Thermo-elastodynamics of finitely-strained multipolar viscous solids with an energy-controlled stress.

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Abstract

The thermodynamical model of viscoelastic deformable solids at finite strains with Kelvin-Voigt rheology with a higher-order viscosity (using the concept of multipolar materials) is formulated in a fully Eulerian way in rates. Assumptions used in this paper allow for a physically justified free energy leading to non-negative entropy that satisfies the 3rd law of thermodynamics, i.e. entropy vanishes at zero temperature, and energy-controlled stress. This last attribute is used advantageously to prove the existence and a certain regularity of weak solutions by a simplified Faedo-Galerkin semi-discretization, based on estimates obtained from the total-energy and the mechanical-energy balances. Some examples that model neo-Hookean-type materials are presented, too.

Mathematics Subject Classification. 35Q74, 35Q79, 74A30, 74Dxx, 80A17, 80M10.

Keywords. Elastodynamics, Kelvin-Voigt viscoelasticity, thermal coupling, large strains, multipolar continua, Eulerian formulation, semi-Galerkin discretization, weak solutions.

1 Introduction

The heat-transfer problem is an "evergreen" problem studies in the mathematical literature for centuries, in recent decades in non-linear variants in a mechanical context and possibly coupled with other physical or physical or chemical processes. Often (or even mostly) the heat capacity is considered to be "uniformly" positive (quite typically just constant), which is only possible if the entropy decays to $-\infty$ as the temperature approaches absolute zero. Although the constant heat capacity is the first choice in most in most mathematical and engineering publications, such a choice is not physical because the entropy $-\infty$ at zero temperature contradicts the sound physical arguments. Rather, the heat capacity should degenerate to zero at zero temperature, so that the entropy remains finite (usually calibrated to zero) at zero temperature, which is a physically desirable (and even ultimate) property.

This degeneracy need not be detrimental to the mathematical analysis. In this article, we want to revisit the *heat-transfer problem at finite strains* in the *Eulerian framework*, which

[1]Support from CSF grant no. 23-06220S, and from the institutional support RVO:61388998 (ČR) is acknowledged.

is probably the most complicated due to L^1-heat sources arising from the mechanical visco-elastodynamic compressible model together with transport equations and convective terms in the heat equation. On the other hand, we confine ourselves on the mathematically simplest viscoelastic *rheology* of the *Kelvin-Voigt* type.

The Eulerian formulation is relatively standard, cf. [13,19,34,36,38]. Mathematical analysis in the conventional simple-material variant is, however, open (as also articulated in [1,2]) except in the fluidic variant where the absence of any shear elastic response is exploited, cf. [10]. For completeness, we report some existing analytical results in isothermal incompressible situations in [17, 18] and compressible in [15, 26] with small data or with (non-physical) convex stored energies. There seems to be a certain agreement that some higher gradients (falling into the concept of so-called *non-simple materials*) are desirable or even necessary for a reasonable analysis (unless some some very week e.g. of measure-valued-type solution concepts like e.g. in Lagrangian formulation in [6–8, 12] are used), and it should be openly admitted that it is the main purpose to use them in this paper. A certain additional motivation could be to open up options for modelling the dispersion of possible elastic waves in some desirable way. In the Eulerian frame, such higher gradients are to be involved rather in the dissipative than the conservative part (giving rise continua which are sometimes referred as *multipolar*), so that their influence is manifested itself only in fast evolution. Then the higher gradients can lead to easier propagation of elastic waves with less dispersion and less attenuation (compared with the usual first-order gradients) especially at frequencies close to the critical frequency above which the waves cannot propagate at all, as pointed out in [31, Sect.3.1] in the 1-dimensional linear variant.

The main features (and differences from [32]) can be summarized as follows:

- usage of the *actual free energy*, so that formally (as opposed to the referential energy as in Remark 2.2) one does not need to control explicitly the determinant of the deformation gradient (although the control of the mass density implicitly includes this determinant),
- estimation based on the *energy-controlled stress*, i.e. the Cauchy stress whose magnitude is dominated by the actual internal energy,
- sufficiently weak qualification of data which, in particular, admits physically justified free energies
  - satisfying the 3rd law of thermodynamics, i.e. yielding the non-negative entropy which vanishes at zero temperature (and is then independent of the mechanical state), although this attribute will not be explicitly exploited in the analysis, and
  - allowing for a natural continuous extension for negative values of temperature (as in Fig. 2 below) exploited advantageously for the Galerkin approximation.

Also, in comparison with [32], the mentioned weaker data qualification will admit free energies describing e.g. thermal expansion and complying with a heat capacity which remains positive for arbitrarily large deformations, not only for conditionally not too large deformations. Moreover, the semi-Galerkin approximation scheme used here is much simpler than in [32] and in particular does not rely on any cut-off and the non-negativity of the limit temperature is obtained more easily, among various other simplifications.

The plan for this paper is as follows: In Section 2, we formulate the thermo-visco-elastic system at finite strains, and briefly derive its thermomechanical consistency, i.e. 1st and 2nd laws of thermodynamics. Then, in Section 3, we present its rigorous analysis as far as the global existence and certain regularity of weak solutions for large data. Finally, in Section 4, we present some examples of neo-Hookean materials complying with the assumptions imposed in Section 3.

Let us emphasize that the approximation and estimation strategy developed in Section 3 seems to be well applicable to the (in some sense simpler) variant in the Lagrangian framework and, a-fortiori, to a small-strain variant.

2 The thermomechanical system and its energetics

We briefly remind the fundamental concepts and formulas that can mostly be found in various monographs, as e.g. [13, 19, 34, 36, 38].

In finite-strain continuum mechanics, the basic geometrical concept is the time-evolving deformation $\mathbf{y} : \varOmega \to \mathbb{R}^d$ as a mapping from a reference configuration of the body $\varOmega \subset \mathbb{R}^d$ into a physical space $\mathbb{R}^d$. The "Lagrangian" space variable in the reference configuration is denoted as $\mathbf{X} \in \varOmega$ while in the "Eulerian" physical-space variable as $\boldsymbol{x} \in \mathbb{R}^d$. The basic geometrical object is the Lagrangian deformation gradient $\mathbf{F} = \nabla_{\mathbf{X}}\mathbf{y}$. We will be interested in deformations $\boldsymbol{x} = \mathbf{y}(t, \mathbf{X})$ evolving in time, which are sometimes called "motions". The referential velocity is $\mathbf{v} = \frac{\partial}{\partial t}\mathbf{y}$. The inverse mapping $\boldsymbol{\xi} : \boldsymbol{x} \mapsto \mathbf{X} = \mathbf{y}^{-1}(t, \boldsymbol{x})$ is called a *return* (sometimes called also a *reference*) *mapping*.

Having $\boldsymbol{\xi}$, we define the Eulerian velocity $\boldsymbol{v} = \mathbf{v}\circ\boldsymbol{\xi}$ and the Eulerian deformation gradient $\boldsymbol{F} = \mathbf{F}\circ\boldsymbol{\xi}$. Having the Eulerian velocity $\boldsymbol{v}$, we define the *convective time derivative* applied to scalars or, component-wise, to vectors or tensors, using the dot-notation $(\cdot)^{\bullet}$. By the chain-rule calculus, we have the *transport equation-and-evolution for the deformation gradient* and its determinant and its inverse as

$$\dot{\boldsymbol{F}} = (\nabla\boldsymbol{v})\boldsymbol{F}\,, \qquad \overset{\bullet}{\overline{\det \boldsymbol{F}}} = (\det \boldsymbol{F})\mathrm{div}\,\boldsymbol{v}\,, \quad \text{and} \quad \overset{\bullet}{\overline{\Big(\frac{1}{\det \boldsymbol{F}}\Big)}} = -\frac{\mathrm{div}\,\boldsymbol{v}}{\det \boldsymbol{F}}\,. \tag{2.1}$$

The reference mapping $\boldsymbol{\xi}$, which itself is well defined by its transport equation $\dot{\boldsymbol{\xi}} = \mathbf{0}$, actually does not explicitly occur in the formulation of the problem if the medium is considered homogeneous in its reference configuration. Here, we will benefit from the boundary condition $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ below, which causes that the actual domain $\varOmega$ does not evolve in time. The same convention applies to temperature θ and thus also $\boldsymbol{T}$, η, and $\boldsymbol{D}$ in (2.5) and (2.6) below, which will make the problem indeed fully Eulerian. Cf. the continuum-mechanics textbooks as e.g. [13, 19]. The mass density (in kg/m^3) transport and the "mass sparsity" as the inverse mass density $1/\varrho$ transport write:

$$\dot{\varrho} = -\varrho\,\mathrm{div}\,\boldsymbol{v} \qquad \text{and} \qquad \overset{\bullet}{\overline{\Big(\frac{1}{\varrho}\Big)}} = \frac{\mathrm{div}\,\boldsymbol{v}}{\varrho}\,. \tag{2.2}$$

The former equation in (2.2) called the *continuity equation* equivalently writes $\frac{\partial}{\partial t}\varrho + \mathrm{div}(\varrho\boldsymbol{v}) = 0$ (expressing that the conservation of mass) and ensures the transport of the momentum $\varrho\boldsymbol{v}$:

$$\frac{\partial}{\partial t}(\varrho\boldsymbol{v}) + \mathrm{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v}) = \varrho\dot{\boldsymbol{v}}\,. \tag{2.3}$$

One can determine the density ϱ instead of the transport equation for mass density (2.2) from the algebraic relation

$$\varrho = \frac{\varrho_{\mathrm{R}}}{\det \boldsymbol{F}}\,, \tag{2.4}$$

where $\varrho_{\rm R}$ is the mass density in the reference configuration. Later, we will consider the initial conditions $\boldsymbol{F}_0$ for (2.1) and ϱ_0 for (2.2). These two should be related by $\varrho(0) = \varrho_{\rm R}/\det \boldsymbol{F}_0$.

The main ingredients of the model are the (volumetric) *free energy* ψ depending on deformation gradient $\boldsymbol{F}$ and temperature θ, and the temperature-dependent *dissipative stress* $\boldsymbol{D}$ not necessarily possessing any underlying potential. The free energy $\psi = \psi(\boldsymbol{F},\theta)$ considered per the *actual volume* (in contrast to a referential energy as in Remark 2.2 below) leads to the conservative part of the (actual) Cauchy stress, the entropy, and the heat capacity respectively

$$\boldsymbol{T} = \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top + \psi(\boldsymbol{F},\theta)\mathbb{I}\,, \qquad \eta = -\psi'_\theta(\boldsymbol{F},\theta)\,, \quad \text{and} \quad c(\boldsymbol{F},\theta) = -\theta\psi''_{\theta\theta}(\boldsymbol{F},\theta)\,, \tag{2.5}$$

where $\mathbb{I}$ denotes the unit matrix. The concept of stresses governed by some potential is called *hyperelasticity*.

In the already anticipated visco-elastodynamic *Kelvin-Voigt rheological model*, we will also use a dissipative contribution to the Cauchy stress, which will make the system of a parabolic type compared to mere elastodynamics which would be of a hyperbolic type. In addition to the usual first-order stress depending on $\boldsymbol{e}(\boldsymbol{v})$, we consider a dissipative contribution to the Cauchy stress involving also a higher-order 2nd-grade *hyper-stress* $\mathcal{H}$, so that the overall dissipative stress is:

$$\boldsymbol{D} = \nu_1\boldsymbol{e}(\boldsymbol{v}) - {\rm div}\,\mathcal{H} \quad \text{with} \quad \text{and} \;\; \mathcal{H} = \nu_2|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v}\,. \tag{2.6}$$

Rather for notational simplicity, we consider $\nu_1 > 0$ and $\nu_2 > 0$ constant, even though it would not pose any analytical difficulties to consider them continuously dependent on $(\boldsymbol{F},\theta)$ and/or tensor-valued. The mentioned concept of the 2nd-grade "hyper-stress" refers to a 3rd-order tensor (here denoted by $\mathcal{H}$) whose divergence yields a contribution to the the Cauchy stress. This modelling concept is inspired by R.A. Toupin [37] and R.D. Mindlin [21] and has quite widely been used for the general nonlinear context of *multipolar materials*, both fluids by J. Nečas at al. [23–25] or solids [27, 35].

The *momentum equilibrium* equation then balances the divergence of the total Cauchy stress with the inertial and gravity force:

$$\varrho\dot{\boldsymbol{v}} - {\rm div}\big(\boldsymbol{T}{+}\boldsymbol{D}\big) = \varrho\boldsymbol{g} \tag{2.7}$$

with $\boldsymbol{T}$ from (2.5) and $\boldsymbol{D}$ from (2.6).

The second ingredient in (2.5) is subjected to the *entropy equation*:

$$\frac{\partial\eta}{\partial t} + {\rm div}\big(\boldsymbol{v}\,\eta\big) = \frac{\xi - {\rm div}\,\boldsymbol{j}}{\theta} \qquad \text{with} \;\; \boldsymbol{j} = -\kappa(\boldsymbol{F},\theta)\nabla\theta \tag{2.8}$$

with ξ the heat production rate and $\boldsymbol{j}$ the heat flux. The latter equality in (2.8) is the *Fourier law* determining phenomenologically the heat flux $\boldsymbol{j}$ proportional to the temperature gradient through the thermal conductivity coefficient $\kappa = \kappa(\boldsymbol{F},\theta)$. Assuming $\xi \geq 0$ and $\kappa \geq 0$ and integrating (2.8) over the domain Ω while imposing the impenetrability of the boundary in the sense that the normal velocity $\boldsymbol{v}{\cdot}\boldsymbol{n}$ vanishes across the boundary Γ of Ω, we obtain the *Clausius-Duhem inequality*:

$$\frac{\rm d}{{\rm d}t}\underbrace{\int_\Omega \eta\,{\rm d}\boldsymbol{x}}_{\text{total entropy}} = \int_\Omega \underbrace{\frac{\xi}{\theta} + \kappa(\boldsymbol{F},\theta)\frac{|\nabla\theta|^2}{\theta^2}}_{\text{entropy production rate}}{\rm d}\boldsymbol{x} + \int_\Gamma \underbrace{\Big(\kappa(\boldsymbol{F},\theta)\frac{\nabla\theta}{\theta} - \eta\boldsymbol{v}\Big)}_{\text{entropy flux}}{\cdot}\,\boldsymbol{n}\,{\rm d}S \geq \int_\Gamma \kappa(\boldsymbol{F},\theta)\frac{\nabla\theta}{\theta}{\cdot}\boldsymbol{n}\,{\rm d}S\,. \tag{2.9}$$

If the system is thermally isolated in the sense that the normal heat flux $\boldsymbol{j}\cdot\boldsymbol{n}$ vanishes across the boundary Γ, we recover the *2nd law of thermodynamics*, i.e. the total entropy in isolated systems is nondecreasing in time.

The *internal energy* is given by the *Gibbs relation* $e = \psi + \theta\eta$. Using the calculus

$$
\begin{aligned}
\frac{\partial e}{\partial t} + \operatorname{div}\big(\boldsymbol{v}\,e\big) &= \dot{e} + e\operatorname{div}\boldsymbol{v} = \overset{\bullet}{\overline{\psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)}} + \big(\psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)\big)\operatorname{div}\boldsymbol{v}\\
&= \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta){:}\dot{\boldsymbol{F}} + \psi'_\theta(\boldsymbol{F},\theta)\dot\theta - \theta\psi''_{\boldsymbol{F}\theta}(\boldsymbol{F},\theta)\dot{\boldsymbol{F}} - \theta\psi''_{\theta\theta}(\boldsymbol{F},\theta)\dot\theta - \dot\theta\psi'_\theta(\boldsymbol{F},\theta)\\
&\qquad\qquad + \big(\psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)\big)\operatorname{div}\boldsymbol{v}\\
&= \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta){:}\dot{\boldsymbol{F}} + \theta\overset{\bullet}{\overline{\eta(\boldsymbol{F},\theta)}} + \big(\psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)\big)\operatorname{div}\boldsymbol{v}\\
&\overset{(2.8)}{=} \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta){:}\dot{\boldsymbol{F}} + \xi - \operatorname{div}\boldsymbol{j} - \theta\eta(\boldsymbol{F},\theta)\operatorname{div}\boldsymbol{v} + \big(\psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)\big)\operatorname{div}\boldsymbol{v}\\
&\overset{(2.1)}{=} \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top{:}\nabla\boldsymbol{v} + \xi - \operatorname{div}\boldsymbol{j} + \psi(\boldsymbol{F},\theta)\operatorname{div}\boldsymbol{v} \overset{(2.5)}{=} \xi - \operatorname{div}\boldsymbol{j} + \boldsymbol{T}{:}\nabla\boldsymbol{v}\,,
\end{aligned}
$$

we obtain the *internal-energy equation*

$$
\begin{aligned}
&\frac{\partial e}{\partial t} + \operatorname{div}\big(\boldsymbol{v}e{+}\boldsymbol{j}\big) = \nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p + \boldsymbol{T}{:}\nabla\boldsymbol{v}\\
&\qquad\text{with}\ \ e = E(\boldsymbol{F},\theta) := \psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)\,.
\end{aligned}
\tag{2.10}
$$

Altogether, let us formulate the thermo-visco-elastodynamic system for the quadruple $(\varrho,\boldsymbol{v},\boldsymbol{F},\theta)$, composed from the equations (2.1), (2.2), (2.7), and (2.10):

$$
\dot\varrho = -(\operatorname{div}\boldsymbol{v})\varrho\,,
\tag{2.11a}
$$

$$
\begin{aligned}
\varrho\dot{\boldsymbol{v}} = \operatorname{div}\big(\boldsymbol{T}(\boldsymbol{F},\theta){+}\boldsymbol{D}\big) + \varrho\boldsymbol{g} \quad &\text{where}\quad \boldsymbol{T}(\boldsymbol{F},\theta) = \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top{+}\,\psi(\boldsymbol{F},\theta)\mathbb{I}\\
&\text{and}\quad \boldsymbol{D} = \nu_1\boldsymbol{e}(\boldsymbol{v}) - \operatorname{div}\mathcal{H} \quad\text{with}\quad \mathcal{H} = \nu_2|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v}\,,
\end{aligned}
\tag{2.11b}
$$

$$
\dot{\boldsymbol{F}} = (\nabla\boldsymbol{v})\boldsymbol{F}\,,
\tag{2.11c}
$$

$$
\begin{aligned}
\dot{e} = \operatorname{div}\big(\kappa(\boldsymbol{F},\theta)\nabla\theta\big) + \nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p - (\operatorname{div}\boldsymbol{v})e + \boldsymbol{T}(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v}&\\
\text{with}\quad e = E(\boldsymbol{F},\theta)\,,\ \ \text{where}\ \ E(\boldsymbol{F},\theta) := \psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)\,.&
\end{aligned}
\tag{2.11d}
$$

Denoting by $\boldsymbol{n}$ the unit outward normal to the (fixed) boundary Γ of the domain Ω, we complete this system by suitable boundary conditions:

$$
\boldsymbol{v}{\cdot}\boldsymbol{n} = 0,\quad \big[(\boldsymbol{T}(\boldsymbol{F},\theta){+}\boldsymbol{D})\boldsymbol{n}{-}\operatorname{div}_{\mathrm{S}}\big(\mathcal{H}\boldsymbol{n}\big)\big]_{\mathrm{T}} = \boldsymbol{0}\,,\quad \nabla^2\boldsymbol{v}{:}(\boldsymbol{n}{\otimes}\boldsymbol{n}) = \boldsymbol{0}\,,\ \ \text{and}\ \ \kappa(\boldsymbol{F},\theta)\nabla\theta{\cdot}\boldsymbol{n} = h
\tag{2.12}
$$

with $[\,\cdot\,]_{\mathrm{T}}$ a tangential part of a vector. Here $\operatorname{div}_{\mathrm{S}} = \operatorname{tr}(\nabla_{\mathrm{S}})$ denotes the $(d{-}1)$-dimensional surface divergence with $\operatorname{tr}(\cdot)$ being the trace of a $(d{-}1){\times}(d{-}1)$-matrix and $\nabla_{\mathrm{S}}z = \nabla z{-}(\nabla z{\cdot}\boldsymbol{n})\boldsymbol{n}$ being the surface gradient of a field z. The first condition (i.e. normal velocity zero) expresses impenetrability of the boundary has already been used for (2.9) and is most frequently adopted in literature for the Eulerian formulation. This simplifying assumption fixes the shape of Ω in its referential configuration allows also for considering the fixed boundary even for such time-evolving Eulerian description. The other conditions in (2.12) model the free slip in the tangential direction in our multipolar variant in a variationally legitimate simplest way, cf. [32, Formula (2.24)]. The last condition in (2.12) prescribes the heat flux through the boundary.

The energetics behind the system is revealed when testing (2.11b) by $\boldsymbol{v}$ and (2.11d) by 1. To execute the former test, let us use (2.11a) tested by $\frac{1}{2}|\boldsymbol{v}|^2$ for $\frac{\partial}{\partial t}(\varrho|\boldsymbol{v}|^2/2) = \varrho\boldsymbol{v}{\cdot}\frac{\partial}{\partial t}\boldsymbol{v}\,-$

$\mathrm{div}(\varrho\boldsymbol{v})|\boldsymbol{v}|^2/2$ and realize that $\int_\Omega \varrho\dot{\boldsymbol{v}}{\cdot}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac12\varrho|\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \varrho|\boldsymbol{v}|^2\boldsymbol{v}{\cdot}\boldsymbol{n}\,\mathrm{d}S$. Thus, by using Green formula twice relying also on (2.12), we obtain the *mechanical-energy dissipation balance*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\ \text{energy}}}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \underbrace{\nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p}_{\text{dissipation rate}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \underbrace{\varrho\boldsymbol{g}{\cdot}\boldsymbol{v}}_{\substack{\text{power of}\\ \text{gravity}}} - \underbrace{\boldsymbol{T}(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v}}_{\substack{\text{power of}\\ \text{conservative}\\ \text{stress}}}\,\mathrm{d}\boldsymbol{x}\,. \tag{2.13}$$

When added (2.11d) tested by 1, we obtain the *total-energy balance*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{E(\boldsymbol{F},\theta)}_{\substack{\text{internal}\\ \text{energy}}}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \underbrace{\varrho\boldsymbol{g}{\cdot}\boldsymbol{v}}_{\substack{\text{power of}\\ \text{gravity field}}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \underbrace{h}_{\substack{\text{external}\\ \text{heat}\\ \text{flux}}}\,\mathrm{d}S\,, \tag{2.14}$$

expressing the *1st law of thermodynamics*: in isolated systems, the total energy is conserved.

Furthermore, for the test of (2.11d) by the so-called coldness $1/\theta$, we need to assume positivity of θ and then use the calculus

$$\begin{aligned}
\frac{1}{\theta}\Big(\frac{\partial e}{\partial t} + \mathrm{div}(e\boldsymbol{v})\Big) &= \frac{1}{\theta}\,\overline{E(\boldsymbol{F},\theta)}^{\boldsymbol{\cdot}} + \frac{E(\boldsymbol{F},\theta)}{\theta}\mathrm{div}\,\boldsymbol{v}\\
&= \frac{E'_\theta(\boldsymbol{F},\theta)}{\theta}\dot\theta + \frac{E'_{\boldsymbol{F}}(\boldsymbol{F},\theta)}{\theta}{:}\dot{\boldsymbol{F}} + \frac{E(\boldsymbol{F},\theta)}{\theta}\mathrm{div}\,\boldsymbol{v}\\
&= \overline{\eta(\boldsymbol{F},\theta)}^{\boldsymbol{\cdot}} + \Big(\frac{E'_{\boldsymbol{F}}(\boldsymbol{F},\theta)}{\theta} - \eta'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\Big){:}\dot{\boldsymbol{F}} + \frac{E(\boldsymbol{F},\theta)}{\theta}\mathrm{div}\,\boldsymbol{v}\\
&= \overline{\eta(\boldsymbol{F},\theta)}^{\boldsymbol{\cdot}} + \eta(\boldsymbol{F},\theta)\mathrm{div}\,\boldsymbol{v} + \frac{\psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)}{\theta}{:}\dot{\boldsymbol{F}} + \frac{\psi(\boldsymbol{F},\theta)}{\theta}\mathrm{div}\,\boldsymbol{v}\\
&= \frac{\partial}{\partial t}\eta(\boldsymbol{F},\theta) + \mathrm{div}(\eta(\boldsymbol{F},\theta)\boldsymbol{v}) + \underbrace{\frac{\psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)}{\theta}{:}\dot{\boldsymbol{F}} + \frac{\psi(\boldsymbol{F},\theta)}{\theta}\mathrm{div}\,\boldsymbol{v}}_{=\boldsymbol{T}(\boldsymbol{F},\theta){:}\boldsymbol{e}(\boldsymbol{v})/\theta}\,.
\end{aligned} \tag{2.15}$$

Here we also used $\eta'_\theta(\boldsymbol{F},\theta) = E'_\theta(\boldsymbol{F},\theta)/\theta$ and $\psi(\boldsymbol{F},\theta)/\theta - E(\boldsymbol{F},\theta)/\theta = \psi'_\theta(\boldsymbol{F},\theta) = -\eta(\boldsymbol{F},\theta)$. Therefore, by the mentioned test of (2.11d), we obtain the *entropy balance*:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\boldsymbol{F},\theta)\,\mathrm{d}\boldsymbol{x}}_{\substack{\text{total}\\ \text{entropy}}} = \int_\Omega \underbrace{\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p}{\theta}}_{\substack{\text{entropy production due to}\\ \text{mechanical viscosity}}} + \underbrace{\kappa(\boldsymbol{F},\theta)\frac{|\nabla\theta|^2}{\theta^2}}_{\substack{\text{entropy produ-}\\ \text{ction due to}\\ \text{heat transfer}}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \underbrace{\frac{h}{\theta}}_{\substack{\text{entropy flux}\\ \text{through}\\ \text{boundary}}}\,\mathrm{d}S\,, \tag{2.16}$$

expressing the *2nd law of thermodynamics*: in isolated systems, the total entropy is not decaying.

Eliminating $e = E(\boldsymbol{F},\theta) = \psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)$ from (2.11d) by using the calculus

$$\begin{aligned}
\overline{E(\boldsymbol{F},\theta)}^{\boldsymbol{\cdot}} &= \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta){:}\dot{\boldsymbol{F}} + \psi'_\theta(\boldsymbol{F},\theta)\dot\theta - \dot\theta\psi'_\theta(\boldsymbol{F},\theta) - \theta\psi''_{\boldsymbol{F}\theta}(\boldsymbol{F},\theta)\dot{\boldsymbol{F}} - \theta\psi''_{\theta\theta}(\boldsymbol{F},\theta)\dot\theta\\
&\overset{(2.11c)}{=} \boldsymbol{T}(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v} - \theta\boldsymbol{T}'_\theta(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v} - e\,\mathrm{div}\,\boldsymbol{v}\,,
\end{aligned}$$

we obtain the *heat-transfer equation* for temperature

$$c(\boldsymbol{F},\theta)\dot\theta = \mathrm{div}\big(\kappa(\boldsymbol{F},\theta)\nabla\theta\big) + \nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p + \theta\big(\psi''_{\boldsymbol{F}\theta}(\boldsymbol{F},\theta)\boldsymbol{F}^\top + \psi'_\theta(\boldsymbol{F},\theta)\mathbb{I}\big){:}\nabla\boldsymbol{v}$$

$$\text{with}\quad c(\boldsymbol{F},\theta)=-\theta\psi''_{\theta\theta}(\boldsymbol{F},\theta)\,, \tag{2.17}$$

where $c(\boldsymbol{F},\theta)$ is in the position of heat capacity. The heat equation (2.17) involves explicitly $\psi''_{\boldsymbol{F}\theta}$ and $\psi''_{\theta\theta}$ and thus needs a more smooth ψ than (2.11d).

Remark 2.1 (*Selecting the mere stored temperature-independent energy out*)**.** Sometimes, it is useful to split the free energy as $\psi(\boldsymbol{F},\theta)=\phi(\boldsymbol{F})+\varphi(\boldsymbol{F},\theta)$, where ϕ plays the role of a stored energy. Such a splitting allows for a calibration $\varphi(\boldsymbol{F},0)=0$, which actually determines this splitting uniquely since then $\phi(\boldsymbol{F})=\psi(\boldsymbol{F},0)$ and will be used in Steps 3 and 4 in the proof of Theorem 3.2 below. This also implies the splitting of the Cauchy stress as

$$\boldsymbol{T}(\boldsymbol{F},\theta)=\boldsymbol{T}_0(\boldsymbol{F})+\boldsymbol{T}_1(\boldsymbol{F},\theta)\quad\text{ with }\quad \boldsymbol{T}_0(\boldsymbol{F}):=\phi'(\boldsymbol{F})\boldsymbol{F}^\top+\phi(\boldsymbol{F})\mathbb{I}=\boldsymbol{T}(\boldsymbol{F},0)\,, \tag{2.18}$$

which allows for usage of the calculus

$$\begin{aligned}\boldsymbol{T}_0(\boldsymbol{F}){:}\nabla\boldsymbol{v}&=\phi'(\boldsymbol{F})\boldsymbol{F}^\top{:}\nabla\boldsymbol{v}+\phi(\boldsymbol{F})\mathrm{div}\,\boldsymbol{v}=\phi'(\boldsymbol{F}){:}((\nabla\boldsymbol{v})\boldsymbol{F})+\phi(\boldsymbol{F})\mathrm{div}\,\boldsymbol{v}\\&\overset{(2.11c)}{=}\phi'(\boldsymbol{F}){:}\dot{\boldsymbol{F}}+\phi(\boldsymbol{F})\mathrm{div}\,\boldsymbol{v}=\overset{\bullet}{\overline{\phi(\boldsymbol{F})}}+\phi(\boldsymbol{F})\mathrm{div}\,\boldsymbol{v}\,.\end{aligned} \tag{2.19}$$

This further allows for the calculus

$$\begin{aligned}\int_\varOmega\boldsymbol{T}_0(\boldsymbol{F}){:}\nabla\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}&=\frac{\mathrm{d}}{\mathrm{d}t}\int_\varOmega\phi(\boldsymbol{F})\,\mathrm{d}\boldsymbol{x}+\int_\varOmega\nabla\phi(\boldsymbol{F}){\cdot}\boldsymbol{v}+\phi(\boldsymbol{F})\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}=\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_\varOmega\phi(\boldsymbol{F})\,\mathrm{d}\boldsymbol{x}+\int_\varOmega\mathrm{div}\big(\phi(\boldsymbol{F})\boldsymbol{v}\big)\,\mathrm{d}\boldsymbol{x}=\frac{\mathrm{d}}{\mathrm{d}t}\int_\varOmega\phi(\boldsymbol{F})\,\mathrm{d}\boldsymbol{x}+\int_\varGamma\phi(\boldsymbol{F})\boldsymbol{v}{\cdot}\boldsymbol{n}\,\mathrm{d}S\,,\end{aligned}$$

where the last term vanishes due to the boundary condition $\boldsymbol{v}{\cdot}\boldsymbol{n}=0$ on $\varGamma$. By this way, one can see a more detailed mechanical-energy dissipation balance than (2.13), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\varOmega\underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\\text{energy}}}+\underbrace{\phi(\boldsymbol{F})}_{\substack{\text{stored}\\\text{energy}}}\,\mathrm{d}\boldsymbol{x}+\int_\varOmega\underbrace{\nu_1|\boldsymbol{e}(\boldsymbol{v})|^2+\nu_2|\nabla^2\boldsymbol{v}|^p}_{\text{dissipation rate}}\,\mathrm{d}\boldsymbol{x}=\int_\varOmega\underbrace{\varrho\boldsymbol{g}{\cdot}\boldsymbol{v}}_{\substack{\text{power of}\\\text{gravity}}}-\underbrace{\boldsymbol{T}_1(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v}}_{\substack{\text{power of}\\\text{adiabatic}\\\text{effects}}}\,\mathrm{d}\boldsymbol{x}\,. \tag{2.20}$$

Also the internal energy E used in the total energy balance (2.14) then sees directly the stored energy, since obviously $E(\boldsymbol{F},\theta)=\phi(\boldsymbol{F})+\varphi(\boldsymbol{F},\theta)-\theta\varphi'_\theta(\boldsymbol{F},\theta)$ and the mentioned calibration $\varphi(\boldsymbol{F},0)=0$ yields that simply $\phi(\boldsymbol{F})=E(\boldsymbol{F},0)=\psi(\boldsymbol{F},0)$. Noteworthy, using (2.19) for the internal-energy equation (2.11d) eliminates the mere stored energy and yields some *heat-type equation* in an enthalpy-like formulation

$$\begin{aligned}\dot{u}&=\mathrm{div}\big(\kappa(\boldsymbol{F},\theta)\nabla\theta\big)+\nu_1|\boldsymbol{e}(\boldsymbol{v})|^2+\nu_2|\nabla^2\boldsymbol{v}|^p-(\mathrm{div}\,\boldsymbol{v})u+\boldsymbol{T}_1(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v}\\&\quad\text{with}\quad u=U(\boldsymbol{F},\theta)\,,\ \text{ where }\ U(\boldsymbol{F},\theta):=\psi(\boldsymbol{F},\theta)-\theta\psi'_\theta(\boldsymbol{F},\theta)-\psi(\boldsymbol{F},0)\,.\end{aligned} \tag{2.21}$$

Here $\boldsymbol{T}_1(\boldsymbol{F},\theta)=\boldsymbol{T}(\boldsymbol{F},\theta)-\boldsymbol{T}(\boldsymbol{F},0)$ is from (2.18). Note that $U(\boldsymbol{F},0)=0$ and $\boldsymbol{T}_1(\boldsymbol{F},0)=\boldsymbol{0}$.

Remark 2.2 (*Thermoelasticity in an "engineering" formulation*)**.** The system (2.11) is well prepared for analysis but may not be found "engineering friendly". There are two aspects: the free energy is more often considered referential rather than actual (let us distinguish it by $\uppsi$) and the heat equation is formulated in temperature rather than internal energy as in (2.11d). The free energy by reference volume is more standard in continuum physics [13,19] than the free energy per actual evolving volume, as it corresponds more directly to experimentally available data. The relation in between ψ and ψ is $\psi(\boldsymbol{F},\theta) = \psi(\boldsymbol{F},\theta)/\det\boldsymbol{F}$ and then the conservative part of the (actual) Cauchy stress, the actual entropy, and the actual heat capacity are respectively

$$\boldsymbol{T}(\boldsymbol{F},\theta) = \frac{\psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top}{\det\boldsymbol{F}}, \qquad \eta(\boldsymbol{F},\theta) = -\frac{\psi'_\theta(\boldsymbol{F},\theta)}{\det\boldsymbol{F}}, \quad \text{and} \quad c(\boldsymbol{F},\theta) = -\theta\frac{\psi''_{\theta\theta}(\boldsymbol{F},\theta)}{\det\boldsymbol{F}}. \tag{2.22}$$

The resulting system then consists from (2.11a-c) but with $\boldsymbol{T}$ from (2.22). Moreover, (2.11d) in the form of the heat equation like (2.17) reads as

$$c(\boldsymbol{F},\theta)\dot{\theta} = \mathrm{div}\big(\kappa(\boldsymbol{F},\theta)\nabla\theta\big) + \nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p + \theta\frac{\psi''_{\boldsymbol{F}\theta}(\boldsymbol{F},\theta)\boldsymbol{F}^\top}{\det\boldsymbol{F}}{:}\nabla\boldsymbol{v}\,. \tag{2.23}$$

Actually, the split from Remark (2.1) allows for a bit specification of the last term which then does not see the mere stored energy contribution to the adiabatic power; in particular, the last term in (2.23) can be written as $\theta[\boldsymbol{T}_1]'_\theta(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v}$. Sometimes, in literature the factor $1/\det\boldsymbol{F}$ is replaced by ϱ if the referential free energy ψ is the physical units not in Pa=J/m^3 but in J/kg.

3 Analytical results: existence of weak solutions

To highlight the nontriviality of the coupled system (2.11), it is perhaps useful to give an overview of various strategies in the proof that can or should to be considered. While the momentum equation is to be ultimately tested by the velocity $\boldsymbol{v}$, the heat equation can be subjected to various tests:

(i) by 1 to obtain energy balance,

(ii) $1/\theta$ to obtain entropy balance (not directly exploited in this paper, see Remark 3.6),

(iii) θ^- to see non-negativity of temperature (not used in this paper),

(iv) θ to see an estimate for $\nabla\theta$ if the heat sources would be regularized, and

(v) $1 - 1/(1{+}\theta)^a$ for $a > 0$ to estimate $\nabla\theta$ for the physically relevant L^1-heat sources.

In (iii), we have used the notation (decomposition)

$$\theta = \theta^+ - \theta^- \qquad \text{with} \quad \theta^+ := \max(0,\theta)\,. \tag{3.1}$$

The construction of an approximate solution by time-discretization is very problematic because the free energy $\psi(\cdot,\theta)$ is highly nonconvex. Also the Galerkin discretization is nontrivial even if applied only to the momentum and the heat equations while keeping the transport equations continuous, as used below. The nontriviality of this scenario consists in the mentioned tests of the heat equation: only the tests (i) and (iv) are legitimate in conformal Galerkin approximations and, for (i), the internal energy is to be bounded from below while, for (iv), the heat capacity should be granted non-negative for negative temperatures. This is slightly contradictory, since it actually requires that the internal energy stays constant and the heat capacity simply vanishes for negative temperatures. This is the scenario which will be exploited below, cf. the illustration in Figure 1, and gives the non-negativity of temperature as a side effect without using the test (iii).

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely $L^p(\varOmega;\mathbb{R}^n)$ for Lebesgue measurable functions $\varOmega \to \mathbb{R}^n$ whose Euclidean norm is integrable with

p-power, and $W^{k,p}(\Omega;\mathbb{R}^n)$ for functions from $L^p(\Omega;\mathbb{R}^n)$ whose all derivatives up to the order k have their Euclidean norm integrable with p-power. We also write briefly $H^k = W^{k,2}$. The notation p' will denote the conjugate exponent $p/(p{-}1)$ while p^* will denote the exponent from the embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, i.e. $p^* = dp/(d{-}p)$ for $p < d$ while $p^* \geq 1$ is arbitrary for $p = d$ or $p^* = +\infty$ for $p > d$. Furthermore, for a Banach space X and for $I = [0,T]$, we will use the notation $L^p(I;X)$ for the Bochner space of Bochner measurable functions $I \to X$ whose norm is in $L^p(I)$ while $W^{1,p}(I;X)$ denotes for functions $I \to X$ whose distributional derivative is in $L^p(I;X)$. Also, $C(\cdot)$ and $C^1(\cdot)$ will denote the spaces of continuous and continuously differentiable functions, respectively, and $\overline{\Omega}$ will denote the closure of Ω. Eventually, the spaces of weakly continuous functions $I \to X$ is denoted by $C_{\rm w}(I;X)$.

Moreover, as usual, we will use C for a generic constant which may vary from estimate to estimate.

We will consider an initial-value problem, prescribing the initial conditions

$$\varrho|_{t=0} = \varrho_0 := \frac{\varrho_{\rm R}}{\det \boldsymbol{F}_0}\,, \qquad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0\,, \qquad \boldsymbol{F}|_{t=0} = \boldsymbol{F}_0\,, \quad \text{and} \quad \theta|_{t=0} = \theta_0\,. \tag{3.2}$$

To devise a weak formulation of the initial-boundary-value problem (2.12) and (3.2) for the system (2.11), we use the by-part integration in time and the Green twice formula for (2.11b) multiplied by a smooth test function $\widetilde{\boldsymbol{v}}$ and once for (2.11d) multiplied by a smooth test function $\widetilde{\theta}$, while (2.12a,c) are considered just a.e. on $I{\times}\Omega$:

Definition 3.1 (Weak solutions to (2.11))**.** *For $p,q \in [1,\infty)$, a quadruple $(\varrho,\boldsymbol{v},\boldsymbol{F},\theta)$ with $\varrho \in L^\infty(I{\times}\Omega){\cap}W^{1,1}(I{\times}\Omega)$, $\boldsymbol{v} \in L^2(I;H^1(\Omega;\mathbb{R}^d)) \cap L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))$, $\boldsymbol{F} \in L^\infty(I{\times}\Omega;\mathbb{R}^{d\times d}){\cap}W^{1,1}(I{\times}\Omega;\mathbb{R}^{d\times d})$, and $\theta \in L^1(I;W^{1,1}(\Omega))$ will be called a weak solution to the system (2.11) with the boundary conditions (2.12) and the initial condition (3.2) if the integral identities*

$$\begin{aligned}\int_0^T\!\!\int_\Omega \bigg(\Big(\psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top{+}\,\nu_1\boldsymbol{e}(\boldsymbol{v}) - \varrho\boldsymbol{v}{\otimes}\boldsymbol{v}\Big){:}\boldsymbol{e}(\widetilde{\boldsymbol{v}}) + \psi(\boldsymbol{F},\theta){\rm div}\,\widetilde{\boldsymbol{v}}\\ + \nu_2|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v}\,{\vdots}\nabla^2\widetilde{\boldsymbol{v}} - \varrho\boldsymbol{v}{\cdot}\frac{\partial\widetilde{\boldsymbol{v}}}{\partial t}\bigg)\,{\rm d}\boldsymbol{x}{\rm d}t = \int_0^T\!\!\int_\Omega \varrho\boldsymbol{g}{\cdot}\widetilde{\boldsymbol{v}}\,{\rm d}\boldsymbol{x}{\rm d}t + \int_\Omega \varrho_0\boldsymbol{v}_0{\cdot}\widetilde{\boldsymbol{v}}(0)\,{\rm d}\boldsymbol{x}\end{aligned} \tag{3.3a}$$

$$\begin{aligned}\int_0^T\!\!\int_\Omega\bigg(E(\boldsymbol{F},\theta)\frac{\partial\widetilde{\theta}}{\partial t} + \Big(E(\boldsymbol{F},\theta)\boldsymbol{v}{+}\kappa(\boldsymbol{F},\theta)\nabla\theta\Big){\cdot}\nabla\widetilde{\theta} + \Big(\nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p\\ + \boldsymbol{T}(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v}\Big)\widetilde{\theta}\bigg){\rm d}\boldsymbol{x}{\rm d}t + \int_0^T\!\!\int_\Gamma h\widetilde{\theta}\,{\rm d}S{\rm d}t + \int_\Omega E(\boldsymbol{F}_0,\theta_0)\widetilde{\theta}(0)\,{\rm d}\boldsymbol{x} = 0\end{aligned} \tag{3.3b}$$

with $\boldsymbol{T}(\cdot,\cdot)$ from (2.11b) and $E(\cdot,\cdot)$ from (2.11d) hold for any $\widetilde{\boldsymbol{v}}$ and smooth with $\widetilde{\boldsymbol{v}}{\cdot}\boldsymbol{n} = \boldsymbol{0}$ and $\widetilde{\boldsymbol{v}}(T) = 0$ and for any $\widetilde{\theta}$ smooth with $\widetilde{\theta}(T) = 0$, and eventually if also (2.11a) and (2.11c) hold a.e. on $I{\times}\Omega$ together with the initial conditions for ϱ and $\boldsymbol{F}$ in (3.2).

Before stating the main analytical result, let us summarize the data qualification. We use the notation $\mathbb{R}^+ := [0,+\infty)$. For some $\delta > 0$, and some $1 < q < \infty$ and $d < p < \infty$, recalling the notation

$$E(\boldsymbol{F},\theta) = \psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta) \quad \text{and} \quad \boldsymbol{T}(\boldsymbol{F},\theta) = \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top{+}\,\psi(\boldsymbol{F},\theta)\mathbb{I}\,, \tag{3.4}$$

and denoting by ${\rm GL}^+(d) = \{F \in \mathbb{R}^{d\times d};\ \det F > 0\}$ the orientation-preserving general linear group, we assume:

$$\Omega \ \text{ a smooth bounded domain of } \mathbb{R}^d,\ \ d = 2,3, \tag{3.5a}$$

$$\psi \in C^2(\mathrm{GL}^+(d) \times \mathbb{R}), \quad \kappa \in C(\mathrm{GL}^+(d)\times\mathbb{R}), \tag{3.5b}$$
$\exists C < +\infty \ \ \forall(\boldsymbol{F},\theta) \in \mathrm{GL}^+(d)\times\mathbb{R}:$
$$\psi(\boldsymbol{F},\theta^-) = E(\boldsymbol{F},0) \quad \text{and} \quad \kappa(\boldsymbol{F},\theta^-) = \kappa(\boldsymbol{F},0), \tag{3.5c}$$
$$|\boldsymbol{T}(\boldsymbol{F},\theta)| \le C\big(1+E(\boldsymbol{F},\theta)\big), \tag{3.5d}$$
$\forall K\subset \mathrm{GL}^+(d)$ compact $\ \exists\, 0 < c_K \le C_K < +\infty \ \ \exists\, 0 \le \alpha < \begin{cases} 1 & \text{for } d=2 \\ 1/2 & \text{for } d=3 \end{cases} \ \ \forall(\boldsymbol{F},\theta)\in K\times\mathbb{R}$:
$$E(\boldsymbol{F},\theta) \ge c_K(\theta^+)^{1+\alpha} \quad \text{and} \quad E_\theta'(\boldsymbol{F},\cdot) > 0 \text{ on } (0,+\infty), \tag{3.5e}$$
$$|E_{\boldsymbol{F}}'(\boldsymbol{F},\theta)| + \theta^+ E_\theta'(\boldsymbol{F},\theta) \le C_K\big(1+(\theta^+)^{1+\alpha}\big) \ \text{ and } \ |E_{\boldsymbol{F}\theta}''(\boldsymbol{F},\theta)| \le C_K\big(1+(\theta^+)^{\alpha}\big), \tag{3.5f}$$
$$c_K \le \kappa(\boldsymbol{F},\theta) \le C_K, \quad \nu_1 > 0, \ \nu_2 > 0, \tag{3.5g}$$
$$\boldsymbol{v}_0 \in L^2(\varOmega;\mathbb{R}^d), \qquad \boldsymbol{F}_0 \in W^{1,\infty}(\varOmega;\mathbb{R}^{d\times d}) \quad \text{with} \quad \min\nolimits_{\overline{\varOmega}} \det \boldsymbol{F}_0 > 0, \tag{3.5h}$$
$$\boldsymbol{g} \in L^1(I;L^\infty(\varOmega;\mathbb{R}^d)), \quad \varrho_{\rm R} \in W^{1,\infty}(\varOmega) \quad \text{with} \quad \min\nolimits_{\overline{\varOmega}}\varrho_{\rm R} > 0, \tag{3.5i}$$
$$h\in L^1(I\times\varGamma), \quad h \ge 0, \quad \theta_0 \in L^1(\varOmega), \quad \theta_0 \ge 0 \ \text{ a.e. on } \ \varOmega, \quad E(\boldsymbol{F}_0,\theta_0) \in L^1(\varOmega). \tag{3.5j}$$

Let us note that we have formally defined the data also for negative temperatures, which will be useful in the proof below. Noting $E_\theta'(\boldsymbol{F},\theta) = -\theta\psi_{\theta\theta}''(\boldsymbol{F},\theta)$, the latter condition in (3.5e) implies $\psi(\boldsymbol{F},\cdot)$ concave on $\mathbb{R}^+$. The relevance of the possibility of controlling the stress via the energy, which is what (3.5d) does, was pointed out by J.M. Ball [1,2] for the Kirchhoff rather than the Cauchy stress, cf. Remark 4.2 below. Actually, (3.5b) allows also for $\sup_{\boldsymbol{F}\in K}|\psi(\boldsymbol{F},0)| \le c_K$ for some C_K since $E(\cdot,0) = \psi(\cdot,0) \in C(\mathrm{GL}^+(d))$, and then (3.5f) gives, for K and α as in (3.5e–g), a bound for the internal energy E as

$$\forall(\boldsymbol{F},\theta) \in K\times\mathbb{R}: \quad E(\boldsymbol{F},\theta) = \psi(\boldsymbol{F},0) + \int_0^\theta E_\theta'(\boldsymbol{F},\widetilde\theta)\,\mathrm{d}\widetilde\theta \le C_K\big(1+(\theta^+)^{1+\alpha}\big) \tag{3.6}$$

so that the energy-controlled-stress condition (3.5d) gives

$$\forall(\boldsymbol{F},\theta) \in K\times\mathbb{R}: \quad |\boldsymbol{T}(\boldsymbol{F},\theta)| \le C_K\big(1+(\theta^+)^{1+\alpha}\big). \tag{3.7}$$

Moreover, the expected *symmetry* of such Cauchy stress $\boldsymbol{T}$ is granted by *frame indifference* of $\psi(\cdot,\theta)$. This means that

$$\forall(\boldsymbol{F},\theta) \in \mathrm{GL}^+(d)\times\mathbb{R}, \ \ Q \in \mathrm{SO}(d): \qquad \psi(\boldsymbol{F},\theta) = \psi(Q\boldsymbol{F},\theta), \tag{3.8}$$

where $Q \in \mathrm{SO}(d) = \{Q \in \mathbb{R}^{d\times d};\ Q^\top Q = QQ^\top = \mathbb{I}\}$ is the special orthogonal group.

Theorem 3.2 (Existence and regularity of weak solutions)**.** *Let $p > d$ and the assumptions (3.5) and (3.8) hold. Then there exists at least one weak solution $(\varrho,\boldsymbol{v},\boldsymbol{F},\theta)$ according Definition 3.1 such that, in addition, $\varrho \in C_{\rm w}(I;W^{1,r}(\varOmega))$ and $\boldsymbol{F} \in C_{\rm w}(I;W^{1,r}(\varOmega;\mathbb{R}^{d\times d}))$ for any $1 \le r < \infty$, and further $\theta \in C_{\rm w}(I;L^{1+\alpha}(\varOmega)) \cap L^\mu(I;W^{1,\mu}(\varOmega))$ with $1 \le \mu < (d{+}2{+}(2{-}d)\alpha)/(d{+}1{+}\alpha)$. Moreover, $\frac{\partial}{\partial t}(\varrho\boldsymbol{v}) \in L^2(I;H^1(\varOmega;\mathbb{R}^d)^*) + L^{p'}(I;W^{2,p}(\varOmega;\mathbb{R}^d)^*)$ and $\inf_{I\times\varOmega}\varrho > 0$, $\inf_{I\times\varOmega}\det\boldsymbol{F} > 0$, and $\theta \ge 0$ a.e. on $I\times\varOmega$.*

Proof. For clarity, we will divide the proof into six steps.

Step 1: Regularization of the system and a Galerkin semi-discretization. For $\varepsilon > 0$ and $k \in \mathbb{N}$, we consider a regularization of the momentum equation (2.11b) and of the internal-energy equation (2.11d). Altogether, the system (2.11) is regularized as

$$\dot\varrho = -(\operatorname{div}\boldsymbol{v})\varrho, \tag{3.9a}$$

$$\varrho\dot{\boldsymbol{v}} = \mathrm{div}\big(\boldsymbol{T}(\boldsymbol{F},\theta){+}\boldsymbol{D}\big) + \varrho\boldsymbol{g} - \frac{1}{k}\boldsymbol{v} \quad \text{where } \ \boldsymbol{T}(\boldsymbol{F},\theta) = \psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top{+}\,\psi(\boldsymbol{F},\theta)\mathbb{I}$$
$$\text{and} \quad \boldsymbol{D} = \nu_1\boldsymbol{e}(\boldsymbol{v}) - \mathrm{div}\big(\nu_2|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v}\big)\,, \tag{3.9b}$$
$$\dot{\boldsymbol{F}} = (\nabla\boldsymbol{v})\boldsymbol{F}\,, \tag{3.9c}$$
$$\dot{e} = \mathrm{div}\big(\kappa(\boldsymbol{F},\theta)\nabla\theta\big) - (\mathrm{div}\,\boldsymbol{v})e + \frac{\nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 + \nu_2|\nabla^2\boldsymbol{v}|^p}{1{+}\varepsilon|\boldsymbol{e}(\boldsymbol{v})|^2{+}\varepsilon|\nabla^2\boldsymbol{v}|^p} + \boldsymbol{T}(\boldsymbol{F},\theta){:}\nabla\boldsymbol{v} - \frac{1}{k}\frac{\partial\theta}{\partial t}$$
$$\text{with} \quad e = \psi(\boldsymbol{F},\theta) - \theta\psi'_\theta(\boldsymbol{F},\theta)\,. \tag{3.9d}$$

The equations (3.9b,d) are completed by the boundary conditions (2.12) but with the last initial condition regularized, specifically

$$\kappa(\boldsymbol{F},\theta)\nabla\theta{\cdot}\boldsymbol{n} = h_\varepsilon \quad \text{ with } \quad h_\varepsilon := \frac{h}{1{+}\varepsilon h} \tag{3.10}$$

and similarly the initial conditions (3.2) are considered with a regularized temperature, namely

$$\varrho|_{t=0} = \varrho_0\,, \qquad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0\,, \qquad \boldsymbol{F}|_{t=0} = \boldsymbol{F}_0\,, \quad \text{and} \quad \theta|_{t=0} = \theta_{0\varepsilon} := \frac{\theta_0}{1{+}\varepsilon\theta_0}\,. \tag{3.11}$$

The purpose of the regularization $\boldsymbol{v}/k$ in (3.9b) is to make values of $\boldsymbol{v}$ controlled still before we establish information about the sparsity $1/\varrho$ even before we will prove a uniform estimate (3.23c) below, cf. the proof in [32, Lemma 5.1], while the ε-regularization in (3.9d) and in (3.10) is to allow its testing by θ. The regularizing term $\frac{\partial}{\partial t}\theta/k$ in (3.9d) will ensure the existence of Galerkin solution by the usual successive-prolongation argument in Step 3 below.

We use a spatial semi-discretization, keeping the transport equations (3.9a) and (3.9c) continuous (i.e. non-discretised). More specifically, we make a conformal Galerkin approximation of (3.9b) by using a collection of nested finite-dimensional subspaces $\{V_k\}_{k=0,1,...}$ whose union is dense in $W^{2,p}(\varOmega;\mathbb{R}^d)$ and a conformal Galerkin approximation of (3.9d) by using a collection of nested finite-dimensional subspaces $\{Z_k\}_{k=0,1,...}$ whose union is dense in $H^1(\varOmega)$. Note that we use the index k occuring also in (3.9b) and in (3.9d).

Let us denote the solution of thus approximated regularized system (3.9) with the initial/boundary conditions (3.11) and (2.12) regularized as (3.10) by $(\varrho_{\varepsilon k},\boldsymbol{v}_{\varepsilon k},\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}) : I \to W^{1,r}(\varOmega)\times V_k\times W^{1,r}(\varOmega;\mathbb{R}^{d\times d})\times Z_k$. Without loss of generality, we can assume $\boldsymbol{v}_0\in V_0$ and $\theta_{0\varepsilon}\in Z_0$. Specifically, $\varrho_{\varepsilon k}\in C_{\rm w}(I;W^{1,r}(\varOmega))\cap W^{1,1}(I;L^1(\varOmega))$ and $\boldsymbol{F}_{\varepsilon k}\in C_{\rm w}(I;W^{1,r}(\varOmega;\mathbb{R}^{d\times d}))\cap W^{1,1}(I;L^1(\varOmega;\mathbb{R}^{d\times d}))$ should satisfy

$$\frac{\partial\varrho_{\varepsilon k}}{\partial t} = -\mathrm{div}(\varrho_{\varepsilon k}\boldsymbol{v}_{\varepsilon k}) \quad \text{ and } \quad \frac{\partial\boldsymbol{F}_{\varepsilon k}}{\partial t} = (\nabla\boldsymbol{v}_{\varepsilon k})\boldsymbol{F}_{\varepsilon k} - (\boldsymbol{v}_{\varepsilon k}{\cdot}\nabla)\boldsymbol{F}_{\varepsilon k} \quad \text{a.e. on } I{\times}\varOmega \tag{3.12}$$

together with the following integral identities

$$\int_0^T\!\!\int_\varOmega\bigg(\Big(\psi'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\boldsymbol{F}_{\varepsilon k}^\top{+}\,\nu_1\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}) - \varrho_{\varepsilon k}\boldsymbol{v}_{\varepsilon k}{\otimes}\boldsymbol{v}_{\varepsilon k}\Big){:}\boldsymbol{e}(\widetilde{\boldsymbol{v}})$$
$$+\,\psi(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\mathrm{div}\widetilde{\boldsymbol{v}} - \varrho_{\varepsilon k}\boldsymbol{v}_{\varepsilon k}{\cdot}\frac{\partial\widetilde{\boldsymbol{v}}}{\partial t} + \nu_2|\nabla^2\boldsymbol{v}_{\varepsilon k}|^{p-2}\nabla^2\boldsymbol{v}_{\varepsilon k}{\vdots}\nabla^2\widetilde{\boldsymbol{v}} + \frac{\boldsymbol{v}_{\varepsilon k}{\cdot}\widetilde{\boldsymbol{v}}}{k}\bigg)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t$$
$$= \int_0^T\!\!\int_\varOmega\varrho_{\varepsilon k}\boldsymbol{g}{\cdot}\widetilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_\varOmega\varrho_0\boldsymbol{v}_0{\cdot}\widetilde{\boldsymbol{v}}(0)\,\mathrm{d}\boldsymbol{x} \tag{3.13a}$$

for any $\widetilde{\boldsymbol{v}}\in L^\infty(I;V_k)$ with $\widetilde{\boldsymbol{v}}{\cdot}\boldsymbol{n}=0$ on $I{\times}\varGamma$ and $\widetilde{\boldsymbol{v}}(T)=\boldsymbol{0}$, and

$$\int_0^T\!\!\int_\varOmega\bigg(\Big(e_{\varepsilon k}{+}\frac{\theta_{\varepsilon k}}{k}\Big)\frac{\partial\widetilde{\theta}}{\partial t} + \big(e_{\varepsilon k}\boldsymbol{v}_{\varepsilon k}{-}\kappa(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\nabla\theta_{\varepsilon k}\big){\cdot}\nabla\widetilde{\theta}$$

$$+\Big(\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2+\nu_2|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}{1{+}\varepsilon|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2{+}\varepsilon|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}+\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{v}_{\varepsilon k}\Big)\widetilde\theta\,\bigg)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t$$

$$+\int_\Omega E(\boldsymbol{F}_0,\theta_{0\varepsilon})\widetilde\theta(0)\,\mathrm{d}\boldsymbol{x}+\int_0^T\!\!\int_\Gamma h_\varepsilon\widetilde\theta\,\mathrm{d}S\mathrm{d}t=0\quad\text{with}\ \ e_{\varepsilon k}=E(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\tag{3.13b}$$

holding for any $\widetilde\theta\in C^1(I;Z_k)$ with $\widetilde\theta(T)=0$, considered for $k=1,2,....$

Existence of this solution is based on the standard theory of systems of ordinary differential equations first locally in time combined here with the abstract $W^{1,r}(\Omega)$-valued differential equations (3.12) based on [32, Sect.5] (or, in a bit less general form, also [30, 33]) and then by successive prolongation on the whole time interval based on the L^∞-estimates below in Step 3.

Step 2: first a-priori estimates. Actually, the fixing initial conditions $\boldsymbol{F}_0\in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d})$ and $\varrho_0\in W^{1,\infty}(\Omega)$ and $\boldsymbol{v}_{\varepsilon k}$ enough regular, the essential point is that the transport equations (3.12) have unique solutions. According [32, Sect.5], it thus defines the weakly-continuous nonlinear operators $\mathfrak{F}:I\times L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))\to W^{1,r}(\Omega;\mathbb{R}^{d\times d})$ and $\mathfrak{R}:I\times L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))\to W^{1,r}(\Omega)$ by

$$\boldsymbol{F}_{\varepsilon k}(t)=\mathfrak{F}\big(t,\boldsymbol{v}_{\varepsilon k}\big)\quad\text{and}\quad\varrho_{\varepsilon k}(t)=\mathfrak{R}\big(t,\boldsymbol{v}_{\varepsilon k}\big)\,.\tag{3.14}$$

Moreover, these nonlinear operators are bounded in the sense that, for any $1\le r<+\infty$, $\mathfrak{F}(\cdot,\boldsymbol{v}_{\varepsilon k})$ is bounded in $L^\infty(I;W^{1,r}(\Omega;\mathbb{R}^{d\times d}))\cap W^{1,1}(I;L^r(\Omega;\mathbb{R}^{d\times d}))$ and $\mathfrak{R}(\cdot,\boldsymbol{v}_{\varepsilon k})$ is bounded in $L^\infty(I;W^{1,r}(\Omega))\cap W^{1,1}(I;L^r(\Omega))$ when $\nabla\boldsymbol{v}_{\varepsilon k}$ ranges over a bounded set in $L^p(I;W^{1,p}(\Omega;\mathbb{R}^{d\times d}))$ provided also $\boldsymbol{v}_{\varepsilon k}\in L^1(I;W^{2,p}(\Omega;\mathbb{R}^d))$ with $\boldsymbol{v}_{\varepsilon k}{\cdot}\boldsymbol{n}=0$. Besides, the *mass conservation* holds:

$$\varrho_{\varepsilon k}\ge0\ \text{ on }\ I{\times}\Omega\quad\text{and}\quad\int_\Omega\varrho_{\varepsilon k}(t)\,\mathrm{d}\boldsymbol{x}=\int_\Omega\varrho_0\,\mathrm{d}\boldsymbol{x}\ \text{ for all }\ t\in I\,.\tag{3.15}$$

Testing (3.9b) discretized as (3.13a) by $\boldsymbol{v}_{\varepsilon k}$ gives the discrete mechanical-energy balance as an analog of (2.13):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\varrho_{\varepsilon k}}{2}|\boldsymbol{v}_{\varepsilon k}|^2\,\mathrm{d}\boldsymbol{x}+\int_\Omega\nu_1|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2+\nu_2|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p+\frac1k|\boldsymbol{v}_{\varepsilon k}|^2\,\mathrm{d}\boldsymbol{x}$$
$$=\int_\Omega\varrho_{\varepsilon k}\boldsymbol{g}{\cdot}\boldsymbol{v}_{\varepsilon k}-\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{v}_{\varepsilon k}\,\mathrm{d}\boldsymbol{x}\,.\tag{3.16}$$

Testing (3.9d) discretized as (3.13b) by 1 and summing it with (3.16), we obtain the variant of the total-energy balance (2.14) as an inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\varrho_{\varepsilon k}}{2}|\boldsymbol{v}_{\varepsilon k}|^2+E(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\,\mathrm{d}\boldsymbol{x}+\int_\Omega\frac1k|\boldsymbol{v}_{\varepsilon k}|^2\,\mathrm{d}\boldsymbol{x}\le\int_\Omega\varrho_{\varepsilon k}\boldsymbol{g}{\cdot}\boldsymbol{v}_{\varepsilon k}\,\mathrm{d}\boldsymbol{x}+\int_\Gamma h_\varepsilon\,\mathrm{d}S\,.\tag{3.17}$$

The inequality (instead of an equality) in (3.17) arises from the regularization of the dissipative heat in (3.9d).

Moreover, using the calculus for the Galerkin solution (2.19), the internal-energy equation (3.9d) can be modified not to see the mere stored energy to a heat equation as (2.21):

$$\overset{\bullet}{u}{}^{\varepsilon k}_{\varepsilon k}=\mathrm{div}\big(\kappa(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\nabla\theta_{\varepsilon k}\big)+\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2+\nu_2|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}{1{+}\varepsilon|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2{+}\varepsilon|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}$$
$$-(\mathrm{div}\,\boldsymbol{v}_{\varepsilon k})u_{\varepsilon k}+\boldsymbol{T}_1(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{v}_{\varepsilon k}-\frac1k\frac{\partial\theta_{\varepsilon k}}{\partial t}\qquad\text{with}\quad u_{\varepsilon k}=U(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\,,\tag{3.18}$$

where we denoted the convective time derivatives related to the velocity field $\boldsymbol{v}_{\varepsilon k}$ as $\overset{\bullet}{u}^{\varepsilon k} := \frac{\partial}{\partial t}u + \boldsymbol{v}_{\varepsilon k}\cdot\nabla u$ and where U is from (2.21) and $\boldsymbol{T}_1(\boldsymbol{F},\theta) = \boldsymbol{T}(\boldsymbol{F},\theta) - \boldsymbol{T}(\boldsymbol{F},0)$, cf. (2.18).

Relying on that E bounded from below due to the assumption (3.5d) which implies in particular that $E(\boldsymbol{F},\theta) \geq -1$, the inequality (3.17) can yield a first set of a-priori estimates. To this aim, we are to estimate the right-hand side in (3.17). The issue is estimation of the gravity force $\varrho\boldsymbol{g}$ when tested by the velocity $\boldsymbol{v}$. For any time instant t, this can be estimated by the Hölder/Young inequality as

$$\begin{aligned}\int_\varOmega \varrho_{\varepsilon k}(t)\boldsymbol{g}(t)\cdot\boldsymbol{v}_{\varepsilon k}(t)\,\mathrm{d}\boldsymbol{x} &\leq \big\|\sqrt{\varrho_{\varepsilon k}(t)}\big\|_{L^2(\varOmega)}\big\|\sqrt{\varrho_{\varepsilon k}(t)}\boldsymbol{v}_{\varepsilon k}(t)\big\|_{L^2(\varOmega;\mathbb{R}^d)}\big\|\boldsymbol{g}(t)\big\|_{L^\infty(\varOmega;\mathbb{R}^d)}\\ &\leq \frac{1}{2}\Big(\big\|\sqrt{\varrho_{\varepsilon k}(t)}\big\|_{L^2(\varOmega)}^2 + \big\|\sqrt{\varrho_{\varepsilon k}(t)}\boldsymbol{v}_{\varepsilon k}(t)\big\|_{L^2(\varOmega;\mathbb{R}^d)}^2\Big)\,\big\|\boldsymbol{g}(t)\big\|_{L^\infty(\varOmega;\mathbb{R}^d)}\\ &\overset{(3.15)}{=} \big\|\boldsymbol{g}(t)\big\|_{L^\infty(\varOmega;\mathbb{R}^d)}\int_\varOmega \frac{\varrho_0}{2} + \frac{\varrho_{\varepsilon k}(t)}{2}|\boldsymbol{v}_{\varepsilon k}(t)|^2\,\mathrm{d}\boldsymbol{x}\,. \end{aligned}\tag{3.19}$$

The integral on the right-hand side of (3.19) can then be treated by the Gronwall lemma, for which one needs the qualification of $\boldsymbol{g}$ in (3.5i). Here we further use also the L^1-qualification of h in (3.5j). As a result, from (3.17) we obtain the a-priori estimates

$$\big\|\sqrt{\varrho_{\varepsilon k}}\boldsymbol{v}_{\varepsilon k}\big\|_{L^\infty(I;L^2(\varOmega;\mathbb{R}^d))} \leq C \quad\text{ and }\quad \big\|\boldsymbol{v}_{\varepsilon k}\big\|_{L^2(I\times\varOmega;\mathbb{R}^d))} \leq C\sqrt{k}\,, \tag{3.20a}$$

$$\big\|E(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\big\|_{L^\infty(I;L^1(\varOmega))} \leq C\,. \tag{3.20b}$$

Using the energy-controlled-stress condition (3.5d), the estimate (3.20b) implies also

$$\big\|\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\big\|_{L^\infty(I;L^1(\varOmega;\mathbb{R}^{d\times d}))} \leq C\,. \tag{3.20c}$$

Now we can use the mechanical-energy balance (3.16) together with the symmetry of $\boldsymbol{T}$ due to (3.8) and estimate the right-hand side at each time instant t as

$$\begin{aligned}&\int_\varOmega -\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{v}_{\varepsilon k}\mathrm{d}\boldsymbol{x} = \int_\varOmega -\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})\mathrm{d}\boldsymbol{x} \leq \|\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\|_{L^1(\varOmega;\mathbb{R}^{d\times d})}\|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})\|_{L^\infty(\varOmega;\mathbb{R}^{d\times d})}\\ &\leq N\|\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\|_{L^1(\varOmega;\mathbb{R}^{d\times d})}\big(\|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})\|_{L^2(\varOmega;\mathbb{R}^{d\times d})} + \|\nabla^2\boldsymbol{v}_{\varepsilon k}\|_{L^p(\varOmega;\mathbb{R}^{d\times d\times d})}\big)\\ &\leq N\|\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\|_{L^1(\varOmega;\mathbb{R}^{d\times d})}\Big(\frac{1}{4\delta} + \frac{p-1}{p^{p/(p-1)}\delta^{1/(p-1)}} + \delta\|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})\|_{L^2(\varOmega;\mathbb{R}^{d\times d})}^2 + \delta\|\nabla^2\boldsymbol{v}_{\varepsilon k}\|_{L^p(\varOmega;\mathbb{R}^{d\times d\times d})}^p\Big),\end{aligned}$$

where N denotes the norm of the continuous embedding $W^{1,p}(\varOmega) \subset L^\infty(\varOmega)$ when the norm $\|\cdot\|_{L^2(\varOmega)} + \|\nabla\cdot\|_{L^p(\varOmega;\mathbb{R}^d)}$ on $W^{1,p}(\varOmega)$ is used. The constant $\delta > 0$ is to be chosen so small that the last two terms can be absorbed in the left-hand side of (3.16), exploiting also the assumption (3.5g). Then (3.16) gives the additional estimates

$$\|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})\|_{L^2(I\times\varOmega;\mathbb{R}^{d\times d})} \leq C \qquad\text{and}\qquad \|\nabla^2\boldsymbol{v}_{\varepsilon k}\|_{L^p(I\times\varOmega;\mathbb{R}^{d\times d\times d})} \leq C\,. \tag{3.21}$$

When assuming $p > d$, the estimate (3.21) is essential by preventing the evolution of singularities of the quantities transported by such a smooth velocity field. The latter estimate in (3.20a), which is not uniform with k, anyhow guarantees that $\boldsymbol{v}_{\varepsilon k} \in L^1(I;W^{2,p}(\varOmega;\mathbb{R}^d))$ for any k, which is needed for the proof of the above mentioned properties of the operators in (3.14).

We have also the following transport-and-evolution equation for $1/\det \boldsymbol{F}_{\varepsilon k}$ and a similar equation holds also for the sparsity $1/\varrho_{\varepsilon k}$, cf. (2.1) and (2.2), namely

$$\frac{\partial}{\partial t}\frac{1}{\det \boldsymbol{F}_{\varepsilon k}} = -\mathrm{div}\Big(\frac{\boldsymbol{v}_{\varepsilon k}}{\det \boldsymbol{F}_{\varepsilon k}}\Big) \quad \text{and} \quad \frac{\partial}{\partial t}\frac{1}{\varrho_{\varepsilon k}} = \frac{\mathrm{div}\, \boldsymbol{v}_{\varepsilon k}}{\varrho_{\varepsilon k}} - \boldsymbol{v}_{\varepsilon k}\cdot\nabla\frac{1}{\varrho_{\varepsilon k}}\,. \tag{3.22}$$

We can apply the same arguments to the solutions to (3.14) as we did for (3.12) based on [32, Sect.5]. Here, due to the qualification of $\boldsymbol{F}_0$ and $\varrho_0 = \varrho_{\mathrm{R}}/\det \boldsymbol{F}_0$ in (3.5h) and (3.5i), it yields $\varrho_{\varepsilon k} > 0$ and $\det \boldsymbol{F}_{\varepsilon k} > 0$ a.e. with the estimates

$$\|\boldsymbol{F}_{\varepsilon k}\|_{L^\infty(I;W^{1,r}(\Omega;\mathbb{R}^{d\times d}))} \le C_r\,, \quad \Big\|\frac{1}{\det \boldsymbol{F}_{\varepsilon k}}\Big\|_{L^\infty(I;W^{1,r}(\Omega))} \le C_r\,, \tag{3.23a}$$

$$\|\varrho_{\varepsilon k}\|_{L^\infty(I;W^{1,r}(\Omega))} \le C_r\,, \quad \text{and} \quad \Big\|\frac{1}{\varrho_{\varepsilon k}}\Big\|_{L^\infty(I;W^{1,r}(\Omega))} \le C_r \quad \text{for any } 1 \le r < +\infty; \tag{3.23b}$$

cf. also the regularity of the initial conditions that follows from the assumptions (3.5i-k):

$$\nabla\Big(\frac{1}{\det \boldsymbol{F}_0}\Big) = -\frac{\det'(\boldsymbol{F}_0){:}\nabla\boldsymbol{F}_0}{(\det \boldsymbol{F}_0)^2} = -\frac{\mathrm{Cof}\boldsymbol{F}_0{:}\nabla\boldsymbol{F}_0}{(\det \boldsymbol{F}_0)^2} \in L^r(\Omega;\mathbb{R}^d)\,,$$
$$\nabla\varrho_0 = \frac{\nabla\varrho_{\mathrm{R}}}{\det \boldsymbol{F}_0} - \varrho_{\mathrm{R}}\frac{\mathrm{Cof}\boldsymbol{F}_0{:}\nabla\boldsymbol{F}_0}{(\det \boldsymbol{F}_0)^2} \in L^r(\Omega;\mathbb{R}^d)\,, \quad \text{and}$$
$$\nabla\Big(\frac{1}{\varrho_0}\Big) = \nabla\Big(\frac{\det \boldsymbol{F}_0}{\varrho_{\mathrm{R}}}\Big) = \frac{\mathrm{Cof}\boldsymbol{F}_0{:}\nabla\boldsymbol{F}_0}{\varrho_{\mathrm{R}}} - \frac{\det \boldsymbol{F}_0\nabla\varrho_{\mathrm{R}}}{\varrho_{\mathrm{R}}^2} \in L^r(\Omega;\mathbb{R}^d)\,.$$

From (3.23b) and the former estimate in (3.20a), we then have also

$$\big\|\boldsymbol{v}_{\varepsilon k}\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \le \big\|\sqrt{\varrho_{\varepsilon k}}\boldsymbol{v}_{\varepsilon k}\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))}\Big\|\frac{1}{\sqrt{\varrho_{\varepsilon k}}}\Big\|_{L^\infty(I\times\Omega)} \le C_r\,, \tag{3.23c}$$

which improves the latter estimate in (3.20a). Realizing the embedding $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ for $p > d$, one can complement (3.21) by the bound $\mathrm{ess\,sup}_{t\in I}\|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}(t))\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \le C_r$.

The estimate (3.23a) also serves for taken a compact set $K \subset \mathrm{GL}^+(d)$ in the assumptions (3.5e–g). In particular, using (3.7), one can complement also (3.20c) by

$$\Big\|\frac{\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})}{1{+}(\theta_{\varepsilon k}^+)^{1+\alpha}}\Big\|_{L^\infty(I\times\Omega;\mathbb{R}^{d\times d})} \le C_{r,K}\,. \tag{3.24}$$

Step 3: existence of semi-Galerkin solution and an L^2-estimate of $\nabla\theta_{\varepsilon k}$. We still test (3.9d) by $\theta_{\varepsilon k}$ which gives an estimate for $\nabla\theta_{\varepsilon k}$ because of the regularization of the right-hand side in (3.9d). We have the pointwise estimate $|\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})| \le C_{r,K}(1{+}(\theta_{\varepsilon k}^+)^{1+\alpha})$ with $C_{r,K}$ from (3.24) and the same type of estimate holds also for $\boldsymbol{T}_1(\boldsymbol{F},\theta) = \boldsymbol{T}(\boldsymbol{F},\theta) - \boldsymbol{T}(\boldsymbol{F},0)$. Therefore, exploiting the embedding $W^{1,p}(\Omega) \subset L^\infty(\Omega)$ and the symmetry of the Cauchy stress $\boldsymbol{T}_1$ granted by (3.8), we have also at each time instant t the pointwise estimate for the power of the conservative part of the Cauchy stress (i.e. the internal-energy source) as

$$\big|\boldsymbol{T}_1(\boldsymbol{F}_{\varepsilon k}(t),\theta_{\varepsilon k}(t)){:}\nabla\boldsymbol{v}_{\varepsilon k}(t)\big| = \big|\boldsymbol{T}_1(\boldsymbol{F}_{\varepsilon k}(t),\theta_{\varepsilon k}(t)){:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}(t))\big| \le C(t)\big(1{+}(\theta_{\varepsilon k}^+(t))^{1+\alpha}\big) \tag{3.25}$$

with some $C \in L^p(I)$ depending on the estimates (3.24).

Let us consider $\widetilde{E}(\boldsymbol{F},\theta)$ so that $\widetilde{E}'_\theta(\boldsymbol{F},\theta) = \theta c(\boldsymbol{F},\theta)$ with the heat capacity $c(\boldsymbol{F},\theta) = E'_\theta(\boldsymbol{F},\theta)$. Specifically, we put $\widetilde{E}(\boldsymbol{F},\theta) := \int_0^\theta \widehat{\theta} c(\boldsymbol{F},\widehat{\theta})\,\mathrm{d}\widehat{\theta}$ or, equivalently,

$$\widetilde{E}(\boldsymbol{F},\theta) = \int_0^1 r\theta^2 c(\boldsymbol{F},r\theta)\,\mathrm{d}r\,. \tag{3.26}$$

Let us further put $\widetilde{U}(\boldsymbol{F},\theta) := \widetilde{E}(\boldsymbol{F},\theta) - \theta U(\boldsymbol{F},\theta)$. From (3.6), we can see that $|\widetilde{U}(\boldsymbol{F},\theta)| \le 2C_K(1{+}(\theta^+)^{2+\alpha})$ with the compact subset of $\mathrm{GL}^+(d)$ respecting the already obtained estimates (3.23a). Then, like in [32], we have the calculus

$$\begin{aligned}
\theta_{\varepsilon k}\overline{U(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})}^{\bullet\varepsilon k} &= \theta_{\varepsilon k}U'_\theta(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\,\overset{\bullet\varepsilon k}{\theta}_{\varepsilon k} + \theta_{\varepsilon k}U'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\overset{\bullet\varepsilon k}{\boldsymbol{F}}_{\varepsilon k}\\
&= \widetilde{E}'_\theta(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\,\overset{\bullet\varepsilon k}{\theta}_{\varepsilon k} + \theta_{\varepsilon k}U'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\overset{\bullet\varepsilon k}{\boldsymbol{F}}_{\varepsilon k}\\
&= \overline{\widetilde{E}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})}^{\bullet\varepsilon k} - \widetilde{E}'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\overset{\bullet\varepsilon k}{\boldsymbol{F}}_{\varepsilon k} + \theta_{\varepsilon k}U'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\overset{\bullet\varepsilon k}{\boldsymbol{F}}_{\varepsilon k}\\
&\overset{(3.12)}{=} \overline{\widetilde{E}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})}^{\bullet\varepsilon k} - \big(\widetilde{E}'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}) - \theta_{\varepsilon k}U'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\big){:}\big((\nabla\boldsymbol{v}_{\varepsilon k})\boldsymbol{F}_{\varepsilon k}\big)\\
&= \overline{\widetilde{E}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})}^{\bullet\varepsilon k} - \widetilde{U}'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\boldsymbol{F}_{\varepsilon k}^\top{:}\nabla\boldsymbol{v}_{\varepsilon k}\,.
\end{aligned}$$

In terms of $\widetilde{E}$ and $\widetilde{U}$, we can write the Galerkin approximation of (3.18) tested by $\theta_{\varepsilon k}$ as

$$\begin{aligned}
&\int_\Omega \widetilde{E}(\boldsymbol{F}_{\varepsilon k}(t),\theta_{\varepsilon k}(t)) + \frac{1}{k}|\theta_{\varepsilon k}(t)|^2\,\mathrm{d}\boldsymbol{x} + \int_0^t\!\!\int_\Omega \kappa(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})|\nabla\theta_{\varepsilon k}|^2\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&= \int_\Omega \widetilde{E}(\boldsymbol{F}_0,\theta_{0\varepsilon})\,\mathrm{d}\boldsymbol{x} + \int_0^t\!\!\int_\Omega \bigg(\Big(\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2 + \nu_2|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}{1{+}\varepsilon|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2{+}\varepsilon|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p} + \boldsymbol{T}_1(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{v}_{\varepsilon k}\Big)\theta_{\varepsilon k}\\
&\qquad\qquad + \widetilde{U}'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\boldsymbol{F}_{\varepsilon k}^\top{:}\nabla\boldsymbol{v}_{\varepsilon k} + \widetilde{U}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\mathrm{div}\,\boldsymbol{v}_{\varepsilon k}\bigg)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^t\!\!\int_\Gamma h_\varepsilon\theta_{\varepsilon k}\,\mathrm{d}S\mathrm{d}t\\
&\le \int_\Omega \widetilde{E}(\boldsymbol{F}_0,\theta_{0\varepsilon})\,\mathrm{d}\boldsymbol{x} + \int_0^t\!\!\int_\Omega \bigg(\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2 + \nu_2|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}{1{+}\varepsilon|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2{+}\varepsilon|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}\,\theta_{\varepsilon k}^+ + \big(\boldsymbol{T}_1(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{v}_{\varepsilon k}\big)\theta_{\varepsilon k}^+\\
&\qquad\qquad + \widetilde{U}'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\boldsymbol{F}_{\varepsilon k}^\top{:}\nabla\boldsymbol{v}_{\varepsilon k} + \widetilde{U}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\mathrm{div}\,\boldsymbol{v}_{\varepsilon k}\bigg)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^t\!\!\int_\Gamma \frac{h\,\theta_{\varepsilon k}^+}{1{+}\varepsilon h}\,\mathrm{d}S\mathrm{d}t\,.
\end{aligned}\tag{3.27}$$

We also used that $h \ge 0$ as assumed in (3.5j) and that $\boldsymbol{T}_1(\cdot,\theta) = \boldsymbol{0}$ for $\theta \le 0$, as noted in Remark 2.1. Here we use the regularization in (3.11) which makes $\widetilde{E}(\boldsymbol{F}_0,\theta_{0\varepsilon})$ integrable. From (3.25), at each time instant, we can read the estimate $|\boldsymbol{T}_1(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}(\nabla\boldsymbol{v}_{\varepsilon k})\theta_{\varepsilon k}| \le C(t)(1{+}(\theta_{\varepsilon k}^+)^{2+\alpha})$ with $C \in L^p(I)$. Since $\widetilde{E}'_{\boldsymbol{F}}(\boldsymbol{F},\theta)$ is a component-wise primitive function (antiderivative) of $\theta \mapsto \theta c'_{\boldsymbol{F}}(\boldsymbol{F},\theta) = \theta E''_{\boldsymbol{F}\theta}(\boldsymbol{F},\theta)$, cf. (3.26), we can rely on the assumptions (3.5f) and therefore $|\widetilde{E}'_{\boldsymbol{F}}(\boldsymbol{F},\theta)| \le C_K(1{+}(\theta^+)^{2+\alpha})$ and thus also $|\widetilde{U}'_{\boldsymbol{F}}(\boldsymbol{F},\theta)| \le 2C_K(1{+}(\theta^+)^{2+\alpha})$. The coercivity (3.5e) yields the coercivity $\widetilde{E}(\boldsymbol{F},\theta) \ge c_K(\theta^+)^{2+\alpha}/(2{+}\alpha)$, which allows for usage of the Gronwall inequality for (3.27). The estimation of the boundary term in (3.27) exploits $\int_\Gamma h\theta_{\varepsilon k}^+/(1{+}\varepsilon h)\,\mathrm{d}S \le \varepsilon^{-1}\|\theta_{\varepsilon k}^+\|_{L^1(\Gamma)} \le C_{\alpha,\varepsilon}(1{+}\|\theta_{\varepsilon k}^+\|_{L^{2+\alpha}(\Omega)}^{2+\alpha}) + \frac{1}{2}c_K\|\nabla\theta_{\varepsilon k}\|_{L^2(\Omega;\mathbb{R}^d)}^2$ with c_K referring to (3.5g) and with some $C_{\alpha,\varepsilon}$ sufficiently large, so that the mentioned coercivity of $\widetilde{E}(\boldsymbol{F},\cdot)$ on $\mathbb{R}^+$ and the Gronwall inequality can be used. From the already obtained estimates for the

Galerkin approximation, we can thus read some a-priori estimated also for $\theta_{\varepsilon k}$, namely

$$\big\|\nabla\theta_{\varepsilon k}\big\|_{L^2(I\times\Omega;\mathbb{R}^d)}\le C,\quad \big\|\theta_{\varepsilon k}^+\big\|_{L^\infty(I;L^{2+\alpha}(\Omega))\cap L^2(I;H^1(\Omega))}\le C,\ \text{ and }\ \big\|\theta_{\varepsilon k}\big\|_{L^\infty(I;L^2(\Omega))}\le C\sqrt{k}\,. \tag{3.28}$$

Note that, disregarding $\nabla\theta^-$, we do not have any uniform control on the negative part of temperature which may be indeed nonvanishing in the Galerkin approximation, but it does not harm the arguments in the proof because the data $\boldsymbol{T}(\boldsymbol{F},\cdot)$ and $\kappa(\boldsymbol{F},\cdot)$ are insensitive to possible negative values of temperature, cf. the assumption (3.5c). Anyhow, the last L^∞-estimate of $\theta_{\varepsilon k}$ in (3.28), although being nonuniform in k, together with the other L^∞-estimates (3.23a–c) allow us to apply the successive-prolongation arguments and to show the existence of a semi-discrete solution $(\varrho_{\varepsilon k},\boldsymbol{v}_{\varepsilon k},\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})$ not only locally in time but globally on the whole time interval I.

Furthermore, from (3.28), we can read an estimate for

$$\begin{aligned}\nabla u_{\varepsilon k}&=U'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{F}_{\varepsilon k}+U'_\theta(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\nabla\theta_{\varepsilon k}\\&=\big(\psi'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})-\theta\psi''_{\boldsymbol{F}\theta}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})-\psi'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},0)\big){:}\nabla\boldsymbol{F}_{\varepsilon k}+c(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\nabla\theta_{\varepsilon k}\end{aligned}$$

with the heat capacity $c(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})=-\theta_{\varepsilon k}\psi''_{\theta\theta}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})$. Specifically, due to (3.5f) with (3.5c), (3.23a), and (3.28), we have $U'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})$ bounded in $L^{(2+\alpha)/(1+\alpha)}(I\times\Omega;\mathbb{R}^{d\times d})$ while $\nabla\boldsymbol{F}_{\varepsilon k}$ is bounded in $L^r(I\times\Omega;\mathbb{R}^{d\times d\times d})$ for arbitrarily big $1\le r<+\infty$. Thus $E'_{\boldsymbol{F}}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{F}_{\varepsilon k}$ is bounded in $L^{r(2+\alpha)/(2+\alpha+r+r\alpha)}(I\times\Omega;\mathbb{R}^d)$. Realizing that $c(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\le C(1+(\theta^+)^\alpha)$ as actually assumed by the first condition in (3.5f), from (3.28) we have $c(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\nabla\theta_{\varepsilon k}$ bounded in $L^{(4+2\alpha)/(2+3\alpha)}(I\times\Omega;\mathbb{R}^d)$, so that altogether (considering r sufficiently big) we have

$$\big\|\nabla u_{\varepsilon k}\big\|_{L^{(4+2\alpha)/(2+3\alpha)}(I\times\Omega;\mathbb{R}^d)}\le C \tag{3.29a}$$

and, due to the first estimate in (3.28), we have also

$$\Big\|\nabla\Big(u_{\varepsilon k}+\frac{\theta_{\varepsilon k}}{k}\Big)\Big\|_{L^{(4+2\alpha)/(2+3\alpha)}(I\times\Omega;\mathbb{R}^d)}\le C\,. \tag{3.29b}$$

Step 4: Convergence for $k\to\infty$. Using the Banach selection principle, we can extract some subsequence of $\{(\varrho_{\varepsilon k},\boldsymbol{v}_{\varepsilon k},\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}^+,u_{\varepsilon k}{+}\theta_{\varepsilon k}/k)\}_{k\in\mathbb{N}}$ and its limit $(\varrho_\varepsilon,\boldsymbol{v}_\varepsilon,\boldsymbol{F}_\varepsilon,\theta_\varepsilon,u_\varepsilon):I\to W^{1,r}(\Omega)\times L^2(\Omega;\mathbb{R}^d)\times W^{1,r}(\Omega;\mathbb{R}^{d\times d})\times L^{2+\alpha}(\Omega)\times L^1(\Omega)$ such that, for any $1\le r<+\infty$, we have

$$\begin{aligned}
&\varrho_{\varepsilon k}\to\varrho_\varepsilon &&\text{weakly* in }\ L^\infty(I;W^{1,r}(\Omega))\cap W^{1,p}(I;L^r(\Omega))\,, &&(3.30\text{a})\\
&\boldsymbol{v}_{\varepsilon k}\to\boldsymbol{v}_\varepsilon &&\text{weakly* in }\ L^\infty(I;L^2(\Omega;\mathbb{R}^d))\cap L^p(I;W^{2,p}(\Omega;\mathbb{R}^d)), &&(3.30\text{b})\\
&\boldsymbol{F}_{\varepsilon k}\to\boldsymbol{F}_\varepsilon &&\text{weakly* in }\ L^\infty(I;W^{1,r}(\Omega;\mathbb{R}^{d\times d}))\cap W^{1,p}(I;L^r(\Omega;\mathbb{R}^{d\times d})), &&(3.30\text{c})\\
&\theta_{\varepsilon k}^+\to\theta_\varepsilon &&\text{weakly* in }\ L^\infty(I;L^{2+\alpha}(\Omega))\cap L^2(I;H^1(\Omega)), &&(3.30\text{d})\\
&u_{\varepsilon k}+\theta_{\varepsilon k}/k\to u_\varepsilon &&\text{weakly in }\ L^{(4+2\alpha)/(2+3\alpha)}(I;W^{(4+2\alpha)/(2+3\alpha)}(\Omega))\,. &&(3.30\text{e})
\end{aligned}$$

In (3.30c), we used also the estimates (3.21) and (3.23a,c), which yields the bound of $\frac{\partial}{\partial t}\boldsymbol{F}_{\varepsilon k}=(\nabla\boldsymbol{v}_{\varepsilon k})\boldsymbol{F}_{\varepsilon k}-(\boldsymbol{v}_{\varepsilon k}{\cdot}\nabla)\boldsymbol{F}_{\varepsilon k}$ in $L^p(I;L^r(\Omega;\mathbb{R}^{d\times d}))$. By the Aubin-Lions lemma here considering $r>d$, we also have that

$$\varrho_{\varepsilon k}\to\varrho_\varepsilon\ \text{ strongly in }\ C(I{\times}\overline{\Omega})\quad\text{and}\quad\boldsymbol{F}_{\varepsilon k}\to\boldsymbol{F}_\varepsilon\ \text{ strongly in } C(I{\times}\overline{\Omega};\mathbb{R}^{d\times d})\,. \tag{3.31}$$

This already allows for the limit passage in the evolution equations (3.12).

Further, by comparison in the equation (3.18) with the boundary condition (3.10) in its Galerkin approximation, we obtain a bound on $\frac{\partial}{\partial t}e_{\varepsilon k}$ in the seminorms $|\cdot|_l$ on $L^2(I;H^1(\Omega)^*)$ arising from this Galerkin approximation:

$$\big|f\big|_l := \sup_{\|\widetilde\theta\|_{L^2(I;H^1(\Omega))}\le 1,\ \widetilde\theta(t)\in Z_l\ \text{ for } t\in I} \int_0^T\!\!\int_\Omega f\widetilde\theta\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\,.$$

Specifically, for any $k\ge l$, we can estimate

$$\bigg|\frac{\partial}{\partial t}\Big(u_{\varepsilon k}+\frac{\theta_{\varepsilon k}}{k}\Big)\bigg|_l = \sup_{\substack{\widetilde\theta(t)\in Z_l\ \text{ for } t\in I\\ \|\widetilde\theta\|_{L^2(I;H^1(\Omega))}\le 1}} \int_0^T\!\!\int_\Omega\bigg(\Big(\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2+\nu_2|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}{1+\varepsilon|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k})|^2+\varepsilon|\nabla^2\boldsymbol{v}_{\varepsilon k}|^p}+\boldsymbol{T}_1(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\nabla\boldsymbol{v}_{\varepsilon k}\Big)\widetilde\theta \\ +\big(u_{\varepsilon k}\boldsymbol{v}_{\varepsilon k}-\kappa(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k})\nabla\theta_{\varepsilon k}\big)\cdot\nabla\widetilde\theta\bigg)\mathrm{d}\boldsymbol{x}\mathrm{d}t+\int_0^T\!\!\int_\Gamma h_\varepsilon\widetilde\theta\,\mathrm{d}S\mathrm{d}t\le C$$

with some C depending on the estimates (3.24) and (3.28) but independent on $l\in N$. Thus, by (3.30e) and by a generalized Aubin-Lions theorem [29, Ch.8], we obtain

$$u_{\varepsilon k}+\frac{\theta_{\varepsilon k}}{k}\to u_\varepsilon \qquad \text{strongly in } L^{(4+2\alpha)/(2+3\alpha)}(I{\times}\Omega); \tag{3.32}$$

actually, the exponent in (3.33) taken from (3.30e) could be improved when interpolating also the estimate (3.20b). Due to the last estimate in (3.28), $\|\theta_{\varepsilon k}/k\|_{L^2(I\times\Omega)}=\mathscr{O}(1/\sqrt{k})\to 0$, and therefore we have also $u_{\varepsilon k}\to u_\varepsilon$ strongly in $L^{(4+2\alpha)/(2+3\alpha)}(I{\times}\Omega)$. Moreover, in view of the continuity of $E(\cdot,0)$ on the compact K where $\boldsymbol{F}_{\varepsilon k}$ are valued, (3.31) implies also

$$e_{\varepsilon k}=u_{\varepsilon k}-E(\boldsymbol{F}_{\varepsilon k},0)\ \to\ u_\varepsilon-E(\boldsymbol{F}_\varepsilon,0)=:e_\varepsilon \quad \text{strongly in } L^{(4+2\alpha)/(2+3\alpha)}(I{\times}\Omega). \tag{3.33}$$

We unfortunately cannot read convergence of $\theta_{\varepsilon k}\in[U(\boldsymbol{F}_{\varepsilon k},\cdot)]^{-1}(u_{\varepsilon k})$ from the mentioned convergence of $u_{\varepsilon k}$ because the inverse $[U(\boldsymbol{F},\cdot)]^{-1}$ of $U(\boldsymbol{F},\cdot):\mathbb{R}\to\mathbb{R}$ is not continuous, being set-valued at 0, specifically $[U(\boldsymbol{F},\cdot)]^{-1}(0)=(-\infty,0]$ while $[U(\boldsymbol{F},\cdot)]^{-1}(u)=\emptyset$ for $u<0$. Anyhow, since $E(\boldsymbol{F},\cdot):\mathbb{R}\to\mathbb{R}^+$ is assumed increasing and coercive on $\mathbb{R}^+$, cf. (3.5c) and (3.5e), so is $U(\boldsymbol{F},\cdot)=E(\boldsymbol{F},\cdot)-E(\boldsymbol{F},0)$, and thus there exists a continuous inverse $[U(\boldsymbol{F},\cdot)]^{-1}:(0,+\infty)\to\mathbb{R}$. This analytical peculiarity can be circumvented by defining a modification for $u<0$ as a continuous extension to obtain the single-valued continuous function

$$\big[U(\boldsymbol{F},\cdot)\big]_{\mathrm{M}}^{-1}:\mathbb{R}\to\mathbb{R}^+:u\mapsto\begin{cases}[U(\boldsymbol{F},\cdot)]^{-1}(u) & \text{for } \ u>0\,,\\ \qquad 0 & \text{for } \ u\le 0\,;\end{cases} \tag{3.34}$$

cf. Figure 1-right for illustration.

Since surely $u_{\varepsilon k}\ge 0$, we can write $\theta_{\varepsilon k}^+=[U(\boldsymbol{F}_{\varepsilon k},\cdot)]_{\mathrm{M}}^{-1}(u_{\varepsilon k})$. Thanks to the continuity of $(\boldsymbol{F},u)\mapsto[U(\boldsymbol{F},\cdot)]_{\mathrm{M}}^{-1}(u):\mathbb{R}^{d\times d}\times\mathbb{R}^+\to\mathbb{R}$ coming from the inverse-function theorem using the assumption $U_\theta'=E_\theta'>0$ in (3.5e), we can improve (3.30d) for

$$\theta_{\varepsilon k}^+\to\theta_\varepsilon=\big[U(\boldsymbol{F}_\varepsilon,\cdot)\big]_{\mathrm{M}}^{-1}(u_\varepsilon) \qquad \text{strongly in } L^{2+4/d}(I{\times}\Omega); \tag{3.35}$$

here we have used the Gagliardo-Nirenberg interpolation for the embedding $L^\infty(I;L^2(\Omega))\cap L^2(I;H^1(\Omega))\subset L^{2+4/d}(I{\times}\Omega)$. Let us emphasize that we do not have any direct information

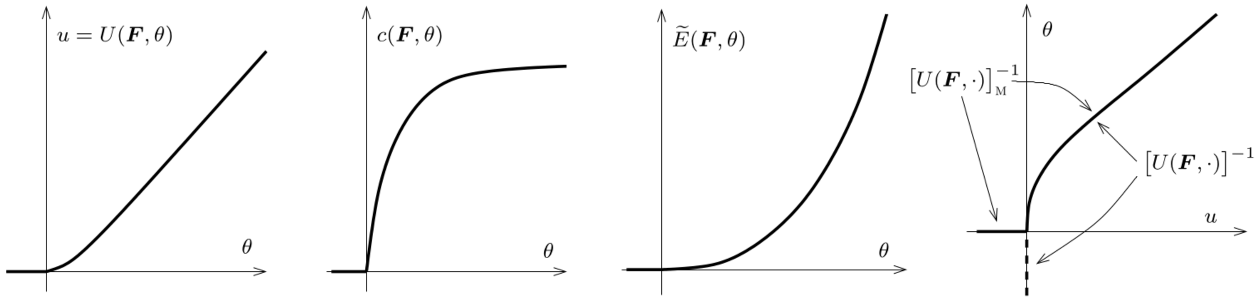

Figure 1: *A schematic illustration of continuous extensions for negative temperatures of the thermal part of the internal energy $u = U(\boldsymbol{F},\theta)$, the heat capacity $c = c(\boldsymbol{F},\theta)$, the nonlinearity $\widetilde{E}$ from (3.26), and the modified continuous inverse $\big[U(\boldsymbol{F},\cdot)\big]_{\rm M}^{-1}$ of $U(\boldsymbol{F},\cdot)$ as defined by (3.34); here $\boldsymbol{F}$ is considered fixed.*

about $\frac{\partial}{\partial t}\theta_{\varepsilon k}$ neither for $\frac{\partial}{\partial t}\theta_{\varepsilon k}^+$ so that we could not use the Aubin-Lions arguments directly for $\{\theta_{\varepsilon k}\}_{k\in\mathbb{N}}$ neither for $\{\theta_{\varepsilon k}^+\}_{k\in\mathbb{N}}$.

Thus, by the continuity of the corresponding Nemytskiĭ (or here simply superposition) mappings, also the conservative part of the Cauchy stress $\boldsymbol{T}$ as well as the coefficient κ converge strongly, namely

$$\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}) \to \boldsymbol{T}(\boldsymbol{F}_\varepsilon,\theta_\varepsilon) \qquad \text{strongly in } L^r(I{\times}\varOmega;\mathbb{R}^{d\times d}_{\rm sym}),\quad 1\le r<\frac{2d+4}{d(1{+}\alpha)},\ \text{and} \tag{3.36a}$$

$$\kappa(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}) \to \kappa(\boldsymbol{F}_\varepsilon,\theta_\varepsilon) \qquad \text{strongly in } L^r(I{\times}\varOmega),\quad 1\le r<\infty. \tag{3.36b}$$

Here the assumption (3.5c) has been employed for $\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}) = \boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}^+)$ and $\kappa(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}) = \kappa(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}^+)$. For the limit passage in the dissipative heat power, we will need also

$$\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}) \to \boldsymbol{e}(\boldsymbol{v}_\varepsilon) \qquad \text{strongly in } \ L^2(I{\times}\varOmega;\mathbb{R}^{d\times d})\ \text{ and} \tag{3.36c}$$

$$\nabla^2\boldsymbol{v}_{\varepsilon k} \to \nabla^2\boldsymbol{v}_\varepsilon \qquad \text{strongly in } \ L^p(I{\times}\varOmega;\mathbb{R}^{d\times d\times d}), \tag{3.36d}$$

which is based on the uniform monotonicity of the dissipative stress, i.e., of the quasilinear operator $\boldsymbol{v}\mapsto{\rm div}({\rm div}(\nu_2|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v})-\nu_1\boldsymbol{e}(\boldsymbol{v})))$. Thus, to prove (3.36c,d), we use the Galerkin approximation of the regularized momentum equation (3.13a) tested by $\widetilde{\boldsymbol{v}} = \boldsymbol{v}_{\varepsilon k} - \widetilde{\boldsymbol{v}}_k$ with $\widetilde{\boldsymbol{v}}_k : I\to V_k$ an approximation of $\boldsymbol{v}_\varepsilon$ in the sense that $\widetilde{\boldsymbol{v}}_k\to\boldsymbol{v}_\varepsilon$ strongly in $L^p(I;W^{2,p}(\varOmega;\mathbb{R}^d))$ and $\widetilde{\boldsymbol{v}}_k(0)\to\boldsymbol{v}_0$ strongly in $L^2(\varOmega;\mathbb{R}^d)$ for $k\to\infty$ while $\widetilde{\boldsymbol{v}}_k{\cdot}\boldsymbol{n}=0$ on $I{\times}\varGamma$ and $\{\frac{\partial}{\partial t}\widetilde{\boldsymbol{v}}_k\}_{k\in\mathbb{N}}$ is bounded in $L^{p'}(I;W^{2,p}(\varOmega;\mathbb{R}^d)^*)$. Using the perturbed continuity equation $\frac{\partial}{\partial t}\varrho_{\varepsilon k} + {\rm div}(\varrho_{\varepsilon k}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)) = -{\rm div}(\varrho_{\varepsilon k}\widetilde{\boldsymbol{v}}_k)$ tested by $|\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k|^2/2$ to derive the identity

$$\begin{aligned}
\int_0^T\!\!\int_\varOmega \varrho_{\varepsilon k}\overset{\varepsilon k}{\dot{\boldsymbol{v}}}_{\varepsilon k}{\cdot}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)\,{\rm d}\boldsymbol{x}{\rm d}t &= \int_0^T\!\!\int_\varOmega\bigg(\varrho_{\varepsilon k}\Big(\frac{\partial}{\partial t}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k){\cdot}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k) + ((\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k){\cdot}\nabla)(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)\Big)\\
&\qquad\qquad + \varrho_{\varepsilon k}\Big(\frac{\partial\widetilde{\boldsymbol{v}}_k}{\partial t}{\cdot}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k) + (\widetilde{\boldsymbol{v}}_k{\cdot}\nabla)(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)\Big)\bigg)\,{\rm d}\boldsymbol{x}{\rm d}t\\
&= \int_0^T\!\!\int_\varOmega \varrho_{\varepsilon k}\Big(\frac{\partial\widetilde{\boldsymbol{v}}_k}{\partial t}{\cdot}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k) + (\widetilde{\boldsymbol{v}}_k{\cdot}\nabla)(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)\Big) + \frac12{\rm div}(\varrho_{\varepsilon k}\widetilde{\boldsymbol{v}}_k)|\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k|^2\,{\rm d}\boldsymbol{x}{\rm d}t\\
&\qquad\qquad + \int_\varOmega \frac{\varrho_{\varepsilon k}(T)}{2}|\boldsymbol{v}_{\varepsilon k}(T){-}\widetilde{\boldsymbol{v}}_k(T)|^2 - \frac{\varrho_0}{2}|\boldsymbol{v}_0{-}\widetilde{\boldsymbol{v}}_k(0)|^2\,{\rm d}\boldsymbol{x}\,,
\end{aligned}$$

we can estimate

$$
\begin{aligned}
&\nu_1\|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}{-}\boldsymbol{v}_\varepsilon)\|^2_{L^2(I\times\Omega;\mathbb{R}^{d\times d})}+\widetilde\nu_2\|\nabla^2(\boldsymbol{v}_{\varepsilon k}{-}\boldsymbol{v}_\varepsilon)\|^p_{L^p(I\times\Omega;\mathbb{R}^{d\times d\times d})}\le\int_\Omega\frac{\varrho_{\varepsilon k}(T)}{2}\big|\boldsymbol{v}_{\varepsilon k}(T){-}\widetilde{\boldsymbol{v}}_k(T)\big|^2\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\int_0^T\!\!\int_\Omega\bigg(\nu_1\big|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}{-}\boldsymbol{v}_\varepsilon)\big|^2+\nu_2\big(|\nabla^2\boldsymbol{v}_{\varepsilon k}|^{p-2}\nabla^2\boldsymbol{v}_{\varepsilon k}-|\nabla^2\boldsymbol{v}_\varepsilon|^{p-2}\nabla^2\boldsymbol{v}_\varepsilon\big)\vdots\nabla^2(\boldsymbol{v}_{\varepsilon k}{-}\boldsymbol{v}_\varepsilon)\bigg)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\int_0^T\!\!\int_\Omega\bigg(\varrho_{\varepsilon k}\boldsymbol{g}\cdot(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)-\boldsymbol{T}(\boldsymbol{F}_{\varepsilon k},\theta_{\varepsilon k}){:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)-\frac{\boldsymbol{v}_{\varepsilon k}\cdot(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)}{k}-\nu_1\boldsymbol{e}(\widetilde{\boldsymbol{v}}_k){:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)\\
&\qquad+\nu_1\boldsymbol{e}(\boldsymbol{v}_{\varepsilon k}){:}\boldsymbol{e}(\widetilde{\boldsymbol{v}}_k{-}\boldsymbol{v}_\varepsilon)-\nu_2\big(|\nabla^2\widetilde{\boldsymbol{v}}_k|^{p-2}\nabla^2\widetilde{\boldsymbol{v}}_k\big)\vdots\nabla^2(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)+\nu_2|\nabla^2(\boldsymbol{v}_{\varepsilon k}|^{p-2}\nabla^2\boldsymbol{v}_{\varepsilon k}\vdots\nabla^2(\widetilde{\boldsymbol{v}}_k{-}\boldsymbol{v}_\varepsilon)\\
&\qquad\qquad-\varrho_{\varepsilon k}\Big(\frac{\partial\widetilde{\boldsymbol{v}}_k}{\partial t}\cdot(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)+(\widetilde{\boldsymbol{v}}_k\cdot\nabla)(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)\Big)-\frac12\mathrm{div}(\varrho_{\varepsilon k}\widetilde{\boldsymbol{v}}_k)|\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k|^2\bigg)\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&\qquad\qquad+\int_\Omega\frac{\varrho_0}{2}|\boldsymbol{v}_0{-}\widetilde{\boldsymbol{v}}_k(0)|^2\,\mathrm{d}\boldsymbol{x}\stackrel{k\to\infty}{\longrightarrow}0
\end{aligned}\tag{3.37}
$$

with some $\widetilde\nu_2>0$ depending on $p\ge2$, related to the inequality $\widetilde\nu_2|G-\widetilde G|^p\le\nu_2(|G|^{p-2}G-|\widetilde G|^{p-2}\widetilde G)\vdots(G-\widetilde G)$. Here it is also important that $\mathrm{div}(\varrho_{\varepsilon k}\boldsymbol{v}_{\varepsilon k})=\nabla\varrho_{\varepsilon k}\cdot\boldsymbol{v}_{\varepsilon k}+\varrho_{\varepsilon k}\mathrm{div}\boldsymbol{v}_{\varepsilon k}$ is bounded in $L^\infty(I;L^r(\Omega))$ due to the already obtained bounds (3.21) and (3.23b). The term $\boldsymbol{v}_{\varepsilon k}\cdot(\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k)/k$ converges to 0 in $L^2(I\times\Omega)$ because, due to (3.20a), $\|\boldsymbol{v}_{\varepsilon k}/k\|_{L^2(I\times\Omega;\mathbb{R}^d)}=\mathscr{O}(1/\sqrt{k})\to0$ while $\boldsymbol{v}_{\varepsilon k}{-}\widetilde{\boldsymbol{v}}_k$ is bounded in $L^2(I\times\Omega;\mathbb{R}^d)$ thanks to (3.23c). Thus (3.36c,d) is proved.

The strong convergences (3.36) allow for limit passage simply by continuity in the Galerkin approximation of the regularized momentum equation (3.13a) towards (3.9b) written without the regularizing term $\boldsymbol{v}_{\varepsilon k}/k$ and in the regularized internal-energy equation (3.13b) (or alternatively modified for (3.18)) towards (3.9d) written without the regularizing term $\frac{\partial}{\partial t}\theta_{\varepsilon k}/k$. Here, in addition to [32], we use that the regularizing term $\boldsymbol{v}_{\varepsilon k}/k$ in (3.9b) converges to zero strongly in $L^2(I\times\Omega;\mathbb{R}^d)$ due to the latter estimate in (3.20a) and also the mentioned convergence $\theta_{\varepsilon k}/k\to0$ used for the weak formulation (3.13b).

Step 5: Further a-priori estimates uniform for $\varepsilon>0$. First, let us emphasize that the estimates (3.20)–(3.21) and (3.23)–(3.25) are uniform not only with respect to $k\in\mathbb{N}$ but also in $\varepsilon>0$. Thus they are inherited also for $(\boldsymbol{v}_\varepsilon,\boldsymbol{F}_\varepsilon,\theta_\varepsilon)$. In particular, since we have $\theta_\varepsilon\ge0$ granted simply by construction of the limit (3.30d), the estimate (3.20b) inherited for $(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)$ with the assumption (3.5e) leads to

$$\|\theta_\varepsilon\|_{L^\infty(I;L^{1+\alpha}(\Omega))}\le C\,.\tag{3.38}$$

We may benefit from that the regularized internal-energy equation is continuous. Therefore, one can employ the L^1-theory for heat-transfer-type equations which is based on the "nonlinear" test by $\chi_a(\theta_\varepsilon)$ applied to the internal-energy equation (3.9d) written for $(\boldsymbol{v}_\varepsilon,\boldsymbol{F}_\varepsilon,\theta_\varepsilon)$ instead of $(\boldsymbol{v},\boldsymbol{F},\theta)$ without the term $\frac{\partial}{\partial t}\theta/k$. Specifically, we will use the nonlinear increasing function $\chi_a:[0,+\infty)\to[0,1]$ defined as

$$\chi_a(\theta):=1-\frac{1}{(1{+}\theta)^a}\,,\qquad a>\alpha\,,\tag{3.39}$$

as also used for $\alpha=0$ in [22], simplifying the original idea of L. Boccardo, T. Gallouët, et al. [3, 4] and extending it for the mechanically coupled systems. Cf. also [5, 9] where other functions were used, namely $\chi_a(\theta) := \theta^a$ or $\chi_a(\theta) := 1/\theta^a$ with $0 < a < 1$. Importantly, here we have $\chi_a(\theta_\varepsilon(t,\cdot)) \in H^1(\Omega)$, so it is a legal test function, because $0 \le \theta_\varepsilon(t,\cdot) \in H^1(\Omega)$ has already been proved and because χ_a is Lipschitz continuous on $[0,+\infty)$.

The rather technical arguments in [32] for $\alpha = 0$ have to be generalized here for $\alpha \ge 0$, also combining the interpolation from [16, Sect.8.2]. Since this generalization is not entirely straightforward at some spots, we will present this part quite in detail. We consider $1 \le \mu < 2$ and estimate the L^μ-norm of $\nabla\theta_\varepsilon$ by Hölder's inequality as

$$\int_0^T\!\!\int_\Omega |\nabla\theta_\varepsilon|^\mu \,\mathrm{d}\boldsymbol{x}\mathrm{d}t \le C_1\Big(\underbrace{\int_0^T \big\|1{+}\theta_\varepsilon(t,\cdot)\big\|_{L^{(1+a)\mu/(2-\mu)}(\Omega)}^{(1+a)\mu/(2-\mu)}\,\mathrm{d}t}_{=:\,I^{(1)}_{\mu,a}(\theta_\varepsilon)}\Big)^{1-\mu/2}\Big(\underbrace{\int_0^T\!\!\int_\Omega \chi_a'(\theta_\varepsilon)|\nabla\theta_\varepsilon|^2}_{=:\,I^{(2)}_a(\theta_\varepsilon)}\Big)^{\mu/2} \tag{3.40}$$

with χ_a from (3.39) so that $\chi_a'(\theta) = a/(1{+}\theta)^{1+a}$ and with a constant C_1 dependent on a and μ. Then we interpolate the Lebesgue space $L^{(1+a)\mu/(2-\mu)}(\Omega)$ between $W^{1,\mu}(\Omega)$ and $L^{1+\alpha}(\Omega)$ to exploit (3.38). More specifically, by the Gagliardo-Nirenberg inequality, we obtain

$$\big\|1{+}\theta_\varepsilon(t,\cdot)\big\|_{L^{(1+a)\mu/(2-\mu)}(\Omega)} \le C_2\Big(1 + \big\|\nabla\theta_\varepsilon(t,\cdot)\big\|_{L^\mu(\Omega;\mathbb{R}^d)}\Big)^\lambda$$
$$\text{for}\quad \frac{2-\mu}{(1{+}a)\mu} \ge \lambda\Big(\frac{1}{\mu} - \frac{1}{d}\Big) + \frac{1{-}\lambda}{1{+}\alpha} \quad\text{with}\quad 0 < \lambda \le 1\,. \tag{3.41}$$

Thus, we obtain $I^{(1)}_{\mu,a}(\theta_\varepsilon) \le C_3(1 + \int_0^T\int_\Omega \big|\nabla\theta_\varepsilon\big|^\mu \,\mathrm{d}\boldsymbol{x}\mathrm{d}t)$ providing

$$\lambda = \frac{2{-}\mu}{1{+}a}\,, \tag{3.42}$$

cf. also [16, Formulas (8.2.14)-(8.2.16)]. Combining it with (3.40), we obtain

$$\|\nabla\theta_\varepsilon\|^\mu_{L^\mu(I\times\Omega;\mathbb{R}^d)} = C_1C_3\big(1 + \|\nabla\theta_\varepsilon\|^\mu_{L^\mu(I\times\Omega)}\big)^{1-\mu/2} I^{(2)}_a(\theta_\varepsilon)^{\mu/2}\,. \tag{3.43}$$

Furthermore, we are to estimate $I^{(2)}_a(\theta_\varepsilon)$ in (3.40). Like (3.26), we now use a primitive function $\mathfrak{X}_a$ to $\theta \mapsto \chi_a(\theta)E_\theta'(\boldsymbol{F},\theta)$ depending smoothly on $\boldsymbol{F}$, specifically

$$\mathfrak{X}_a(\boldsymbol{F},\theta) = \int_0^1 \theta\chi_a(r\theta)E_\theta'(\boldsymbol{F},r\theta)\,\mathrm{d}r\,. \tag{3.44}$$

To modify (3.27) appropriately, we use now the calculus

$$\begin{aligned}\int_\Omega \chi_a(\theta)\frac{\partial e}{\partial t}\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \chi_a(\theta)E_\theta'(\boldsymbol{F},\theta)\frac{\partial\theta}{\partial t} + \chi_a(\theta)E_{\boldsymbol{F}}'(\boldsymbol{F},\theta){:}\frac{\partial\boldsymbol{F}}{\partial t}\,\mathrm{d}\boldsymbol{x}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathfrak{X}_a(\boldsymbol{F},\theta)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \big[\mathscr{X}_a\big]_{\boldsymbol{F}}'(\boldsymbol{F},\theta){:}\frac{\partial\boldsymbol{F}}{\partial t}\,\mathrm{d}\boldsymbol{x}\\ &\qquad\text{where}\quad \mathscr{X}_a(\boldsymbol{F},\theta) := \mathfrak{X}_a(\boldsymbol{F},\theta) - \chi_a(\theta)E(\boldsymbol{F},\theta)\,.\end{aligned} \tag{3.45}$$

In view of (3.44), it holds $[\mathscr{X}_a]_{\boldsymbol{F}}'(\boldsymbol{F},\theta) = \int_0^1 \theta\chi_a(r\theta)E_{\boldsymbol{F}\theta}''(\boldsymbol{F},r\theta)\,\mathrm{d}r - \chi_a(\theta)E_{\boldsymbol{F}}'(\boldsymbol{F},\theta)$. Altogether, testing (3.9d) with (3.10) by $\chi_a(\theta_\varepsilon)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathfrak{X}_a(\boldsymbol{F}_\varepsilon,\boldsymbol{m}_\varepsilon,\theta_\varepsilon)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \chi_a'(\theta_\varepsilon)\kappa(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)|\nabla\theta_\varepsilon|^2\,\mathrm{d}\boldsymbol{x} = \int_\Omega\bigg(\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^2 + \nu_2|\nabla^2\boldsymbol{v}_\varepsilon|^p}{1{+}\varepsilon|\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^q{+}\varepsilon|\nabla\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^p}\chi_a(\theta_\varepsilon)$$

$$
- E(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\chi_a'(\theta_\varepsilon)\boldsymbol{v}_\varepsilon{\cdot}\nabla\theta_\varepsilon + \big[\mathscr{X}_a\big]_{\boldsymbol{F}}'(\boldsymbol{F}_\varepsilon,\theta_\varepsilon){:}\frac{\partial\boldsymbol{F}_\varepsilon}{\partial t} + \chi_a(\theta_\varepsilon)\boldsymbol{T}(\boldsymbol{F}_\varepsilon,\theta_\varepsilon){:}\boldsymbol{e}(\boldsymbol{v}_\varepsilon)\bigg)\,\mathrm{d}\boldsymbol{x} + \int_\Gamma h_\varepsilon\chi_a(\theta_\varepsilon)\mathrm{d}S. \tag{3.46}
$$

We realize that $\chi_a'(\theta) = a/(1{+}\theta)^{1+a}$ as used already in (3.40) and that $\mathcal{X}_a(\boldsymbol{F}_\varepsilon,\theta_\varepsilon) \ge c_K\theta_\varepsilon^{1+\alpha}$ with some c_K due to (3.5f); again K is a compact subset of $\mathrm{GL}^+(d)$ related here with the already proved estimates (3.23a) inherited for $\boldsymbol{F}_\varepsilon$. The convective term in (3.46) can be estimated, for any $\delta > 0$, as

$$
\begin{aligned}
\int_\Omega e_\varepsilon\chi_a'(\theta_\varepsilon)\boldsymbol{v}_\varepsilon{\cdot}\nabla\theta_\varepsilon\,\mathrm{d}\boldsymbol{x} &\le \frac{1}{\delta}\int_\Omega \chi_a'(\theta_\varepsilon)|\boldsymbol{v}_\varepsilon|^2 e_\varepsilon^2\,\mathrm{d}\boldsymbol{x} + \delta\int_\Omega \chi_a'(\theta_\varepsilon)|\nabla\theta_\varepsilon|^2\,\mathrm{d}\boldsymbol{x}\\
&= \frac{1}{\delta}\int_\Omega \chi_a'(\theta_\varepsilon)|\boldsymbol{v}_\varepsilon|^2 e_\varepsilon^2\,\mathrm{d}\boldsymbol{x} + \delta I_a^{(2)}(\theta_\varepsilon)\,.
\end{aligned} \tag{3.47}
$$

For c_K acting in (3.5g), using (3.46) integrated over $I = [0,T]$, we further estimate:

$$
\begin{aligned}
&I_a^{(2)}(\theta_\varepsilon) \le \frac{1}{ac_K}\int_0^T\!\!\int_\Omega \kappa(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon{\cdot}\nabla\chi_a(\theta_\varepsilon)\,\mathrm{d}\boldsymbol{x}\\
&\le \frac{1}{ac_K}\int_0^T\!\!\int_\Omega \kappa(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\nabla\theta_\varepsilon{\cdot}\nabla\chi_a(\theta_\varepsilon)\,\mathrm{d}\boldsymbol{x} + \frac{1}{ac_K}\int_\Omega \mathcal{X}_a(\boldsymbol{F}_\varepsilon(T),\theta_\varepsilon(T))\,\mathrm{d}\boldsymbol{x}\\
&\overset{(3.46)}{=}\frac{1}{ac_K}\bigg(\int_\Omega \mathcal{X}_a(\boldsymbol{F}_0,\theta_{0,\varepsilon})\,\mathrm{d}\boldsymbol{x} + \int_0^T\!\!\int_\Omega\bigg(\frac{\nu_1|\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^2+\nu_2|\nabla^2\boldsymbol{v}_\varepsilon|^p}{1{+}\varepsilon|\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^q{+}\varepsilon|\nabla\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^p}\chi_a(\theta_\varepsilon) - E(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\chi_a'(\theta_\varepsilon)\boldsymbol{v}_\varepsilon{\cdot}\nabla\theta_\varepsilon\\
&\qquad + \big[\mathscr{X}_a\big]_{\boldsymbol{F}}'(\boldsymbol{F}_\varepsilon,\theta_\varepsilon){:}\frac{\partial\boldsymbol{F}_\varepsilon}{\partial t} + \boldsymbol{T}(\boldsymbol{F}_\varepsilon,\theta_\varepsilon){:}\boldsymbol{e}(\boldsymbol{v}_\varepsilon)\chi_a(\theta_\varepsilon)\bigg)\,\mathrm{d}\boldsymbol{x} + \int_0^T\!\!\int_\Gamma h_\varepsilon\chi_a(\theta_\varepsilon)\,\mathrm{d}S\mathrm{d}t\bigg)\\
&\le \frac{1}{ac_K}\bigg(\big\|\mathcal{X}_a(\boldsymbol{F}_0,\theta_{0,\varepsilon})\big\|_{L^1(\Omega)} + \big\|\nu_1|\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^2+\nu_2|\nabla^2\boldsymbol{v}_\varepsilon|^p\big\|_{L^1(I\times\Omega)}\\
&\qquad + \int_0^T\big\|\big[\mathscr{X}_a\big]_{\boldsymbol{F}}'(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\big\|_{L^{r'}(\Omega;\mathbb{R}^{d\times d})}^{r'} + \bigg\|\frac{\partial\boldsymbol{F}_\varepsilon}{\partial t}\bigg\|_{L^r(\Omega;\mathbb{R}^{d\times d})}^r\mathrm{d}t + \big\|\boldsymbol{T}(\boldsymbol{F}_\varepsilon,\theta_\varepsilon){:}\boldsymbol{e}(\boldsymbol{v}_\varepsilon)\big\|_{L^1(I\times\Omega)}\\
&\qquad + \big\|h\big\|_{L^1(I\times\Gamma)} + \frac{1}{\delta}\|\boldsymbol{v}_\varepsilon\|_{L^2(I;L^\infty(\Omega;\mathbb{R}^d))}^2\|\chi_a'(\theta_\varepsilon)E^2(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\|_{L^\infty(I;L^1(\Omega))}\bigg) + \frac{\delta}{c_K}I_a^{(2)}(\theta_\varepsilon)\,.
\end{aligned} \tag{3.48}
$$

Here we used the bound of $\frac{\partial}{\partial t}\boldsymbol{F}_\varepsilon$ in $L^1(I;L^r(\Omega;\mathbb{R}^{d\times d}))$, as follows from due to (3.30c) with the exponent r arbitrarily large. By the qualification (3.5f), we have $|[\mathscr{X}_a]_{\boldsymbol{F}}'(\boldsymbol{F},\theta)| \le C(1{+}\theta^{1+\alpha})$, so that we need the estimation of the term $\|[\mathscr{X}_a]_{\boldsymbol{F}}'(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\|_{L^{r'}(\Omega;\mathbb{R}^{d\times d})}^{r'} \le C\|1{+}\theta_\varepsilon\|_{L^{r'(1+\alpha)}(\Omega)}^{r'(1+\alpha)}$ by using the already obtained estimate (3.38) and by the Gagliardo-Nirenberg inequality to interpolate $L^{r'(1+\alpha)}(\Omega)$ between $L^{1+\alpha}(\Omega)$ and $W^{1,\mu}(\Omega)$. Since $r' > 1$ is arbitrarily small, this estimate yields $\|\nabla\theta_\varepsilon\|_{L^\mu(I\times\Omega;\mathbb{R}^d)}^b$ with arbitrarily small $b > 0$. Due to (3.7), we can estimate the power of the conservative stress $\boldsymbol{T}(\boldsymbol{F}_\varepsilon,\theta_\varepsilon){:}\boldsymbol{e}(\boldsymbol{v}_\varepsilon)$ in (3.48) in $L^\infty(I;L^1(\Omega))$. The penultimate term in (3.48) is a-priori bounded independently of ε for $a > \alpha$ fixed because, as $E(\boldsymbol{F},\theta) = \mathscr{O}(\theta^{1+\alpha})$ due to (3.5f) and due to $\chi_a'(\theta) = \mathscr{O}(1/\theta^{1+\alpha})$ uniformly for $a > \alpha$. Let us emphasize that here we rely on that $a > \alpha$ as stated in (3.39). As a result, we have $[\chi_a'(\cdot)E^2(\boldsymbol{F},\cdot)](\theta) = \mathscr{O}(\theta^{1+\alpha})$. Thus the estimate (3.38) guarantees $\chi_a'(\theta_\varepsilon)E^2(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)$ bounded in $L^\infty(I;L^1(\Omega))$ while $|\boldsymbol{v}_\varepsilon|^2$ is surely bounded in $L^1(I;L^\infty(\Omega))$, cf. (3.23c) inherited for $\boldsymbol{v}_\varepsilon$ independently of $\varepsilon > 0$. Eventually, when choosing $\delta < c_K$, we can absorb the last term in the left-hand side. Similarly, combining it with (3.43), also the mentioned term $\|\nabla\theta_\varepsilon\|_{L^\mu(I\times\Omega;\mathbb{R}^d)}^b$ can be absorbed in the left-hand side of (3.43) because $b > 0$ is arbitrarily small.

Substituting λ from (3.42) into (3.41), by some algebra we obtain the bound $\mu \le (d{+}2{+}2\alpha{-}ad)/(d{+}1{+}\alpha)$, cf. also [16, Formula (8.2.18)]. Taking into account $a > \alpha$ in (3.39), we obtain the restriction on the integrability exponent for the temperature gradient:

$$\mu < \frac{d+2+(2{-}d)\alpha}{d+1+\alpha}\,. \tag{3.49}$$

Simultaneously, $\mu \ge 1$ is desirable, which needs $\alpha < 1/2$ if $d = 3$ or $\alpha < 1$ if $d = 2$ as assumed in (3.5e,f). Realizing the embedding $L^\infty(I; L^{1+\alpha}(\Omega)) \cap L^\mu(I; W^{1,\mu}(\Omega)) \subset L^s(I{\times}\Omega)$ with $1 \le s < \mu(1 + (1{+}\alpha)/d) = (d + 2 + (2{-}d)\alpha)/d$, in total we have proved

$$\big\|\theta_\varepsilon\big\|_{L^\infty(I;L^{1+\alpha}(\Omega))\cap L^r(I\times\Omega)} \le C \quad \text{with } \ 1 \le s < \frac{d{+}2{+}(2{-}d)\alpha}{d} \quad \text{ and} \tag{3.50a}$$

$$\big\|\nabla\theta_\varepsilon\big\|_{L^\mu(I\times\Omega;\mathbb{R}^d)} \le C \quad \text{ with } \ 1 \le \mu < \frac{d{+}2{+}(2{-}d)\alpha}{d{+}1{+}\alpha}\,. \tag{3.50b}$$

Exploiting the calculus $\nabla e_\varepsilon = E'_\theta(\boldsymbol{F}_\varepsilon, \theta_\varepsilon)\nabla\theta_\varepsilon + E'_{\boldsymbol{F}}(\boldsymbol{F}_\varepsilon, \theta_\varepsilon)\nabla\boldsymbol{F}_\varepsilon$ with $\nabla\boldsymbol{F}_\varepsilon$ bounded in $L^\infty(I; L^r(\Omega; \mathbb{R}^{d\times d\times d}))$ for and $1 \le r < +\infty$ and relying on the assumption (3.5f), we have also the bound on ∇e_ε in $L^\mu(I; L^{\mu(1+\alpha)/(1+\alpha+\alpha\mu)}(\Omega; \mathbb{R}^d))$, so that

$$\|e_\varepsilon\|_{L^\infty(I;L^1(\Omega))\cap L^\mu(I;W^{1,\mu(1+\alpha)/(1+\alpha+\alpha\mu)}(\Omega))} \le C_\mu \quad \text{with } \ \mu \ \text{ from (3.50b)}\,. \tag{3.50c}$$

Since $e_\varepsilon = E(\boldsymbol{F}_\varepsilon, \theta_\varepsilon)$ and $E(\boldsymbol{F}, \cdot)$ is bounded from below due to (3.5d) and has at most $(1{+}\alpha)$-polynomial growth (3.6), uniformly for $\boldsymbol{F} \in K$, we also have

$$\|e_\varepsilon\|_{L^{s/(1+\alpha)}I\times\Omega)} \le C \quad \text{with } \ s \ \text{ from (3.50a)}\,. \tag{3.50d}$$

Step 6: Convergence for $\varepsilon \to 0$. Using the Banach selection principle as in Step 4, now also taking the estimates (3.20)–(3.21) and (3.23)–(3.25) inherited for $(\boldsymbol{v}_\varepsilon, \boldsymbol{F}_\varepsilon, \theta_\varepsilon)$ and (3.50) into account, we can extract some subsequence of $\{(\varrho_\varepsilon, \boldsymbol{v}_\varepsilon, \boldsymbol{F}_\varepsilon, \theta_\varepsilon, e_\varepsilon)\}_{\varepsilon>0}$ and its limit $(\varrho, \boldsymbol{v}, \boldsymbol{F}, \theta, e)$: $I \to H^1(\Omega) \times L^2(\Omega; \mathbb{R}^d) \times H^1(\Omega; \mathbb{R}^{d\times d}) \times L^{1+\alpha}(\Omega) \times L^1(\Omega)$ such that, for any $1 \le r < +\infty$,

$$\varrho_\varepsilon \to \varrho \qquad \text{weakly* in } \ L^\infty(I; W^{1,r}(\Omega)) \cap W^{1,p}(I; L^r(\Omega)) \quad \text{and strongly in } C(I{\times}\overline{\Omega})\,, \tag{3.51a}$$

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v} \qquad \text{weakly* in } \ L^\infty(I; L^2(\Omega; \mathbb{R}^d)) \cap L^p(I; W^{2,p}(\Omega; \mathbb{R}^d))\,, \tag{3.51b}$$

$$\boldsymbol{F}_\varepsilon \to \boldsymbol{F} \qquad \text{weakly* in } \ L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d\times d})) \cap W^{1,p}(I; L^r(\Omega; \mathbb{R}^{d\times d})) \quad \text{and strongly in } C(I{\times}\overline{\Omega}; \mathbb{R}^{d\times d})\,, \tag{3.51c}$$

$$e_\varepsilon \to e \qquad \text{weakly in } \ L^\mu(I; W^{1,\mu(1+\alpha)/(1+\alpha+\alpha\mu)}(\Omega))\,, \text{ and} \tag{3.51d}$$

$$\theta_\varepsilon \to \theta \qquad \text{weakly in } \ L^\mu(I; W^{1,\mu}(\Omega)) \ \text{ with } \ \mu \ \text{ from (3.50b).} \tag{3.51e}$$

We now also have the estimate of $\frac{\partial}{\partial t}e_\varepsilon$ in $L^1(I; H^3(\Omega)^*)$ relying on $H^3(\Omega) \subset W^{1,\infty}(\Omega)$ for $d \le 3$. So, like (3.33) and using (3.50d), by the Aubin-Lions theorem we now have

$$e_\varepsilon \to e \qquad \text{strongly in } L^{s/(1+\alpha)}(I{\times}\Omega) \ \text{ with } \ s \ \text{ from (3.50a)}\,. \tag{3.52a}$$

Then, we realize that $u_\varepsilon = U(\boldsymbol{F}_\varepsilon, \theta_\varepsilon)$ with $u_\varepsilon = e_\varepsilon - E(\boldsymbol{F}_\varepsilon, 0)$ converges to $u = e - E(\boldsymbol{F}, 0) = U(\boldsymbol{F}, \theta)$ strongly in $L^{s/(1+\alpha)}(I{\times}\Omega)$ and, like in (3.35), we use again continuity of $(\boldsymbol{F}, u) \mapsto [U_{\mathrm{M}}(\boldsymbol{F}, \cdot)]^{-1}(u)$ and realize that actually $[U_{\mathrm{M}}(\boldsymbol{F}, \cdot)]^{-1}(u_\varepsilon) = [U(\boldsymbol{F}, \cdot)]^{-1}(u_\varepsilon)$ since $u_\varepsilon \ge 0$. By this arguments, we also have

$$\theta_\varepsilon \to \theta = [U(\boldsymbol{F}, \cdot)]^{-1}(u) \ \text{ strongly in } \ L^s(I{\times}\Omega) \ \text{ with } \ s \ \text{ again from (3.50a).} \tag{3.52b}$$

By the continuity of $\boldsymbol{T}(\cdot,\cdot)$ and of κ, we have also

$$\boldsymbol{T}(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\to\boldsymbol{T}(\boldsymbol{F},\theta)\ \text{strongly in}\ L^r(I{\times}\varOmega;\mathbb{R}^{d\times d}_{\rm sym})\ \text{for any}\ 1\le r<\frac{d{+}2{+}(2{-}d)\alpha}{d(1{+}\alpha)},\ \text{and}\tag{3.52c}$$

$$\kappa(\boldsymbol{F}_\varepsilon,\theta_\varepsilon)\to\kappa(\boldsymbol{F},\theta)\quad\text{strongly in}\ L^r(I{\times}\varOmega)\ \text{for any}\ 1\le r<\infty.\tag{3.52d}$$

For the exponent in (3.52c), we used the estimate (3.50a) together with the energy-controlled-stress condition (3.5d) exploiting also (3.6). Also the strong convergence of velocity gradients like (3.36c,d) can be proved for $\boldsymbol{v}_\varepsilon$ instead of $\boldsymbol{v}_{\varepsilon k}$ analogously as in Step 4; for details concerning a slight analytical modification see [32] where, in particular, the information about the boundedness of $\frac{\partial}{\partial t}(\varrho_\varepsilon\boldsymbol{v}_\varepsilon)$ in $L^2(I;H^1(\varOmega;\mathbb{R}^d)^*)+L^{p'}(I;W^{2,p}(\varOmega;\mathbb{R}^d)^*)$ is exploited.

The limit passage in the weak formulation of the system (3.9) without the terms $\boldsymbol{v}/k$ and $\frac{\partial}{\partial t}\theta/k$ towards the solution of the system (2.11) according Definition 3.1 is then easy by continuity and convergence $h_\varepsilon\to h$ in $L^1(I{\times}\varGamma)$. $\square$

Remark 3.3 (*Fourier boundary conditions)*. The last boundary condition (2.12) can be generalized towards the Robin-type (in the heat context also called Fourier) condition $\kappa(\boldsymbol{F},\theta)\nabla\theta{\cdot}\boldsymbol{n}+b\theta=h$ with $b\in L^\infty(\varGamma)$ non-negative. Then (3.10) should be modified as $\kappa(\boldsymbol{F},\theta)\nabla\theta{\cdot}\boldsymbol{n}+b\theta^+=h_\varepsilon$, which would work similarly as ψ and κ satisfying (3.5c) in the Galerkin approximation where also the negative values of temperature may occur. The limit passage in Step 6 would then rely on the compactness of the trace operator $W^{1,\mu}(\varOmega)\to L^1(\varGamma)$ with $\mu>1$ from (3.49).

Remark 3.4 (*Energy balances)*. In fact, any solution according Definition 3.1 that possesses the regularity as specified in Theorem 3.2 satisfies the energy balances (2.13), (2.14), and (2.20) integrated over a current time interval $[0,t]$. It is important that the above tests of the four equations in (2.11) successively by $\frac{1}{2}|\boldsymbol{v}|^2$, $\boldsymbol{v}$, $\phi'(\boldsymbol{F})$, and 1 are analytically legitimate; here we again refer to [32]. On the other hand, the non-negativity of the temperature requires the test of the internal-energy equation (2.11d) by θ^- which is not in duality with $\dot{e}$ in general so that, we should realize that, rigorously, we have proved this non-negativity only for at least one solution. Finally, note that the entropy balance (2.16) needs positivity of θ, which is not obvious and likely very nontrivial to prove even for the non-negative temperature mentioned above, because the heat capacity in our Eulerian formulation inevitably also depends on $\boldsymbol{F}$, which complicates the usual arguments.

Remark 3.5 (*Smoothness of ψ)*. Noteworthy, the assumptions (3.5e) needs existence of $\psi''_{\theta\theta}$ and (3.5f) needs $\psi''_{\boldsymbol{F}\theta}$ and even $\psi'''_{\boldsymbol{F}\theta\theta}$ while the system (2.11) itself involves only $\psi'_{\boldsymbol{F}}$ and ψ'_θ. In fact, more careful formulation of the assumptions (3.5)e,f) and a suitable mollification of such less smooth ψ would allow for using Theorem 3.2 followed by another limit passage might allow for a discontinuous $\psi''_{\theta\theta}$, which could describe 2nd-order phase transitions.

Remark 3.6 (*Exploiting the entropy balance – an open problem)*. One may be tempted to use the entropy equation (2.16) for a direct estimation of the temperature gradient in order to avoid, like [10] does, the use of the technically demanding L^1-theory for the heat equation based on [3,4]. We recall that (2.16) was obtained by using the test of (2.11d) by $1/\theta$, i.e. by a so-called *coldness*. However, such nonlinear test is incompatible with a Galerkin approximation, so one is tempted to formulate the system directly in terms of the coldness, following the idea of Truesdell [38] who articulated that 'in continuum mechanics the reciprocal of the temperature, which may be called the coldness ϑ, is often more convenient for the mathematical theory'. This would potentially allow for omitting the intermediate test by θ in Step 3 and the technically demanding L^1-theory for estimating $\nabla\theta$ in Step 5 in the above proof. In terms of $\vartheta := 1/\theta$, one can define $\widehat{\psi}(\boldsymbol{F},\vartheta) := \psi(\boldsymbol{F},1/\vartheta)$, $\widehat{E}(\boldsymbol{F},\vartheta) := E(\boldsymbol{F},1/\vartheta)$, $\widehat{\eta}(\boldsymbol{F},\vartheta) := \eta(\boldsymbol{F},1/\vartheta)$, $\widehat{\kappa}(\boldsymbol{F},\vartheta) := \kappa(\boldsymbol{F},1/\vartheta)$, and the Cauchy stress $\widehat{\boldsymbol{T}}(\boldsymbol{F},\vartheta) := \widehat{\psi}_{\boldsymbol{F}}'(\boldsymbol{F},\vartheta)\boldsymbol{F}^\top + \widehat{\psi}(\boldsymbol{F},\vartheta)\mathbb{I}$. The relation between $\widehat{E}$ and $\widehat{\psi}$ is

$$\begin{aligned}\widehat{E}(\boldsymbol{F},\vartheta) := E\Big(\boldsymbol{F},\frac{1}{\vartheta}\Big) &= \psi\Big(\boldsymbol{F},\frac{1}{\vartheta}\Big) - \frac{1}{\vartheta}\psi_\theta'\Big(\boldsymbol{F},\frac{1}{\vartheta}\Big) = \widehat{\psi}(\boldsymbol{F},\vartheta) - \frac{1}{\vartheta}\psi_\theta'\Big(\boldsymbol{F},\frac{1}{\vartheta}\Big)\\ &= \widehat{\psi}(\boldsymbol{F},\vartheta) - \vartheta\frac{1}{\vartheta^2}\psi_\theta'\Big(\boldsymbol{F},\frac{1}{\vartheta}\Big) = \widehat{\psi}(\boldsymbol{F},\vartheta) + \vartheta\widehat{\psi}_\vartheta'(\boldsymbol{F},\vartheta)\,.\end{aligned}$$

The heat equation in terms of the internal energy (2.11d) is then transformed to

$$\dot{e} = \mathrm{div}\big(\widehat{\kappa}(\boldsymbol{F},\vartheta)\nabla\vartheta\big) - (\mathrm{div}\,\boldsymbol{v})e - \nu_1|\boldsymbol{e}(\boldsymbol{v})|^2 - \nu_2|\nabla^2\boldsymbol{v}|^p - \widehat{\boldsymbol{T}}(\boldsymbol{F},\vartheta){:}\nabla\boldsymbol{v}$$

with $e = \widehat{E}(\boldsymbol{F},\vartheta)$. This equation well respects physics and can be discretized by the Galerkin method, allowing the test by 1 to obtain the total-energy equation (from which one can obtain $\vartheta \ge 0$ even in the discrete approximation due to the blowup of $\widehat{E}(\boldsymbol{F},\cdot)$ at $\vartheta = 0$ and also an estimate for $\widehat{\eta}$) and by ϑ relying on non-negativity of entropy, from which one can get an L^2-estimate of $\sqrt{\widehat{\kappa}(\boldsymbol{F},\vartheta)}\nabla\vartheta$ and, in some sense, also the dissipation rate. Then, from the mechanical-energy balance as (2.13), one can improve the estimate of the dissipation rate. This seems a promissing strategy but, having in mind Example 4.1 below, $\widehat{E}(\boldsymbol{F},\cdot)$ has a singularity like $\vartheta^{-1-\alpha}$ so that estimation of $\nabla e = \widehat{E}_{\boldsymbol{F}}'(\boldsymbol{F},\vartheta){:}\nabla\boldsymbol{F} + \widehat{E}_\vartheta'(\boldsymbol{F},\vartheta)\nabla\vartheta$ calls for a singularity of $\widehat{\kappa}(\boldsymbol{F},\cdot)$ like $\vartheta^{-3-\alpha}$, which eventually causes problems with convergence of the Galerkin approximation.

4 Examples and remarks

Let us complete this article with physically relevant examples that comply with the assumptions (3.5). We will formulate them in terms of the referential free energy $\uppsi$ as used in Remark 2.2, which will also be defined for negative temperature for the sake of the Galerkin method as used in the proof of Theorem 3.2. In general, such that $\uppsi(\boldsymbol{F},\cdot)$ is to be concave not to conflict with non-negativity of the heat capacity. We distinguish the actual free energy, the actual internal energy, and the actual heat capacity, i.e. respectively

$$\psi(\boldsymbol{F},\theta) = \frac{\uppsi(\boldsymbol{F},\theta)}{\det\boldsymbol{F}}\,,\qquad E(\boldsymbol{F},\theta) = \frac{\upomega(\boldsymbol{F},\theta)}{\det\boldsymbol{F}}\,,\quad\text{and}\quad c(\boldsymbol{F},\theta) = \frac{\mathrm{c}(\boldsymbol{F},\theta)}{\det\boldsymbol{F}} \tag{4.1}$$

for $\det\boldsymbol{F} > 0$. In (4.1), we have used the referential internal energy $\upomega(\boldsymbol{F},\theta) = \uppsi(\boldsymbol{F},\theta) + \theta\upeta(\boldsymbol{F},\theta)$ with the referential entropy $\upeta(\boldsymbol{F},\theta) = -\uppsi_\theta'(\boldsymbol{F},\theta)$, and the referential heat capacity $\mathrm{c}(\boldsymbol{F},\theta) = \theta\upeta_\theta'(\boldsymbol{F},\theta) = -\theta\uppsi_{\theta\theta}''(\boldsymbol{F},\theta)$.

Example 4.1 (*A free energy of the neo-Hookean type*)**.** Denoting by K_E and G_E the elastic bulk and shear moduli, respectively, $c_\mathrm{V} > 0$ a referential heat capacity at a reference temperature $\theta_0 > 0$ at constant volume, and the exponent $\alpha > 0$, a physically relevant example is

$$\uppsi(\boldsymbol{F},\theta) = \upphi(\boldsymbol{F}) - \frac{c_\mathrm{V}\theta}{\alpha(1{+}\alpha)}\Big(\frac{\theta^+}{\theta_0}\Big)^\alpha \quad\text{with}\quad \upphi(\boldsymbol{F}) = \frac{1}{2}K_\mathrm{E}\big(\det\boldsymbol{F}{-}1\big)^2 + G_\mathrm{E}\frac{\mathrm{tr}(\boldsymbol{F}\boldsymbol{F}^\top)}{(\det\boldsymbol{F})^{2/d}}\,. \tag{4.2}$$

The actual entropy $\eta(\boldsymbol{F},\theta)=\upeta(\boldsymbol{F},\theta)/\det\boldsymbol{F}=-\psi'_\theta(\boldsymbol{F},\theta)$ is then

$$\eta(\boldsymbol{F},\theta)=\frac{c_{\rm V}}{\alpha\det\boldsymbol{F}}\Big(\frac{\theta^+}{\theta_0}\Big)^\alpha. \tag{4.3}$$

In particular, $\eta(\boldsymbol{F},0)=0$, i.e. it complies with the 3rd law of thermodynamics, which stays that entropy is zero and, in particular, independent of the mechanical state at zero temperature. Furthermore, the actual heat capacity $c(\boldsymbol{F},\theta)=\theta\eta'_\theta(\boldsymbol{F},\theta)$ is

$$c(\boldsymbol{F},\theta)=\frac{c_{\rm V}}{\det\boldsymbol{F}}\Big(\frac{\theta^+}{\theta_0}\Big)^\alpha \tag{4.4}$$

and obviously vanishes at zero temperature. Note that we defined $\uppsi$ also for negative temperatures (cf. Figure 2-left) in order to cope with the Galerkin approximation of the internal-energy equation in the above proof, which gives vanishing heat capacity for $\theta<0$ without affecting the sensibility of the limit problem where one can prove that temperature will actually not go below 0. The actual internal energy $E(\boldsymbol{F},\theta)=\psi(\boldsymbol{F},\theta)+\theta\eta(\boldsymbol{F},\theta)$ is then

$$E(\boldsymbol{F},\theta)=\frac{K_{\rm E}}{2\det\boldsymbol{F}}\big(\det\boldsymbol{F}{-}1\big)^2+\frac{G_{\rm E}{\rm tr}(\boldsymbol{F}\boldsymbol{F}^\top)}{(\det\boldsymbol{F})^{1+2/d}}+\frac{c_{\rm V}\theta}{(1{+}\alpha)\det\boldsymbol{F}}\Big(\frac{\theta^+}{\theta_0}\Big)^\alpha\quad\text{for}\quad\det\boldsymbol{F}>0 \tag{4.5}$$

while $E(\boldsymbol{F},\theta)=+\infty$ for $\det\boldsymbol{F}\le0$. In contrast to the referential internal energy $\upomega$, this actual internal energy contains the term $K_{\rm E}/(2\det\boldsymbol{F})$ and thus exhibits the physically relevant blow-up

$$E(\boldsymbol{F},\theta)\to+\infty\quad\text{ if }\quad\det\boldsymbol{F}\to0+\,. \tag{4.6}$$

This ensures $\det\boldsymbol{F}>0$ and thus to prevent local non-interpenetration directly from the stored energy (uniformly in time) and not only indirectly by the estimate (3.23a) which degenerates if the time horizon $T\to\infty$. It is also important that the internal energy (4.5) is non-negative even for negative values of temperature. The Cauchy stress $\boldsymbol{T}$ is

$$\boldsymbol{T}=\frac{\uppsi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top}{\det\boldsymbol{F}}=\psi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top{+}\psi(\boldsymbol{F},\theta)\mathbb{I}=K_{\rm E}(\det\boldsymbol{F}{-}1)\mathbb{I}+2G_{\rm E}\frac{\boldsymbol{F}\boldsymbol{F}^\top{-}\,{\rm tr}(\boldsymbol{F}\boldsymbol{F}^\top)\mathbb{I}/d}{(\det\boldsymbol{F})^{1+2/d}}\,; \tag{4.7}$$

here we used Cramer's rule $F^{-1}={\rm Cof}\,F^\top/\det F$ together with the calculus $\det'(F)={\rm Cof}\,F$. This stress complies trivially with the energy-controlled-stress condition (3.5d) and $\boldsymbol{T}_1\equiv\boldsymbol{0}$.

Remark 4.2 (*The energy-controlled Kirchhoff stress*)**.** Since additive constants in $\uppsi$ and then in E are actually irrelevant in the thermodynamical system (2.11), the energy control (3.5d) can equally be written simply as $|\boldsymbol{T}(\boldsymbol{F},\theta)|\le CE(\boldsymbol{F},\theta)$ when assuming a suitable choice of an additive constant in $\uppsi$. In terms of the referential free energy, this condition means $|\uppsi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top/\det\boldsymbol{F}|\le C\upomega(\boldsymbol{F},\theta)/\det\boldsymbol{F}$ with the referential internal energy $\upomega(\boldsymbol{F},\theta)=\uppsi(\boldsymbol{F},\theta)-\theta\uppsi'_\theta(\boldsymbol{F},\theta)$. Avoiding a suitable choice of the mentioned additive constants, this can be written (with a certain tolerance to physical dimensions) as

$$\big|\underbrace{\uppsi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)\boldsymbol{F}^\top}_{\text{Kirchhoff stress}}\big|\le C\big(1+\upomega(\boldsymbol{F},\theta)\big)\,, \tag{4.8}$$

as formulated (for the isothermal case) in by J.M. Ball [1, 2], being devised for usage in the referential Lagrangian frame and employed e.g. in [11, 20]. It was shown in [2, Prop. 2.3] that (4.8) is implied by such energy control of the Mandel stress $\boldsymbol{F}^\top\uppsi'_{\boldsymbol{F}}(\boldsymbol{F},\theta)$. In [20], such conditions have been used under the name *multiplicative stress control.*

Remark 4.3 (*The free energy blowup (4.6)*). Sometimes, (4.2) is expanded by a term $-K_0\ln(\det \boldsymbol{F})$ with some (presumably small) modulus $K_0 > 0$ in J/m^3=Pa to articulate the blow-up (4.6) not only in the actual Eulerian frame but even in the reference frame. Specifically,

$$\psi(\boldsymbol{F},\theta) = \frac{1}{2}K_{\rm E}\Big(\det \boldsymbol{F}{-}1{+}\frac{K_0}{K_{\rm E}}\Big)^2 + G_{\rm E}\frac{{\rm tr}(\boldsymbol{F}\boldsymbol{F}^\top)}{(\det \boldsymbol{F})^{2/d}} - \frac{c_{\rm V}\theta}{\alpha(1{+}\alpha)}\Big(\frac{\theta^+}{\theta_0}\Big)^\alpha - K_0\ln(\det \boldsymbol{F}) \tag{4.9}$$

for $\det \boldsymbol{F} > 0$ while $\psi(\boldsymbol{F},\theta) = +\infty$ for $\det \boldsymbol{F} \le 0$. Note that $\psi(\cdot,\theta)$ from (4.9) is minimized on the orbit SO(d) as also (4.2) is. This additional term $-K_0\ln(\det \boldsymbol{F})$ in (4.9) also expands the internal energy (4.5) and yields the additional hydrostatic-pressure-type stress $K_0\mathbb{I}/\det \boldsymbol{F}$, which complies with the energy-controlled-stress condition (3.5d). Sometimes, a similar term $K_0/\det \boldsymbol{F}^m$ with $m > 0$ in ψ is considered instead, referring to the Ogden hyperelastic model and leading to similar effects and again complying with (3.5d), cf. [11, Sec. 5]. This is often used in the Lagrangian formulation where local non-interpenetration is ensured differently than here, specifically by a higher-order gradient in the stored energy, and a faster blowup than only $1/\det \boldsymbol{F}$ in Example 4.1 is needed to exploit the results by T.J. Healey and S. Krömer [14], cf. also [16].

Example 4.4 (*Volumetric thermal expansion*). Augmenting the free energy (4.2) by the term such as $-\beta K_{\rm E}\theta^+\ln(\det \boldsymbol{F})$ with $\beta > 0$ a volumetric-expansion coefficient in K^{-1} does not violate concavity of $\psi(\boldsymbol{F},\cdot)$ and allows for modelling thermal expansion when one realizes that the minimum with respect to $\det \boldsymbol{F}$ of $\psi(\cdot,\theta)$ is at $\det \boldsymbol{F} = 1 + \beta\theta + o(\theta)$. Yet, such $\psi(\cdot,\theta)$ is nonsmooth at $\theta = 0$, so we should rather consider some smooth ansatz, say

$$\psi(\boldsymbol{F},\theta) = \phi(\boldsymbol{F}) - \frac{c_{\rm V}\theta}{\alpha(1{+}\alpha)}\Big(\frac{\theta^+}{\theta_0}\Big)^\alpha - \beta K_{\rm E}\frac{(\theta^+)^2}{\theta^+{+}\theta_{\rm R}}\ln(\det \boldsymbol{F}) \tag{4.10}$$

with the stored energy ϕ from (4.2) and with some fixed (presumably small) parameter $\theta_{\rm R} > 0$. The last term in (4.10) contributes to the referential entropy by $\beta K_{\rm E}\theta^+(\theta{+}2\theta_{\rm R})\ln(\det \boldsymbol{F})/(\theta^+{+}\theta_{\rm R})^2$ so that the 3rd law of thermodynamics is not corrupted. Its contribution to the referential internal energy is $-\beta K_{\rm E}(\theta^+)^2(\theta{+}2\theta_{\rm R})\ln(\det \boldsymbol{F})/(\theta^+{+}\theta_{\rm R})^2$, which is small if $\theta_{\rm R} > 0$ is small and which does not violate non-negativity of the heat part of the internal energy $U(\boldsymbol{F},\cdot)$. The contribution to the referential heat capacity $2\beta K_{\rm E}\theta^+\theta_{\rm R}\ln(\det \boldsymbol{F})/(\theta^+{+}\theta_{\rm R})^2$ is bounded (uniformly for $\theta_{\rm R} > 0$ and, if $\theta_{\rm R} > 0$ is small, manifests itself rather only on a small neighbourhood of the zero temperature. Eventually, its contribution to the Cauchy stress would be of the hydrostatic-pressure character, namely $\boldsymbol{T}_1 = -\beta K_{\rm E}(\theta^+\theta/(\theta^+{+}\theta_{\rm R}))\mathbb{I}$.

Example 4.5 (*A bounded heat capacity*). An interesting modification of a concave referential free energy (4.2) satisfying $\psi'_\theta(\boldsymbol{F},0) = 0$ and complying with the assumptions (3.5c–f) for $\alpha = 0$ is

$$\psi(\boldsymbol{F},\theta) = \phi(\boldsymbol{F}) - c_{\rm V}\Big((\theta^+{+}\theta_{\rm R})\ln(\theta^+{+}\theta_{\rm R}) - \theta^+\big(1{+}\ln(\theta_{\rm R})\big) - \theta_{\rm R}\ln(\theta_{\rm R})\Big)\,, \tag{4.11}$$

where $c_{\rm V} > 0$ represents a finite heat capacity asymptotical for $\theta \to \infty$ for undeformed medium. Like in Example 4.4, $\theta_{\rm R} > 0$ is a fixed (presumably small) temperature-like parameter. The corresponding actual entropy is

$$\eta(\boldsymbol{F},\theta) = -\psi'_\theta(\boldsymbol{F},\theta) = \frac{c_{\rm V}}{\det \boldsymbol{F}}\ln\Big(1 + \frac{\theta^+}{\theta_{\rm R}}\Big)$$

and the actual internal energy $E(\boldsymbol{F},\theta) = \psi(\boldsymbol{F},\theta) + \theta\eta(\boldsymbol{F},\theta)$ is

$$E(\boldsymbol{F},\theta) = \frac{\phi(\boldsymbol{F})}{\det \boldsymbol{F}} + \frac{c_{\rm V}}{\det \boldsymbol{F}}\bigg(\theta^+ + \theta_{\rm R}\ln\Big(1 + \frac{\theta^+}{\theta_{\rm R}}\Big)\bigg)\,. \tag{4.12}$$

Realizing that $\psi''_{\theta\theta}(\boldsymbol{F},\theta) = -(c_{\rm V}/\theta_{\rm R})/(1{+}\theta^+/\theta_{\rm R})$, we obtain a bounded increasing actual heat capacity $c = c(\boldsymbol{F},\cdot)$ with $c(\boldsymbol{F},0) = 0$ and $c(\boldsymbol{F},+\infty) = c_{\rm V}/\det \boldsymbol{F}$, namely

$$c(\boldsymbol{F},\theta) = \frac{c_{\rm V}\theta^+}{(\theta^+{+}\theta_{\rm R})\det \boldsymbol{F}}\,. \tag{4.13}$$

A comparison of this model with Example 4.1 is in Figure 2.

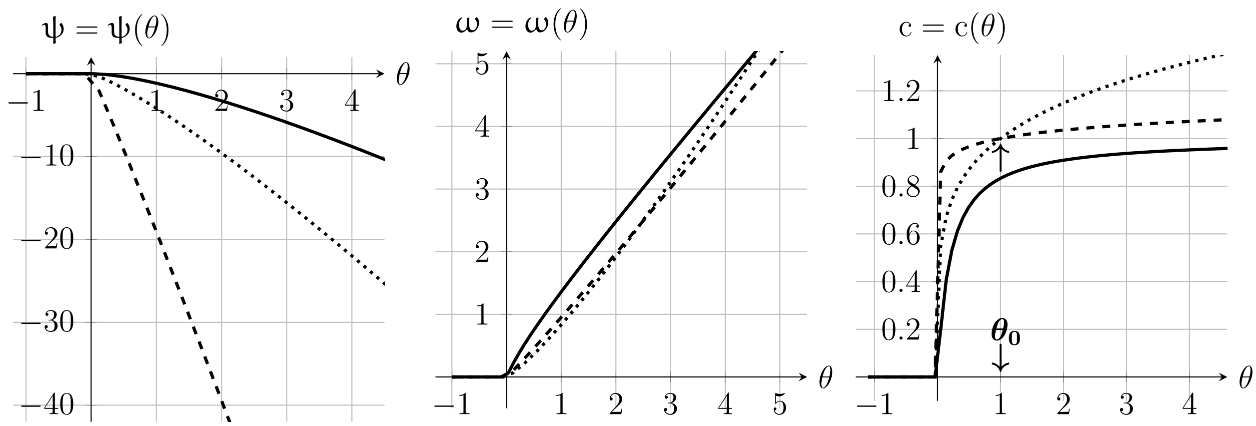

Figure 2: *A comparison of Examples 4.1 and 4.5 with* $c_{\rm V} = 1$ *ignoring the mechanical part, i.e.* $\phi = 0$. *The former example is depicted for* $\alpha = 0.05$ *(dashed lines) and for* $\alpha = 0.2$ *(dotted lines) while the latter example is depicted for* $\theta_{\rm R} = 0.2$ *(solid lines). Both examples yield a similar internal energy* E *and a similar heat capacity* c *for* $\alpha \to 0$ *and* $\theta_{\rm R} \to 0$.

Example 4.6 (*The phase transition in shape-memory alloys*)**.** The energy ψ from (4.2) has a single well in the sense that the minimum $\psi(\cdot,\theta)$ is attained on the orbit $\mathrm{SO}(d)$. A multi-well modification is used to model a so-called martensitic transformation in shape-memory alloys on the level of single crystals, cf. [28] for an overview. One considers, beside one so-called austenitic well as in (4.2), also L lower-symmetrical wells to models variants of so-called martensite. In particular, $L = 3$ for the tetragonal martensite while $L = 6$ in case of the orthorhombic martensite or $L = 12$ for the monoclinic martensite. The stress-free geometric configuration of particular phases is characterized by the distortion $\mathbb{F}_\ell$ with $\ell = 0, 1, ..., L$. For particular phases, the ansatz (4.2) is the modified as

$$\psi_\ell(\boldsymbol{F},\theta) = \phi_\ell(\boldsymbol{F}) - \frac{c_\ell\theta}{\alpha(1{+}\alpha)}\Big(\frac{\theta^+}{\theta_0}\Big)^\alpha \quad \text{with} \quad \phi_\ell(\boldsymbol{F}) = \frac{1}{2}K_\ell\big(\det \boldsymbol{F}{-}1\big)^2 + G_\ell\frac{\mathrm{tr}(\mathbb{F}_\ell^{-\top}\boldsymbol{F}\boldsymbol{F}^\top\mathbb{F}_\ell^{-1})}{(\det \boldsymbol{F})^{2/d}}\,.$$

The minimum of $\psi_\ell(\cdot,\theta)$ is attained on the orbit $\mathrm{SO}(d)\boldsymbol{F}_\ell$. For austenite, $\mathbb{F}_0 = \mathbb{I}$ while $\mathbb{F}_\ell \neq \mathbb{I}$ for $\ell = 1, ..., L$ and, since martensitic transformation is essentially isochoric, $\det \mathbb{F}_\ell = 1$ for $\ell = 1, ..., L$. The overall referential multi-well free energy is

$$\psi(\boldsymbol{F},\theta) = -\varkappa \ln\bigg(\sum_{\ell=0}^{L} \mathrm{e}^{-\psi_\ell(\boldsymbol{F},\theta)/\varkappa}\bigg)\,, \tag{4.14}$$

where $\varkappa$ is a constant with the physical dimension J/m^3=Pa; one can think about the Boltzmann constant (related per unit volume) times a reference temperature. Such $\uppsi(\cdot,\theta)$ exhibits the *multi-well* character and is backed up by statistical physics. The so-called shape-memory effect is modelled by considering $\mathrm{c}_0 > \mathrm{c}_1 = ... = \mathrm{c}_L$, which causes that the well of austenite falls energetically faster than the well of martensitic variants within increasing temperature, which makes austenite energetically dominant at high temperatures while martensite is dominant at lower temperatures. The Cauchy stress corresponding to (4.14) is

$$\boldsymbol{T}(\boldsymbol{F},\theta) = \frac{\sum_{\ell=0}^{L} \mathrm{e}^{-\uppsi_\ell(\boldsymbol{F},\theta)/\varkappa} \upphi'_\ell(\boldsymbol{F})}{\sum_{\ell=0}^{L} \mathrm{e}^{-\uppsi_\ell(\boldsymbol{F},\theta)/\varkappa} \det \boldsymbol{F}} \boldsymbol{F}^\top \tag{4.15}$$

while the actual entropy $\eta = -\psi'_\theta(\boldsymbol{F},\theta) = -\uppsi'_\theta(\boldsymbol{F},\theta)/\det \boldsymbol{F}$ can be evaluated as

$$\eta(\boldsymbol{F},\theta) = \frac{\sum_{\ell=0}^{L} \mathrm{e}^{-\psi_\ell(\boldsymbol{F},\theta)/\varkappa} \eta_\ell(\boldsymbol{F},\theta)}{\sum_{\ell=0}^{L} \mathrm{e}^{-\psi_\ell(\boldsymbol{F},\theta)/\varkappa}} \quad \text{with} \quad \eta_\ell(\boldsymbol{F},\theta) = -\frac{\mathrm{c}_\ell}{\alpha \det \boldsymbol{F}} \Big(\frac{\theta^+}{\theta_0}\Big)^\alpha, \tag{4.16}$$

and the actual internal energy $e = \psi + \theta\eta$ is

$$e = E(\boldsymbol{F},\theta) = \frac{\sum_{\ell=0}^{L} \mathrm{e}^{-\uppsi_\ell(\boldsymbol{F},\theta)/\varkappa} E_\ell(\boldsymbol{F},\theta)}{\sum_{\ell=0}^{L} \mathrm{e}^{-\uppsi_\ell(\boldsymbol{F},\theta)/\varkappa}} \quad \text{with} \quad E_\ell(\boldsymbol{F},\theta) = \psi_\ell(\boldsymbol{F},\theta) + \theta\eta_\ell(\boldsymbol{F},\theta)\,, \tag{4.17}$$

where naturally $\psi_\ell(\boldsymbol{F},\theta) = \uppsi_\ell(\boldsymbol{F},\theta)/\det \boldsymbol{F}$. Let us remark that the above isotropic neo-Hookean $\upphi_\ell$ is typically considered rather anisotropic, reflecting lower-symmetrical character of particular martensitic variants for $\ell = 1, ..., L$.

Remark 4.7 (*Nonphysical temperature-independent heat capacity*)**.** Often, the specific heat capacity is considered nearly constant for temperatures far from the absolute zero (so-called *Dulong–Petit*'s *law*), which suggests $\alpha > 0$ in (4.2) or $\theta_\mathrm{R} > 0$ in (4.10) or in (4.11) to be very small, cf. also Figure 2-right. On the other hand, the above examples do not work for $\alpha = 0$ nor for $\theta_\mathrm{R} = 0$. To obtain the truly temperature-independent heat capacity, the thermal term in the free energy (4.2) or in (4.11) is to be modified as

$$\uppsi(\boldsymbol{F},\theta) = \upphi(\boldsymbol{F}) - c_\mathrm{v}\theta \ln\Big(\frac{\theta}{\theta_0}\Big)\,. \tag{4.18}$$

Although this is the frequently considered ansatz in engineering and mathematical literature, the corresponding actual entropy $c_\mathrm{v}\ln(\theta/\theta_0)/\det \boldsymbol{F}$ obviously conflicts with the 3rd law of thermodynamics.

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