On the rectilinear crossing number of complete balanced multipartite graphs and layered graphs [*]

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Abstract

A rectilinear drawing of a graph is a drawing of the graph in the plane in which the edges are drawn as straight-line segments. The rectilinear crossing number of a graph is the minimum number of pairs of edges that cross over all rectilinear drawings of the graph. Let $n \geq r$ be positive integers. The graph K_n^r, is the complete r-partite graph on n vertices, in which every set of the partition has at least $\lfloor n/r \rfloor$ vertices. The layered graph, L_n^r, is an r-partite graph on n vertices, in which for every $1 \leq i \leq r-1$, all the vertices in the i-th partition are adjacent to all the vertices in the $(i+1)$-th partition. In this paper, we give upper bounds on the rectilinear crossing numbers of K_n^r and L_n^r.

1 Introduction

Let G be a graph on n vertices and let D be a drawing of G. The crossing number of D is the number, $\operatorname{cr}(D)$, of pairs of edges that cross in D. The *crossing number* of G is the minimum crossing number, $\operatorname{cr}(G)$, over all drawings of G in the plane. A *rectilinear drawing* of G is a drawing of G in the plane in which its vertices are points in general position, and its edges are drawn as straight-line segments joining these points. The *rectilinear crossing number* of G, is the minimum crossing number, $\overline{\operatorname{cr}}(G)$, over all rectilinear drawings of G in the plane. Computing crossing and rectilinear crossing numbers of graphs are important problems in Graph Theory and Combinatorial Geometry. For a comprehensive review of the literature on crossing numbers, we refer the reader to Schaefer's book [2].

Most of the research on crossing numbers have been focused around the complete graph, K_n, and the complete bipartite graph $K_{n,m}$. For the complete graph, Hill [3] gave the following drawing of K_n; see Figure 1 for an example. Place half of the vertices equidistantly on the top circle of a cylinder, and the other half equidistantly on the bottom circle. Join the vertices with geodesics on the cylinder. Hill showed that the

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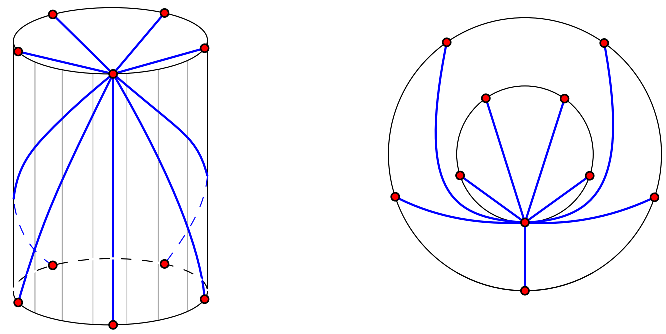

Figure 1: An example of Hill's drawings of K_8, where here for convenience only the edges of one vertex are drawn. Left: The drawing on a cylinder. Right: An equivalent representation of Hill's drawings via concentric cycles.

following number, $H(n)$, is the crossing number of this drawing, and it is now conjectured to be optimal. Let

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Conjecture 1 (Harary-Hill [4])

$$\operatorname{cr}(K_n) = H(n).$$

For the complete bipartite graph, Zarankiewicz gave a rectilinear drawing with the following number, $Z(n,m)$, as crossing number of this drawing, and it is now conjectured to be optimal. Let

$$Z(n,m) := \left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor$$

and

$$Z(n) := Z(n,n).$$

Conjecture 2 (Zarankiewicz [5])

$$\operatorname{cr}(K_{n,m}) = Z(n,m).$$

The number $Z(n,m)$ is also conjectured to be the general optimal crossing number, directly implying the following conjecture.

Conjecture 3

$$\overline{\operatorname{cr}}(K_{n,m}) = \operatorname{cr}(K_{n,m}).$$

Much less is known for the rectilinear crossing number of the complete graph. For $n \geq 10$, it is known that

$$\operatorname{cr}(K_n) < \overline{\operatorname{cr}}(K_n).$$

In contrast to the case of the complete bipartite graph, there is no conjectured value for $\overline{\operatorname{cr}}(K_n)$, nor drawings conjectured to be optimal. The best bounds to date are

$$0.379972\binom{n}{4} < \overline{\operatorname{cr}}(K_n) < 0.380445\binom{n}{4} + O(n^3).$$

The lower bound is due to Ábrego, Fernández-Merchant, Leaños, and Salazar [6], and the upper bound to Aichholzer, Duque, Fabila-Monroy, García-Quintero, and Hidalgo-Toscano [7]. It is known that

$$\lim_{n\to\infty}\frac{\overline{\operatorname{cr}}(K_n)}{\binom{n}{4}} = \overline{q},$$

for some positive constant $\overline{q}$; this constant is known as the *rectilinear crossing constant.*

Let $K_{n_1,n_2,\ldots,n_r}$ be the complete r-partite graph with n_i vertices in the i-th set of the partition; and let K_n^r be the complete balanced r-partite graph in which there are at least $\lfloor n/r \rfloor$ vertices in every partition set. Harborth [8] gave a drawing that provides an upper bound for $\mathrm{cr}(K_{n_1,n_2,\ldots,n_r})$; and gave an explicit formula for this number. He claims that for the case of $r = 3$, his drawing can be made rectilinear. More recently, Gethner, Hogben, Lidický, Pfender, Ruiz and Young [9] independently studied the problem of the crossing number and rectilinear crossing number of complete balanced r-partite graphs. For $r = 3$, they obtain the same bound as Harborth; and their drawing is rectilinear.

Let r be a positive integer and let n be a multiple of r. The *layered graph*, L_n^r, is the graph defined as follows. Its vertex set is partitioned into sets $V_1, \ldots, V_r$, each consisting of n/r vertices. We call the set V_i, the i-th layer of L_n^r. The edge set of L_n^r is given by

$$\{uv : u \in L_i \text{ and } v \in L_{i+1}, \text{ for } i = 1, \ldots, r-1\};$$

that is, the edges are exactly all possible edges between vertices on consecutive layers.

In this paper, we mainly focus on the rectilinear crossing numbers of K_n^r and L_n^r . If n is fixed and r tends to n, then K_n^r tends to K_n. We believe that studying the rectilinear crossing number of K_n^r might shed some light on how optimal rectilinear drawings of K_n look like.

This paper is organized as follows. In Section 2, we give a general technique to obtain non-rectilinear and rectilinear drawings of a given graph G on n vertices. It simply consists of mapping randomly the vertices of G to optimal drawings of K_n. We show how this technique upper bounds $\mathrm{cr}(K_n^r)$ and $\overline{\mathrm{cr}}(K_n^r)$. The bounds obtained in this way are very close to being optimal. However, for the layered graphs this technique gives rather poor upper bounds. In Section 3, we give a technique were given an specific drawing of a graph, we use this drawing as a "seed" to produce larger drawings by replacing each vertex u with a cluster of collinear vertices S_u arbitrarily close to u. In the new drawing two vertices in different clusters S_u and S_v are adjacent whenever u and v are adjacent in the original drawing. We call the new larger drawing a "planted drawing". The conjectured crossing optimal drawings of $K_{n,n}$ and $K_{n,n,n}$ mentioned above are actually planted drawings with drawings of $K_{2,2}$ and $K_{2,2,2}$ as seeds, respectively. However, we show that there is no rectilinear drawing of K_4 or K_8^4 that can be the seed of a crossing optimal planted drawing of K_n^4. For the layered graph, we give a rectilinear planar drawing of L_{2r}^r. When used as a seed this drawing produces a planted drawing of L_n^r, with significantly smaller crossing number, than those produced by the random embedding technique. The proofs of many of our results are long and technical; for the sake of clarity, we have relocated most of the proofs and constructions to an appendix.

2 Random Embeddings into Drawings of K_n with Small Crossing Number

Suppose that we have a drawing (that can be rectilinear but doesn't have to be) D' of K_n. If $\mathrm{cr}(D')$ is small, it might be a good idea to use this drawing to produce a drawing of a graph G on n vertices. Let D be the drawing of G that is produced by mapping the vertices of G randomly to the vertices of D', and where the edges are drawn as their corresponding edges of D'. We call D a *random embedding* of G into D'.

In every 4-tuple of vertices of D', there are three pairs of independent edges, which could cross. Of these three pairs at most one pair is crossing. For every pair of independent edges of G, we have a possible crossing in D; thus, the probability that this pair of edges is mapped to a pair of crossing edges is equal to

$$\frac{1}{3} \cdot \frac{\mathrm{cr}(D')}{\binom{n}{4}}.$$

By defining, for every pair of independent edges of G, an indicator random variable with value equal to one if the edges cross and zero otherwise, we obtain the following result where $||G||$ is the number of edges in G and $d(v)$ is the degree of a vertex v of G.

Lemma 4

$$\mathrm{E}(\mathrm{cr}(D)) = \frac{cr(D')}{3\binom{n}{4}} \left(\binom{||G||}{2} - \sum_{v \in V(G)} \binom{d(v)}{2} \right).$$

Complete r-partite Graphs

For an upper bound on the crossing number of K_n^r, we use Lemma 4 and Hill's drawing of K_n.

Theorem 5 *Suppose that n is a multiple of r. Let D be a random embedding of K_n^r into Hill's drawing of K_n. Then,*

$$\mathrm{cr}(K_n^r) \leq E(\mathrm{cr}(D)) \leq \frac{1}{16} \left(\frac{r-1}{r} \right)^2 \left(\frac{n^4}{4} - \frac{3n^3}{2} \right) + O(n^2).$$

In [9], the authors obtain the same bound on $\mathrm{cr}(K_n^r)$ by considering a random mapping of the vertices of K_n^r into a sphere, and then joining the corresponding vertices with geodesics. This type of drawing is called a *random geodesic spherical drawing.* In 1965, Moon [10], showed that the expected number of crossings of a random geodesic spherical drawing of K_n is equal to

$$\frac{1}{16} \binom{n}{2} \binom{n-2}{2} = H(n) - O(n^3);$$

which explains why the bound of Theorem 5 matches the bound of [9].

Let $H(n,r)$ be the number of crossings in Harborth's drawing for $\mathrm{cr}(K_n^r)$. Due to the complexity of the formula, we use the following approximation to $H(n,r)$ instead.

Lemma 6 *If n is a multiple of r, then*

$$H(n,r) \leq \frac{1}{16} \left(\frac{r-1}{r} \right)^2 \left(\frac{n^4}{4} - 2n^3 \right) + O(n^2).$$

Let D be as in Theorem 5; note that by Lemma 6, it holds that

$$E(\mathrm{cr}(D)) - H(n,r) \leq \frac{1}{32} \left(\frac{r-1}{r} \right)^2 n^3 + O(n^2) = O(n^3).$$

Thus, the random embedding gives an upper bound on $\mathrm{cr}(D)$ that matches the conjectured value up to the leading term, but it is a little worse in the lower terms.

We now upper bound $\overline{\mathrm{cr}}(K_n^r)$, with this technique.

Theorem 7 *Let r be a positive integer and n a multiple of r. Let $\overline{D}$ be a random embedding of K_n^r into an optimal rectilinear drawing of K_n. Then*

$$\begin{aligned} \overline{\mathrm{cr}}(K_n^r) &\leq E(\mathrm{cr}(\overline{D})) \\ &\leq \frac{\overline{q}}{4!} \left(\frac{r-1}{r} \right)^2 n^4 + o(n^4) \\ &< 0.015852 \left(\frac{r-1}{r} \right)^2 n^4 + o(n^4). \end{aligned}$$

For a lower bound we have the following.

Theorem 8 *Let r be a positive integer and n a multiple of r. Then*

$$\overline{\mathrm{cr}}(K_n^r) \geq \overline{\mathrm{cr}}(K_r) \left(\frac{n}{r} \right)^4.$$

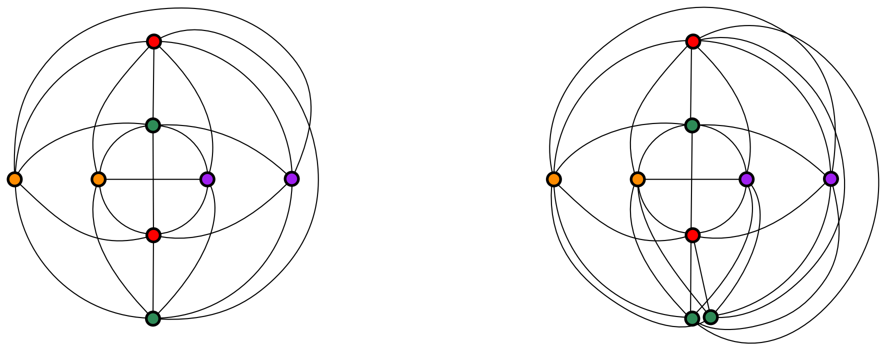

Figure 2: A drawing of K_8^4 with 6 crossings (left) and K_9^4 with 15 crossings (right).

Theorems 7 and 8 imply the following.

Corollary 9 *Let $r = r(n)$ be a monotone function of n such that $r \to \infty$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n^r)}{\binom{n}{4}} = \overline{q}.$$

In both [8] and [9], it is conjectured that

$$\mathrm{cr}\left(K_n^3\right) = \overline{\mathrm{cr}}\left(K_n^3\right).$$

Using the order type database [11], we have verified that

$$\overline{\mathrm{cr}}\left(K_8^4\right) = 8 \text{ and } \overline{\mathrm{cr}}\left(K_9^4\right) = 15.$$

On the other hand

$$\mathrm{cr}\left(K_8^4\right) \le H(8,4) = 6 \text{ and } \mathrm{cr}\left(K_9^4\right) \le H(9,4) = 15.$$

See Figure 2 for an example. From the above results we conjecture the following.

Conjecture 10 *There exists a natural number $n_0 > 9$ such that for all $n \ge n_0$,*

$$\mathrm{cr}\left(K_n^4\right) < \overline{\mathrm{cr}}\left(K_n^4\right).$$

Layered Graphs

Using the random embedding technique into Hill's drawing of K_n, we obtain the following upper bound for $\mathrm{cr}(L_n^r)$.

Theorem 11

$$\mathrm{cr}(L_n^r) \le \frac{(r-1)^2}{16r^4} n^4 + O(n^3).$$

We improve this upper bound in Section 3.

3 Planted Rectilinear Drawings

Let D be a rectilinear drawing of a graph G. For every vertex v of G, let ℓ_v, be a directed straight line passing through v and no other vertex of D, such that to left of ℓ_v there are $\lfloor d(v)/2 \rfloor$ neighbors of v and to the right of ℓ_v there are the remaining $\lceil d(v)/2 \rceil$ neighbors of v. Let G^s be the graph whose vertex set is equal to

$$\{(v,i) : i = 1, \dots, s \text{ and } v \in V(G)\},$$

and in which (v,i) is adjacent to (w,j) whenever vw is an edge of G. We say that the set $\{(v,1), \dots, (v,s)\}$ is the *cluster* of v. Let D^s be the rectilinear drawing of G^s in which for every vertex v of G, the vertices its cluster are placed arbitrarily close to ℓ_v and arbitrarily close to v (in D). We say that D^s is a *planted drawing* of G^s with *seed* D.

Lemma 12

$$\operatorname{cr}(D^s) = \operatorname{cr}(D)s^4 + \sum_{v \in V(G)} \left(\binom{\lfloor d(v)/2 \rfloor}{2} + \binom{\lceil d(v)/2 \rceil}{2} \right) \frac{s^3(s-1)}{2} + ||G|| \frac{s^2(s-1)^2}{4}.$$

Seeds and planted drawings were first used by Ábrego and Fernández-Merchant [12][1] to upper bound the rectilinear crossing number of K_n. The current best upper bound on $\overline{\operatorname{cr}}(K_n)$ is obtained via a seed of 2643 vertices and 771218714414 crossings.

Complete r-partite Graphs

Note that if we use K^r_{tr} as a seed for a planted drawing of K^r_n, we have that $s = \frac{n}{tr}$. Thus, from Lemma 12 we obtain the following.

Corollary 13 *Let D be a rectilinear drawing of K^r_{tr}. Then using D as a seed we obtain a planted drawing of K^r_n with*

$$\left(\frac{\operatorname{cr}(D) + \frac{rt}{2} \left(\binom{\lfloor (r-1)t/2 \rfloor}{2} + \binom{\lceil (r-1)t/2 \rceil}{2} \right) + \frac{r(r-1)t^2}{8}}{(rt)^4} \right) n^4 - O(n^3)$$

crossings.

Using the seeds in Figure 3, we obtain planted rectilinear drawings of K^2_n and K^3_n, with the conjectured minimum number of crossings.

Using the random embedding technique and Theorem 5 we obtain a rectilinear drawing of K^4_n with at most

$$0.0089676n^4 + o(n^4) \tag{1}$$

crossings; and since $\overline{q} > 0.379972$, the best we can hope to achieve with the random embedding technique is a rectilinear drawing of K^4_n with

$$0.0089055n^4 + o(n^4) \tag{2}$$

crossings.

Using a planar drawing of K_4 as a seed, we obtain a rectilinear planted drawing of K^r_n with

$$\left(\frac{2\left(\binom{1}{2} + \binom{2}{2}\right) + \frac{3}{2}}{4^4} \right) n^4 - O(n^3) = \frac{7}{2^9} n^4 - O(n^3) = 0.013671875n^4 - O(n^3)$$

[1] They do it in a different way as presented here; first they duplicate each vertex along halving lines; then they choose halving lines for the original and new vertices and duplicate a new. They iterate this process.

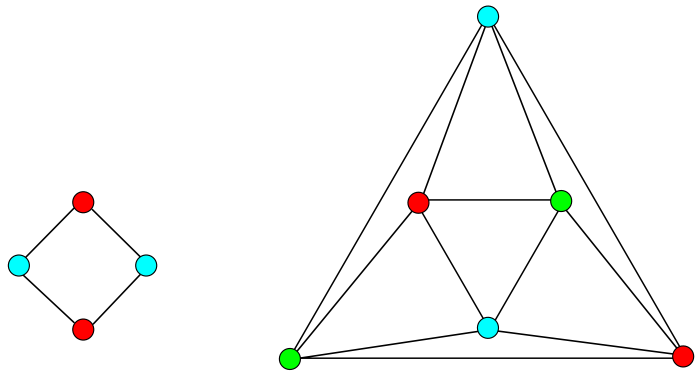

Figure 3: The seeds for the planted drawings of K_n^2 and K_n^3

crossings. Using a rectilinear drawing of K_8^4 with 8 crossings, we obtain a planted rectilinear drawing with

$$\left(\frac{8+4\left(\binom{3}{2}+\binom{3}{2}\right)+6}{8^4}\right)n^4 - O(n^3) = \frac{38}{8^4}n^4 - O(n^3) = 0.009277344n^4 - O(n^3)$$

crossings.

Fabila-Monroy and López [13] used an heuristic of randomly moving vertices to obtain a rectilinear drawing of K_{75} with 45049 crossings. This was used as a seed for a previous best upper bound on $\overline{q}$. In [14] Duque, Fabila-Monroy, Hernández-Vélez and Hidalgo-Toscano gave an $O(n^2 \log n)$ time algorithm to compute the crossing number of a rectilinear drawing of a graph on n vertices. Using a similar heuristic as in [13] and the algorithm of [14], we obtained a rectilinear drawing of K_{24}^4 with 2033 crossings. Using this as a seed we obtain a planted rectilinear drawing of K_n^4 with

$$\left(\frac{2033+12\left(\binom{9}{2}+\binom{9}{2}\right)+54}{24^4}\right)n^4 - O(n^3) = \frac{2951}{24^4}n^4 - O(n^3) = 0.0088946n^4 - O(n^3)$$

crossings. This is better than the best possible upper bound obtainable with the random embedding technique. However, for $r \geq 5$, we have not found seeds that provide planted drawings with less crossings than the drawings obtained from the random embedding technique.

Layered Graphs

We now show a rectilinear planar drawing D_r of L_{2r}^r. For $i = 1, \ldots, r$, let $\{u_i, v_i\}$ be the two vertices on layer i of L_{2r}^r. Place u_i and v_i at the points p_i and q_i, respectively; where

$$p_i := \begin{cases} (i,0) \text{ if } i \text{ is odd,} \\ (0,i) \text{ if } i \text{ is even,} \end{cases} \quad \text{and} \quad q_i := \begin{cases} (-i,0) \text{ if } i \text{ is odd,} \\ (0,-i) \text{ if } i \text{ is even.} \end{cases}$$

See Figure 4 for the drawing of L_{12}^6.

Using this drawing as a seed for a planted drawing of L_n^r, we obtain a rectilinear drawing with

$$\begin{aligned}
&\sum_{v \in V(D_r)} \left(\binom{\lfloor d(v)/2 \rfloor}{2} + \binom{\lceil d(v)/2 \rceil}{2}\right)\frac{s^4}{2} + ||D_r||\frac{s^4}{4} - O(s^3) \\
&= (2(r-2)\cdot 2)\frac{n^4}{2\cdot(2r)^4} + 4\cdot(r-1)\frac{n^4}{4\cdot(2r)^4} - O(n^3) \\
&= \frac{3r-5}{16r^4}n^4 - O(n^3)
\end{aligned}$$

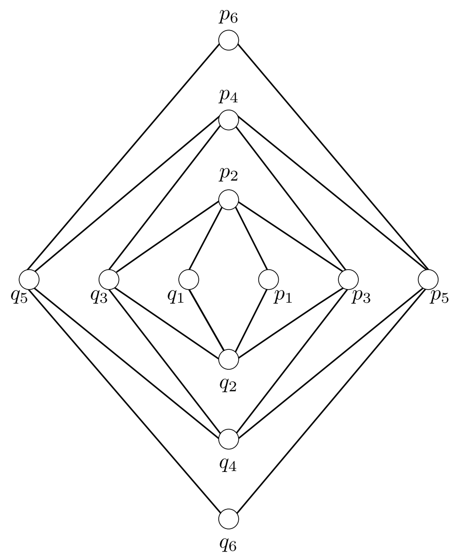

Figure 4: The rectilinear D_6 drawing of L_{12}^6

crossings. For $r \geq 4$, this is better than the upper bound obtained with the random embedding technique.

For $i = 2, \ldots, r-1$, let H_i be the subgraph of L_n^r induced by the vertices in layers $i-1, i$ and $i+1$. Note that this graph isomorphic to $K_{n/r, 2n/r}$. Thus, assuming that Zarankiewicz's conjecture holds, in every drawing of L_n^r, H_i produces at least $Z(n/r, 2n/r)$ crossings. Each of these crossings is produced by at most two such H_i's. Therefore, assuming that Zarankiewicz's conjecture is true, we have that

$$\operatorname{cr}(L_n^r) \geq \frac{(r-2)}{2} Z(n/r, 2n/r) = \frac{2r-4}{16r^4} n^4 - O(n^3).$$

References

[1] R. Fabila-Monroy, R. Paul, J. Viafara-Chanchi, and A. Weinberger, "On the rectilinear crossing number of complete balanced multipartite graphs and layered graphs," in *XX Encuentros de Geometrıa Computacional (EGC'23), Santiago de Compostela, Spain*, pp. 33–36, 2023.

[2] M. Schaefer, *Crossing numbers of graphs.* CRC Press, 2018.

[3] F. Harary and A. Hill, "On the number of crossings in a complete graph," *Proceedings of the Edinburgh Mathematical Society*, vol. 13, no. 4, pp. 333–338, 1963.

[4] R. K. Guy, "A combinatorial problem," *Nabla (Bulletin of the Malayan Mathematical Society)*, vol. 7, pp. 68–72, 1960.

[5] K. Zarankiewicz, "On a problem of P. Turan concerning graphs," *Fund. Math.*, vol. 41, pp. 137–145, 1954.

[6] B. M. Ábrego, S. Fernández-Merchant, J. Leaños, and G. Salazar, "A central approach to bound the number of crossings in a generalized configuration," in *The IV Latin-American Algorithms, Graphs, and Optimization Symposium*, vol. 30 of *Electron. Notes Discrete Math.*, pp. 273–278, Elsevier Sci. B. V., Amsterdam, 2008.

[7] O. Aichholzer, F. Duque, R. Fabila Monroy, O. E. García-Quintero, and C. Hidalgo-Toscano, "An ongoing project to improve the rectilinear and the pseudolinear crossing constants." Preprint.

[8] H. Harborth, "Über die Kreuzungszahl vollständiger, n-geteilter Graphen," *Mathematische Nachrichten*, vol. 48, no. 1-6, pp. 179–188, 1971.

[9] E. Gethner, L. Hogben, B. Lidický, F. Pfender, A. Ruiz, and M. Young, "On crossing numbers of complete tripartite and balanced complete multipartite graphs," *Journal of Graph Theory*, vol. 4, no. 84, pp. 552–565, 2017.

[10] J. W. Moon, “On the distribution of crossings in random complete graphs,” *J. Soc. Indust. Appl. Math.*, vol. 13, pp. 506–510, 1965.

[11] O. Aichholzer, F. Aurenhammer, and H. Krasser, “Enumerating order types for small point sets with applications,” *Order*, vol. 19, no. 3, pp. 265–281, 2002.

[12] B. M. Ábrego and S. Fernández-Merchant, “Geometric drawings of K_n with few crossings,” *J. Combin. Theory Ser. A*, vol. 114, no. 2, pp. 373–379, 2007.

[13] R. Fabila-Monroy and J. López, “Computational search of small point sets with small rectilinear crossing number,” *Journal of Graph Algorithms and Applications*, vol. 18, no. 3, pp. 393–399, 2014.

[14] F. Duque, R. Fabila-Monroy, C. Hernández-Vélez, and C. Hidalgo-Toscano, “Counting the number of crossings in geometric graphs,” *Inform. Process. Lett.*, vol. 165, pp. Paper No. 106028, 5, 2021.

4 Appendix

We continuously use that:

- if x is an even integer, then $\lfloor (x-1)/2 \rfloor = (x-2)/2 = x/2 - 1$;
- and if x is an odd integer, then $\lfloor x/2 \rfloor = (x-1)/2$.

Lemma 14

$$\frac{H(n)}{3\binom{n}{4}} \leq \frac{1}{8}\left(1 - \frac{2}{n}\right).$$

Proof. If n is even, then

$$\begin{aligned}
\frac{H(n)}{3\binom{n}{4}} &= \frac{2}{n(n-1)(n-2)(n-3)} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \\
&= \frac{2}{n(n-1)(n-2)(n-3)} \left(\frac{n}{2}\right) \left(\frac{n-2}{2}\right) \left(\frac{n-2}{2}\right) \left(\frac{n-4}{2}\right) \\
&= \frac{1}{(n-1)(n-3)} \left(\frac{1}{2}\right) \left(\frac{n-2}{2}\right) \left(\frac{n-4}{2}\right) \\
&= \frac{1}{8(n-1)(n-3)} ((n-2)(n-4)) \\
&= \frac{n-2}{8} \left(\frac{(n-4)}{(n-1)(n-3)}\right) \\
&= \frac{n-2}{8} \left(\frac{(n-4)}{n^2-4n+3}\right) \\
&= \frac{n-2}{8} \left(\frac{(n-4)n}{(n^2-4n+3)n}\right) \\
&= \frac{n-2}{8} \left(\frac{n^2-4n}{(n^2-4n+3)n}\right) \\
&< \frac{n-2}{8} \left(\frac{1}{n}\right) \\
&= \frac{1}{8}\left(1 - \frac{2}{n}\right).
\end{aligned}$$

If n is odd, then

$$
\begin{aligned}
\frac{H(n)}{3\binom{n}{4}} &= \frac{2}{n(n-1)(n-2)(n-3)} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor \\
&= \frac{2}{n(n-1)(n-2)(n-3)} \left(\frac{n-1}{2} \right) \left(\frac{n-1}{2} \right) \left(\frac{n-3}{2} \right) \left(\frac{n-3}{2} \right) \\
&= \frac{1}{8n} \left(\frac{(n-1)(n-3)}{n-2} \right) \\
&= \frac{n-2}{8n} \left(\frac{(n-1)(n-3)}{(n-2)(n-2)} \right) \\
&= \frac{n-2}{8n} \left(\frac{n^2-4n+3}{n^2-4n+4} \right) \\
&< \frac{1}{8} \left(\frac{n-2}{n} \right) \\
&= \frac{1}{8} \left(1 - \frac{2}{n} \right).
\end{aligned}
$$

□

According to Harborth [8], if n is a multiple of r, then

$$
H(n,r) \leq \frac{3}{8}\binom{r}{4}\frac{n^4}{r^4} + r \left\lfloor \frac{n/r}{2} \right\rfloor \left\lfloor \frac{n/r-1}{2} \right\rfloor \left\lfloor \frac{n-n/r}{2} \right\rfloor \left\lfloor \frac{n-n/r-1}{2} \right\rfloor - \binom{r}{2} \left(\left\lfloor \frac{n/r}{2} \right\rfloor^2 \right) \left(\left\lfloor \frac{n/r-1}{2} \right\rfloor^2 \right) + O(n^2).
$$

We bound this formula to obtain Lemma 6.

Proof. [Lemma 6] Let

$$
A := \frac{3}{8}\binom{r}{4}\frac{n^4}{r^4} = \frac{1}{64} \cdot \frac{(r-1)(r-2)(r-3)}{r^3} n^4,
$$

$$
B := r \left\lfloor \frac{n/r}{2} \right\rfloor \left\lfloor \frac{n/r-1}{2} \right\rfloor \left\lfloor \frac{n-n/r}{2} \right\rfloor \left\lfloor \frac{n-n/r-1}{2} \right\rfloor,
$$

and

$$
C := \binom{r}{2} \left\lfloor \frac{n/r}{2} \right\rfloor^2 \left\lfloor \frac{n/r-1}{2} \right\rfloor^2.
$$

If n/r is even, then

$$
\begin{aligned}
B &= r \left(\frac{n}{2r} \right) \left(\frac{n}{2r} - 1 \right) \left(\frac{n}{2} - \frac{n}{2r} \right) \left(\frac{n}{2} - \frac{n}{2r} - 1 \right) \\
&= \frac{n}{2} \left(\frac{n}{2r} - 1 \right) \left(\frac{n}{2} \left(\frac{r-1}{r} \right) \right) \left(\frac{n}{2} \left(\frac{r-1}{r} \right) - 1 \right) \\
&= \left(\frac{n^2}{4r} - \frac{n}{2} \right) \left(\frac{n^2}{4} \left(\frac{r-1}{r} \right)^2 - \frac{n}{2} \left(\frac{r-1}{r} \right) \right) \\
&= \frac{n^4}{16r} \left(\frac{r-1}{r} \right)^2 - \frac{n^3}{8r} \left(\frac{r-1}{r} \right) - \frac{n^3}{8} \left(\frac{r-1}{r} \right)^2 + O(n^2);
\end{aligned}
$$

If n/r is odd, and n is even, then

$$\begin{aligned} B &= r\left\lfloor\frac{n/r}{2}\right\rfloor\left\lfloor\frac{n/r-1}{2}\right\rfloor\left\lfloor\frac{n-n/r}{2}\right\rfloor\left\lfloor\frac{n-n/r-1}{2}\right\rfloor \\ &= r\left(\frac{n/r-1}{2}\right)\left(\frac{n/r-1}{2}\right)\left(\frac{n-n/r-1}{2}\right)\left(\frac{n-n/r-1}{2}\right) \\ &= \frac{r}{16}\left(\frac{n^2}{r^2}-\frac{2n}{r}+1\right)\left(\left(\frac{r-1}{r}\right)^2 n^2 - 2\left(\frac{r-1}{r}\right)n+1\right) \\ &= \frac{n^4}{16r}\left(\frac{r-1}{r}\right)^2 - \frac{n^3}{8r}\left(\frac{r-1}{r}\right) - \frac{n^3}{8}\left(\frac{r-1}{r}\right)^2 + O(n^2) \end{aligned}$$

If n/r is odd and n is odd, then

$$\begin{aligned} B &= r\left\lfloor\frac{n/r}{2}\right\rfloor\left\lfloor\frac{n/r-1}{2}\right\rfloor\left\lfloor\frac{n-n/r}{2}\right\rfloor\left\lfloor\frac{n-n/r-1}{2}\right\rfloor \\ &= r\left(\frac{n/r-1}{2}\right)\left(\frac{n/r-1}{2}\right)\left(\frac{n-n/r}{2}\right)\left(\frac{n-n/r-2}{2}\right) \\ &= \frac{r}{16}\left(\frac{n^2}{r^2}-\frac{2n}{r}+1\right)\left(\left(\frac{r-1}{r}\right)^2 n^2 - 2\left(\frac{r-1}{r}\right)n\right) \\ &= \frac{n^4}{16r}\left(\frac{r-1}{r}\right)^2 - \frac{n^3}{8r}\left(\frac{r-1}{r}\right) - \frac{n^3}{8}\left(\frac{r-1}{r}\right)^2 + O(n^2) \end{aligned}$$

If n is even, then

$$\begin{aligned} C &= \binom{r}{2}\left\lfloor\frac{n/r}{2}\right\rfloor^2\left\lfloor\frac{n/r-1}{2}\right\rfloor^2 \\ & \frac{r(r-1)}{2}\cdot\frac{n^2}{4r^2}\left(\frac{n}{2r}-1\right)^2 \\ &= \frac{r-1}{32r^3}n^4 - \frac{r-1}{8r^2}n^3 + O(n^2); \end{aligned}$$

if n is odd, then

$$\begin{aligned} C &= \binom{r}{2}\left\lfloor\frac{n/r}{2}\right\rfloor^2\left\lfloor\frac{n/r-1}{2}\right\rfloor^2 \\ &= \frac{r(r-1)}{2}\left(\frac{n/r-1}{2}\right)^2\left(\frac{n/r-1}{2}\right)^2 \\ &= \frac{r(r-1)}{32}\left(\frac{n^4}{r^4}-\frac{4n^3}{r^3}+O(n^2)\right) \\ &= \frac{r-1}{32r^3}n^4 - \frac{r-1}{8r^2}n^3 + O(n^2). \end{aligned}$$

Therefore,

$$
\begin{aligned}
H(n,r) &\le A+B-C+O(n^2) \\
&= \frac{1}{16}\left(\frac{(r-1)(r-2)(r-3)}{4r^3}+\frac{(r-1)^2}{r^3}-\frac{r-1}{2r^3}\right)n^4 \\
&+ \frac{1}{8}\left(-\frac{r-1}{r^2}-\frac{(r-1)^2}{r^2}+\frac{r-1}{r^2}\right)n^3 \\
&+ O(n^2) \\
&= \frac{1}{16}\left(\frac{r^3-2r+r}{4r^3}\right)n^4+n^3+O(n^2)-\frac{1}{16}\left(\frac{r-1}{r}\right)^2 2n^3+O(n^2) \\
&= \frac{1}{16}\left(\frac{r-1}{r}\right)^2\left(\frac{n^4}{4}-2n^3\right)+O(n^2).
\end{aligned}
$$

□

Lemma 15 *If n is a multiple of r, then*

$$
\left(\binom{||K_n^r||}{2}-\sum_{v\in V(K_n^r)}\binom{d(v)}{2}\right)=\frac{1}{2}\left(\frac{r-1}{r}\right)^2\left(\frac{n^4}{4}-n^3\right)+O(n^2).
$$

Proof. Every set in the partition has n/r vertices. Thus, the number of edges between two different sets is equal to n^2/r^2. Therefore,

$$
||K_n^r|| = \frac{n^2}{r^2}\binom{r}{2} = \frac{n^2}{2}\cdot\frac{r-1}{r},
$$

and

$$
\binom{||K_n^r||}{2} = \frac{n^4}{8}\left(\frac{r-1}{r}\right)^2-\frac{n^2}{4}\left(\frac{r-1}{r}\right).
$$

For every vertex v of K_n^r, it holds that

$$
d(v) = \frac{r-1}{r}n.
$$

Thus,

$$
\sum_{v\in V(K_n^r)}\binom{d(v)}{2} = \frac{n^3}{2}\left(\frac{r-1}{r}\right)^2-\frac{r-1}{r}\cdot\frac{n^2}{2}.
$$

It follows that

$$
\begin{aligned}
\left(\binom{||K_n^r||}{2}-\sum_{v\in V(K_n^r)}\binom{d(v)}{2}\right) &= \frac{n^4}{8}\left(\frac{r-1}{r}\right)^2-\frac{n^2}{4}\left(\frac{r-1}{r}\right)-\frac{n^3}{2}\left(\frac{r-1}{r}\right)^2+\frac{n^2}{2}\left(\frac{r-1}{r}\right) \\
&= \frac{1}{2}\left(\frac{r-1}{r}\right)\left(\frac{n^4}{4}\left(\frac{r-1}{r}\right)-\frac{n^2}{2}-n^3\left(\frac{r-1}{r}\right)+n^2\right) \\
&= \frac{1}{2}\left(\frac{r-1}{r}\right)^2\left(\frac{n^4}{4}-n^3\right)+O(n^2).
\end{aligned}
$$

□

Using Lemmas 4, 14 and 15, we can prove Theorem 5.

Proof. [Theorem 5] By Lemma 4, it holds that

$$E(\mathrm{cr}(D)) = \frac{H(n)}{3\binom{n}{4}} \left(\binom{||K_n^r||}{2} - \sum_{v \in V(K_n^r)} \binom{d(v)}{2} \right).$$

Applying Lemmas 14 and 15 on the equality above yields

$$\begin{aligned} E(\mathrm{cr}(D)) &\leq \frac{1}{8}\left(1 - \frac{2}{n}\right)\left(\frac{1}{2}\left(\frac{r-1}{r}\right)^2\left(\frac{n^4}{4} - n^3\right) + O(n^2)\right) \\ &\leq \frac{1}{16}\left(\frac{r-1}{r}\right)^2\left(\frac{n^4}{4} - \frac{3n^3}{2}\right) + O(n^2). \end{aligned}$$

□

Proof. [Theorem 7] From Lemma 4 and the best upper bound known for $\overline{\mathrm{cr}}(K_n)$, it follows that

$$\begin{aligned} E(\overline{\mathrm{cr}}(\overline{D})) &= \frac{\overline{\mathrm{cr}}(K_n)}{3\binom{n}{4}} \left(\binom{||K_n^r||}{2} - \sum_{v \in V(K_n^r)} \binom{d(v)}{2} \right) \\ &= \frac{\overline{q}\binom{n}{4} + o(n^4)}{3\binom{n}{4}} \left(\frac{1}{2}\left(\frac{r-1}{r}\right)^2\left(\frac{n^4}{4} - n^3\right) + O(n^2)\right) \\ &\leq \frac{\overline{q}}{4!}\left(\frac{r-1}{r}\right)^2 n^4 + o(n^4). \end{aligned}$$

□

Proof. [Theorem 8] Let D be a rectilinear drawing of K_n^r. Let D' be a rectilinear drawing of K_r obtained by choosing one point from each color class of D. There are $(n/r)^r$ such choices; and each choice provides at least $\overline{\mathrm{cr}}(K_r)$ crossings. Each such crossing is counted exactly $(n/r)^{r-4}$ times. □

Proof. [Corollary 9] We have that

$$\begin{aligned} \lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n^r)}{\binom{n}{4}} &\geq \lim_{n\to\infty} \overline{\mathrm{cr}}(K_r) \cdot \left(\frac{n}{r}\right)^4 \cdot \frac{4!}{n(n-1)(n-2)(n-3)} \\ &\geq \lim_{r\to\infty} \frac{\overline{\mathrm{cr}}(K_r)}{\left(\frac{r^4}{4!}\right)} \\ &= \overline{q}. \end{aligned}$$

By Theorem 7, $\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n^r)}{\binom{n}{4}} \leq \overline{q}\left(\frac{r-1}{r}\right)^2$.

As $\left(\frac{r-1}{r}\right)^2 < 1$, it follows that

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n^r)}{\binom{n}{4}} = \overline{q}.$$

□

To prove Theorem 11, we use the following proposition.

Proposition 16 *Let r be a positive integer and let n be a multiple of r. Then*

$$\left(\binom{||L_n^r||}{2} - \sum_{v \in V(L_n^r)} \binom{d(v)}{2} \right) = \frac{(r-1)^2}{2r^4}n^4 - \frac{2r-3}{r^3}n^3 + \frac{r-1}{r^2}n^2.$$

Proof. Note that

$$||L_n^r|| = (r-1)\left(\frac{n}{r}\right)^2;$$

and

$$\binom{||L_n^r||}{2} = \frac{(r-1)^2}{2r^4}n^4 - \frac{r-1}{2r^2}n^2.$$

We have that

$$\begin{aligned}
\sum_{v \in V(L_n^r)} \binom{d(v)}{2} &= \frac{2n}{r}\binom{n/r}{2} + \frac{(r-2)n}{r}\binom{2n/r}{2} \\
&= \frac{2n}{r}\left(\frac{n^2}{2r^2} - \frac{n}{2r}\right) + \frac{(r-2)n}{r}\left(\frac{2n^2}{r^2} - \frac{n}{r}\right) \\
&= \frac{2r-3}{r^3}n^3 - \frac{r-1}{r^2}n^2.
\end{aligned}$$

Thus,

$$\left(\binom{||L_n^r||}{2} - \sum_{v \in V(L_n^r)} \binom{d(v)}{2}\right) = \frac{(r-1)^2}{2r^4}n^4 - \frac{2r-3}{r^3}n^3 + \frac{r-1}{r^2}n^2.$$

$\square$

Combining Proposition 16 with Lemmas 4 and 14, we obtain Theorem 11.

Proof. [Theorem 11]

$$\begin{aligned}
\operatorname{cr}(L_n^r) \leq & E(\operatorname{cr}(D)) = \frac{H(n)}{3\binom{n}{4}}\left(\binom{||L_n^r||}{2} - \sum_{v \in V(L_n^r)} \binom{d(v)}{2}\right) \leq \\
& \frac{1}{8}\left(1 - \frac{2}{n}\right)\left(\frac{(r-1)^2}{2r^4}n^4 + O(n^3)\right) \leq \frac{(r-1)^2}{16r^4}n^4 + O(n^3).
\end{aligned}$$

$\square$

Proof. [Lemma 12] We classify the crossings of D^s depending on the number of different clusters in which the endpoints of the edges defining the crossing appear. Let e_1 and e_2 be a pair of edges of D^s that cross.

Suppose that the endpoints of e_1 and e_2 appear in four different clusters. We have that $e_1 = (u,i)(v,j)$ and $e_2 = (w,k)(x,l)$ for some four distinct vertices u, v, w, l of D and indices $1 \leq i, j, k, l \leq s$. Thus, uv, wx is a pair of crossing edges in D; and for each pair of crossing edges in D we obtain s^4 pairs of crossing edges of D^s, such that its endpoints lie in four different clusters. Therefore, the number of crossings of D^s generated by pairs of edges whose endpoints lie in four different clusters is equal to

$$\operatorname{cr}(D)s^4.$$

Suppose that the endpoints of e_1 and e_2 lie in three different clusters. Without loss of generality $e_1 = (u,i)(v,j)$ and $e_2 = (u,k)(w,l)$ for some three distinct vertices u, v, w of D and indices $1 \leq i, j, k, l \leq s$. Thus, v and w lie on the same side of ℓ_u; and for every pair of vertices of D lying on the same side of ℓ_u we obtain $\binom{s}{2}s^2$ crossings in D^s generated by pairs of edges whose endpoints lie in three different clusters. Therefore, the number of crossings of D^s generated by pairs of edges whose endpoints lie in three different clusters is equal to

$$\sum_{v \in V(G)} \left(\binom{\lfloor d(v)/2 \rfloor}{2} + \binom{\lceil d(v)/2 \rceil}{2}\right) \frac{s^3(s-1)}{2}.$$

Suppose that the endpoints of e_1 and e_2 lie in two different clusters. We have that $e_1 = (u,i)(v,j)$ and $e_2 = (u,k)(v,l)$ for some edge uv of D and indices $1 \leq i,j,k,l \leq s$; and for every edge of D we obtain $\binom{s}{2}\binom{s}{2}$ crossings in D^s generated by pairs of edges whose endpoints lie in two different clusters. Therefore, the number of of crossings of D^s generated by pairs of edges whose endpoints lie in two different clusters is equal to

$$||G||\frac{s^2(s-1)^2}{4}.$$

□

We now give the coordinates of the rectilinear drawing D of K_{24}^4 with 2033 crossings. The colors are $0,1,2$ and 3. We have appended the color of each point as a third coordinate.

$$\begin{aligned} V(D) = \{&(-59260959, 44970123, 0), (261261347, -43693014, 0), (158829052, -28658158, 0), \\ &(-20273112, -23913465, 0), (20602644, -8343316, 0), (-8148611, -63519416, 0), \\ &(30209164, 4850528, 1), (12317574, -161508817, 1), (46649346, -344926319, 1), \\ &(-11015825, -47872739, 1), (-26347789, 22655563, 1), (-46729617, 35472331, 1), \\ &(-74136586, 66127255, 2), (-278900322, 316137789, 2), (14791528, -20163276, 2), \\ &(-140757971, 147565111, 2), (14081248, -20874215, 2), (9903931, -24183515, 2), \\ &(-38516867, 27953341, 3), (-60922797, 47350463, 3), (8267623, -135305393, 3), \\ &(-15043716, -39580158, 3), (41831995, 797354, 3), (181333931, -34086725, 3)\}. \end{aligned}$$

The vertices of this drawing can be seen in Figure 5.

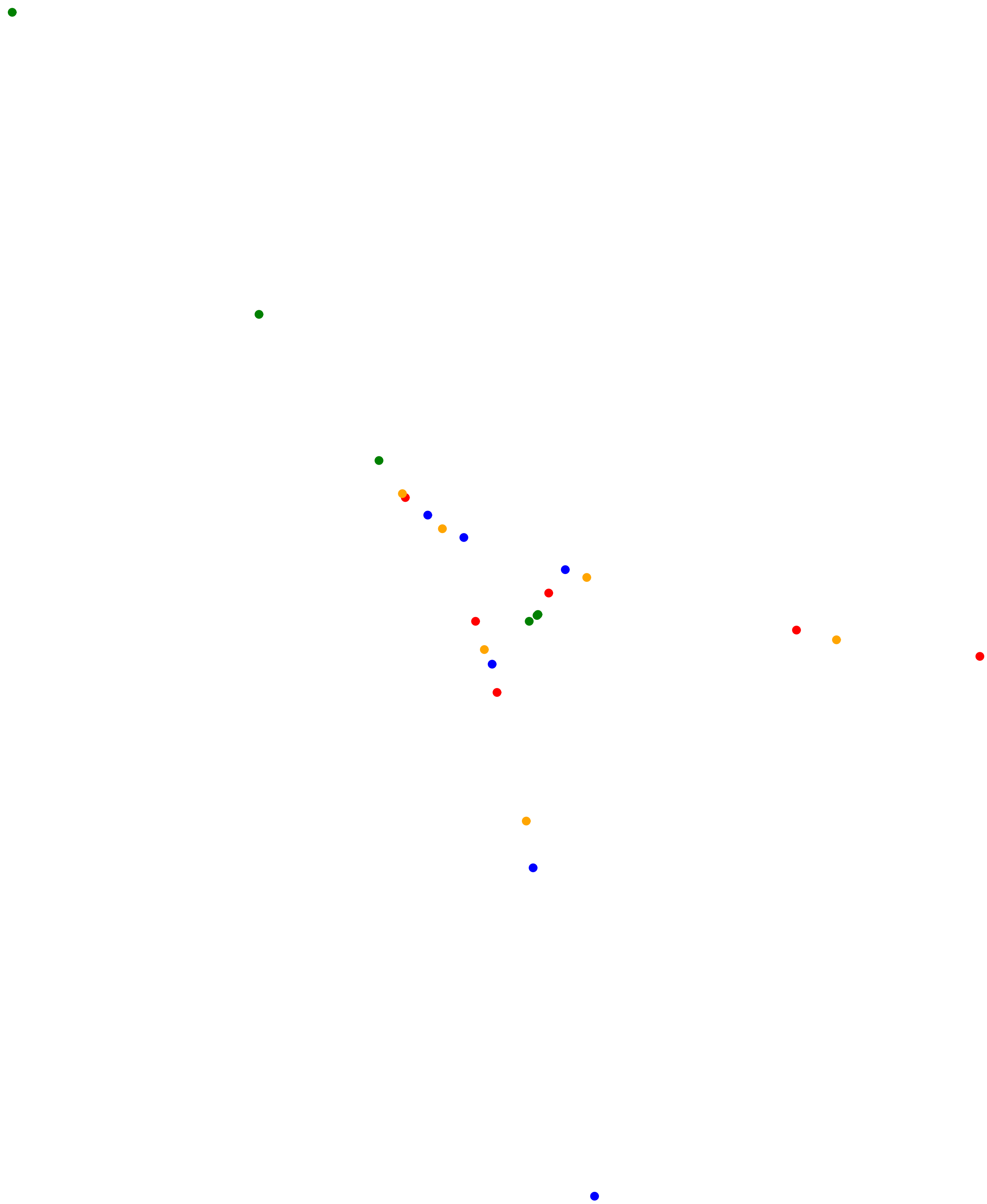

Figure 5: The vertices of a rectilinear drawing of K_{24}^4