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THE MOTIVIC AND ÉTALE BECKER-GOTTLIEB TRANSFER AND SPLITTINGS

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Abstract. In this paper, we establish the key properties of the motivic and étale Becker-Gottlieb transfer including compatibility with étale and Betti realization and show how to obtain various splittings using the transfer.

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1. Introduction

This paper is a continuation of the paper [CJ23-T1], where the authors defined the Becker-Gottlieb transfer in the motivic and étale framework for the Borel construction. The goal of the present paper is to establish the basic properties of this transfer and then establish various splittings making use of it. Therefore, we will adopt the basic framework and terminology of [CJ23-T1, section 1]. We will briefly recall these here.

Basic assumptions on the base field.

(1) A *standing assumption throughout* is that the base field k is a perfect field of arbitrary characteristic.
(2) When considering actions by linear algebraic groups G that are *not special*, we *will also assume the base field is infinite* to prevent certain unpleasant situations.
(3) On considering étale realizations of the transfer, it is important to assume that the base field

(1.0.1)
$$k \text{ has finite } \ell-\text{cohomological dimension, for } \ell \neq char(k) \text{ and satisfies the finiteness conditions that}$$
$$\mathrm{H}^{\mathrm{n}}_{\mathrm{et}}(\mathrm{Spec}\, k, \mathbb{Z}/\ell^{\nu}) \text{ is finitely generated in each degree n and vanishes for all n} >> 0, \text{ all } \nu > 0.$$
(Such an assumption is not needed on dealing with the motivic transfer alone.)

One should be able to see that such an assumption is necessary to get any theory of Spanier-Whitehead duality on the étale site of $\mathrm{Spec}\, k$.

Basic assumptions on the linear algebraic groups considered:

(1) we allow any linear algebraic group over k, *irrespective* of whether it is *connected or not* and
(2) we are not assuming it is special in the sense of Grothendieck (see [Ch]). This means, in particular, we allow groups such as all orthogonal groups and finite groups, which are all known to be non-special.

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2. Basic properties of the transfer

We may start with a G-torsor $E \to B$, with both E and B smooth quasi-projective schemes over S. We will further assume that B is *always connected*.

The Borel construction. Given a simplicial presheaf X with an action by G, one forms the quotient $E \times_G X$ of the product $E \times X$ by the diagonal action: here G acts on the right on E through the involution of G given by $g \mapsto g^{-1}$ and on the left on X in the usual manner. The construction of such a quotient is the Borel construction and it needs to be carried out carefully so that if X is a smooth scheme, one obtains the correct object. This construction is discussed in detail in [CJ23-T1, section 8.3].

Let X and Y denote two simplicial presheaves provided with G-actions. We will consider the following three *basic contexts* for the transfer:

(a) $p : E \to B$ is a G-torsor for the action of a linear algebraic group G with both E and B smooth quasi-projective schemes over k, with B *connected* and

$$\pi_Y : E \times_G (Y \times X) \to E \times_G Y$$

the induced map, where G acts diagonally on $Y \times X$. One may observe that, on taking $Y = \mathrm{Spec}\, k$ with the trivial action of G, the map π_Y becomes $\pi : E \times_G X \to B$ (the induced projection), which is an important special case.

(b) A basic example of such a G-torsor is $EG^{gm,m} \to BG^{gm,m}$, where $BG^{gm,m}$ denotes the m-th degree approximation to the geometric classifying space of the linear algebraic group G as in [MV99] (see also [Tot99]), $p : EG^{gm,m} \to BG^{gm,m}$ is the corresponding universal G-torsor and

$$\pi_Y : EG^{gm,m} \times_G (Y \times X) \to EG^{gm,m} \times_G Y$$

is the induced map.

(c) If p_m ($\pi_{Y,m}$) denotes the map denoted p (π_Y) in (b), here we let $p = \lim_{m\to\infty} p_m$ and let

$$\pi_Y = \lim_{m\to\infty} \pi_{Y,m} : EG^{gm} \times_G (Y \times X) = \lim_{m\to\infty} EG^{gm,m} \times_G (Y \times X) \to \lim_{m\to\infty} EG^{gm,m} \times_G Y = EG^{gm} \times_G Y.$$

Strictly speaking, the above definitions apply only to the case where G is special. When G is not special, the above objects will in fact need to be replaced by the derived push-forward of the above objects viewed as sheaves on the big étale site of k to the corresponding big Nisnevich site of k, as discussed in [CJ23-T1, (8.3.6)]. *However, we will denote these new objects also by the same notation throughout, except when it is necessary to distinguish between them.* Recall that, for G not special, we *will assume the base field is also infinite* to prevent certain unpleasant situations.

Throughout the following discussion, $\mathcal{E}^G$ will denote any one of the G-equivariant ring spectra considered in [CJ23-T1, (4.0.24)], with $\mathcal{E}$ denoting the corresponding non-equivariant spectrum: see [CJ23-T1, Definition 4.13, (4.0.29)].

Definition 2.1. (Weak ring and module spectra over commutative ring spectra) Let A denote a spectrum in $\mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$ ($\mathbf{Spt}(k_{et}, \epsilon^*(\mathcal{E}))$). Then we call A a *weak ring spectrum* if there is given a pairing

$$\mu : A \wedge A \to A$$

in $\mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$ ($\mathbf{Spt}(k_{et}, \epsilon^*(\mathcal{E}))$) that is homotopy associative. A spectrum $M \in \mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$ ($\mathbf{Spt}(k_{et}, \epsilon^*(\mathcal{E}))$) is then called a *a weak module spectrum* over A if it comes equipped with a pairing $A \wedge M \to M$ that is homotopy associative in the sense that the obvious square involving μ and the last pairing homotopy commutes.

Then we recall from [CJ23-T1, Definition 8.8, (8.4.18), (8.4.19) and (8.4.20)] the transfer map is defined as follows. Let $f : X \to X$ denote a G-equivariant map and let $\pi_Y : E \times_G (Y \times X) \to E \times_G Y$ denote any one of the maps considered in (a) through (c) above. Let $f_Y = id_Y \times f : Y \times X \to Y \times X$ denote the induced map.

Then in case (a) and when the group G is special, we obtain a map (called *the transfer*)

$$tr(f_Y) : \Sigma^\infty_{\mathbf{T}}(E \times_G Y)_+ \to \Sigma^\infty_{\mathbf{T}}(E \times_G (Y \times X))_+ \quad (tr(f_Y) : \mathcal{E} \wedge (E \times_G Y)_+ \to \mathcal{E} \wedge (E \times_G (Y \times X))_+)$$

in $\mathcal{SH}(k)$ ($\mathcal{SH}(k, \mathcal{E})$, respectively) if $\Sigma^\infty_{\mathbf{T}} X_+$ is dualizable in $\mathcal{SH}(k)$ (if $\mathcal{E} \wedge X_+$ is dualizable in $\mathcal{SH}(k, \mathcal{E})$, respectively). In case (a) and when G is not special, the corresponding transfer is the map

$$tr(f_Y) : R\epsilon_*(\epsilon^*\mathbb{S}_k) \wedge R\epsilon_*(\widetilde{E} \times^{et}_G Y)_+ \to R\epsilon_*(\epsilon^*\mathbb{S}_k) \wedge R\epsilon_*(\widetilde{E} \times^{et}_G (Y \times X))_+$$

$$(tr(\mathrm{f_Y}) : \mathrm{R}\epsilon_*(\epsilon^*\mathcal{E}) \wedge \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times^{\mathrm{et}}_{\mathrm{G}} \mathrm{Y})_+ \to \mathrm{R}\epsilon_*(\epsilon^*\mathcal{E}) \wedge \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times^{\mathrm{et}}_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+, \text{respectively}).$$

Definition 2.2. Let A denote a weak ring spectrum in $\mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$, and let M denote a weak module spectrum over A. When G is special, and Z is a simplicial presheaf with G-action, we will let $\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Z}, \mathrm{M})$ denote the generalized motivic cohomology of $\mathrm{E} \times_{\mathrm{G}} \mathrm{Z}$ with respect to the motivic spectrum M.

When G is non-special and Z is a simplicial presheaf with G-action, we will let

$$\mathrm{h}^{\mathrm{i,j}}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Z}, \mathrm{M}) = [\mathrm{R}\epsilon_*(\epsilon^*(\mathcal{E})) \wedge \mathrm{R}\epsilon_*(\mathrm{E} \times_{\mathrm{G}} \mathrm{Z})_+, \mathrm{M(j)[i]}],$$

and where $[\quad,\quad]$ denotes hom in $\mathcal{SH}(k, \mathcal{E})$.

Theorem 2.3. *The transfer has the following properties.*

(i) *If* $tr(\mathrm{f_Y})^{\mathrm{m}} : \Sigma^\infty_{\mathbf{T}}(\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y})_+ \to \Sigma^\infty_{\mathbf{T}}(\mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+$ $(tr(\mathrm{f_Y})^{\mathrm{m}} : \mathcal{E} \wedge (\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{Y})_+ \to \mathcal{E} \wedge (\mathrm{EG}^{gm,m} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+)$ *denotes the corresponding transfer maps in case* (b), *the maps* $\{tr(\mathrm{f_Y})^{\mathrm{m}}|\mathrm{m}\}$ *are compatible as m varies. The corresponding induced map* $\lim_{m\to\infty} tr(\mathrm{f_Y})^{\mathrm{m}}$ *will be denoted* $tr(\mathrm{f_Y})$.

For items (ii) *through* (v) *we will assume that* A *is a weak ring spectrum in* $\mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$ *and that* M *is a weak module spectrum over* A. *Moreover, for items* (ii) *through* (iv) *we may assume any one of the above contexts* (a) *through* (c).

(ii) *If* $\mathrm{h}^{*,\bullet}(\quad, \mathrm{A})$ $(\mathrm{h}^{*,\bullet}(\quad, \mathrm{M}))$ *denotes the generalized motivic cohomology theory defined as in Definition 2.2 with respect to the weak ring spectrum* A *(the weak module spectrum* M *over* A*, respectively) then,*

$$tr(\mathrm{f_Y})^*(\pi^*_{\mathrm{Y}}(\alpha).\beta) = \alpha.tr(\mathrm{f_Y})^*(\beta), \quad \alpha \in \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M}), \beta \in \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{A}).$$

Here $tr(\mathrm{f_Y})^*$ (π^*_{Y}) *denotes the map induced on generalized cohomology by the map* $tr(\mathrm{f_Y})$ $(\pi_{\mathrm{Y}}$*, respectively). Both* $tr(\mathrm{f_Y})^*$ *and* π^*_{Y} *preserve the degree as well as the weight.*

(iii) *The composition* $tr(\mathrm{f_Y})^* \circ \pi^*_{\mathrm{Y}} : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{A}) \to \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{A})$ *is an isomorphism if and only if* $tr(\mathrm{f_Y})^*(1) = tr(\mathrm{f_Y})^*(\pi^*_{\mathrm{Y}}(1))$ *is a unit in* $\mathrm{h}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{A})$*, where* $1 \in \mathrm{h}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{A})$ *is the unit of the graded ring* $\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{A})$.

In particular, $\pi^*_{\mathrm{Y}} : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M}) \to \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M})$ *is split injective if* $tr(\mathrm{f_Y})^*(1) = tr(\mathrm{f_Y})^*(\pi^*_{\mathrm{Y}}(1))$ *is a unit, where* $1 \in \mathrm{h}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{A})$ *is again the unit of the graded ring* $\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{A})$.

(iv) *The transfer* $tr(\mathrm{f_Y})$ *is compatible with respect to restriction to subgroups of a given group. It is also compatible with respect to change of base fields.* [1]

(v) *The map* $tr(\mathrm{f_Y})^* : \mathrm{h}^{*,\bullet}(\mathrm{EG}^{gm} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M}) \to \mathrm{h}^{*,\bullet}(\mathrm{EG}^{gm} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M})$ *is independent of the choice of a geometric classifying space that satisfies certain basic assumptions (as in 2.1: Proof of Theorem 2.3), and depends only on* X, Y *and the* G*-equivariant map* f.

(vi) *Assume the base field* k *satisfies the finiteness conditions in* (1.0.1). *Assume* $\mathcal{E}$ *(which belongs to* $\mathbf{Spt}(k_{\mathrm{mot}})$*) is* ℓ*-complete, in the sense of Definition 3.6, for some prime* $\ell \neq char(k)$. *Let* $\epsilon^* : \mathbf{Spt}(k_{\mathrm{mot}}) \to \mathbf{Spt}(k_{et})$ *denote the functor induced by the obvious map sites from the étale site of* k *to the Nisnevich site of* k.

If $\epsilon^*(\mathcal{E} \wedge \mathrm{X}_+)$ *is dualizable in* $\mathcal{SH}(k_{et}, \epsilon^*(\mathcal{E}))$*, and* A *is a weak ring spectrum in* $\mathbf{Spt}(k_{et}, \epsilon^*(\mathcal{E}))$ *with* M *a weak-module spectrum over* A*, then there exists a transfer* $tr(\mathrm{f_Y})$ *in* $\mathcal{SH}(k_{et}, \epsilon^*(\mathcal{E}))$ *satisfying similar properties.*

(vii) *Assume the base field* k *satisfies the finiteness conditions in* (1.0.1). *Let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}(k_{\mathrm{mot}})$ *which is* ℓ*-complete. Then if* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathcal{SH}(k, \mathcal{E})$*,* $\epsilon^*(\mathcal{E} \wedge \mathrm{X}_+)$ *is dualizable in* $\mathcal{SH}(k_{et}, \epsilon^*(\mathcal{E}))$*, the transfer map* $tr(\mathrm{f_Y})$ *is compatible with étale realizations, and for groups* G *that are special,* $\epsilon^*(tr(\mathrm{f_Y})) = tr(\epsilon^*(\mathrm{f_Y}))$.

The above theorem is proven by first establishing various key properties of the transfer as in the following theorems. We will adopt the terminology and notation as in [CJ23-T1, Terminology 8.5].

Proposition 2.4. *(Naturality with respect to base-change and change of groups) Let* G *denote a linear algebraic group over* k *and let* X, Y *denote smooth quasi-projective* G*-schemes over* k *or unpointed simplicial presheaves on* Sm_k *provided with* G*-actions. Let* $\mathrm{p} : \mathrm{E} \to \mathrm{B}$ *denote a* G*-torsor with* E *and* B *smooth quasi-projective schemes over* k*, with* B *connected and let* $\mathrm{f} : \mathrm{X} \to \mathrm{X}$ *denote a* G*-equivariant map.*

[1] We prove in Theorem 2.4 that the transfer is compatible with a more general base-change which includes this as a special case.

Let G' *denote a closed linear algebraic subgroup of* G, $\mathrm{p}' : \mathrm{E}' \to \mathrm{B}'$ *a* G'*-torsor with* B' *connected, and* Y' *a* G'*-quasi-projective scheme over* k *or an unpointed simplicial sheaf* Sm_k *provided with a* G'*-action, so that it comes equipped with a map* $\mathrm{Y}' \to \mathrm{Y}$ *that is compatible with the* G'*-action on* Y' *and the* G*-action on* Y*. Further, we assume that one is provided with a commutative square*

$$\begin{tikzcd} \mathrm{E}' \arrow[r] \arrow[d] & \mathrm{E} \arrow[d] \\ \mathrm{B}' \arrow[r] & \mathrm{B} \end{tikzcd}$$

compatible with the action of G' *(*G*) on* E' *(*E*, respectively).*

(i) *Then if* $\Sigma^\infty_{\mathbf{T}}\mathrm{X}_+$ *is dualizable in* $\mathcal{SH}(k)$*, the square*

$$\begin{tikzcd} \Sigma^\infty_{\mathbf{T}}(\mathrm{E}' \times_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}))_+ \arrow[r] & \Sigma^\infty{}_{\mathbf{T}}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{Y} \times \mathrm{X}))_+ \\ \Sigma^\infty_{\mathbf{T}}(\mathrm{E}' \times_{\mathrm{G}'} \mathrm{Y}')_+ \arrow[r] \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}'})"] & \Sigma^\infty_{\mathbf{T}}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}})"'] \end{tikzcd}$$

$$\begin{tikzcd} (\ \mathrm{R}\epsilon_*(\epsilon^*\Sigma^\infty_{\mathbf{T}}) \wedge \mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}))_+ \arrow[r] & \mathrm{R}\epsilon_*(\epsilon^*\Sigma^\infty_{\mathbf{T}}) \wedge \mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+ \\ \mathrm{R}\epsilon_*(\epsilon^*\Sigma^\infty_{\mathbf{T}}) \wedge \mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} \mathrm{Y}')_+ \arrow[r] \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}'})"] & \mathrm{R}\epsilon_*(\epsilon^*\Sigma^\infty_{\mathbf{T}}) \wedge \mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} \mathrm{Y})_+) \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}})"'] \end{tikzcd}$$

commutes in the motivic stable homotopy category when G *is special (*G *is not necessarily special, respectively).*

(ii) *Next let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}(k_{\mathrm{mot}})$ *with* $\mathcal{E} \wedge X_+$ *dualizable in* $\mathcal{SH}(k_{\mathrm{mot}}, \mathcal{E})$*. Then the square*

$$\begin{tikzcd} \mathcal{E} \wedge (\mathrm{E}' \times_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}))_+ \arrow[r] & \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+ \\ \mathcal{E} \wedge (\mathrm{E}' \times_{\mathrm{G}'} \mathrm{Y}')_+ \arrow[r] \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}'})"] & \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_+ \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}})"'] \end{tikzcd}$$

$$\begin{tikzcd} (\mathrm{R}\epsilon_*(\epsilon^*\mathcal{E}) \wedge \mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}))_+ \arrow[r] & \mathrm{R}\epsilon_*(\epsilon^*\mathcal{E}) \wedge \mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+ \\ \mathrm{R}\epsilon_*(\epsilon^*\mathcal{E}) \wedge \mathrm{R}\epsilon_*(\mathrm{E}' \times^{\mathrm{et}}_{\mathrm{G}'} \mathrm{Y}')_+ \arrow[r] \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}'})"] & \mathrm{R}\epsilon_*(\epsilon^*\mathcal{E}) \wedge \mathrm{R}\epsilon_*(\mathrm{E} \times^{\mathrm{et}}_{\mathrm{G}} \mathrm{Y})_+) \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}})"'] \end{tikzcd}$$

commutes in the motivic stable homotopy category $\mathcal{SH}(k, \mathcal{E})$ *when* G *is special (*G *is not necessarily special, respectively). (In this case we may require* ℓ *is a prime* $\neq char(k)$ *so that* $\mathcal{E}$ *is* $Z_{(\ell)}$*-local or that* $\mathcal{E}$ *is* ℓ*-complete.)*

(iii) *In case* $\mathcal{E}$ *denotes a commutative ring spectrum in* $\mathbf{Spt}(k_{et})$ *which is* ℓ*-complete for some prime* ℓ *and* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathbf{Spt}(k_{et}, \mathcal{E})$*, one obtains a homotopy commutative diagram as in the first square in (ii) (with all the objects there replaced by their pull-backs to the big étale site) in the corresponding étale stable homotopy category* $\mathcal{SH}(k_{et}, \mathcal{E})$*.*

Proof. We will discuss this explicitly only in the case G is special, since the other case follows along similar lines with the obvious modifications. For each fixed representation V of G, let ξ^{V} (η^{V}) denote the vector bundles on $\widetilde{\mathrm{B}}$ chosen as in [CJ23-T1, 8.4, Step 2] ([CJ23-T1, (8.4.7)], respectively). Let ${\xi'}^{\mathrm{V}}$ (${\eta'}^{\mathrm{V}}$) denote the pull-back of these bundles to $\widetilde{\mathrm{B}}'$. Since $\xi^{\mathrm{V}} \oplus \eta^{\mathrm{V}}$ is trivial, so is ${\xi'}^{\mathrm{V}} \oplus {\eta'}^{\mathrm{V}}$. Now the required property follows readily in view of this observation and the definition of the transfer as in [CJ23-T1, Definition 8.8], in view of the cartesian square, where $\pi_{\rm Y}$ ($\pi_{\rm Y'}$) is induced by the projection $\rm Y \times X \rightarrow Y$ ($\rm Y' \times X \rightarrow Y'$, respectively):

$$\begin{tikzcd} \mathrm{E}' \times_{\mathrm{G}'} (\mathrm{Y}' \times \mathrm{X}) \arrow[r] \arrow[d, "\pi_{\mathrm{Y}'}"] & \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \arrow[d, "\pi_{\mathrm{Y}}"] \\ \mathrm{E}' \times_{\mathrm{G}'} \mathrm{Y}' \arrow[r] & \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}. \end{tikzcd} \tag{2.0.1}$$

The main observation here is that the diagrams in [CJ23-T1, (8.4.3)] through [CJ23-T1, (8.4.17)] for $\widetilde{\rm E}' \times_{\rm G'} {\rm Y}'$ map to the corresponding diagrams for $\widetilde{\rm E} \times_{\rm G} {\rm Y}$ making the resulting diagrams commute, since all the transfers are constructed from the pre-transfers $tr_{\rm G}({\rm f})'$ and $tr_{\rm G'}({\rm f})'$. □

Remark 2.5. Taking different choices for $\rm p'$ and $\rm Y'$ provides many examples where the last Proposition applies. For example, let $\rm B' \rightarrow B$ denote a map from another smooth quasi-projective scheme that is also connected, let $\rm E' = E \underset{B}{\times} B'$ and let $\pi'_{\rm Y} : \rm E' \times_G (Y \times X) \rightarrow E' \times_G Y$ denote the induced map: this corresponds to base-change. (In particular, $\rm B'$ could be given by $\rm E/H \cong E \times_G G/H$ for a closed subgroup H so that E/H is connected.) Moreover, the above proposition readily provides the following key multiplicative property of the transfer.

Proposition 2.6. *(Multiplicative property)*

(i) *Let* H *denote a linear algebraic group. Let* X, Y *denote smooth quasi-projective schemes over k or unpointed simplicial presheaves on* Sm_k *provided with* H*-actions, so that $\Sigma^\infty_{\mathbf{T}} \mathrm{X}_+$ is a dualizable in $\mathcal{SH}(k)$. Let $\rm p : E \rightarrow B$ denote an* H*-torsor and let $\pi_{\rm Y} : \rm E \underset{H}{\times} X \rightarrow B$ denote the induced map. Then the diagram*

$$\begin{tikzcd} \Sigma^\infty_{\mathbf{T}} \mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X})_+ \arrow[r, "\mathrm{d}"] & \Sigma^\infty_{\mathbf{T}} (\mathrm{E} \times \mathrm{E}) \underset{\mathrm{H} \times \mathrm{H}}{\times} ((\mathrm{Y} \times \mathrm{X}) \times (\mathrm{Y} \times \mathrm{X}))_+ \arrow[r, "\mathrm{id} \wedge \mathrm{q} \wedge \mathrm{id}"] & \Sigma^\infty_{\mathbf{T}} (\mathrm{E} \times \mathrm{E}) \underset{\mathrm{H} \times \mathrm{H}}{\times} (\mathrm{Y} \times (\mathrm{Y} \times \mathrm{X}))_+ \\ \Sigma^\infty_{\mathbf{T}} \mathrm{E} \times_{\mathrm{H}} \mathrm{Y}_+ \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}})"] \arrow[rr, "\mathrm{d}"] & & \Sigma^\infty_{\mathbf{T}} (\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \wedge (\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \arrow[u, "\mathrm{id} \wedge tr(\mathrm{f}_{\mathrm{Y}})"'] \end{tikzcd}$$

commutes in $\mathcal{SH}(k)$, when H *is special. Here* d *denotes the diagonal map induced by the diagonal map $\rm Y \times X \rightarrow (Y \times X) \times (Y \times X)$ and* q *denotes the map induced by the projection $\rm Y \times X \rightarrow Y$. In case* H *is not necessarily special, one obtains a corresponding commutative diagram which is obtained by replacing any term of the form $\Sigma^\infty_{\rm T} \mathrm{E} \underset{\mathrm{H}}{\times} \mathrm{Z}_+$ with $\mathrm{R}\epsilon_*(\epsilon^* \mathbb{S}_k) \wedge \mathrm{R}\epsilon_*(\mathrm{E} \underset{\mathrm{H}}{\overset{\mathrm{et}}{\times}} \epsilon^*(\mathrm{Z}))_+$.*

(ii) *In case $\mathcal{E}$ is a commutative ring spectrum in $\mathbf{Spt}(k_{\rm mot})$ with $\mathcal{E} \wedge \mathrm{X}_+$ dualizable in $\mathcal{SH}(k_{\rm mot}, \mathcal{E})$ and* X *is as in (i), the square*

$$\begin{tikzcd} \mathcal{E} \wedge \mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X})_+ \arrow[r, "\mathrm{d}"] & \mathcal{E} \wedge (\mathrm{E} \times \mathrm{E}) \underset{\mathrm{H} \times \mathrm{H}}{\times} ((\mathrm{Y} \times \mathrm{X}) \times (\mathrm{Y} \times \mathrm{X}))_+ \arrow[r, "\mathrm{id} \wedge \mathrm{q} \wedge \mathrm{id}"] & \mathcal{E} \wedge (\mathrm{E} \times \mathrm{E}) \underset{\mathrm{H} \times \mathrm{H}}{\times} (\mathrm{Y} \times (\mathrm{Y} \times \mathrm{X}))_+ \\ \mathcal{E} \wedge \mathrm{E} \times_{\mathrm{H}} \mathrm{Y}_+ \arrow[u, "tr(\mathrm{f}_{\mathrm{Y}})"] \arrow[rr, "\mathrm{d}"] & & \mathcal{E} \wedge (\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \wedge (\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})_+ \arrow[u, "\mathrm{id} \wedge tr(\mathrm{f}_{\mathrm{Y}})"'] \end{tikzcd}$$

also commutes $\mathcal{SH}(k_{\rm mot}, \mathcal{E})$, when H *is special. (Here we may also require that ℓ is a prime $\neq char(k)$ so that $\mathcal{E}$ is $Z_{(\ell)}$-local or that $\mathcal{E}$ is ℓ-complete.) In case* H *is not necessarily special, one obtains a corresponding commutative diagram with all the terms above replaced as in (i).*

(iii) *In case $\mathcal{E}$ denotes a commutative ring spectrum in $\mathbf{Spt}(k_{et})$ which is ℓ-complete for some prime $\ell \neq char(k)$, and $\mathcal{E} \wedge \mathrm{X}_+$ is dualizable in $\mathbf{Spt}(k_{et}, \mathcal{E})$, the corresponding square commutes in the étale stable homotopy category $\mathcal{SH}(k_{et}, \mathcal{E})$.*

Proof. We apply Proposition 2.4 with the following choices:

(i) for G, we take $\rm H \times H$, for $\rm G'$ we take the diagonal H in $\rm H \times H$,
(ii) for p ($\rm p'$) we take $\rm p \times p : E \times E \rightarrow B \times B$ (the given $\rm p : E \rightarrow B$, respectively),
(iii) for Y we take $\rm Y \times Y$ provided with the obvious action of $\rm H \times H$ and for $\rm Y'$ we take Y provided with the given action of H.

Then we obtain the cartesian square as in (2.0.1) and the map in the corresponding top row is given by

$\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}) \rightarrow (\mathrm{E} \times \mathrm{E}) \underset{\mathrm{H}\times\mathrm{H}}{\times} (\mathrm{Y} \times (\mathrm{Y} \times \mathrm{X}))$.

This factors as $\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}) \overset{\mathrm{d}}{\rightarrow} (\mathrm{E} \times \mathrm{E}) \underset{\mathrm{H}\times\mathrm{H}}{\times} ((\mathrm{Y} \times \mathrm{X}) \times (\mathrm{Y} \times \mathrm{X})) \overset{\mathrm{id}\times\mathrm{q}\times\mathrm{id}}{\rightarrow} (\mathrm{E} \times \mathrm{E}) \underset{\mathrm{H}\times\mathrm{H}}{\times} (\mathrm{Y} \times (\mathrm{Y} \times \mathrm{X}))$. Therefore, Proposition 2.4 applies. ☐

Next one may recall the definition of weak-module spectra from Definition 2.1.

Corollary 2.7. *(Multiplicative property of transfer in generalized cohomology theories) Let* H *denote a linear algebraic group.*

(i) *Assume that* A *is a weak ring spectrum in* $\mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$*, and that* M *is a weak module spectrum over* A*. Assume that* X*,* Y *are smooth quasi-projective schemes over* k *or unpointed simplicial presheaves on* Sm_k *provided with an action by* H *and that* $\mathcal{E} \wedge \mathrm{X}_+$ *is dualizable in* $\mathcal{SH}(k_{\mathrm{mot}}, \mathcal{E})$*.*

Let $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{E} \times_{\mathrm{H}} \mathrm{Y}$ *(*$\pi_{\mathrm{Y}} : \mathrm{R}\epsilon_*(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{R}\epsilon_*(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y})$*) denote the (obvious) map induced by the structure map* $\mathrm{X} \rightarrow \mathrm{Spec}\, k$ *when* H *is special (when* H *is not necessarily special, respectively).*

Then

$$tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(\alpha).\beta) = \alpha.tr(\mathrm{f_Y})^*(\beta), \quad \alpha \in \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M}), \beta \in \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X})), \mathrm{A}),$$

irrespective of whether H *is special or not, and where the generalized motivic cohomology is defined as in Definition 2.2. Here* $tr(\mathrm{f_Y})^*$ *(*π_{Y}^**) denotes the map induced on generalized cohomology by the map* $tr(\mathrm{f_Y})$ *(*π_{Y}*, respectively). In particular,*

$$\pi_{\mathrm{Y}}^* : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M}) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M})$$

is split injective when H *if* $tr(\mathrm{f_Y})^*(1) = tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))$ *is a unit, where* $1 \in \mathrm{h}^{0,0}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{A})$ *is the unit of the graded ring* $\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{A})$*.*

(ii) *Assume that* ℓ *is a prime* $\neq char(k)$ *so that* $\mathcal{E}$ *is* ℓ*-complete, with* $\epsilon^*(\mathcal{E} \wedge \mathrm{X}_+)$ *dualizable in* $\mathcal{SH}(k_{et}, \epsilon^*(\mathcal{E}))$*. Assume further that* A *is a weak ring spectrum in* $\mathbf{Spt}(k_{et}, \epsilon^*(\mathcal{E}))$*, and that* M *is a weak module spectrum over* A*. Then the same conclusions as in (i) hold for the generalized cohomology with respect to* A *and* M*.*

Proof. The proof of both statements follow by applying the cohomology theory $\mathrm{h}^{*,\bullet}$ to all terms in the commutative diagram in Proposition 2.6. We will provide details only for the case H is special, as the general case follows similarly. Moreover, as (ii) may be proven in the same manner as (i), we will only discuss the proof of (i).

In the proof of (i), one needs to start with $\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M})$ and $\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M})$ along with the pairings:

$$\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y}, \mathrm{M}) \otimes \mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}), \mathrm{A}) \overset{\pi_{\mathrm{Y}}^*\otimes\mathrm{id}}{\rightarrow} \mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}), \mathrm{M}) \otimes \mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}), \mathrm{A}) \rightarrow \mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}), \mathrm{M}),$$

$$\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M}) \otimes \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{A}) \rightarrow \mathrm{h}^*(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M}).$$

This provides us with the commutative diagram:
(2.0.2)

$$\begin{tikzcd}
\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M}) \otimes \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}), \mathrm{A}) \arrow[rr, "id\otimes tr(\mathrm{f_Y})^*"] \arrow[d, "\pi_{\mathrm{Y}}^*\otimes id"] & & \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M}) \otimes \mathrm{h}^*(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{A}) \arrow[d, "d^*"] \\
\mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M}) \otimes \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}), \mathrm{A}) \arrow[r] & \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M}) \arrow[r, "tr(\mathrm{f_Y})^*"] & \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{H}} \mathrm{Y}, \mathrm{M})
\end{tikzcd}$$

Since the multiplicative property is the key to obtaining splittings in the motivic stable homotopy category (see Theorems 5.1 and 5.7), we will provide details on how one deduces commutativity of the above diagram. The commutativity of the above diagram follows from the commutativity of a large diagram which we break up into three squares as follows, where RHom denotes the derived external Hom in the category $\mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$.

(2.0.3)
$$\begin{CD}
\mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M})\otimes\mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A}) @>>> \mathrm{RHom}((\mathrm{E}\times\mathrm{E})_{\mathrm{H}\times\mathrm{H}}(\mathrm{Y}\times(\mathrm{Y}\times\mathrm{X})),\mathrm{M}\wedge\mathrm{A})\\
@V{id\otimes tr(\mathrm{f_Y})^*}VV @VV{(id\wedge tr(\mathrm{f_Y}))^*}V\\
\mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M})\otimes\mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{A}) @>>> \mathrm{RHom}((\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+\wedge(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+,\mathrm{M}\wedge\mathrm{A})
\end{CD}$$

$$\begin{CD}
\mathrm{RHom}((\mathrm{E}\times\mathrm{E})_{\mathrm{H}\times\mathrm{H}}(\mathrm{Y}\times(\mathrm{Y}\times\mathrm{X})),\mathrm{M}\wedge\mathrm{A}) @>{d^*\circ(\pi_{\mathrm{Y}}^*\wedge id)}>> \mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M}\wedge\mathrm{A})\\
@V{(id\wedge tr(\mathrm{f_Y}))^*}VV @VV{tr(\mathrm{f_Y})^*}V\\
\mathrm{RHom}((\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+\wedge(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y})_+,\mathrm{M}\wedge\mathrm{A}) @>{d^*}>> \mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M}\wedge\mathrm{A})
\end{CD}$$

$$\begin{CD}
\mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M}\wedge\mathrm{A}) @>{\mu}>> \mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M})\\
@V{tr(\mathrm{f_Y})^*}VV @VV{tr(\mathrm{f_Y})^*}V\\
\mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M}\wedge\mathrm{A}) @>{\mu}>> \mathrm{RHom}(\mathrm{E}\times_{\mathrm{H}}\mathrm{Y},\mathrm{M})
\end{CD}$$

The commutativity of the first square is clear from the observation that the transfer $tr(\mathrm{f_Y})$ is a stable map. The commutativity of the second square is essentially the multiplicative property proved in Proposition 2.6. The map μ in the third square is the map induced by the pairing $\mathrm{M}\wedge\mathrm{A}\rightarrow\mathrm{M}$. The commutativity of this square again follows readily from the observation that $tr(\mathrm{f_Y})$ is a stable map. The commutativity of the square in (2.0.2) results by composing the appropriate maps in the first square followed by the second and then the third square. More precisely, one can see that the composition of the maps in the top rows of the three squares followed by the right vertical map in the last square equals $tr(\mathrm{f_Y})^*\circ\mathrm{d}^*\circ(\pi_{\mathrm{Y}}^*\otimes\mathrm{id})$ which is the composition of the left vertical map and the bottom row in the square (2.0.2). Similarly the composition of the left vertical map in the first square and the maps in the bottom rows of the three squares above equals $d^*\circ(id\otimes tr(\mathrm{f_Y})^*)$ which is the composition of the top row and the right vertical map in (2.0.2). This completes the proof of the first statement in (i).

The last statement in (i) follows by taking $\beta=1=\pi_{\mathrm{Y}}^*(1)\in\mathrm{h}^{0,0}(\mathrm{E}\times_{\mathrm{H}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A})$. The statement in (ii) follows from an entirely similar argument in the étale case. ☐

Next, we proceed to interpret the condition that $tr(\mathrm{f_Y})^*(1)$ be a unit in $\mathrm{h}^{0,0}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{A})$, for a weak ring spectrum A in terms of the trace $\tau_{\mathrm{X}}(\mathrm{f})$. We will do this only under the assumption the group G is also *special*. Recall that we have a standing assumption that B is *connected*. In this context, we will *assume* the following: Y is a connected smooth scheme and $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}=\lim_{\mathrm{n}\rightarrow\infty}\{\mathrm{E_n}\times_{\mathrm{G}}\mathrm{Y}|\mathrm{n}\geq 1\}$, where $\{\mathrm{E_n}|\mathrm{n}\geq 1\}$ is an ind-scheme with each $\mathrm{E_n}\times_{\mathrm{G}}\mathrm{Y}$ is a connected smooth scheme so that each $\mathrm{B_n}=\mathrm{E_n}/\mathrm{G}$ for $n>>0$ has k-rational points. We let $\mathrm{B}=\lim_{\infty\rightarrow\mathrm{n}}\mathrm{B_n}$.

We already observed in [CJ23-T1, 8.2.4. Convention] that when B denotes a finite degree approximation, $\mathrm{BG}^{gm,m}$, (with m sufficiently large) of the classifying spaces for a linear algebraic group G, it has k-rational points and is therefore a geometrically connected smooth scheme of finite type over k: see [EGA, Tome 24, Chapitre 4, Corollaire 4.5.14]. Furthermore, we will assume that the generalized cohomology theory $\mathrm{h}^{*,\bullet}(\quad,\mathrm{A})$ (defined with respect to the weak ring spectrum A as in Definition 2.2) is such that the restriction

$$\mathrm{h}^{0,0}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{A})\rightarrow\mathrm{h}^{0,0}(\mathrm{Y},\mathrm{A}) \tag{2.0.4}$$

is an isomorphism, where $\mathrm{Spec}\,k\rightarrow\mathrm{B}$ is any k-rational point of B, and Y is viewed as the fiber of $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$ over $\mathrm{Spec}\,k$.

Proposition 2.8. *Assume that both* Y *and* X *are smooth schemes of finite type over* k *and* G *is special.*

Under the above assumptions

$$tr(\mathrm{f_Y})^*(1)=tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))=(\mathrm{id_Y}\wedge\tau_{\mathrm{X}}(\mathrm{f}))^*(1)$$

where $\tau_{\mathrm{X}}(\mathrm{f})$ *is the trace defined in* [CJ23-T1, Definition 8.4(iv)] *and* $\pi_{\mathrm{Y}}^*:\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{A})\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A})$, $tr(\mathrm{f_Y})^*:\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}),\mathrm{A})\rightarrow\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{A})$ *denote the induced maps.*

Proof. We discuss explicitly only the case where $char(k) = 0$. In positive characteristics p, one needs to replace the sphere spectrum $\mathbb{S}_k$ everywhere by the corresponding sphere spectrum with the prime p inverted, or completed away from p.

Then the first equality is clear since $\pi_{\rm Y}^*$ is a ring homomorphism and therefore, $\pi_{\rm Y}^*(1) = 1$. Recall that we are assuming the group G is special. The naturality with respect to base-change as in Proposition 2.4, together with the assumption that the restriction $\rm h^{0,0}(E \times_G^{et} Y, A) \rightarrow h^{0,0}(Y, A)$ is an isomorphism shows that $tr(\rm f_Y)^*\pi_Y^*(1)$ is the same for $\rm E \times_G Y$ as well as for Y. When G is special and the scheme $\rm B = Spec\, \mathit{k}$, $tr(\rm f) : \mathcal{E} \wedge (Spec\, \mathit{k})_+ \rightarrow \mathcal{E} \wedge X_+$, (which also identifies with the corresponding pre-transfer $tr(\rm f)'$) so that, $tr(\rm f_Y) = id_Y \wedge \mathit{tr}(f)'$. Therefore, it is clear that $\pi_{\rm Y} \circ tr(\rm f_Y) = id_Y \wedge \tau_X(f)$ as defined in [CJ23-T1, Definition 8.4 (iii), (iv)]. Therefore, the equality $tr(\rm f_Y)^*(\pi_Y^*(1)) = (id_Y \wedge \tau_X(f))^*(1)$ follows, and completes the proof. ∎

Proposition 2.9. *The hypotheses in* (2.0.4) *are satisfied when* $\rm h^{*,\bullet}$ *denotes motivic cohomology with respect to any commutative ring* R*,* Y *is a connected smooth scheme,* $\rm E \times_G^{et} Y = \underset{n\rightarrow\infty}{\mathit{lim}} \{E_n \times_G Y | n \geq 1\}$*, where* $\rm \{E_n | n \geq 1\}$ *is an ind-scheme with each* $\rm E_n \times_G Y$ *a connected smooth scheme, and when* $\rm B_n = E_n/G$*, for* $n >> 0$ *has* k*-rational points.*

Proof. Recall that Y and $\rm E \times_G Y$ are assumed to be connected. Observe that now, $\rm h^{0,0} = H_M^{0,0}$, which denotes motivic cohomology in degree 0 and weight 0. The motivic complex R(0) is the constant sheaf associated to the ring R. Therefore, since $\rm E \times_G Y$ and Y are connected, the restriction map $\rm H^{0,0}(E \times_G Y, R) \rightarrow H^{0,0}(Y, R)$ is an isomorphism, where Y is the fiber of $\rm E \times_G Y \rightarrow B$ over any k-rational point of B. It follows that the hypothesis in (2.0.4) is always satisfied, when $\rm h^{*,\bullet}(\quad, A)$ denotes motivic cohomology with respect to any commutative ring R. This proves the proposition. ∎

2.1. **Proof of Theorem 2.3.** We will first clarify the terminology used. Recall that $\rm BG^{\mathit{gm,m}}$ ($\rm EG^{\mathit{gm,m}}$) denotes the m-th degree approximation to the classifying space of the group G (its principal G-bundle, respectively) as in [MV99]: see also [Tot99]. If X is a scheme with G-action, one can form the scheme $\rm EG^{\mathit{gm,m}} \times_G X$, which is called the Borel construction. In case G is not special, the torsor $\rm EG^{\mathit{gm,m}} \rightarrow BG^{\mathit{gm,m}}$ is locally trivial only in the étale topology, so that in this case we replace the Borel construction above by $\rm R\epsilon_*(EG^{\mathit{gm,m}} \times_G^{et} X)$ as discussed in [CJ23-T1, section 8.3, Case 2]. However, we will continue to denote $\rm R\epsilon_*(EG^{\mathit{gm,m}} \times_G^{et} X)$ by $\rm EG^{\mathit{gm,m}} \times_G X$ mainly for the sake of simplicity of notation.

The first statement in the Theorem is the compatibility of the transfer with various degrees of finite dimensional approximations to the classifying space: this has been discussed in [CJ23-T1, 8.4, Step 2] in the construction of the transfer. The second statement in the Theorem is the multiplicative property proven in Corollary 2.7. Taking $\beta = 1$ in (ii) proves that if $tr(\rm f_Y)^*(1)$ is a unit, then the composition $tr(\rm f_Y)^* \circ \pi_Y^*$ is an isomorphism for any weak module spectrum M over A. Conversely if $tr(\rm f_Y)^* \circ \pi_Y^*$ is an isomorphism, then the multiplicative property in (ii) shows that, there exists some class $\alpha \in \rm h^{0,0}(E \times_G Y, A)$ so that $\alpha.tr(\rm f_Y)^*(\pi_Y^*(1)) = 1 \in h^{0,0}(E \times_G Y, A)$, which shows $tr(\rm f_Y)^*(\pi_Y^*(1))$ is a unit in $\rm h^{0,0}(E \times_G Y, A)$. This proves the first statement in (iii).

The second statement in (iii) is now clear, and the first statement in (iv) follows from the naturality property for the transfer with respect to base-change as in Proposition 2.4. (See Remark 2.5.) That the transfer is compatible with change of base fields follows from the corresponding property for the pre-transfer: see [CJ23-T1, Proposition 7.3]. The second statement in (ii) follows from the fact that the transfer is defined using the pre-transfer, which is a stable map that involves no degree or weight shifts.

Next we will sketch an argument to prove Theorem 2.3(v). Let $\rm \{BG^{\mathit{gm,m}}(1)|m\}$, $\rm \{BG^{\mathit{gm,m}}(2)|m\}$ denote two sequences of finite degree approximations to the classifying space of the given group G satisfying certain basic assumptions as in [MV99]. (See also [Tot99].) Let $\rm \{EG^{\mathit{gm,m}}(1), EG^{\mathit{gm,m}}(2)|m\}$ denote the corresponding universal G-bundles: the main requirements here are that both these have free actions by G and that as $m \rightarrow \infty$, these are $\mathbb{A}^1$-acyclic.

Then a key observation is that $\rm \{EG^{\mathit{gm,m}}(1) \times EG^{\mathit{gm,m}}(2)|m\}$ with the diagonal action of the group G also satisfies the same hypotheses so that their quotient by the diagonal action of G will also define approximations to the classifying space of the G. Therefore, after replacing $\rm \{EG^{\mathit{gm,m}}(1)|m\}$ with

$\{\mathrm{EG}^{gm,m}(1) \times \mathrm{EG}^{gm,m}(2)|\mathrm{m}\}$, we may assume that one has a direct system of smooth surjective maps $\{\mathrm{EG}^{gm,m}(1) \rightarrow \mathrm{EG}^{gm,m}(2)|\mathrm{m}\}$. Now it is straightforward to verify that all the constructions discussed in the above steps for the transfer are compatible with the maps $\{\mathrm{EG}^{gm,m}(1) \rightarrow \mathrm{EG}^{gm,m}(2)|\mathrm{m}\}$. Therefore, by Proposition 2.4, one obtains a direct system of homotopy commutative diagrams, $m \geq 1$:

$$\begin{tikzcd}
\Sigma^{\infty}_{\mathbf{T}}(\mathrm{EG}^{gm,m}(1) \times_{\mathrm{G}} \mathrm{X})_+ \arrow[r] & \Sigma^{\infty}_{\mathbf{T}}(\mathrm{EG}^{gm,m}(2) \times_{\mathrm{G}} \mathrm{X})_+ \\
\Sigma^{\infty}_{\mathbf{T}}\mathrm{BG}^{gm,m}(1)_+ \arrow[r] \arrow[u, "tr(f)^m(1)"] & \Sigma^{\infty}_{\mathbf{T}}\mathrm{BG}^{gm,m}(2)_+. \arrow[u, "tr(f)^m(2)"']
\end{tikzcd}$$

Finally, one may also verify that the maps $\{\mathrm{EG}^{gm,m}(1) \times_{\mathrm{G}} \mathrm{X} \rightarrow \mathrm{EG}^{gm,m}(2) \times_{\mathrm{G}} \mathrm{X}|\mathrm{m}\}$ and $\{\mathrm{BG}^{gm,m}(1) \rightarrow \mathrm{BG}^{gm,m}(2)|\mathrm{m}\}$ induce isomorphisms on generalized motivic cohomology as one takes the $\lim_{m \rightarrow \infty}$: see, for example [MV99, §4, Proposition 2.6]. These complete the proof of (v) when $\mathrm{Y} = \operatorname{Spec} k$: the case when Y is a general smooth G-scheme or a simplicial presheaf with G-action is similar.

The construction of the transfer in the étale framework is entirely similar, though care has to be taken to ensure that affine spaces are contractible in this framework, which accounts partly for the hypothesis in (vi) and in (1.0.1). Property (vii) is proved in the next section. □

3. Étale and Betti realization

3.1. *The motivation for considering étale and Betti realization.* We hope this short discussion clarifies the role of étale and Betti realization in the context of the transfer.

We make use of the following *two distinct strategies* for obtaining splittings using the transfer:

• One of these is to show certain classes are units in the Grothendieck-Witt ring of the base field: this is quite difficult and is do-able only for very special cases as explained later: see the discussion following the proof of Theorem 5.2.

• The second technique is to restrict to what are slice-completed generalized motivic cohomology theories (defined in Definition 5.5): since several of the well-known generalized motivic cohomology theories, such as Algebraic K-Theory and Algebraic Cobordism are slice completed, this is not a major restriction.

Since the transfer is a stable map, it induces a map of the motivic Atiyah-Hirzebruch spectral sequences for the above generalized cohomology theories. The E_2-terms of these spectral sequences are modules over the motivic cohomology of the corresponding motivic spaces. By the multiplicative property of the transfer (see Corollary 2.7), we may then reduce to obtaining splittings at the level of motivic cohomology.

It is precisely at this point that it becomes quite convenient to know that the pre-transfer and transfer are compatible with both étale and Betti realization (see section 4), so that one may reduce to showing the transfer in étale cohomology or Betti cohomology produces the desired splittings. Clearly proving such splittings at the level of étale or Betti cohomology is considerably easier than showing this at the level of motivic cohomology. Nevertheless, we show that such splittings at the level of the étale or Betti realization is enough to show the desired splitting exists at the level of motivic cohomology: see Proposition 5.8.

In view of the above observations, our primary interest is in considering étale realization and Betti realization *only* with respect to Eilenberg-Maclane spectra: this justifies the following short discussion. A more detailed discussion of étale realization in general is left to the Appendix.

3.2. *Passage to spectra on the étale site.* We start with the following morphism of sites: $\epsilon : \mathrm{Sm}_{k,et} \rightarrow \mathrm{Sm}_{k,\mathrm{Nis}}$, $\bar{\epsilon} : \mathrm{Sm}_{\bar{k},et} \rightarrow \mathrm{Sm}_{\bar{k},\mathrm{Nis}}$ and $\eta : \mathrm{Sm}_{\bar{k},et} \rightarrow \mathrm{Sm}_{k,et}$, where $\bar{k}$ denotes the algebraic closure of k.

In view of the discussion of the alternate model structure on Nisnevich presheaves defined by inverting hypercovers and the discussion of the model structure on étale presheaves as in [CJ23-T1, 2.1.7], one can see that these induce the following functors:

(3.2.1)
$$\epsilon^* : \mathbf{Spt}^{\mathrm{G}}(k_{\mathrm{mot}}) \rightarrow \mathbf{Spt}^{\mathrm{G}}(k_{et}), \bar{\epsilon}^* : \mathbf{Spt}^{\mathrm{G}}(\bar{k}_{\mathrm{mot}}) \rightarrow \mathbf{Spt}^{\mathrm{G}}(\bar{k}_{et}), \eta^* : \mathbf{Spt}^{\mathrm{G}}(k_{et}) \rightarrow \mathbf{Spt}^{\mathrm{G}}(\bar{k}_{et}),$$
$$\epsilon^* : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}) \rightarrow \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}), \bar{\epsilon}^* : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(\bar{k}_{\mathrm{mot}}) \rightarrow \widetilde{\mathbf{Spt}}^{\mathrm{G}}(\bar{k}_{et}) \text{ and } \eta^* : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}) \rightarrow \widetilde{\mathbf{Spt}}^{\mathrm{G}}(\bar{k}_{et}),$$

as well as corresponding functors for the categories $\widetilde{\mathbf{Spt}}(k_{\mathrm{mot}})$, $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et})$, $\mathbf{Spt}(k_{\mathrm{mot}})$, $\mathbf{Spt}(k_{et})$ and similar categories of spectra defined in [CJ23-T1, Definitions 4.6 through 4.9]. Here the above categories are provided with the stable injective model structures as mentioned in [CJ23-T1, Terminology]. Since we work with the injective model structures on all of the above categories, it is clear that these are left Quillen functors.

Proposition 3.1. *The above functors are weakly monoidal functors.*

Proof. We merely observe that this follows readily from the definition of the smash product as a co-end in [CJ23-T1, Definition 4.2, 4.8(ii) and 4.9(iii)]. □

Definition 3.2. We call the functor ϵ^* ($\bar{\epsilon}^*$), *étale realization over* k (*étale realization over* $\bar{k}$, respectively). □

The justification for the above definition will be clear from Proposition 3.4.

We proceed to consider étale Eilenberg-Maclane spectra. We start with the simplicial 2-sphere S^2 identified with $\Delta[2]/\delta\Delta[2]$. Recall that the free $\mathbb{Z}/\ell$-module on the above S^2 followed by the forgetful functor sending a simplicial abelian presheaf to the underlying simplicial presheaf provides the Eilenberg-Maclane space $\mathrm{K}(\mathbb{Z}/\ell, 2)$. Therefore, one obtains a natural map $\mathrm{S}^2 \to \mathrm{K}(\mathbb{Z}/\ell, 2)$.

Definition 3.3. Let ℓ be different from the characteristic of the base field k and n a fixed positive integer. We let the *étale Eilenberg-Maclane spectrum* denote the sheaf of S^2-spectra defined as follows: the space in degree $2m$, for a non-negative integer m is the constant sheaf $\mathrm{K}(\mathbb{Z}/\ell^n, 2m)$ (whose sheaf of homotopy groups are trivial in all degrees except $2m$, where it is $\mathbb{Z}/\ell^n$), and whose structure maps are defined by the pairing $\mathrm{S}^2 \wedge \mathrm{K}(\mathbb{Z}/\ell^{\mathrm{n}}, 2\mathrm{m}) \to \mathrm{K}(\mathbb{Z}/\ell^{\mathrm{n}}, 2) \wedge \mathrm{K}(\mathbb{Z}/\ell^{\mathrm{n}}, 2\mathrm{m}) \to \mathrm{K}(\mathbb{Z}/\ell^{\mathrm{n}}, 2\mathrm{m}+2)$.

Let $\mathrm{S} = \operatorname{Spec} k$. Recall that the Motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z}/\ell^n)_{\mathrm{S}}$ has as its m-th space $\mathcal{U}(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m))$, where where $\mathcal{U}$ denotes the forgetful functor sending a simplicial abelian presheaf (with transfers) to the underlying simplicial presheaf. Next we apply ϵ^* to the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z}/\ell^n)_{\mathrm{S}}$ to obtain the following object: the sequence $\{\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m))|m \geq 0\}$ with the structure maps

$$\begin{aligned}\epsilon^*(\mathbf{T}) \wedge \mathcal{U}(\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m))) &\to \mathcal{U}(\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}))) \wedge \mathcal{U}(\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m))) \to \\ &\mathcal{U}(\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}) \otimes^{tr} \mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m))) \cong \mathcal{U}(\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^{m+1}))).\end{aligned} \tag{3.2.2}$$

Proposition 3.4. *Let* $\mathrm{S} = \operatorname{Spec} k$*, where* k *is a perfect field. Then the étale realization of the Motivic Eilenberg-Maclane spectrum* $\mathbb{H}(\mathbb{Z}/\ell^n)_{\mathrm{S}}$*, where* ℓ *is different from char*(k)*, is the spectrum whose* n*-th term is the presheaf of Eilenberg-Maclane spaces* $\mathrm{K}(\mu_{\ell^{\mathrm{n}}}(\mathrm{m}), 2\mathrm{m})$ *and whose structure maps are given by the pairings:*

$$\epsilon^*(\mathbf{T}) \wedge \mathrm{K}(\mu_{\ell^{\mathrm{n}}}(\mathrm{m}), 2\mathrm{m}) \to \mathrm{K}(\mu_{\ell^{\mathrm{n}}}(1), 2) \wedge \mathrm{K}(\mu_{\ell^{\mathrm{n}}}(\mathrm{m}), 2\mathrm{m}) \to \mathrm{K}(\mu_{\ell^{\mathrm{n}}}(\mathrm{m}+1), 2\mathrm{m}+1). \tag{3.2.3}$$

Assume in addition that the base field k *has a primitive* ℓ^n*-th root of unity. Then the above spectrum identifies with the étale Eilenberg-Maclane spectrum considered in Definition 3.3.*

Proof. A key observation here is the following:

$$\mathcal{U}(\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m))) \simeq \mathrm{K}(\mu_{\ell^{\mathrm{n}}}(\mathrm{m}), 2\mathrm{m}), \tag{3.2.4}$$

This is discussed in [MVW, Theorem 10.3 and Theorem 15.2]. Recall the structure maps of the spectrum $\mathbb{H}(\mathbb{Z}/\ell^m)_{\mathrm{S}}$ are given by the pairing

$$\mathcal{U}(\mathbb{Z}/\ell^{n,tr}(\mathbf{T})) \wedge \mathcal{U}(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m)) \to \mathcal{U}(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}) \otimes^{tr} \mathbb{Z}/\ell^{n,tr}(\mathbf{T}^m)) \to \mathcal{U}(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}^{m+1}).$$

Therefore, the observation that $\mathcal{U}(\epsilon^*(\mathbb{Z}/\ell^{n,tr}(\mathbf{T}))) \simeq \mathrm{K}(\mu_{\ell^n}(1), 2)$ proves the first statement.

Since k is assumed to have a primitive ℓ^n-th root of unity, $\mu_{\ell^n}(1) \cong \mathbb{Z}/\ell^n$ and therefore $\mathrm{K}(\mu_{\ell^{\mathrm{n}}}(1), 2) \cong \mathrm{K}(\mathbb{Z}/\ell^{\mathrm{n}}, 2)$. Similarly $\mathrm{K}(\mu_{\ell^{\mathrm{n}}}(\mathrm{m}), 2\mathrm{m})$ identifies with $\mathrm{K}(\mathbb{Z}/\ell^{\mathrm{n}}, 2\mathrm{m})$. This proves the second statement and completes the proof of the Proposition. □

Remark 3.5. (Further remarks on étale realization) Étale realization in *the unstable setting* has been discussed in [Isak] and also [Schm] where they consider étale realization of motivic spaces to take values in the category of pro-simplicial sets, following the setting of étale homotopy theory as in [AM69]. So far there has been no discussion of étale realization of motivic spectra. A discussion of étale realization for motivic spectra in general, i.e., apart from what is obtained by pull-back to the étale site and also apart from the special case of the Eilenberg-Maclane spectra is extraneous to our goals. In fact the only place where we need to show that the transfer is compatible with étale realization is at the level of motivic and étale cohomology over algebraically closed fields and away from the characteristic of the base field: see Proposition 4.1, Corollary 4.2, as well as Theorem 5.7 and Proposition 5.8. Therefore, the above comparison results suffice to show that often splitting at the level of etale cohomology by means of the transfer implies splitting at the level of motivic cohomology.

Nevertheless, in view of general interests in obtaining a stable version of étale realization, we provide a short discussion on this topic in the Appendix, though it is not used in the rest of the paper. ☐

As is well-known, étale cohomology is well-behaved only with respect to torsion coefficients prime to the residue characteristics. This makes it necessary to consider completions away from the characteristic of the base field on considering spectra on the étale site. This justifies the following definitions.

Definition 3.6. Let $\mathrm{M} \in \mathcal{SH}(k)$ ($\mathcal{SH}(k_{et})$. For each prime number ℓ, let $\mathbb{Z}_{(\ell)}$ denote the localization of the integers at the prime ideal corresponding to ℓ and let $\mathbb{Z}\hat{}_\ell = \lim_{\infty \leftarrow n} \mathbb{Z}/\ell^n$. Then we say M is $\mathbb{Z}_{(\ell)}$-local (ℓ-complete, ℓ-primary torsion), if each $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma^\infty_{\mathbf{T}} \mathrm{U}_+, \mathrm{M}]$ is a $\mathbb{Z}_{(\ell)}$-module ($\mathbb{Z}\hat{}_\ell$-module, $\mathbb{Z}\hat{}_\ell$-module which is torsion, respectively) as U varies among the objects of the given site, where $[\mathrm{S}^{1^{\wedge s}} \wedge \mathbf{T}^t \wedge \Sigma^\infty_{\mathbf{T}} \mathrm{U}_+, \mathrm{M}]$ denotes *Hom* in the stable homotopy category $\mathcal{SH}(k)$ ($\mathcal{SH}(k_{et})$, respectively).

Let $\mathrm{M} \in \mathcal{SH}(k)$ ($\mathcal{SH}(k_{et})$). Then one may observe that if ℓ is a prime number, and M is ℓ-complete, then M is $\mathbb{Z}_{(\ell)}$-local. This follows readily by observing that the natural map $\mathbb{Z} \to \mathbb{Z}\hat{}_\ell$ factors through $\mathbb{Z}_{(\ell)}$ since every prime different from ℓ is inverted in $\mathbb{Z}\hat{}_\ell$. One may also observe that if $\mathcal{E}$ is a commutative ring spectrum which is $\mathbb{Z}_{(\ell)}$-local (ℓ-complete), then any module spectrum M over $\mathcal{E}$ is also $\mathbb{Z}_{(\ell)}$-local (ℓ-complete, respectively). ℓ-completion in the motivic framework is discussed in some detail in the Appendix.

We conclude this section with the following remarks on Betti realization.

3.3. **Betti realization.** When the base scheme is the field of complex numbers, there is a fairly extensive discussion on Betti realization both in the unstable and stable settings: see [Ay], and [PPR].

4. Computing traces: compatibility of the transfer with realizations

Assume the situation as in Theorem 2.3. Then, very often the main application of the transfer is to prove that π_{Y}^* is a split injection in generalized cohomology, i.e., one needs to verify that $tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))$ is a unit. In order to verify that $tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))$ is a unit, one may adopt the following strategy. First we will show that the transfer constructed above is compatible with passage to a simpler situation, for example passage from over a given base field to its algebraic or separable closure and/or passage to a suitable *realization* functor: we will often use the étale realization. Then, often, $\mathrm{h}^{0,0}(\mathrm{B}) \simeq \mathrm{h}^{0,0}_{\mathrm{real}}(\mathrm{B})$ where $\mathrm{h}^{*,\bullet}_{\mathrm{real}}(\mathrm{B})$ denotes the corresponding generalized cohomology of the realization and B denotes any smooth scheme. Therefore, it will suffice to show that $tr(\mathrm{f_Y})^*_{\mathrm{real}}(\pi_{\mathrm{Y}}^*(1))$ is a unit: here $tr(\mathrm{f_Y})_{\mathrm{real}}$ denotes the corresponding transfer on the realization. We devote all of this section to a detailed discussion of this technique.

As before we will assume the base scheme is the spectrum of a perfect field k satisfying the assumption (1.0.1). $\bar{k}$ will denote a fixed algebraic closure of k and ℓ is a prime different from $char(k)$. Accordingly $\mathrm{S} = \mathrm{Spec}\, k$ and $\bar{\mathrm{S}} = \mathrm{Spec}\, \bar{k}$. We first recall the following functors (from [CJ23-T1, (7.0.14)]):

(4.0.1) $$\epsilon^* : \mathbf{Spt}(k_{\mathrm{mot}}) \to \mathbf{Spt}(k_{et}), \bar{\epsilon}^* : \mathbf{Spt}(\bar{k}_{\mathrm{mot}}) \to \mathbf{Spt}(\bar{k}_{\mathrm{et}}), \text{ and } \eta^* : \mathbf{Spt}(k_{et}) \to \mathbf{Spt}(\bar{k}_{\mathrm{et}}).$$

Let θ and $\phi_{\mathcal{E}}$ denote the functors considered in [CJ23-T1, Proposition 7.3]. We let $\mathcal{E} \in \mathbf{Spt}(k_{\mathrm{mot}})$ denote a commutative ring spectrum which is ℓ-complete for a prime $\ell \neq char(k)$.

Proposition 4.1. *(Commutativity of the pre-transfer with étale realization) Assume the above situation. Then denoting by $tr'(\mathrm{f})$ the pre-transfer (as in* [CJ23-T1, Definition 8.2 (ii)]*) (with* G *trivial),* $\epsilon^*(tr'(\mathrm{f})) \simeq tr'(\epsilon^*(\mathrm{f}))$ *and* $\bar{\epsilon}^*(tr'(\mathrm{f})) \simeq tr'(\bar{\epsilon}^*(\mathrm{f}))$ *when applied to dualizable objects of the form* $\mathcal{E} \wedge X_+$ *appearing in* [CJ23-T1, Theorem 7.1]. *The same conclusion holds for* ϵ^* *and* $\bar{\epsilon}^*$ *replaced by* η^* *or any of the two functors* θ *and* $\phi_{\mathcal{E}}$.

Proof. Implicitly assumed in the proof is the fact that the above functors all send dualizable objects to dualizable objects. This is already proved in [CJ23-T1, Proposition 7.3]. Moreover, as pointed out earlier, [DP84, 2.2 Theorem and 2.4 Corollary] seems to provide a quick proof of the assertion above, so that at least in principle, the results in this proposition should be deducible from op. cit. Nevertheless, it seems best to provide a proof of Proposition 4.1, at least for ϵ^*: the proof for the other functors will be similar. First observe that there is a natural map $\epsilon^*\mathcal{RH}om(\mathrm{K},\mathrm{L}) \rightarrow \mathcal{RH}\mathrm{om}(\epsilon^*(\mathrm{K}), \epsilon^*(\mathrm{L}))$ for any two objects $\mathrm{K},\mathrm{L} \in \mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$. If one takes $\mathrm{L} = \mathcal{E}$, $\mathcal{RH}om(\mathrm{K},\mathrm{L})$ will denote $\mathrm{D}(\mathrm{K})$. Similarly $\mathcal{RH}om(\epsilon^*(\mathrm{K}), \epsilon^*(\mathrm{L}))$ will then denote $\mathrm{D}(\epsilon^*(\mathrm{K}))$.

Now the proof of the assertion for the pre-transfer follows from the commutativity of the following diagrams where the composition of maps in the top row (bottom row) is $\epsilon^*(tr'(\mathrm{f}))$ $(tr'(\epsilon^*(\mathrm{f}))$, respectively), with the smash products denoting their derived versions, and $\mathrm{K} = \mathcal{E} \wedge \mathrm{X}_+$ as in the Proposition:

$$\begin{tikzcd}
\epsilon^*(\mathcal{E}) \arrow[r] \arrow[d, "id"] & \epsilon^*(\mathrm{K} \wedge_{\mathcal{E}} \mathrm{DK}) \arrow[r, "\cong"] & \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{DK}) \arrow[dl] \\
\epsilon^*(\mathcal{E}) \arrow[r] & \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \mathrm{D}(\epsilon^*(\mathrm{K})) &
\end{tikzcd}$$

$$\begin{tikzcd}
\epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{DK}) \arrow[r, "(id\wedge_{\epsilon^*(\mathcal{E})}\epsilon^*(\mathrm{f})\wedge_{\epsilon^*(\mathcal{E})}\epsilon^*(\mathrm{f}))\circ(\mathrm{id}\wedge_{\epsilon^*(\mathcal{E})}\Delta)\circ\tau"] \arrow[d] & \epsilon^*(\mathrm{DK}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) \arrow[r, "\epsilon^*(e)\wedge_{\epsilon^*(\mathcal{E})}id"] \arrow[d] & \epsilon^*(\mathcal{E}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(K) \arrow[d, "id"] \\
\epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \mathrm{D}(\epsilon^*(\mathrm{K})) \arrow[r, "(id\wedge_{\epsilon^*(\mathcal{E})}\epsilon^*(\mathrm{f})\wedge_{\epsilon^*(\mathcal{E})}\epsilon^*(\mathrm{f}))\circ(\mathrm{id}\wedge_{\epsilon^*(\mathcal{E})}\Delta)\circ\tau"] & \mathrm{D}(\epsilon^*(\mathrm{K})) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K}) \arrow[r, "e\wedge_{\epsilon^*(\mathcal{E})}id"] & \epsilon^*(\mathcal{E}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{K})
\end{tikzcd}$$

The isomorphism $\epsilon^*(\mathrm{K} \wedge_{\mathcal{E}} \mathrm{DK}) \xrightarrow{\cong} \epsilon^*(\mathrm{K}) \wedge_{\epsilon^*(\mathcal{E})} \epsilon^*(\mathrm{DK})$ in the top row may be obtained as follows. First, using the fact that ϵ^* is pull-back from the big Nisnevich site of $\mathrm{Spec}\, k$ to the corresponding big étale site, one observes that ϵ^* commutes with smash products of pointed simplicial presheaves. Then, recalling the definition of the smash product of spectra as a left Kan extension (see: [CJ23-T1, Definition 4.2]), one may see that ϵ^* commutes with smash products and then smash products of module spectra over ring spectra. $\square$

Corollary 4.2. *Assume that the group* G *is special and that* $\mathrm{f} : \mathrm{X} \rightarrow \mathrm{X}$ *is a* G*-equivariant map and let* $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *denote any one of the three cases considered in Theorem 2.3. Then* $\epsilon^*(tr(\mathrm{f}_{\mathrm{Y}})) \simeq tr(\epsilon^*(\mathrm{f}_{\mathrm{Y}}))$, *where* $tr(\mathrm{f})$ *denotes the transfer defined with respect to a motivic ring spectrum* $\mathcal{E}$ *that is* ℓ*-complete for a prime* $\ell \neq char(k)$.

Proof. Proposition 4.1 proves the corresponding statement for the pre-transfer when $\mathrm{Y} = \mathrm{Spec}\, k$. Now the corresponding result holds for a general Y, since the corresponding pre-transfer $tr'(\mathrm{f}_{\mathrm{Y}}) = \mathrm{id}_{\mathrm{Y}_+} \wedge tr'(\mathrm{f})$. Next a detailed examination of the various steps in the construction of the transfer (see [CJ23-T1, (8.3.3)] through [CJ23-T1, (8.3.17)]) show that they all pull-back to define the corresponding construction on the étale site. (In fact, $tr(\mathrm{f}_{\mathrm{Y}})$ as in [CJ23-T1, Definition 8.8] is defined by first taking $id \wedge_G tr'^{\mathrm{G}}(\mathrm{f}_{\mathrm{Y}})$, where $tr'^{\mathrm{G}}(\mathrm{f}_{\mathrm{Y}})$ is the G-equivariant pre-transfer.) $\square$

Remark 4.3. Several results on how to deduce splittings on generalized motivic cohomology theories, from splittings produced by the transfer at the level of étale cohomology are discussed in Propositions 5.7 and 5.8. These make use of the compatibility of the pre-transfer with étale realization as proven in Proposition 4.1.

5. **Transfer and stable splittings in the Motivic Stable Homotopy category**

We will briefly recall the context and framework for the discussion in this section. G is a linear algebraic group over a perfect field k, $\mathrm{p} : \mathrm{E} \rightarrow \mathrm{B}$ is a G-torsor over k, and $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{X} \times \mathrm{Y}) \rightarrow \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ is the projection considered in the contexts (a) or (b) in Theorem 2.3. (The splitting results for the context (c) considered in Theorem 2.3(c) can be deduced from the case in Theorem 2.3(b) in view of the compatibility of the transfer maps proved in Theorem 2.3(i).) Recall that X, Y denote unpointed simplicial presheaves (defined on Sm_k) provided with actions by G and so that $\Sigma_{\mathbf{T}}\mathrm{X}_+$ is dualizable in $\mathcal{SH}(k)$ (or $\mathcal{E} \wedge X_+$ is dualizable in $\mathcal{SH}(k, \mathcal{E})$ when considering ring spectra $\mathcal{E}$ other than the motivic sphere spectrum). $\mathrm{f} : \mathrm{X} \rightarrow \mathrm{X}$ is a G-equivariant map.

We next provide a quick review of the results on splitting obtained by making use of the transfer: these form some of the main results of the present paper. In order to obtain splittings in the motivic stable homotopy category, there are essentially *two distinct techniques* we pursue here making use of the transfer as a stable map, each with its own advantages. Both of these apply to actions of all linear algebraic groups, *irrespective of whether they are special*. Both start with the observation that the multiplicative property of the transfer as in Theorem 2.3(ii) and (iii) shows that in order to prove π_{Y}^* is a split monomorphism, it suffices to show $tr(\mathrm{f_Y})^*(1) = tr(\mathrm{f_Y})^*\pi_{\mathrm{Y}}^*(1)$ is a unit. Both approaches also make use of the base-change property of the transfer as in Proposition 2.4 and then reduce to checking this for simpler situations. They both apply to all linear algebraic groups, irrespective of whether they are special (that is, in Grothendieck's classification as in [Ch]), and in particular to all split orthogonal groups, which are known to be non-special. However, for the sake of keeping the discussion simpler, we will restrict to groups that are special while considering the second approach.

5.1. **Splittings via the Grothendieck-Witt ring of the base field k.** In this approach we assume that the class $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt ring (or the Grothendieck-Witt ring with characteristic exponent of the base field inverted) and use that to obtain splittings directly, first at the level of the pre-transfer. This method is rather limited to those schemes X for which it is possible to compute $\tau_{\mathrm{X}}^*(1)$ in the Grothendieck-Witt ring. Such a computation is carried out in [JP23, Theorem 1.2 and Corollary 1.3], where $\mathrm{X} = \mathrm{G/N_G(T)}$, for a connected split reductive group (or more generally a split linear algebraic group) G over a perfect field and $\mathrm{N_G(T)}$ the normalizer of a split maximal torus in G. Therefore, at present this technique only applies to the above case. In the discussion below, we will only consider the case where $char(k) = 0$. In positive characteristics p, the same discussion applies by replacing the sphere spectra everywhere by the corresponding sphere spectra with the prime p inverted, or completed away from p as discussed in [CJ23-T1, Definition 1.1].

Case 1: Here we will assume the group G is *special*. Since the group G is assumed to be special, for each fixed integer $m \geq 1$, the map $\mathrm{p} : \mathrm{E} \to \mathrm{B}$ is a Zariski locally trivial principal G-bundle and let $\tilde{\mathrm{p}} : \widetilde{\mathrm{E}} \to \widetilde{\mathrm{B}}$ denote the induced map where $\widetilde{\mathrm{B}}$ is the affine replacement. Let $\{\mathrm{U_i}|\mathrm{i}\}$ denote a Zariski open cover of $\widetilde{\mathrm{B}}$ over which the map $\tilde{\mathrm{p}}$ trivializes so that $\pi_{\mathrm{Y}|\mathrm{U_i}} = \mathrm{U_i} \times (\mathrm{Y} \times \mathrm{X}) \to \mathrm{U_i} \times \mathrm{Y}$.

Let $tr : \Sigma^\infty_{\mathbf{T}}(\widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathrm{Y})_+ \to \Sigma^\infty_{\mathbf{T}}(\widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}))_+$ denote the transfer defined in [CJ23-T1, Definition 8.6]. Then one may observe that $tr_{|\mathrm{U_i}} : \Sigma^\infty_{\mathbf{T}}(\mathrm{U_i} \times \mathrm{Y})_+ \to \Sigma^\infty_{\mathbf{T}}(\mathrm{U_i} \times \mathrm{Y})_+ \wedge \Sigma^\infty_{\mathbf{T}}\mathrm{X}_+$ is just $id_{\Sigma^\infty_{\mathbf{T}}(\mathrm{U_i}\times\mathrm{Y})_+} \wedge tr'_{\mathrm{X}}$, where tr'_{X} denotes the pre-transfer considered in [CJ23-T1, Definition 8.2(ii)]. Therefore, if $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt group (or in the Grothendieck-Witt group with the characteristic exponent of the base field inverted), (where τ_{X} is the trace defined in [CJ23-T1, Definition 8.2(iv)]), then the composition, $\Sigma^\infty_{\mathbf{T}}\pi_{\mathrm{Y},+} \circ tr_{\mathrm{X}}$, where π_{Y} is the projection $\widetilde{\mathrm{E}} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \widetilde{\mathrm{E}} \times_{\mathrm{G}} \mathrm{Y}$, will be homotopic to the identity over each $\mathrm{U_i}$.

Now let $\mathrm{h}^{*,\bullet}$ denote a generalized motivic cohomology theory defined with respect to a motivic spectrum. Then using a Mayer-Vietoris argument and observing that each $\widetilde{\mathrm{B}}$ is quasi-compact, the splitting over $\mathrm{U_i}$ of the map $\pi_{\mathrm{Y}|\mathrm{U_i}}$ shows that the composite map $tr^* \circ \pi_{\mathrm{Y}}^* : \mathrm{h}^{*,\bullet}(\widetilde{\mathrm{E}}\underset{\mathrm{G}}{\times}\mathrm{Y}) \xrightarrow{\pi_{\mathrm{Y}}^*} \mathrm{h}^{*,\bullet}(\widetilde{\mathrm{E}}\underset{\mathrm{G}}{\times}(\mathrm{Y} \times \mathrm{X})) \xrightarrow{\mathrm{tr}^*} \mathrm{h}^{*,\bullet}(\widetilde{\mathrm{E}}\underset{\mathrm{G}}{\times}\mathrm{Y})$ is an isomorphism. When we vary B over finite dimensional approximations $\{\mathrm{BG}^{gm,m}|\mathrm{m}\}$, the same therefore holds on taking the colimit of the $\mathrm{BG}^{gm,m}$ over m as $m \to \infty$, as we have shown the transfer maps are compatible as m varies: see Theorem 2.3(i). (Here the colimit of the $\mathrm{BG}^{gm,m}$ will pullout of the generalized cohomology spectrum as a homotopy inverse limit, and then one uses the usual lim^1-exact sequence to draw the desired conclusion.)

Case 2: Here we will let G denote *any linear algebraic group*. We first recall from [CJ23-T1, Definition 8.2(iii)], that the G-equivariant pre-transfer is given by $id_{\mathrm{Y}} \times tr'^{\mathrm{G}}(id) : \mathrm{Y} \times \mathbb{S}_k^{\mathrm{G}} \to \mathrm{Y} \times (\mathbb{S}_k^{\mathrm{G}} \wedge \mathrm{X}_+)$, with $tr'^{\mathrm{G}}(id) : \mathbb{S}_k^{\mathrm{G}} \to \mathbb{S}_k^{\mathrm{G}} \wedge \mathrm{X}_+$ the G-equivariant pre-transfer in [CJ23-T1, Definition 8.2(ii)]. Moreover, the composition of the above pre-transfer and the projection $\mathrm{Y} \times (\mathbb{S}_k^{\mathrm{G}} \wedge \mathrm{X}_+) \to \mathrm{Y} \times \mathbb{S}_k^{\mathrm{G}}$ is $id_{\mathrm{Y}} \times \tau_{\mathrm{X}}$, *where we view* $\mathbb{S}_k^{\mathrm{G}}$ *as a spectrum in* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$. In view of the assumption that $\tau_{\mathrm{X}}^*(1)$ is a unit in the Grothendieck-Witt ring of k, this composite map is a weak-equivalence mapping $\mathrm{Y} \times \mathbb{S}_k^{\mathrm{G}}$ to itself, again viewing $\mathbb{S}_k^{\mathrm{G}}$ as a spectrum in $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$. Next consider the composite map

$$\widetilde{\mathrm{E}} \times^{et}_{\mathrm{G}} (\mathrm{a}\epsilon^*(\mathrm{Y} \times \mathbb{S}_k^{\mathrm{G}})) \xrightarrow{\widetilde{\mathrm{E}}\times^{\mathrm{et}}_{\mathrm{G}}(\mathrm{a}\epsilon^* tr'^{\mathrm{G}}(id_{\mathrm{Y}+}))} \widetilde{\mathrm{E}} \times^{\mathrm{et}}_{\mathrm{G}} (\mathrm{a}\epsilon^*(\mathrm{Y} \times (\mathbb{S}_k^{\mathrm{G}} \wedge \mathrm{X}_+))) \xrightarrow{\widetilde{\mathrm{E}}\times^{\mathrm{et}}_{\mathrm{G}}(\mathrm{a}\epsilon^*(\mathrm{pr}))} \widetilde{\mathrm{E}} \times^{\mathrm{et}}_{\mathrm{G}} (\mathrm{a}\epsilon^*(\mathrm{Y} \times \mathbb{S}_k^{\mathrm{G}}))$$

which lives over the small étale site of $\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{et} \mathrm{Y}$. Next we smash each term of the above spectra indexed by $\mathrm{T_V}$ with the Thom-space of the complimentary bundle $\epsilon^*(\eta^{\mathrm{V}})$ over the base $\epsilon^*(\widetilde{\mathrm{E}}_{\mathrm{Y}})$. Working locally on the small étale site of $\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{et} \mathrm{Y}_+$, one can see that the resulting composite map of spectra identifies with $\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{et} (id_{\mathrm{Y}} \times \epsilon^*(\tau_{\mathrm{X}}))$. This is a weak-equivalence since $\tau_{\mathrm{X}} : \mathbb{S}_k^{\mathrm{G}} \to \mathbb{S}_k^{\mathrm{G}}$ is a weak-equivalence, when $\mathbb{S}_k^{\mathrm{G}}$ is viewed as a spectrum in $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$. Therefore, it remains a weak-equivalence of spectra on applying $\mathrm{R}\epsilon_*$ and collapsing the section from $\tilde{\mathrm{E}}_{\mathrm{Y}}$ as in [CJ23-T1, Steps 3-5 of 8.3:Construction of the transfer]. In fact, now the corresponding map is

$$\mathrm{R}\epsilon_*(\mathrm{id}_{(\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{\mathrm{et}} \mathrm{a}\epsilon^*(\mathrm{Y}))_+} \wedge \epsilon^*(\tau_{\mathrm{X}})) : \mathrm{R}\epsilon_*(\epsilon^*\mathbb{S}_k) \wedge \mathrm{R}\epsilon_*((\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{\mathrm{et}} \mathrm{a}\epsilon^*(\mathrm{Y}))_+ \to \mathrm{R}\epsilon_*(\epsilon^*\mathbb{S}_k) \wedge \mathrm{R}\epsilon_*((\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{\mathrm{et}} \mathrm{Y})_+. \tag{5.1.1}$$

Finally we apply a generalized motivic cohomology theory $\mathrm{h}^{*,\bullet}$ to the above maps to observe that the composite map

$$\begin{aligned}\mathrm{h}^{*,\bullet}(\mathrm{R}\epsilon_*(\epsilon^*\mathbb{S}_k) \wedge \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{\mathrm{et}} (\mathrm{a}\epsilon^*(\mathrm{Y})))_+) &\xrightarrow{\pi_{\mathrm{Y}}^*} \mathrm{h}^{*,\bullet}(\mathrm{R}\epsilon_*(\epsilon^*\mathbb{S}_k) \wedge \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{\mathrm{et}} (\mathrm{a}\epsilon^*(\mathrm{Y} \times \mathrm{X}))))_+) \\ &\xrightarrow{tr(id_{\mathrm{Y}}^*)} \mathrm{h}^{*,\bullet}(\mathrm{R}\epsilon_*(\epsilon^*\mathbb{S}_k) \wedge \mathrm{R}\epsilon_*(\widetilde{\mathrm{E}} \times_{\mathrm{G}}^{\mathrm{et}} (\mathrm{a}\epsilon^*(\mathrm{Y})))_+)\end{aligned}$$

is an isomorphism.

The *main advantage* of this method is that it provides splittings for all generalized motivic cohomology theories, whenever the above computation of the trace $\tau_{\mathrm{X}}^*(1)$ can be carried out in the Grothendieck-Witt ring of the base field, independent of whether the group G is special. One may see a detailed discussion of splittings obtained this way in Theorems 5.1 and 5.2 discussed below as well as Corollary 5.3.

Theorem 5.1. *Let* $\pi_{\mathrm{Y}} : \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \to \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *denote a map as in one of the three cases considered in Theorem 2.3. Assume* G *is a split linear algebraic group. In case* G *is not special, we will also assume the field k is infinite and we will also assume the field k satisfies the hypothesis* (1.0.1). *Let* M *denote a motivic spectrum.*

Then the map induced by $tr(id_{\mathrm{Y}})^*$ *provides a splitting to the map*

$$\pi_{\mathrm{Y}}^* : \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M}) \to \mathrm{h}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M})$$

in the following cases:

(i) $\Sigma_{\mathbf{T}}^{\infty}\mathrm{X}_+$ *is dualizable in the motivic homotopy category* $\mathcal{SH}(k)$*, the trace* $\tau_X^*(1)$ *is a unit in the Grothendieck-Witt ring of the base field k and* M *denotes any motivic spectrum. In particular, this holds if* X *and* Y *are smooth schemes of finite type over k and* $char(k) = 0$*, provided* $\tau_{\mathrm{X}}^*(1)$ *is a unit in the Grothendieck-Witt ring of k.*

(ii) $Char(k) = p > 0$*,* $\mathcal{E}$ *denotes any one of the ring spectra, (a)* $\mathbb{S}_k[p^{-1}]$*, (b)* $\mathbb{S}_{k,(\ell)}$ *or (c)* $\mathcal{E} = \widehat{\mathbb{S}}_{k,\ell}$ *for some prime* $\ell \neq p$ *and* $\mathcal{E} \wedge X_+$ *is dualizable in* $\mathcal{SH}(k, \mathcal{E})$, $\mathrm{M} \in \mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$. *We will further assume that the corresponding trace* $\tau_{\mathrm{X}} : \mathcal{E} \to \mathcal{E}$ *is a unit in the corresponding variant of the Grothendieck-Witt ring, that is,* $[\mathcal{E}, \mathcal{E}]$*, which denotes stable homotopy classes of maps from* $\mathcal{E}$ *to* $\mathcal{E}$*. In particular, this holds if* X *and* Y *are smooth schemes of finite type over k, provided* $\tau_{\mathrm{X}}^*(1)$ *is a unit in the above variant of Grothendieck-Witt ring of k.*

The main example, where one is able to compute the motivic trace in the Grothendieck-Witt group is for $\mathrm{G}/\mathrm{N_G}(\mathrm{T})$, where G is a split linear algebraic group and $\mathrm{N_G}(\mathrm{T})$ denotes the normalizer of a split maximal torus in G: see [Lev18] for partial results in this direction and see [JP23, Theorem 1.2 and Corollary 1.3] for a computation in the general case. This yields the following result.

Theorem 5.2. *(See* [JP23, Theorem 1.2 and Corollary 1.3] *and also* [JP22, Proposition 2.2]*.) Let* G *denote a split linear algebraic group over the given base field k: we will assume k is infinite when* G *is not special. Let* $\mathrm{h}^{*,\bullet}$ *denote a generalized motivic cohomology theory defined with respect to a motivic spectrum (with p inverted, if* $char(k) = p > 0$*.) Then, with* $\mathrm{N_G}(\mathrm{T})$ *denoting the normalizer of a split maximal torus in* G*, and with* $\mathrm{p} : \Sigma_{\mathbf{T}}^{\infty}\mathrm{BN_G}(\mathrm{T})_+ \to \Sigma_{\mathbf{T}}^{\infty}\mathrm{BG}_+$ *denoting the map induced by the inclusion* $\mathrm{N_G}(\mathrm{T}) \to \mathrm{G}$*, the induced map*

$$\mathrm{p}^* : \mathrm{h}^{*,\bullet}(\mathrm{BG}_+) \to \mathrm{h}^{*,\bullet}(\mathrm{BN_G}(\mathrm{T})_+)$$

is split injective. In particular, when the group G *is special, the above splitting holds for Algebraic K-theory (integral in characteristic* 0 *and with finite coefficients prime to the characteristic, in general.)*

Corollary 5.3. *Assume in addition to the the hypotheses of Theorem 5.2, that the following hypotheses hold for (i) through (iii). Let* M *denote any motivic spectrum if the base field is of characteristic* 0 *and let* M *denote a motivic spectrum in* $\mathbf{Spt}(k_{\mathrm{mot}}, \mathbb{S}[p^{-1}])$ *if the base field is of characteristic* $p > 0$*. Let* $\mathrm{p}: \mathrm{E} \to \mathrm{B}$ *denote the map appearing in one of the three cases (a) through (c) considered in Theorem 2.3.*

(i) *Let* $\pi : \mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})) \to \mathrm{B}$ *denote the map induced by the projection* $\mathrm{G}/\mathrm{N}(\mathrm{T}) \to \operatorname{Spec} k$*. Then the corresponding induced map*
$$\pi^* : \mathrm{h}^{\bullet,*}(\mathrm{B}, \mathrm{M}) \to \mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})), \mathrm{M})$$
is a split monomorphism, where $\mathrm{h}^{*,\bullet}(\quad, \mathrm{M})$ *denotes the generalized motivic cohomology theory defined with respect to the spectrum* M.

(ii) *Let* Y *denote a* G*-scheme or an unpointed simplicial presheaf provided with a* G*-action. Let* $\mathrm{q}: \mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N}_{\mathrm{G}}(\mathrm{T})}{\times}\mathrm{Y}) \to \mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{Y}$ *denote the map induced by the map* $\mathrm{G}\underset{\mathrm{N}_{\mathrm{G}}(\mathrm{T})}{\times}\mathrm{Y} \to \mathrm{Y}$ *sending* $(g, y) \mapsto gy$*. Then, the induced map*
$$\mathrm{q}^* : \mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}\mathrm{Y}, \mathrm{M}) \to \mathrm{h}^{\bullet,*}(\mathrm{E}\underset{\mathrm{G}}{\times}(\mathrm{G}\underset{\mathrm{N}_{\mathrm{G}}(\mathrm{T})}{\times}\mathrm{Y}), \mathrm{M})$$
is also a split injection.

(iii) *Assume the base field* k *satisfies the finiteness hypotheses in* (1.0.1)*. Let* $\mathcal{E}$ *denote a commutative ring spectrum in* $\mathbf{Spt}(k_{\mathrm{mot}})$*, whose presheaves of homotopy groups are all* ℓ*-primary torsion for a fixed prime* $\ell \neq char(k)$*, and let* $\epsilon^*(\mathcal{E})$ *denote the corresponding spectrum in* $\mathbf{Spt}(k_{et})$*. We will further assume that* M *is a module spectrum over* $\mathcal{E}$*. Then the results corresponding to (i) through (iii) hold for* $\mathrm{h}^*(\quad, \epsilon^*(\mathrm{M}))$ *which is the generalized étale cohomology with respect to the étale spectrum* $\epsilon^*(\mathrm{M})$.

Proofs of Theorems 5.1, 5.2 and Corollary 5.3. Clearly the statements in Theorem 5.1(i) and Theorem 5.2 follow readily in view of the above discussion in 5.1. The proof of the statement in Theorem 5.1(ii) is entirely similar with the motivic sphere spectrum $\mathbb{S}_k$ replaced by the given motivic ring spectrum $\mathcal{E}$. ☐

In view of the results on motivic Euler characteristic of $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ proved in [JP23, Theorem 1.2 and Corollary 1.3], Theorem 5.2 and Corollary 5.3 follow from the above discussion. ☐

Remark 5.4. Special cases of the above theorem and Corollary, such as for Algebraic K-theory are particularly interesting. Theorem 5.2, for Algebraic K-Theory in fact enables one to restrict the structure group from G to $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ (and then to T by ad-hoc arguments) in several situations. Taking $\mathrm{G} = \mathrm{GL}_{\mathrm{n}}$, this becomes a *splitting principle* reducing problems on vector bundles to corresponding problems on line bundles.

The main *disadvantages* of this method are as follows. Computing the trace associated to a scheme in the Grothendieck-Witt ring is extremely difficult for many schemes, and possibly not do-able with the present technology. As pointed out above, the only case where this seems do-able at present is for $\mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$, where G is a split linear algebraic group and $\mathrm{N}_{\mathrm{G}}(\mathrm{T})$ denotes the normalizer of a split maximal torus in G. Moreover, the above discussion is only for the case the self-map $\mathrm{f}: \mathrm{X} \to \mathrm{X}$, (with $\mathrm{X} = \mathrm{G}/\mathrm{N}_{\mathrm{G}}(\mathrm{T})$) is the identity: it is far from clear how to carry out a similar computation in the Grothendieck-Witt ring for a general self-map f, even for the same X.

5.2. **Splittings for slice-completed generalized motivic cohomology theories.** First we define slice-completed generalized motivic cohomology theories.

Definition 5.5. For a smooth scheme Y (smooth ind-scheme $\mathcal{Y} = \{\mathrm{Y}_{\mathrm{m}}|\mathrm{m}\}$), we define *the slice completed generalized motivic cohomology spectrum* with respect to a motivic spectrum M to be
$$\hat{\mathrm{h}}(\mathrm{Y}, \mathrm{M}) = \underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}, \mathrm{s}_{\leq \mathrm{n}}\mathrm{M}) \simeq \mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}, \underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathrm{s}_{\leq \mathrm{n}}\mathrm{M})$$
$$(\hat{\mathrm{h}}(\mathcal{Y}, \mathrm{M}) = \underset{\infty\leftarrow \mathrm{m}}{\mathrm{holim}}\underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}_{\mathrm{m}}, \mathrm{s}_{\leq \mathrm{n}}\mathrm{M}) \simeq \underset{\infty\leftarrow \mathrm{m}}{\mathrm{holim}}\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}_{\mathrm{m}}, \underset{\infty\leftarrow \mathrm{n}}{\mathrm{holim}}\mathrm{s}_{\leq \mathrm{n}}\mathrm{M})),$$
where $s_{\leq n}\mathrm{M}$ is the homotopy cofiber of the map $f_{n+1}\mathrm{M} \to \mathrm{M}$ and $\{f_n\mathrm{M}|\mathrm{n}\}$ is the *slice tower* for M with $f_{n+1}\mathrm{M}$ being the $n+1$-th connective cover of M. ($\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}, \mathrm{F})$ and $\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y}_{\mathrm{m}}, \mathrm{F})$ denote the generalized hypercohomology spectrum with respect to a motivic spectrum F computed on the Nisnevich site.) The corresponding homotopy groups for maps from $\Sigma^{\infty}_{\mathbf{T}}(\mathrm{S}^u \wedge \mathbb{G}_m^v)$ to the above spectra will be denoted $\hat{\mathrm{h}}^{u+v,v}(\mathcal{Y}, \mathrm{M})$. One may

define the completed generalized étale cohomology spectrum of a scheme with respect to an S^1-spectrum by using the Postnikov tower in the place of the slice tower in a similar manner.

Remark 5.6. By [Hirsch02, 18.1.8], the homotopy inverse limit $\underset{\infty \leftarrow n}{\text{holim}} s_{\le n}\mathrm{M}$ belongs to $\mathbf{Spt}(k)$. We may, therefore, define the *slice completion* of the spectrum M to be $\text{holim}\, s_{\le n}\mathrm{M}$ (denoted henceforth by $\hat{\mathrm{M}}$) and define M to be slice-complete, if the natural map $\mathrm{M} \to \hat{\mathrm{M}}$ is a weak-equivalence. Therefore, one may see that $\hat{h}(\mathrm{Y}, \mathrm{M}) = h(\mathrm{Y}, \hat{\mathrm{M}})$ and $\hat{h}(\mathcal{Y}, \mathrm{M}) = h(\mathcal{Y}, \hat{\mathrm{M}})$. Several important spectra, like the spectrum representing algebraic K-theory and algebraic cobordism, are known to be slice-complete.

This method makes strong use of the fact that the transfer map we have constructed is a map in the appropriate stable homotopy category and, therefore induces a map of the corresponding motivic (or étale) Atiyah-Hirzebruch spectral sequences: therefore, it suffices to show the transfer induces a splitting at the E_2-terms of this spectral sequence. The spectral sequence can be shown to convergence strongly once we replace a given spectrum M with one of its slices, as shown in the discussion in the proof of Theorem 5.7.

The multiplicative properties of the slice filtration that a natural pairing of the slices of a motivic spectrum lift to the category $\mathbf{Spt}(k_{\mathrm{mot}})$ from the corresponding motivic stable homotopy category were verified in [Pel08]. Therefore, it follows that the motivic spectra that define the E_2-terms of the motivic Atiyah-Hirzebruch spectral sequence, that is the slices of the given motivic spectrum, are modules over the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z})$. Then the multiplicative property of the transfer as in Proposition 2.6 and Corollary 2.7 reduce to checking that we obtain a splitting for motivic cohomology.

Next, we make use of the base-change property of the transfer as in Proposition 2.4 and then reduce to checking this for the action of trivial groups, that is for the pre-transfer. See, for example, Proposition 2.8.

At this point it is often very convenient, as well as necessary, to know that the transfer is compatible with passage to simpler situations, for example, to a change of the base field to one that is separably or algebraically closed and with suitable realizations, that is either the étale realization or the Betti realization. A *main advantage* of this approach is that it would be only necessary to compute $tr(\mathrm{f})^*(1)$ and the trace $\tau_{\mathrm{X}}^*(1)$ after such reductions and realizations, which are readily do-able for a large number of schemes X: see Proposition 4.1 and Corollary 4.2. *Another advantage* is that it addresses affirmatively the important question if the pre-transfer and transfer are compatible with such reductions and realizations. Moreover, by this method, one can allow any self-map $\mathrm{f} : \mathrm{X} \to \mathrm{X}$ and compute the corresponding trace $\tau_{\mathrm{X}}(\mathrm{f})$. One can consult Theorem 5.7 and Corollary 5.10 discussed below, for examples of splittings obtained this way. Though a variant of the following theorem holds for groups that are non-special, with certain modifications, we will restrict to groups that are special, mainly to keep the discussion simpler.

Theorem 5.7. *Assume in addition to the hypotheses of Theorem 5.1 that the following hold. Assume the group* G *is special and let* $\mathrm{f} : \mathrm{X} \to \mathrm{X}$ *denote a* G*-equivariant map and let* $\mathrm{f}_{\mathrm{Y}} = \mathrm{id}_{\mathrm{Y}_+} \wedge \mathrm{f} : \mathrm{Y}_+ \wedge \mathrm{X}_+ \to \mathrm{Y}_+ \wedge \mathrm{X}_+$. *The map induced by* $tr(\mathrm{f}_{\mathrm{Y}})^*$ *provides a splitting to the map* $\pi_{\mathrm{Y}}^* : \hat{\mathrm{h}}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{M}) \to \hat{\mathrm{h}}^{*,\bullet}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}), \mathrm{M})$ *in the following cases:*

(i) $\Sigma_{\mathbf{T}}^{\infty}\mathrm{X}_+$ *is dualizable in* $\mathcal{SH}(k)$ *and* $tr(\mathrm{f}_{\mathrm{Y}})^*(1)$ *is a unit in* $\mathrm{H}^{0,0}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}, \mathbb{Z}) \cong \mathrm{CH}^0(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y})$. *In particular, this holds if* X *and* Y *are smooth schemes of finite type over* k *and* $char(k) = 0$, *provided* $tr(\mathrm{f}_{\mathrm{Y}})^*(1)$ *is a unit in* $\mathrm{H}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathbb{Z})$.

(ii) *A corresponding result also holds for the following alternate scenario:*

(a) The field k *is of positive characteristic* p, $\mathbb{S}_k[\mathrm{p}^{-1}] \wedge \mathrm{X}_+$ *is dualizable in* $\mathcal{SH}(k, \mathbb{S}_k[\mathrm{p}^{-1}])$, $\mathrm{M} \in \mathbf{Spt}(k_{\mathrm{mot}}, \mathbb{S}_k[\mathrm{p}^{-1}])$ *and* $tr(\mathrm{f}_{\mathrm{Y}})^*(1)$ *is a unit in* $\mathrm{H}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathbb{Z}[\mathrm{p}^{-1}]) \cong \mathrm{CH}^0(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{Z}[\mathrm{p}^{-1}])$.

(b) The field k *is of positive characteristic* p, $\mathcal{E} = \mathbb{S}_{k,(\ell)}$ *(or* $\mathcal{E} = \widehat{\mathbb{S}}_{k,\ell}$*) for some prime* $\ell \neq p$ *and* $\mathcal{E} \wedge X_+$ *is dualizable in* $\mathcal{SH}(k, \mathcal{E})$, $\mathrm{M} \in \mathbf{Spt}(k_{\mathrm{mot}}, \mathcal{E})$ *and* $tr(\mathrm{f}_{\mathrm{Y}})^*(1)$ *is a unit in* $\mathrm{H}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathbb{Z}_{(\ell)}) \cong \mathrm{CH}^0(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{Z}_{(\ell)})$ *(*$\mathbb{H}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathbb{Z}\hat{}_{\ell}) \cong \mathrm{CH}^0(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathrm{Z}\hat{}_{\ell})$*, respectively). (Here* $\mathbb{S}_{k,(\ell)}$ *(*$\widehat{\mathbb{S}}_{k,\ell}$*) denotes the localization of the motivic spectrum* $\mathbb{S}_k$ *at the prime ideal* (ℓ) *(the completion at* ℓ*, respectively).)*

(iii) *Assume the following: (a) both* X *and* Y *are smooth schemes of finite type over* k *provided with* G*-actions, and (b) both* Y *and* $\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *are connected. Let* $\mathrm{H}^{*,\bullet}(\quad, \mathrm{R})$ *denote motivic cohomology with* *coefficients in the commutative Noetherian ring* R *with a unit. Then under the isomorphism*

$$\mathrm{H}^{0,0}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{R})\overset{\cong}{\rightarrow}\mathrm{H}^{0,0}(\mathrm{Y},\mathrm{R})$$

$tr(\mathrm{f_Y})^*(1)$ *identifies with* $(id_{\mathrm{Y}}\times\tau_{\mathrm{X}}(\mathrm{f}))^*(1)$. *Therefore, the former is a unit if and only if the latter is. Under the assumption that* $tr(\mathrm{f_Y})^*(1)$ *is a unit, then* $tr(\mathrm{f_Y})^*$ *provides a splitting to the map* π_{Y}^* : $\hat{\mathrm{h}}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{M})\rightarrow\hat{\mathrm{h}}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M})$, *provided the slices of the motivic spectrum* M *are modules over the motivic Eilenberg-Maclane spectrum* H(R).

Proof. One key observation here is that the map $tr(\mathrm{f_Y})$ being a stable map, it induces a map of the stable slice spectral sequences for $\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{X}\times\mathrm{Y}),\mathrm{M})$ and $\mathrm{h}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{M})$. (One may observe that the slice spectral sequences converge only conditionally, in general: the convergence issues will be discussed below.) Next we will show, under the hypotheses of the theorem, that multiplication by $tr(\mathrm{f_Y})^*(\pi_{\mathrm{Y}}^*(1))$ induces a splitting of the corresponding E_2-terms of the above spectral sequences. For this, recall that the multiplicative properties of the slice filtration, verified in [Pel08], shows that these E_2-terms are modules over the motivic cohomology and that, in fact these E_2-terms are defined by motivic spectra (that is, the slices) that are module spectra over the motivic Eilenberg-Maclane spectrum.

Therefore, under these assumptions, the multiplicative property of the transfer as in Corollary 2.7 with A there denoting the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z})$ and the module spectrum M there denoting the module spectra defining the above E_2-terms, shows that $tr(\mathrm{f_Y})^*\circ\pi_{\mathrm{Y}}^*$ induces a map of the E_2-terms of the above motivic Atiyah-Hirzebruch spectral sequences. That is, we reduce to proving $tr(\mathrm{f_Y})^*\circ\pi_{\mathrm{Y}}^*$ induces an isomorphism on the motivic cohomology of $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$, modulo the convergence issues of the spectral sequence.

Next we discuss convergence issues of the spectral sequence. Since the map $tr(\mathrm{f_Y})^*\circ\pi_{\mathrm{Y}}^*$ induces an isomorphism at the E_2-terms, and therefore at all the $\mathrm{E_r}$-terms for $r\geq 2$, it follows that it induces an isomorphism of the inverse systems $\{\mathrm{E_r}|\mathrm{r}\}$ and therefore an isomorphism of the E_∞-terms and the derived E_∞-terms. (See [Board98, (5.1)] for a description of the derived E_∞-terms. It is shown in [CE, Chapter XV, section 2] that both the E_∞-terms and the derived E_∞-terms are determined by the sequence $\mathrm{E_r}$, $r\geq 2$.)

Next, let $\{\mathrm{Y_m}|\mathrm{m}\}$ denote either one of the following ind-schemes: $\mathrm{Y_m}=\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$, for all $m\geq 1$ or $\mathrm{Y_m}=\mathrm{EG}^{gm,m}\times_{\mathrm{G}}\mathrm{Y}$, $m\geq 1$. The next observation is that for every fixed integer n and m, on replacing the spectrum M by $s_{\leq n}\mathrm{M}$, the corresponding slice spectral sequence for the schemes $\mathrm{Y_m}$ converge strongly: this is clear since the $\mathrm{E}_1^{u,v}$-terms will vanish for all $u>n$ and also for $u<0$. (See [Board98, Theorem 7.1].) That $\mathrm{E}_1^{u,v}=0$ for $u<0$ or $u>n$ follows from the identification of the E_1-terms of the spectral sequence in terms of the slices of the S^1-spectrum forming the 0-th term in the associated Ω-$\mathbb{P}^1$-spectrum: see [Lev08, Proof of Theorem 11.3.3]. Moreover, the abutment of the spectral sequence are the homotopy groups of the slice-completion of the S^1-spectrum forming the corresponding 0-th term. Then, it follows therefore that for each fixed integer m and n, the composite map

$$tr(\mathrm{f_Y})^*\circ\pi_{\mathrm{Y}}^*:\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m},\mathrm{s_{\leq n}M})\rightarrow\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m},\mathrm{s_{\leq n}M})$$

is a weak-equivalence, provided $tr(\mathrm{f_Y})^*\circ\pi_{\mathrm{Y}}^*$ induces an isomorphism on the motivic cohomology of $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$, where $\mathbb{H}_{Nis}$ denotes the hypercohomology spectrum on the Nisnevich site.

Therefore, from the compatibility of the transfer as m varies as proven in [CJ23-T1, 8.3: Step 2], it follows that the composite map

$$tr(\mathrm{f_Y})^*\circ\pi_{\mathrm{Y}}^*:\underset{\infty\leftarrow\mathrm{m}}{\mathrm{holim}}\,\underset{\infty\leftarrow\mathrm{n}}{\mathrm{holim}}\,\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m},\mathrm{s_{\leq n}M})\rightarrow\underset{\infty\leftarrow\mathrm{m}}{\mathrm{holim}}\,\underset{\infty\leftarrow\mathrm{n}}{\mathrm{holim}}\,\mathbb{H}_{\mathrm{Nis}}(\mathrm{Y_m},\mathrm{s_{\leq n}M})$$

is a weak-equivalence.

Next observe that the above discussion already proves the first statement in Theorem 5.7(i). Then the second statement there follows by observing that $\Sigma_{\mathbf{T}}^\infty\mathrm{X}_+$ is dualizable in $\mathbf{Spt}(k_{\mathrm{mot}})$. These complete the proof of Theorem 5.7(i).

In order to prove the variant in *Theorem 5.7(ii)(a)*, it suffices to observe that the slices of the module spectrum M are now module spectra over $\mathbb{S}_k[p^{-1}]$ and that the zero-th slice of $\mathbb{S}_k[p^{-1}]=\mathbb{H}(\mathbb{Z}[p^{-1}])$. Then essentially the same arguments as above apply, along with [CJ23-T1, Theorem 7.1] to complete the proof of statement (a). In order to prove the variant in Theorem 5.7(ii)(b), it suffices to observe that the slices of the module spectrum M are module spectra over $\mathbb{S}_{k(\ell)}$ ($\widehat{\mathbb{S}}_{k,\ell}$) and that the zero-th slice of $\mathbb{S}_{k,(\ell)}$ is $\mathbb{H}(\mathbb{Z}_{(\ell)})$ (the zero-th slice of $\widehat{\mathbb{S}}_{k,\ell}$ is $\mathbb{H}(\mathbb{Z}\hat{_\ell})$, respectively). In fact, both these statements follow readily by identifying the slice tower with the coniveau tower as in [Lev08]. These complete the proof of Theorem 5.7(ii)(a) and (b).

We next discuss the statement Theorem 5.7(iii). Observe that Y and X are assumed to be smooth schemes of finite type over k and that the group G is assumed to be special. Therefore, we invoke Propositions 2.8, 2.9. Proposition 2.9 shows that the hypotheses of Proposition 2.8 are satisfied by the motivic cohomology with respect to a commutative ring R, so that Proposition 2.8 proves the statement in Theorem 5.7(iii).

□

Proposition 5.8. *Assume in addition to the hypotheses of Theorem 5.7(iii) that the scheme* Y *is geometrically connected and that the base field k satisfies the hypothesis* (1.0.1)*, where ℓ is prime to the characteristic of k and ν is a positive integer.*

(i) *Then denoting by $\mathrm{Y}_{\bar{k}}$ the base change of* Y *to the algebraic closure $\bar{k}$, the class $(id_{\mathrm{Y}}\times\tau_{\mathrm{X}}(\mathrm{f}))^*(1)$ identifies with the class $(id_{\mathrm{Y}_{\bar{k}}}\times\tau_{\mathrm{X}}(\mathrm{f}))^*(1)\in \mathrm{H}^0_{\mathrm{et}}(\mathrm{Y}_{\bar{k}},\mathbb{Z}/\ell^\nu)$ under the isomorphisms*
$$\mathrm{H}^{0,0}(\mathrm{Y},\mathbb{Z}/\ell^\nu)\xrightarrow{\cong}\mathrm{H}^{0,0}(\mathrm{Y}_{\bar{k}},\mathbb{Z}/\ell^\nu)\xrightarrow{\cong}\mathrm{H}^0_{\mathrm{et}}(\mathrm{Y}_{\bar{k}},\mathbb{Z}/\ell^\nu).$$
Therefore, under the hypotheses that $(id_{\mathrm{Y}_{\bar{k}}}\times\tau_{\mathrm{X}}(\mathrm{f}))^*(1)\in \mathrm{H}^0_{\mathrm{et}}(\mathrm{Y}_{\bar{k}},\mathbb{Z}/\ell^\nu)$ is a unit, $tr(\mathrm{f_Y})^*$ provides a splitting to to the map $\pi_{\mathrm{Y}}^*:\hat{\mathrm{h}}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{M})\to\hat{\mathrm{h}}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M})$, provided the slices of the motivic spectrum M *are module spectra over the motivic Eilenberg-Maclane spectrum* $\mathrm{H}(\mathbb{Z}/\ell^\nu)$.

(ii) *If* Y *is a geometrically connected smooth scheme and* $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$ *is direct limit of an ind-scheme* $\{\mathrm{E_n}\times_{\mathrm{G}}\mathrm{Y}|\mathrm{n}\geq 1\}$ *with each* $\mathrm{E_n}\times_{\mathrm{G}}\mathrm{Y}$ *geometrically connected smooth scheme, and ℓ is a fixed prime different from char(k), $(id_{\mathrm{Y}_{\bar{k}}}\times\tau_{\mathrm{X}}(\mathrm{f}))^*(1)$ identifies with $tr(\mathrm{f_Y})^*(1)$ under the isomorphism:*
$$\mathrm{H}^0_{\mathrm{et}}((\mathrm{E}\times_{\mathrm{G}}\mathrm{Y})_{\bar{k}},\mathbb{Z}/\ell^\nu)\cong\underset{\infty\leftarrow\mathrm{n}}{lim}\,\mathrm{H}^0_{\mathrm{et}}((\mathrm{E_n}\times_{\mathrm{G}}\mathrm{Y})_{\bar{k}},\mathbb{Z}/\ell^\nu)\xrightarrow{\cong}\mathrm{H}^0_{\mathrm{et}}(\mathrm{Y}_{\bar{k}},\mathbb{Z}/\ell^\nu)$$
where $(\mathrm{E_n}\times_{\mathrm{G}}\mathrm{Y})_{\bar{k}}$ denotes the base change of $\mathrm{E_n}\times_{\mathrm{G}}\mathrm{Y}$ to Spec $\bar{k}$.

(iii) *Therefore, under the above assumptions on* Y *and* $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$ *as in (ii), and on the spectrum* M *as in (i), the condition that $tr(\mathrm{f_Y})^*(1)\in\mathrm{H}^0_{\mathrm{et}}((\mathrm{E}\times_{\mathrm{G}}\mathrm{Y})_{\bar{k}},\mathbb{Z}/\ell^\nu)$ is a unit proves that $tr(\mathrm{f_Y})^*$ provides a splitting to to the map $\pi_{\mathrm{Y}}^*:\hat{\mathrm{h}}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathrm{M})\to\hat{\mathrm{h}}^{*,\bullet}(\mathrm{E}\times_{\mathrm{G}}(\mathrm{Y}\times\mathrm{X}),\mathrm{M})$.*

Proof. We already proved in Proposition 4.1 that the pre-transfer is compatible with étale realization. In fact, one may take the commutative motivic ring spectrum $\mathcal{E}$ to be the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z}/\ell^\nu)$. Then [CJ23-T1, Theorem 7.3] shows $\mathcal{E}\wedge X_+$ is dualizable in $\mathbf{Spt}(k_{\mathrm{mot}},\mathcal{E})$ and Proposition 4.1 shows that the pre-transfer is compatible with étale realizations. (Here are some more details on the above arguments: Let $\mathbb{H}(\mathrm{R})$ denote the motivic spectrum representing motivic cohomology with respect to the commutative ring R. Now invoking [CJ23-T1, Proposition 8.3], we see that

$$\begin{aligned}(id_{\mathrm{Y}_+}\wedge\tau_{\mathrm{X}}(\mathrm{f}))^*(1)&=(id_{\mathrm{Y}_+}\wedge\mathbb{H}(\mathrm{R})\wedge\tau_{\mathrm{X}}(\mathrm{f}))^*(1)\\&=(id_{\mathrm{Y}_+}\wedge\tau_{\mathbb{H}(\mathrm{R})\wedge\mathrm{X}_+}(id\wedge\mathrm{f}))^*(1).\end{aligned}\tag{5.2.1}$$

Next we take $\mathrm{R}=\mathbb{Z}/\ell^\nu$, for ℓ prime to *char*(k).)

These prove the first statement and the second statement is clear. The third statement then follows from the second in view of Theorem 5.7(iii). □

Remark 5.9. If the base field is the field of complex numbers $\mathbb{C}$, one obtains similar results with respect to Betti realization: the corresponding hypotheses will be that the schemes Y and $\mathrm{E}\times_{\mathrm{G}}\mathrm{Y}$ and the spaces $\mathrm{Y}(\mathbb{C})$ and $(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y})(\mathbb{C})$ are connected, so that restriction maps $\mathrm{H}^{0,0}(\mathrm{E}\times_{\mathrm{G}}\mathrm{Y},\mathbb{Z})\to\mathrm{H}^{0,0}(\mathrm{Y},\mathbb{Z})$ and $\mathrm{H}^0((\mathrm{E}\times_{\mathrm{G}}\mathrm{Y})(\mathbb{C}),\mathbb{Z})\to\mathrm{H}^0(\mathrm{Y}(\mathbb{C}),\mathbb{Z})$ are isomorphisms, where the first (second) map is on motivic (singular) cohomology. Then, the conclusion will be that the map π_{Y}^* on motivic cohomology will be a split injection, provided $tr(\mathrm{f}_{\mathrm{Y}(\mathbb{C})})^*(1)$ is a unit in $\mathrm{H}^0((\mathrm{E}\times_{\mathrm{G}}\mathrm{Y})(\mathbb{C}),\mathbb{Z})$. This is under the assumption that the transfer is compatible with Betti realization, which we have not proved in detail: see [Bain18] for a proof in the case of the pre-transfer.

For the motivic spectrum representing Algebraic K-theory, the slice completed generalized motivic cohomology identifies with Algebraic K-theory. For a smooth ind-scheme $\mathcal{Y}=\{\mathrm{Y_m}|\mathrm{m}\}$, we let its algebraic K-theory spectrum be defined as $\mathbf{K}(\mathcal{Y})=\underset{\infty\leftarrow m}{\mathrm{holim}}\{\mathbf{K}(\mathrm{Y_m})|\mathrm{m}\}$. This provides the following corollary.

Corollary 5.10. *Let* $\pi_{\mathrm{Y}}: \mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X}) \rightarrow \mathrm{E} \times_{\mathrm{G}} \mathrm{Y}$ *and* f *denote maps as in Theorem 5.7, with the group* G *any linear algebraic group that is special.*

(i) *Then,* $\pi_{\mathrm{Y}}^*: \mathbf{K}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}) \rightarrow \mathbf{K}(\mathrm{E} \underset{\mathrm{G}}{\times} (\mathrm{Y} \times \mathrm{X}))$ *is a split injection on homotopy groups, where* $\mathbf{K}$ *denotes the motivic spectrum representing Algebraic K-theory, provided* $\Sigma_{\mathbf{T}}^{\infty}\mathrm{X}_+$ *is dualizable in* $\mathcal{SH}(k)$ *and* $tr(\mathrm{f}_{\mathrm{Y}})^*(1)$ *is a unit in* $\mathrm{H}^{0,0}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}, \mathbb{Z}) \cong \mathrm{CH}^0(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})$. *In particular, this holds for smooth quasi-projective schemes* X, Y *defined over the field* k *with* $char(k) = 0$, *provided the above condition on* $tr(\mathrm{f}_{\mathrm{Y}})^*(1)$ *holds.*

(ii) *Assume in addition to the above situation that the hypotheses in Proposition 5.8(ii) are satisfied. Then,*

$$\pi_{\mathrm{Y}}^*: \mathbf{K}(\mathrm{E} \times_{\mathrm{G}} \mathrm{Y}) \wedge \mathrm{M}(\ell^{\nu}) \rightarrow \mathbf{K}(\mathrm{E} \times_{\mathrm{G}} (\mathrm{Y} \times \mathrm{X})) \wedge \mathrm{M}(\ell^{\nu})$$

is a split injection on homotopy groups, where $\mathrm{M}(\ell^{\nu})$ *denotes the Moore spectrum defined as the homotopy cofiber* $\Sigma_{\mathbf{T}}^{\infty} \xrightarrow{\ell^{\nu}} \Sigma_{\mathbf{T}}^{\infty}$, *provided the following hold:* $\Sigma_{\mathbf{T}}^{\infty}\mathrm{X}_+$ *is dualizable in* $\mathcal{SH}(k, \mathbb{S}[\mathrm{p}^{-1}])$, *where* p *is the characteristic exponent of* k, *and* $tr(\mathrm{f}_{\mathrm{Y}_{\bar{k}}})^*(1)$ *is a unit in* $\mathrm{H}^0_{\mathrm{et}}((\mathrm{E} \times_{\mathrm{G}} \mathrm{Y})_{\bar{k}}, \mathrm{Z}/\ell^{\nu})$. *In particular, this holds for smooth quasi-projective schemes* X *defined over the field* k, *with* $char(k) = p$, *provided the above hypotheses hold.*

Proof. The slice completed generalized motivic cohomology of any smooth scheme with respect to the motivic spectrum representing Algebraic K-theory, identifies with Algebraic K-theory itself. This proves the first statement in Corollary 5.10. The second statement in Corollary 5.10 now follows from the following observations.

First we observe the weak-equivalence for any motivic spectrum $\mathcal{E}$: $s_p(\mathcal{E}) \wedge \Sigma_{\mathbf{T}}^{\infty}\mathrm{M}(\ell^{\nu}) \simeq \mathrm{s_p}(\mathcal{E} \wedge_{\Sigma^{\infty}{}_{\mathbf{T}}} \mathrm{M}(\ell^{\nu}))$, where $\mathrm{M}(\ell^{\nu})$ is defined as the homotopy cofiber of the map $\Sigma_{\mathbf{T}}^{\infty} \xrightarrow{\ell^{\nu}} \Sigma_{\mathbf{T}}^{\infty}$, and where s_p denotes the p-th slice. This follows from the identification of the slices, with the slices obtained from the coniveau tower as in [Lev08, Theorem 9.0.3]. Let $\mathbf{K}$ denote the motivic spectrum representing algebraic K-theory. Next we recall (see [Lev08, section 11.3]) that the slice $s_0(\mathbf{K}) = \mathbb{H}(\mathbb{Z})$ = the motivic Eilenberg-Maclane spectrum and that the p-th slice $s_p(\mathbf{K}) = \mathbb{H}(\mathbb{Z}(p)[2p])$, which is the corresponding shifted motivic Eilenberg-Maclane spectrum. Therefore, each $s_p(\mathbf{K})$ has the structure of a module spectrum over the commutative ring spectrum $\mathbb{H}(\mathbb{Z})$. In view of this, one may also observe that the natural map $s_p(\mathbf{K}) \wedge_{\Sigma_{\mathbf{T}}^{\infty}} \mathrm{M}(\ell^{\nu}) \rightarrow \mathrm{s_p}(\mathbf{K}) \wedge_{\mathbb{HZ}} \mathbb{H}(\mathbb{Z}/\ell^{\nu}) = \mathbb{H}(\mathbb{Z}/\ell^{\nu}(\mathrm{p})[2\mathrm{p}])$ is a weak-equivalence, where $\mathbb{H}(\mathbb{Z}/\ell^{\nu})$ denotes the mod$-\ell^{\nu}$ motivic Eilenberg-Maclane spectrum. Therefore, the slices $s_p(\mathbf{K} \wedge_{\Sigma_{\mathbf{T}}^{\infty}} \mathrm{M}(\ell^{\nu})) \simeq \mathrm{s_p}(\mathbf{K}) \wedge_{\Sigma_{\mathbf{T}}^{\infty}} \mathrm{M}(\ell^{\nu})$ have the structure of weak-module spectra over the motivic Eilenberg-Maclane spectrum $\mathbb{H}(\mathbb{Z}/\ell^{\nu})$. Therefore, the hypotheses of the statement in Theorem 5.7(iii) are met.

The additional assumptions then verify that the hypotheses of Proposition 5.8(iii) are also met thereby completing the proof of the second statement in Corollary 5.10. (One may also want to observe that the spectrum $\mathbf{K} \wedge_{\Sigma_{\mathbf{T}}^{\infty}} \mathrm{M}(\ell^{\nu})$ has cohomological descent on the Nisnevich site of smooth schemes of finite type over k so that the generalized cohomology $\mathrm{h}(\mathrm{X}, \mathbf{K} \wedge_{\Sigma_{\mathbf{T}}^{\infty}} \mathrm{M}(\ell^{\nu})) \simeq \mathbf{K}(\mathrm{X}) \wedge_{\Sigma_{\mathbf{T}}^{\infty}} \mathrm{M}(\ell^{\nu})$ for any smooth scheme X of finite type over k.) $\square$

The only *disadvantages* for this method seems to be that we need to assume that the base B of the torsor is connected, the object Y is a geometrically connected smooth scheme of finite type over k, G is assumed to be special, and also because this method applies only to slice-completed generalized motivic cohomology theories. However, as several important examples of generalized motivic cohomology theories, such as Algebraic K-theory and Algebraic Cobordism are slice-complete, there do not seem to be any serious disadvantages.

6. Appendix: Further details on Étale realization

As we pointed out already in Remark 3.5, there is a discussion of étale realization in the unstable setting in [Isak] and also [Schm]. Though the discussions there have the expected good properties, these do not seem to extend to define an étale realization in the stable setting with good properties, partially because of the following: the target of the realization functors discussed in [Isak] and [Schm] is the pro-category of inverse systems of simplicial sets provided by applying the connected component functor degree-wise to the inverse systems of (rigid) étale hypercoverings. Unfortunately, the model structure on pro-simplicial sets (or for pro-objects in general model categories) that have been discussed in the literature, such as those in [Isak2] are *not cofibrantly generated* as in fact pointed out in op. cit. As a result, obtaining a suitable stable model structure for the category of spectra in such pro-categories with good properties seems doubtful.

We will circumvent these problems, and will sketch here a different process of étale realization that works stably without any of these difficulties stemming from having to deal with pro-objects. This will also be a straightforward extension of the étale realization functors briefly discussed earlier in section 3.

As is well-known, étale cohomology is not well-behaved unless one restricts to torsion coefficients, with torsion prime to the residue characteristics. This shows therefore, the key role played by Bousfield-Kan type completion for spectra in étale setting. We will first discuss such completions in the $\mathbb{A}^1$-setting.

6.1. **$\mathbb{Z}/\ell$-completions and spectra of $\mathbb{Z}/\ell$-vector spaces.** We will assume the framework of [CJ23-T1, section 1] in what follows. Throughout the following discussion, we will let $\mathbb{Z}$ denote the ring of integers.

(i) For a pointed simplicial presheaf $\mathrm{P}\,\varepsilon\,\mathbf{Spc}_*^{\mathrm{G}}(k_{\mathrm{mot}})$, $\mathbb{Z}/\ell(\mathrm{P})$ will denote the presheaf of simplicial $\mathbb{Z}/\ell$-vector spaces defined in [B-K, Chapter, 2.1]. Observe that for each $n \geq 0$, $\mathbb{Z}/\ell(\mathrm{P})_{\mathrm{n}}$ is the quotient of the free $\mathbb{Z}/\ell$-module on $\mathrm{P_n}$ by $\mathbb{Z}/\ell(*)$, where $*$ is the base point of $\mathrm{P_n}$. It follows readily that

$$\mathbb{Z}/\ell(\mathrm{P} \wedge \mathrm{Q}) \cong \mathbb{Z}/\ell(\mathrm{P}) \otimes_{\mathbb{Z}/\ell} \mathbb{Z}/\ell(\mathrm{Q}). \tag{6.1.1}$$

(ii) Observe also that the presheaves of homotopy groups of $\mathbb{Z}/\ell(\mathrm{P})$ identify with the reduced homology presheaves of P with respect to the ring $\mathbb{Z}/\ell$. Hence these are all ℓ-primary torsion. The functoriality of this construction shows that if P has an action by the group G, then $\mathbb{Z}/\ell(\mathrm{P})$ inherits this action.

(iii) Next one extends the construction in the previous step to spectra: as we work largely with the category of spectra $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ and $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et})$ we will only consider these categories. Let $\mathcal{X} \in \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$. Then first one applies the functor $(\quad) \mapsto \mathbb{Z}/\ell(\quad)$ to each $\mathcal{X}(\mathrm{T_W})$, $\mathrm{T_W}\,\varepsilon\,\mathrm{Sph}^{\mathrm{G}}$. In view of (6.1.1), there exists a natural map

$$\mathbb{Z}/\ell(\mathcal{H}om_{\mathbf{Spc}_*^{\mathrm{G}}(k_{\mathrm{mot}})}(\mathrm{T_U}, \mathrm{T_U} \wedge \mathrm{T_V})) \underset{\mathbb{Z}/\ell}{\otimes} \mathbb{Z}/\ell(\mathcal{X}(\mathrm{T_W})) \to \mathbb{Z}/\ell(\mathcal{H}\mathrm{om}_{\mathbf{Spc}_*^{\mathrm{G}}(k_{\mathrm{mot}})}(\mathrm{T_U}, \mathrm{T_U} \wedge \mathrm{T_V}) \wedge \mathcal{X}(\mathrm{T_W})),$$

where $\mathcal{H}om_{(\mathbf{Spc}_*^{\mathrm{G}}(k_{\mathrm{mot}}))}$ denotes the internal hom in the category $\mathbf{Spc}_*^{\mathrm{G}}(k)$. Therefore, one may compose the above maps with the obvious map $\mathbb{Z}/\ell(\mathcal{H}om_{\mathbf{Spc}_*^{\mathrm{G}}(k_{\mathrm{mot}})}(\mathrm{T_U}, \mathrm{T_U} \wedge \mathrm{T_V}) \wedge \mathcal{X}(\mathrm{T_W})) \to \mathbb{Z}/\ell(\mathcal{X}(\mathrm{T_V} \wedge \mathrm{T_W}))$ to define an object in $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}^{\mathrm{G}}))$.

(iv) A pairing $\mathrm{M} \wedge \mathrm{N} \to \mathrm{P}$ in $\mathbf{Spc}_*^{\mathrm{G}}(k_{\mathrm{mot}})$ induces a pairing $\mathbb{Z}/\ell(\mathrm{M}) \underset{\mathbb{Z}/\ell}{\otimes} \mathbb{Z}/\ell(\mathrm{N}) \cong \mathbb{Z}/\ell(\mathrm{M} \wedge \mathrm{N}) \to \mathbb{Z}/\ell(\mathrm{P})$. Similarly a pairing $\mathcal{X} \wedge \mathcal{Y} \to \mathcal{Z}$ in $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ induces a similar pairing $\mathbb{Z}/\ell(\mathcal{X}) \underset{\mathbb{Z}/\ell}{\otimes} \mathbb{Z}/\ell(\mathcal{Y}) \to \mathbb{Z}/\ell(\mathcal{Z})$. (To see this, one needs to recall the construction of the smash-product of spectra from [CJ23-T1, Definition 4.2] as a left-Kan extension. The functor $\mathbb{Z}/\ell$ commutes with this left-Kan extension.) This shows that if $\mathcal{E} \in \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ is a ring spectrum so is $\mathbb{Z}/\ell(\mathcal{E})$ and that if $\mathcal{M} \in \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ is a module spectrum over the ring spectrum $\mathcal{E}$, $\mathbb{Z}/\ell(\mathcal{M})$ is a module spectrum over $\mathbb{Z}/\ell(\mathcal{E})$.

(v) We will assume that $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ is provided with the stable injective model structure, so that every object in $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ is cofibrant. If $\{\mathrm{f} : \mathrm{A} \to \mathrm{B}\} = \mathrm{J_{Sp}}$ is a set of generating cofibrations (trivial cofibrations) for $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$, then, we will let $\{\mathbb{Z}/\ell(\mathrm{f}) : \mathbb{Z}/\ell(\mathrm{A}) \to \mathbb{Z}/\ell(\mathrm{B}) | \mathrm{f}\,\varepsilon\,\mathrm{J_{Sp}}\}$ denote the set of generating cofibrations (trivial cofibrations, respectively) for $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}^{\mathrm{G}}))$. This defines a model structure on $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}^{\mathrm{G}}))$.

The corresponding category of G-equivariant spectra on the big étale site of $\mathrm{Spec}\, k$ will be denoted $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}, \mathbb{Z}/\ell(\mathbb{S}^{\mathrm{G}}_{k,et}))$. These categories are also locally presentable and hence the model categories are combinatorial. One observes that the functor $\mathbb{Z}/\ell(\quad)$ induces a functor

$$\mathbb{Z}/\ell(\quad) : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}) = \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{S}^{\mathrm{G}}_k) \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}^{\mathrm{G}}_k)) \text{ and}$$

$$\mathbb{Z}/\ell(\quad) : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}) = \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}, \mathbb{S}^{\mathrm{G}}_{k,et}) \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}, \mathbb{Z}/\ell(\mathbb{S}^{\mathrm{G}}_{k,et})).$$

Using the adjunction between the functors $\mathbb{Z}/\ell(\quad)$ and U, one may show readily that the functor $\mathbb{Z}/\ell(\quad)$ is a left Quillen functor and preserves weak-equivalences between cofibrant objects. However, in order to produce a triple, using the above functors $\mathbb{Z}/\ell(\quad)$ and U, one needs to compose the functor $\mathbb{Z}/\ell(\quad)$ with a fibrant replacement functor. This composition will be denoted by $\widetilde{\mathbb{Z}/\ell}(\quad)$.

Proposition 6.1. *Let* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}_k^{\mathrm{G}}))_f$ *denote the fibrant objects in* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}_k^{\mathrm{G}}))$.

(i) *There exists a functor* $\mathcal{X} \to \widetilde{\mathbb{Z}/\ell}(\mathcal{X}) : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{S}_k^{\mathrm{G}}) \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}_k^{\mathrm{G}}))_f$ *left-adjoint to the forgetful functor* $\mathrm{U} : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}_k^{\mathrm{G}}))_f \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{S}_k^{\mathrm{G}})$. *This functor preserves* $\mathbb{A}^1$*-fibrant spectra and* $\mathbb{A}^1$*-stable weak-equivalences.*

(ii) *The two adjoint functors* U *and* $\widetilde{\mathbb{Z}/\ell}(\quad)$ *define a triple and hence a cosimplicial object with* $\widetilde{\mathbb{Z}/\ell}(\quad)$ *in degree* 0. *This cosimplicial object is group-like, i.e., each* $\widetilde{\mathbb{Z}/\ell}^n(\mathcal{X})$ *belongs to* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}_k^{\mathrm{G}}))$ *and all the cosimplicial structure maps are maps of* $\mathbb{Z}/\ell$*-vector spaces, except for* d^0. *Therefore, it is fibrant in the Reedy-model structure on cosimplicial objects in* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}_k^{\mathrm{G}}))$.

(iii) *The corresponding results also hold for the functor* $\widetilde{\mathbb{Z}/\ell}(\quad) : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}, \mathbb{S}_{k,et}^{\mathrm{G}}) \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}, \mathbb{Z}/\ell(\mathbb{S}_{k,et}^{\mathrm{G}}))$.

Proof. We will only sketch the outlines of a proof as the above discussion should be known to the experts. One defines the functor $\mathcal{X} \to \widetilde{\mathbb{Z}/\ell}(\mathcal{X})$ by taking a functorial fibrant replacement of the functor $\mathcal{X} \to \mathbb{Z}/\ell(\mathcal{X})$. Then given a map $\mathrm{A} \to \mathrm{U}(\tilde{\mathrm{E}})$, with $\tilde{\mathrm{E}}$ in $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}, \mathbb{Z}/\ell(\mathbb{S}_k^{\mathrm{G}}))_f$, this corresponds by adjunction to a map $\mathbb{Z}/\ell(\mathrm{A}) \to \tilde{\mathrm{E}}$. Then one shows that this map admits a lifting to a map $\widetilde{\mathbb{Z}/\ell}(\mathrm{A}) \to \tilde{\mathrm{E}}$ using a standard construction of the functorial fibrant replacement using a small object argument: see, for example, [Hov99, Theorem 2.1.14]. This proves (i) and the remaining statements follow. □

Definition 6.2. $\widetilde{\mathbb{Z}/\ell}_\infty(\mathcal{X}) = \underset{\Delta}{\mathrm{holim}}\{\widetilde{\mathbb{Z}/\ell}^n(\mathcal{X})|n\}$.

Proposition 6.3. *The functor* $\mathcal{X} \mapsto \widetilde{\mathbb{Z}/\ell}_\infty(\mathcal{X})$, $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}) \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ *has the following properties:*

(i) *For* $\mathcal{X} \in \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$, $\widetilde{Z/l_\infty(\mathcal{X})}$ *is* $\mathbb{Z}/\ell$*-complete, i.e., (i) it is fibrant in* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ *and for every map* $\phi : \mathrm{A} \to \mathrm{B}$ *in* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ *between cofibrant objects which induces an isomorphism*

$$\pi_*(\widetilde{\mathbb{Z}/\ell}(\mathrm{A})) \overset{\cong}{\to} \pi_*(\widetilde{\mathbb{Z}/\ell}(\mathrm{B}))$$

of the reduced homology presheaves, the induced map $\phi^* : Map(\mathrm{B}, \widetilde{\mathbb{Z}/\ell_\infty(\mathcal{X})}) \to \mathrm{Map}(\mathrm{A}, \widetilde{\mathbb{Z}/\ell_\infty(\mathcal{X})})$ *is a weak-equivalence of simplicial sets, where* $\mathcal{M}ap$ *denotes the simplicial mapping space functor. In particular, this applies to the case where* $\mathrm{B} = \widetilde{\mathbb{Z}/\ell}_\infty(\mathrm{A})$ *and* $\phi : \mathrm{A} \to \mathrm{B}$ *is the obvious Bousfield-Kan completion map.*

(ii) *If* $\mathcal{E}\,\varepsilon\,\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ *is a ring spectrum so are each* $\widetilde{\mathbb{Z}/\ell_n(\mathcal{E})}$ *and* $\mathbb{Z}/\ell_\infty(\mathcal{E})$. *If* $\mathcal{E} \in \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ *is a ring spectrum and* $\mathcal{M} \in \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$ *is a module spectrum over* $\mathcal{E}$, *then each* $\widetilde{\mathbb{Z}/\ell_n(\mathcal{M})}$ *is a* $\widetilde{\mathbb{Z}/\ell_n(\mathcal{E})}$*-module spectrum and* $\widetilde{\mathbb{Z}/\ell}_\infty(\mathcal{M})$ *is a* $\widetilde{\mathbb{Z}/\ell}_\infty(\mathcal{E})$*-module spectrum.*

(iii) *For each* $\mathcal{E}\,\varepsilon\,\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$, *each* $\widetilde{\mathbb{Z}/\ell_n(\mathcal{E})}$ *is* $\mathbb{A}^1$*-local and belongs to* $\widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}})$.

(iv) *The corresponding results also hold for the functor* $\widetilde{\mathbb{Z}/\ell}_\infty : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}) \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et})$.

Proof. Here we will again provide only a sketch of (i), as the remaining statements should be clear. A key observation is that the map ϕ induces a weak-equivalence $\mathbb{Z}/\ell(\phi) : \mathbb{Z}/\ell(\mathrm{A}) \to \mathbb{Z}/\ell(\mathrm{B})$, so that the induced map $\phi^* : Map(\mathbb{Z}/\ell(\mathrm{B}), \widetilde{\mathbb{Z}/\ell(\mathcal{X})}) \to \mathrm{Map}(\mathbb{Z}/\ell(\mathrm{A}), \widetilde{\mathbb{Z}/\ell(\mathcal{X})})$ is a weak-equivalence. Now (i) follows readily from this observation. □

6.2. The refined étale realization.

Definition 6.4. (The refined étale realization functor) We define the refined étale realization functor as the composition $Et = \widetilde{\mathbb{Z}/\ell}_\infty \circ \epsilon^* : \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{\mathrm{mot}}) \to \widetilde{\mathbf{Spt}}^{\mathrm{G}}(k_{et}, \widetilde{\mathbb{Z}/\ell}_\infty(\mathbb{S}_{k,et}^{\mathrm{G}}))$.

Proposition 6.5. (i) *The above étale realization functor is weakly-monoidal, i.e., for each pairing* $\mathcal{X} \wedge \mathcal{Y} \rightarrow \mathcal{Z}$ *of spectra in* $\mathbf{Spt}(k_{\rm mot})$*, there is a natural induced pairing* $Et(\mathcal{X}) \wedge Et(\mathcal{Y}) \rightarrow Et(\mathcal{Z})$.

(ii) *Assume the base field* k *is separably closed, with characteristic* p. *Then for any prime* $\ell \neq p$*, the two categories of spectra* $\widetilde{\mathbf{Spt}}(k_{et}, \widetilde{\mathbb{Z}/\ell}_{\infty}(\mathbb{S}_{\mathbf{T},et}))$ *and* $\widetilde{\mathbf{Spt}}(k_{et}, \widetilde{\mathbb{Z}/\ell}_{\infty}(\mathbb{S}_{\mathrm{S}^2,\mathrm{et}}))$ *are equivalent, where* $\mathbb{S}_{\mathbf{T},et}$ *(*$\mathbb{S}_{\mathrm{S}^2,\mathrm{et}}$*) denotes the sphere spectrum defined as the* $\mathbf{T}$*-suspension spectrum (the* S^2*-suspension spectrum, respectively) of* S^0.

Proof. (i) is clear in view of the properties of the completion functor discussed in Proposition 6.3. To see (ii), one uses the following well-known argument. Let $\mathrm{W}(k)$ denote the ring of Witt vectors of k. This is a Hensel ring with closed point corresponding to k and the generic point in characteristic 0. Therefore, one may imbed $\mathrm{W}(k)$ into the field of complex numbers $\mathbb{C}$. Now the projective space $\mathbb{P}^1$ over k lifts to the projective space $\mathbb{P}^1$ over $\mathbb{C}$. On taking the $\mathbb{Z}/\ell$-completion, one sees that $\widetilde{\mathbb{Z}/\ell}_{\infty}(\mathbb{P}^1) \simeq \widetilde{\mathbb{Z}/\ell}_{\infty}(\mathrm{S}^2)$. □

We conclude with the following result.

Proposition 6.6. *Assume the base scheme is a separably closed field* k *and* $char(k) = p$. *Let* $E \,\varepsilon\, \mathbf{Spt}(k_{et})$ *be a constant sheaf of spectra so that all the (sheaves of) homotopy groups* $\pi_n(E)$ *are* ℓ*-primary torsion, for some* $\ell \neq p$. *Then* E *is* $\mathbb{A}^1$*-local in* $\mathbf{Spt}(k_{et})$*, i.e., the projection* $\mathrm{P} \wedge \mathbb{A}^1_+ \rightarrow \mathrm{P}$ *induces a weak-equivalence:* $Map(\mathrm{P},\mathrm{E}) \simeq \mathrm{Map}(\mathrm{P} \wedge \mathbb{A}^1_+, \mathrm{E})$, $\mathrm{P} \,\varepsilon\, \mathbf{Spt}(k_{et})$.

Proof. First let P denote the suspension spectrum associated to some smooth scheme $\mathrm{X} \in \mathrm{Sm}_k$. Then $\mathrm{Map}(\mathrm{P},\mathrm{E})$ ($\mathrm{Map}(\mathrm{P} \wedge \mathbb{A}^1_+, \mathrm{E})$) identifies with the spectrum defining the generalized étale cohomology of X (of $\mathrm{X} \times \mathbb{A}^1$, respectively) with respect to the spectrum E. There exist Atiyah-Hirzebruch spectral sequences that converge to these generalized étale cohomology groups with the $E_2^{s,t}$-terms being $H^s_{et}(\mathrm{X}, \pi_{-\mathrm{t}}(\mathrm{E}))$ and $H^s_{et}(\mathrm{X} \times \mathbb{A}^1, \pi_{-\mathrm{t}}(\mathrm{E}))$, respectively. Since the sheaves of homotopy groups $\pi_{-t}(E)$ are all ℓ-torsion with $\ell \neq p$, and k is assumed to be a separably closed field, X and $\mathrm{X} \times \mathbb{A}^1$ have finite ℓ-cohomological dimension. Therefore these spectral sequences converge strongly and the conclusion of the proposition holds in this case. For a general simplicial presheaf P, one may find a simplicial resolution where each term is a disjoint union of schemes as above (indexed by a small set). Therefore, the conclusion of the proposition holds also for suspension spectra of all simplicial presheaves and therefore for all spectra P. □

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