl} = 0, \\ e_i^a e_j^b R_{ab}^{\ \ ij}(\hat{w}) &= \bar{R}(\bar{w}) + e_i^a e_j^b [D_{[a}(\bar{w}) K_{b]}^{\ \ ij} + K_{[a}^{\ \ ik} K_{b]k}^{\ \ \ j} - \sigma M^i_{\ k} M^j_{\ l} e^k_{[a} e^l_{b]}] = 0 \end{aligned} \tag{4}$$

where $\bar{R}(\bar{w})$ denotes the torsionless Ricci scalar depending only upon the tetrads e_a^i through $\bar{w}(e)$.

Here, we shall consider the simplest possible class of solutions to eq.(4) with trivial four-torsion:

$$K_a^{\ ij} = 0, \ \ M^{ij} = \lambda \eta^{ij} \tag{5}$$

where $\eta_{ij} = diag[-1, 1, 1, 1]$ is the internal four-metric. The constant λ above defines the effective cosmological constant $\chi = -12\sigma\lambda^2$ in the 4D emergent field equation, obtained from the last one among the set (4) as:

$$\bar{R}(\bar{w}) + \chi = 0 \tag{6}$$

The general solution to the above is: $\bar{R}_{ab}(\bar{w}) = \bar{t}_{ab} - \frac{1}{4}\chi g_{ab}$, where $\bar{t}_{ab}$ is an arbitrary symmetric traceless tensor subject to $\nabla_a \bar{t}^{ab} = 0$ owing to the Bianchi identities.

Next, we shall obtain the spherically symmetric static solutions to the equations of motion. Assuming the four-geometry to be of the standard form ($G = 1 = c$):

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

the equation of motion (6) translates to:

$$\frac{f''}{f} - \frac{f'}{2f}\left(\frac{f'}{f} + \frac{g'}{g}\right) + \frac{2}{r}\left(\frac{f'}{f} - \frac{g'}{g}\right) - \frac{2}{r^2}(g-1) - \chi g = 0. \tag{7}$$

In the following, we shall consider two classes of vacuum solutions of this full nonlinear equation above.

A) Newtonian solutions: $f(r)g(r) = 1$

In this case, the solution to eq.(7) is given by:

$$f(r) = 1 + \frac{B_1}{r} + \frac{B_2}{r^2} + \frac{\chi}{12}r^2, \tag{8}$$

where $B_{1,2}$ are arbitrary integration constants. This is a generalization of the Einstein-Maxwell-(A)dS solution of Einstein's theory, in the sense that the charge B_2 could have either sign [15]. Note that the associated (effective) pressure originates purely due to the spacetime geometry in our case.

B) Non-Newtonian solutions: $f(r)g(r) = 1 + \delta(r)$ where $|\delta| << 1$

To be specific, we parametrize the behaviour above as:

$$f(r)g(r) = \left(\frac{r}{R}\right)^{2\alpha}, \tag{9}$$

where R is some length scale and $\alpha << 1$ is a small parameter to be interpreted later. The explicit solution in this case reads:

$$\begin{aligned} f(r) &= \left(\frac{r}{R}\right)^{2\alpha}\left[\frac{1}{1+\alpha+\alpha^2} + \frac{\chi r^2}{2(6+4\alpha+\alpha^2)}\right] + C_1\left(\frac{r}{R}\right)^{\gamma_+} + C_2\left(\frac{r}{R}\right)^{\gamma_-}; \\ g^{-1}(r) &= \frac{1}{1+\alpha+\alpha^2} + \frac{\chi r^2}{2(6+4\alpha+\alpha^2)} + C_1\left(\frac{r}{R}\right)^{\gamma_+-2\alpha} + C_2\left(\frac{r}{R}\right)^{\gamma_--2\alpha}, \end{aligned} \tag{10}$$

where $\gamma_\pm = \frac{-(3-\alpha)\pm(1+10\alpha+\alpha^2)^{\frac{1}{2}}}{2}$ and $C_{1,2}$ are constants.

Asymptotically flat rotation curves

To understand the implications of the vacuum solutions (10), which have no analogue in Einstein gravity with (at least) standard matter couplings, let us explore if these could be used to model the galactic halo which is assumed to be spherically symmetric. In particular, the intriguing question is, if these exact solutions could reproduce the non-Newtonian behaviour of rotation curves for spiral galaxies.

The circular velocity $v^2(r) = \frac{rf'}{2f}$ [16] of a massive test particle moving in this geometry (at the equatorial plane $\theta = \frac{\pi}{2}$), under the practical assumption that the effect of the cosmological constant is negligible within the halo ($\chi \approx 0$), is found to be:

$$v^2(r) = \frac{\frac{\alpha}{1+\alpha+\alpha^2} + \frac{1}{2}C_1\gamma_+\left(\frac{r}{R}\right)^{\gamma_+-2\alpha} + \frac{1}{2}C_2\gamma_-\left(\frac{r}{R}\right)^{\gamma_--2\alpha}}{\frac{1}{1+\alpha+\alpha^2} + C_1\left(\frac{r}{R}\right)^{\gamma_+-2\alpha} + C_2\left(\frac{r}{R}\right)^{\gamma_--2\alpha}}$$

Noting that $\gamma_+ = -1 + 3\alpha + o(\alpha^2)$, $\gamma_- = -2 - 2\alpha + o(\alpha^2)$, the expression above implies: $v^2(r) \to \alpha$ as $r \to \infty$. This corresponds precisely to an asymptotically flat rotation curve, where the small parameter α in the halo metric should be interpreted as the limiting rotation velocity of the test particle at the flat region.

Effective density, pressure and the mass function

It is worthwhile studying what the solution (10) implies from the perspective of an emergent 4D Einstein gravity. Defining an energy-momentum tensor in an effective sense as $\bar{R}_{ab} - \frac{1}{2} g_{ab} \bar{R} = T_{ab}^{(eff)}$, we find the effective density and pressures as:

$$\begin{aligned}
8\pi\rho^{(eff)} &= \frac{\alpha(1+\alpha)}{1+\alpha+\alpha^2}\frac{1}{r^2} - (1+\gamma_+ - 2\alpha)\frac{C_1}{R^2}\left(\frac{r}{R}\right)^{-2+\gamma_+-2\alpha} \\
&- (1+\gamma_- - 2\alpha)\frac{C_2}{R^2}\left(\frac{r}{R}\right)^{-2+\gamma_--2\alpha}, \\
8\pi P_r^{(eff)} &= \frac{\alpha(1-\alpha)}{1+\alpha+\alpha^2}\frac{1}{r^2} + (1+\gamma_+)\frac{C_1}{R^2}\left(\frac{r}{R}\right)^{-2+\gamma_+-2\alpha} \\
&+ (1+\gamma_-)\frac{C_2}{R^2}\left(\frac{r}{R}\right)^{-2+\gamma_--2\alpha}, \\
8\pi P_\theta^{(eff)} &= \frac{\alpha^2}{1+\alpha+\alpha^2}\frac{1}{r^2} - (1+\gamma_+ - \alpha)\frac{C_1}{R^2}\left(\frac{r}{R}\right)^{-2+\gamma_+-2\alpha} \\
&- (1+\gamma_- - \alpha)\frac{C_2}{R^2}\left(\frac{r}{R}\right)^{-2+\gamma_--2\alpha} = 8\pi P_\phi^{(eff)}
\end{aligned} \tag{11}$$

Thus, the halo metric corresponds to anisotropic (effective) pressure. In fact, it could be shown in this context that a solution with asymptotically flat rotation curve cannot exhibit an isotropic (effective) pressure, since the (Schwarzschild) metric (8) with $B_2 = 0$ turns out to be the unique solution of eq.(7) under the demand $P_r = P_\theta = P_\phi$.

If we define a spatially averaged pressure as $\bar{P} = \frac{1}{3}(P_r + P_\theta + P_\phi)$, then we find: $\bar{P} = \frac{1}{3}\rho$. This equation of state correctly captures the traceless nature of $T_{ab}^{(eff)}$. The effective mass function is obtained below by integrating the density over 3-space:

$$m(r) = \frac{\alpha(1+\alpha)}{2(1+\alpha+\alpha^2)} r - \frac{C_1 R}{2}\left(\frac{r}{R}\right)^{1+\gamma_+-2\alpha} - \frac{C_2 R}{2}\left(\frac{r}{R}\right)^{1+\gamma_--2\alpha} + m_0, \tag{12}$$

where the constant m_0 depends on the lowest value of r till which the solution (10) is acceptable. Since the metric must be joined to some interior solution at this small radius, the apparent divergence of $m(r)$ at $r = 0$ is not relevant. Note that the second and third terms go as r^α and $r^{-1-4\alpha}$ respectively (upto $o(\alpha^2)$), leaving the linear term as the leading one at sufficiently large distances. This precisely reflects the behaviour expected for flat rotation curves.

III. OBSERVATIONAL PREDICTIONS: DEFLECTION OF LIGHT

In reality, the galactic halo does not extend to infinity. It would be more practical to join the halo metric to an asymptotically flat geometry, such that both these metrics are vacuum solutions to the fundamental equations of motion (6). We choose the vacuum metric beyond the halo to be Schwarzschild, obtained by setting $B_1 = -2M,\ B_2 = 0 = \chi$ in (8) where M is the total mass enclosed within the halo radius R_H.

Since the term involving C_2 in the halo metric (10) has the fastest fall-off, it is reasonable to assume $C_2 = 0$ (along with $\chi = 0$) in what follows next. Demanding continuity at the boundary of the halo fixes the two constants C_1 and R as:

$$C_1 = -\frac{2M}{R_H} + \frac{\alpha(1+\alpha)}{1+\alpha+\alpha^2},\ \ R = R_H \tag{13}$$

From the null geodesic equation obeyed by a light ray at the equatorial plane ($\theta = \frac{\pi}{2}$) of the halo, the angular distance covered in going from the radial distance (r_0) at closest approach from the centre of the halo to infinity is given by [16]:

$$\begin{aligned}\Delta\phi &= \int_{r_0}^{\infty} dr\,\frac{d\phi}{dr} = \int_{r_0}^{\infty} \frac{dr}{r}\,\left[\frac{E^2}{L^2}\frac{r^2}{fg} - \frac{1}{g}\right]^{-\frac{1}{2}} \\ &= \Delta\phi_1 + \Delta\phi_2 \end{aligned} \tag{14}$$

where E, L are the conserved quantities associated with the t and ϕ motions, respectively, satisfying $\frac{E^2}{L^2} = \frac{f(r_0)}{r_0^2} = \frac{f(r_*)}{r_*^2}$ (r_* denotes the distance at closest approach in the Schwarzschild geometry). The last line in (14) reflects the fact that the integral is to be evaluated in two parts, for the two regions $r_0 \le r \le R_H$ and $R_H \le r < \infty$, respectively. Under the weak-field approximation, we have the condition: $r_0 \approx (1-\alpha+\frac{2M}{r_*})^{\frac{1}{2}} r_*$, which is equivalent to $r_0 \approx r_*$ for $r_0 >> M$ ($2M$ being the Schwarzschild radius of the mass within the halo) at the leading order. The total angle of deflection of the ray is then given by: $\delta = |2\Delta\phi - \pi|$.

Next, using the identity:

$$\begin{aligned}&\frac{E^2}{L^2}\frac{r^2}{fg} - \frac{1}{g} = \frac{1}{1+\alpha+\alpha^2}\left[\left(\frac{r}{r_0}\right)^{2-2\alpha} - 1\right] \times \\ &\left[1 + (1+\alpha+\alpha^2)C_1\left(\frac{r}{R}\right)^{-1+\alpha} \frac{\left(\frac{r}{r_0}\right)^{2-2\alpha} + \left(\frac{r}{r_0}\right)^{1-\alpha} + 1}{\left(\frac{r}{r_0}\right)^{1-\alpha} + 1}\right],\end{aligned} \tag{15}$$

the result for $\Delta\phi$ for the passage of the light ray within the halo becomes:

$$\Delta\phi_1 \approx \int_{r_0}^{R_H} \frac{dr}{r} \, (1+\alpha)^{\frac{1}{2}} \left[\left(\frac{r}{r_0} \right)^{2-2\alpha} - 1 \right]^{-\frac{1}{2}} \times$$
$$\left[1 - \frac{(1+\alpha+\alpha^2)C_1}{2} \left(\frac{r}{R} \right)^{-1+\alpha} \frac{\left(\frac{r}{r_0}\right)^{2-2\alpha} + \left(\frac{r}{r_0}\right)^{1-\alpha} + 1}{\left(\frac{r}{r_0}\right)^{1-\alpha} + 1} \right]$$
$$= \frac{(1+\alpha+\alpha^2)^{\frac{1}{2}}}{1-\alpha} \tan^{-1} \left[\left(\frac{R_H}{r_0} \right)^{2-2\alpha} - 1 \right]^{\frac{1}{2}}$$
$$- \frac{(1+\alpha+\alpha^2)^{\frac{3}{2}} C_1}{2(1-\alpha)} \left[\left(\frac{R_H}{r_0} \right)^{2-2\alpha} - 1 \right]^{\frac{1}{2}} \frac{2\left(\frac{R_H}{r_0}\right)^{1-\alpha} + 1}{\left(\frac{R_H}{r_0}\right)^{1-\alpha} + 1}, \tag{16}$$

In writing the first line above, we have used the weak-field approximation for the term involving C_1, since such a limit is applicable to galactic spacetimes. The remaining contribution to the deflection from the halo boundary to infinity, upon using the expansion for a weak Schwarzschild potential ($\frac{2M}{r} << 1$), is evaluated as:

$$\Delta\phi_2 \approx \int_{R_H}^{\infty} \frac{dr}{r} \left(\frac{r^2}{r_*^2} - 1 \right)^{-\frac{1}{2}} \left[1 + \frac{M}{r} + \frac{Mr}{r_*(r+r_*)} \right]$$
$$= \frac{\pi}{2} + \frac{2M}{r_*} - \tan^{-1} \left[\left(\frac{R_H}{r_*} \right)^2 - 1 \right]^{\frac{1}{2}} - \frac{M}{R_H} \left[\left(\frac{R_H}{r_*} \right)^2 - 1 \right]^{\frac{1}{2}} \left(\frac{\frac{2R_H}{r_*} + 1}{\frac{R_H}{r_*} + 1} \right), \tag{17}$$

As a result, the total deflection angle upon using the expression for C_1 reads:

$$\delta = \frac{4M}{r_*} + \frac{2(1+\alpha+\alpha^2)^{\frac{1}{2}}}{1-\alpha} \tan^{-1} \left[\left(\frac{R_H}{r_0} \right)^{2-2\alpha} - 1 \right]^{\frac{1}{2}}$$
$$+ \frac{(1+\alpha+\alpha^2)^{\frac{3}{2}}}{(1-\alpha)} \left(\frac{2M}{R_H} - \frac{\alpha(1+\alpha)}{1+\alpha+\alpha^2} \right) \times \left[\left(\frac{R_H}{r_0} \right)^{2-2\alpha} - 1 \right]^{\frac{1}{2}} \frac{2\left(\frac{R_H}{r_0}\right)^{1-\alpha} + 1}{\left(\frac{R_H}{r_0}\right)^{1-\alpha} + 1}$$
$$- 2\tan^{-1} \left[\left(\frac{R_H}{r_*} \right)^2 - 1 \right]^{\frac{1}{2}} - \frac{2M}{R_H} \left[\left(\frac{R_H}{r_*} \right)^2 - 1 \right]^{\frac{1}{2}} \left(\frac{\frac{2R_H}{r_*} + 1}{\frac{R_H}{r_*} + 1} \right) \tag{18}$$

Note that for $r_0 = r_* = R_H$, which implies that the light ray passes only through the Schwarzschild geometry without entering the halo, we recover the standard result: $\delta = \frac{4M}{r_*}$. Evidently, the presence of the halo leads to an enhancement in the deflection compared to the Schwarzschild bending. This effect is characterized completely by the ratio $\frac{R_H}{r_0} \approx \frac{R_H}{r_*}$, given the asymptotic rotation velocity. For $C_1 = 0$, reflecting a scenario where the light ray enters only the outer region of the halo where the rotation velocity becomes flat, the total deflection reads:

$$\delta = \frac{4M}{r_*} + \frac{2(1+\alpha+\alpha^2)^{\frac{1}{2}}}{1-\alpha} \tan^{-1}\left[\left(\frac{R_H}{r_0}\right)^{2-2\alpha} - 1\right]^{\frac{1}{2}} - 2\tan^{-1}\left[\left(\frac{R_H}{r_*}\right)^2 - 1\right]^{\frac{1}{2}} - \frac{2M}{R_H}\left[\left(\frac{R_H}{r_*}\right)^2 - 1\right]^{\frac{1}{2}} \left(\frac{\frac{2R_H}{r_*}+1}{\frac{R_H}{r_*}+1}\right) \tag{19}$$

We observe that upon using $r_0 = r_*$ this matches the $o(\alpha)$ result presented in ref.[16], which considers only the asymptotic part of halo geometry.

Infinitely extended halo

Next, we consider the limit of very large galactic halo radii preserving the earlier assumptions. Using the fact that $\alpha << 1$ (the rotational velocities being nonrelativistic), the term involving C_1 in the halo metric (10) may be identified as the baryonic mass term with a small radial variation:

$$m(r) \approx -\frac{C_1 R}{2}\left(\frac{r}{R}\right)^{\alpha} \tag{20}$$

Next, using the general expression (14) for $\Delta\phi$ along with the identity (15) and performing the integration from r_0 to infinity for the halo metric, we obtain the deflection angle in this case as:

$$\delta \approx \left[\frac{(1+\alpha)^{\frac{1}{2}}}{1-\alpha} - 1\right]\pi + \frac{4(1+\alpha)^{\frac{3}{2}} m(r_0)}{(1-\alpha)r_0} \approx \frac{4m(r_0)}{r_0} + \frac{3}{2}\pi\alpha + \frac{10\alpha m(r_0)}{r_0}, \tag{21}$$

ignoring $o(\alpha^2)$ corrections. Thus, for a very large halo radius, the model predicts a constant leading correction $\frac{3}{2}\pi\alpha$ to the Einsteinian bending encoded by the first term. Notably, this value is lesser than the nonbaryonic contribution ($2\pi\alpha$) predicted for a singular isothermal halo within cold 'dark matter' model [17].

To emphasize, the results in (18) and (21) make the formulation here amenable to lensing observations, and also to comparisons with standard 'dark matter' scenarios as well as with other extra dimensional formulations such as the Braneworld model [18].

IV. CONCLUSIONS

We have shown that asymptotically flat rotation curves emerge as exact dynamical solutions of a purely geometric theory, which reflects a number of attractive features compared to some other gravitational alternatives to 'dark matter'. This theory is based on a recent formulation of five dimensional gravity, where the fifth dimension has vanishing proper length. This property is encoded through a degenerate vielbein (with one zero eigenvalue).

Among the static spherically symmetric solutions, the ones corresponding to asymptotically flat rotation curves are associated with anisotropic effective pressure. Though in a different context, the possibility of 'dark matter' models with nontrivial stresses have been considered in the literature [19]. We note that the spatially averaged equation of state is $\frac{\bar{P}}{\rho} = \frac{1}{3}$. The Newtonian class, on the other hand, corresponds to the generalized Einstein-Maxwell-de Sitter geometries as the unique solution, providing a connect with Einstein gravity in presence of a traceless energy-momentum tensor with cosmological constant.

We observe that the associated effective density within the halo, apart from the $r^{-}2$ behaviour required for the flatness of rotation curves at sufficiently large distance, also exhibits r^{-3} and r^{-4} contributions (upto relativistic corrections) which should dominate at scales significantly smaller than the halo radii. These encode the substructure of the rotation curves. Notably, these resemble the large distance behaviour of two well-known density profiles motivated from CDM simulation results, e.g. NFW (r^{-3}) [20] and Hernquist (r^{-4}) [21], respectively. A deeper study in this regard could be pursued elsewhere.

Finally, we obtain the deflection angle of a light ray as it passes through the halo metric joined to an asymptotically flat spacetime. The correction with respect to the standard Schwarzschild result is obtained as a function of $\alpha = \frac{v^2}{c^2}$ and $\frac{r_0}{R_H}$, the ratio of the distance of closest approach to the halo radius. Since α could be determined empirically from the rotation curve, this provides the hope that lensing observations for a given $\frac{r_0}{R_H}$ could be used to test the model set up here. In the case of very large halo radii, we obtain the leading correction to the Einstein bending to be $\frac{3}{2}\pi\alpha$. This provides an intriguing contrast with the predicted value for the standard CDM halo modelled as an isothermal sphere. Furthermore, the full correction including the subleading ones could be used to distinguish this formulation of five-dimensional gravity built upon a noninvertible vielbein from other existing extra dimensional scenarios, in particular from those which could lead to similar 4D emergent theories.

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[1] V. C. Rubin and W. K. Ford, Jr., Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions, Astrophys. J. **159**, 379 (1970); V. C. Rubin, N. Thonnard, and W. K. Ford, Jr., Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 /R = 4kpc/ to UGC 2885 /R = 122 kpc/, Astrophys. J. **238**, 471 (1980).

[2] M. S. Roberts and R. N. Whitehurst, The rotation curve and geometry of M31 at large galactocentric distances., Astrophys. J. **201**, 327 (1975).

[3] M. Milgrom, A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis., Astrophys. J. **270**, 365 (1983); A modification of the Newtonian dynamics - Implications for galaxies., Astrophys. J. **270**, 371 (1983).

[4] J. Bekenstein and M. Milgrom, Does the missing mass problem signal the breakdown of Newtonian gravity?, Astrophys. J. **286**, 7 (1984).

[5] P. D. Mannheim and D. Kazanas, Exact Vacuum Solution to Conformal Weyl Gravity and Galactic Rotation Curves, Astrophys. J. **342**, 635 (1989); P. D. Mannheim, Linear Potentials and Galactic Rotation Curves, Astrophys. J. **419**, 150 (1993), arXiv:hep-ph/9212304 [hep-ph].

[6] R. H. Sanders, A Stratified Framework for Scalar-Tensor Theories of Modified Dynamics, Astrophys. J. **480**, 492 (1997), arXiv:astro-ph/9612099 [astro-ph].

[7] M. Milgrom, Noncovariance at low accelerations as a route to mond, Phys. Rev. D **100**, 084039 (2019).

[8] K. Horne, Conformal Gravity rotation curves with a conformal Higgs halo, Monthly Notices of the Royal Astronomical Society **458**, 4122 (2016).

[9] M. P. Hobson and A. N. Lasenby, Conformal gravity does not predict flat galaxy rotation curves, Phys. Rev. D **104**, 064014 (2021), arXiv:2103.13451 [gr-qc].

[10] M. K. Mak and T. Harko, Can the galactic rotation curves be explained in brane world models?, Phys. Rev. D **70**, 024010 (2004).

[11] S. Pal, S. Bharadwaj, and S. Kar, Can extra dimensional effects replace dark matter?, Phys. Lett. B **609**, 194 (2005), arXiv:gr-qc/0409023.

[12] Y. Sobouti, An f(r) gravitation instead of dark matter, Astron. Astrophys. **464**, 921 (2007), [Erratum: Astron.Astrophys. 472, 833 (2007)], arXiv:0704.3345 [astro-ph]; S. Mendoza and Y. M. Rosas-Guevara, Gravitational waves and lensing of the metric theory proposed by Sobouti, Astron. Astrophys. **472**, 367 (2007), arXiv:astro-ph/0610390.

[13] S. Sengupta, Gravity theory with a dark extra dimension, Phys. Rev. D **101**, 104040 (2020).

[14] S. Sengupta, 4D Einstein-Gauss-Bonnet gravity from non-Einsteinian phase, JCAP **02** (02), 020, arXiv:2109.10388 [gr-qc].

[15] N. Dadhich, R. Maartens, P. Papadopoulos, and V. Rezania, Black holes on the brane, Physics Letters B **487**, 1–6 (2000).

[16] U. Nucamendi, M. Salgado, and D. Sudarsky, Alternative approach to the galactic dark matter problem, Phys. Rev. D **63**, 125016 (2001).

[17] R. D. Blandford and R. Narayan, Cosmological applications of gravitational lensing, Annual Review of Astronomy and Astrophysics **30**, 311 (1992).

[18] T. Harko and K. S. Cheng, Galactic metric, dark radiation, dark pressure, and gravitational lensing in brane world models, The Astrophysical Journal **636**, 8–20 (2006); S. Pal and S. Kar, Gravitational lensing in braneworld gravity: formalism and applications, Classical and Quantum Gravity **25**, 045003 (2008).

[19] S. Bharadwaj and S. Kar, Modeling galaxy halos using dark matter with pressure, Phys. Rev. D **68**, 023516 (2003).

[20] J. F. Navarro, C. S. Frenk, and S. D. M. White, The Structure of Cold Dark Matter Halos, Astrophys. J. **462**, 563 (1996), arXiv:astro-ph/9508025 [astro-ph].

[21] L. Hernquist, An Analytical Model for Spherical Galaxies and Bulges, Astrophys. J. **356**, 359 (1990).