EQUIVARIANT ALGEBRAIC K-THEORY AND DERIVED COMPLETIONS III: APPLICATIONS

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ABSTRACT. In the present paper, we discuss applications of the derived completion theorems proven in our previous two papers. One of the main applications is to Riemann-Roch problems for forms of higher equivariant K-theory, which we are able to establish in great generality both for equivariant G-theory and equivariant homotopy K-theory with respect to actions of linear algebraic groups on normal quasi-projective schemes over a given field. We show such Riemann-Roch theorems apply to all toric and spherical varieties.

We also obtain Lefschetz-Riemann-Roch theorems involving the fixed point schemes with respect to actions of diagonalizable group schemes. We also show the existence of certain spectral sequences that compute the homotopy groups of the derived completions of equivariant G-theory starting with equivariant Borel-Moore motivic cohomology.

CONTENTS

1. Introduction and statement of results

This paper is a sequel to the papers [CJ23] and [CJP23] and is devoted to applications of the derived completion theorems proven in the above two papers. In fact we discuss several applications. One class of applications we discuss in detail is to various different forms of Equivariant Riemann-Roch theorems: a significant part of the paper is devoted to these. In fact, it may be important to point out that the derived completion theorems proved in [CJ23] and [CJP23] are such that strong forms of various equivariant Riemann-Roch theorems can be derived as important consequences. The main observation here is that one can prove Riemann-Roch theorems first as natural transformations, with source the G-theory of the category of equivariant coherent sheaves (or the Homotopy K-theory of the category of equivariant perfect complexes) and taking values in the corresponding G-theory (or the Homotopy K-theory) on a suitable Borel construction associated to the given action of a linear algebraic group.

Once this is done, various other forms of equivariant Riemann-Roch theorems can be obtained. For example, we compose the above Riemann-Roch transformations with Riemann-Roch transformations into various forms of Borel-Moore homology theories on the Borel construction: since the source and the target of these latter Riemann-Roch transformations are both defined on the Borel construction, these can be deduced from the corresponding non-equivariant Riemann-Roch transformation from the G-theory or the homotopy K-theory of a scheme to a suitable form of Borel-Moore homology. We also consider Lefschetz-Riemann-Roch theorems when the given group is a split torus, which relate the Riemann-Roch transformations on the given scheme to those on the fixed points of the torus. These extend various results and formulae previously known, such as the classical results in [AB], [AS68] as well as the Riemann-Roch theorems of Gillet in [Gi] and the coherent trace formulae of Thomason in [Th86-1, Theorem 6.4].

Our main results will be stated only under the following basic assumptions.

Standing Hypothesis 1.1.   (i) We will assume the base scheme S is a perfect infinite field k of arbitrary characteristic p. Let G denote a not-necessarily-connected linear algebraic group defined over k.

  (ii) We will fix an *ambient bigger group*, denoted $\tilde{G}$ throughout, and which contains the given linear algebraic group G as a closed sub-group-scheme and satisfies the following strong conditions: $\tilde{G}$ will denote a connected split reductive group over the field k so that it is *special* (in the sense of Grothendieck: see [Ch]), and if $\tilde{T}$ denotes a maximal torus in $\tilde{G}$, then $R(\tilde{T})$ is free of finite rank over $R(\tilde{G})$ and $R(\tilde{G})$ is Noetherian. (Here $R(\tilde{T})$ and $R(\tilde{G})$ denote the corresponding representation rings.)

 (iii) The above hypothesis is satisfied by $\tilde{G} = GL_n$ or SL_n, for any n, or any finite product of these groups. It is also trivially satisfied by all split tori.(A basic hypothesis that guarantees this condition is that the algebraic fundamental group $\pi_1(\tilde{G})$ is torsion-free: see [CJ23, 2.3].)

 (iv) In general, we will restrict to *normal* quasi-projective schemes of finite type over S, and provided with an action by a linear algebraic group in [CJP23, Theorem 1.4].

      However, in order to consider Riemann-Roch transformations with values in suitable equivariant Borel-Moore homology theories, we need to restrict to the following class of schemes: X is a *normal* G-scheme over S so that it is *either* G-*quasi-projective* (that is, admits a G-equivariant locally closed immersion into a projective space over S on which G-acts linearly: see Definition 2.4), *or* G is connected and X is a normal quasi-projective scheme over S (in which case it is also G-quasi-projective by [SumII, Theorem 2.5]).

The target of the Riemann-Roch transformations will be, in general, Borel-Moore homology theories applied to the Borel-construction. To carry this out in some generality, we start with the framework in [Gi] and then expand it a bit to suit the equivariant framework: we summarize this here, and more details may be found in section 2, especially in Definitions 2.4 through Lemma 2.9. Accordingly we will assume such cohomology theories are defined by graded complexes $\Gamma = \oplus_r \Gamma(r)$ of abelian presheaves on one of the following big sites on the base scheme $S = \operatorname{Spec} k$, such as either the big Zariski site, the restricted big Zariski site (where the objects are the same as in the big Zariski site, but where morphisms are required to be flat maps), or the corresponding sites associated to the big étale site of S. Then the cohomology

groups are defined as the corresponding hypercohomology with respect to the above complexes, that is, $H^{i,j}(X, \Gamma(*)) = \mathbb{H}^i(X, \Gamma(j))$. We may also additionally require that such bi-graded cohomology theories are contravariantly functorial for all maps between smooth schemes. Moreover, for a fixed linear algebraic group G, we will restrict to G-quasi-projective schemes.

The Borel-Moore homology theories we consider will be defined by graded complexes $\Gamma' = \oplus_r \Gamma'(r)$ of abelian presheaves on the same site, and come equipped with pairings: $\Gamma'(r) \otimes \Gamma(s) \to \Gamma'(r-s)$. The homology groups are defined by $H_{s,t}(X, \Gamma') = \mathbb{H}^{-s}(X, \Gamma'(t))$. We will assume such homology groups are *covariantly functorial* for proper maps. We require that the above complexes $\{\Gamma(r)|r\}$ and $\{\Gamma'(s)|s\}$ satisfy a number of properties discussed in detail in Definition 2.5. We extend such cohomology (homology) theories to the equivariant framework by defining them on a suitable form of the Borel construction for a given linear algebraic group G. One important additional hypothesis we impose then is the following *stability condition*: given a scheme X of finite type over k and provided with an action by the linear algebraic group G, and with $E\tilde{G}^{gm,m}$ as in [CJP23, section 3.1], and for each pair of fixed integers s,t, the inverse systems $\{H_{s,t}(E\tilde{G}^{gm,m} \times_G X, \Gamma')|m\}$ and $\{H^{s,t}(E\tilde{G}^{gm,m} \times_G X, \Gamma)|m\}$ stabilize as $m \to \infty$. Then we verify in Lemma 2.9 that a number of cohomology (homology theories) satisfy the above properties. Now we obtain the following theorems which are representative of the Riemann-Roch theorems discussed more fully in section 2.

In view of the difficulties with completions, Equivariant Riemann-Roch theorems such as the following theorem could be proven till now only in very restrictive settings, for example, either only at the level of the Grothendieck groups or only for very restrictive classes of algebraic varieties and schemes (such as projective smooth schemes), or in the non-equivariant framework as in [Gi]. One may observe that the hypotheses of the theorem are satisfied by all normal quasi-projective schemes when the group G is assumed to be connected (in view of Sumihiro's theorem: see [SumII, Theorem 2.5]), and therefore apply to all toric and spherical varieties.

Theorem 1.2. *Let* $X \to \{H_{s,t}(X, \Gamma')|s,t\}$ *denote a homology theory defined in the framework discussed in Definition 2.5 which extends to define an equivariant homology theory satisfying the hypotheses in Definition 2.6.*

(i) Let $f : X \to Y$ *denote a proper G-equivariant map between two normal G-quasi-projective schemes. Let* $\tilde{G}$ *denote either a* GL_n *or a finite product of groups of the form* GL_n *containing G as a closed subgroup-scheme. Then one obtains the commutative diagram, where* τ_X, τ_Y *are suitable Riemann-Roch transformations:*

$$(1.0.1) \quad \begin{array}{ccccc} \pi_* \mathbf{G}(X, G) & \longrightarrow & \lim_{\infty \leftarrow m} \pi_*(\mathbf{G}(E\tilde{G}^{gm,m} \underset{G}{\times} X)) & \xrightarrow{\tau_X} & \lim_{\infty \leftarrow m} H_{*,\bullet}(E\tilde{G}^{gm,m} \underset{G}{\times} X, \Gamma')_{\mathbb{Q}} \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle f_*} \\ \pi_* \mathbf{G}(Y, G) & \longrightarrow & \lim_{\infty \leftarrow m} \pi_*(\mathbf{G}(E\tilde{G}^{gm,m} \underset{G}{\times} Y)) & \xrightarrow{\tau_Y} & \lim_{\infty \leftarrow m} H_{*,\bullet}(E\tilde{G}^{gm,m} \underset{G}{\times} X, \Gamma')_{\mathbb{Q}}. \end{array}$$

(ii) Assume in addition to the hypotheses in (i) that the map f *is also perfect, for example, a map that is a local complete intersection morphism. Then one obtains the commutative diagram, where* τ_X, τ_Y *are suitable Riemann-Roch transformations:*

$$(1.0.2) \quad \begin{array}{ccccc} \pi_* \mathbf{KH}(X, G) & \longrightarrow & \lim_{\infty \leftarrow m} \pi_*(\mathbf{KH}(E\tilde{G}^{gm,m} \underset{G}{\times} X)) & \xrightarrow{\tau_X} & \lim_{\infty \leftarrow m} H_{*,\bullet}(E\tilde{G}^{gm,m} \underset{G}{\times} X, \Gamma')_{\mathbb{Q}} \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle f_*} \\ \pi_* \mathbf{KH}(Y, G) & \longrightarrow & \lim_{\infty \leftarrow m} \pi_*(\mathbf{KH}(E\tilde{G}^{gm,m} \underset{G}{\times} Y)) & \xrightarrow{\tau_Y} & \lim_{\infty \leftarrow m} H_{*,\bullet}(E\tilde{G}^{gm,m} \underset{G}{\times} X, \Gamma')_{\mathbb{Q}}. \end{array}$$

Observe that any finite group G may be imbedded in a symmetric group, Σ_n, for some $n > 0$, as a subgroup, and the symmetric group Σ_n imbeds into $\tilde{G} = GL_n$ as the closed subgroup of all permutation matrices. Then we obtain:

Corollary 1.3. *(Finite group actions) Assume* G *is a finite group imbedded as a closed subgroup of some* GL_n. *Let* $\tilde{\mathrm{G}} = \mathrm{GL}_n$ *and let* X *denote a quasi-projective normal scheme as in 1.1(iv) provided with an action by* G. *Then Theorems* [CJP23, Theorem 1.4]*, and 1.2 apply.*

The following *Lefschetz-Riemann-Roch and the Coherent trace formula* extends a similar result proven by Thomason with the target in his case being Borel-style *equivariant topological G-theory* with finite coefficients, that is, the Borel style equivariant algebraic G-theory with finite coefficients and with the Bott-element inverted. See [Th86-1, Theorem 6.4]. We are able to lift such theorems to Borel-style equivariant algebraic G-theory and to Borel-style equivariant homotopy K-theory, in general.

Theorem 1.4. *(Lefschetz-Riemann-Roch and the Coherent trace formula) Let* $f : \mathrm{X} \to \mathrm{Y}$ *denote a* T-*equivariant proper map between two* T-*schemes of finite type over k for the action of a split torus* T. *Assume that f factors* T-*equivariantly as* $\pi \circ i_1$, *where* $i_1 : \mathrm{X} \to \mathrm{Z}$ *is closed immersion into a regular* T-*scheme* Z, *followed by a proper* T-*equivariant map* $\pi : \mathrm{Z} \to \mathrm{Y}$. *Let* $i : \mathrm{Z}^{\mathrm{T}} \to \mathrm{Z}$ *denote the closed immersion of the fixed point scheme of* Z *into* Z, *with* $\mathcal{N}$ *denoting the corresponding co-normal sheaf. Let* $\pi^{\mathrm{T}} : \mathrm{Z}^{\mathrm{T}} \to \mathrm{Y}^{\mathrm{T}}$ *denote the induced map. Let* $\mathcal{F}$ *denote a* T-*equivariant coherent sheaf on* X *and let*

$$\mathcal{G} = Li^*(i_{1*}(\mathcal{F})) \underset{\mathcal{O}_{\mathrm{Z}^{\mathrm{T}}}}{\otimes} (\lambda_{-1}(\mathcal{N}))^{-1}.$$

(i) *Then one has equality of the classes*

(1.0.3) $$Rf_*(\mathcal{F}) = \Sigma_i (-1)^i R^i f_*(\mathcal{F}) = i_{\mathrm{Y}*} R\pi_*^{\mathrm{T}}(\mathcal{G}) = \Sigma_i (-1)^i i_{\mathrm{Y}*} R^i \pi_*^{\mathrm{T}}(\mathcal{G})$$

in $\pi_0(\underset{\infty \leftarrow m}{holim} \mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}} \times_{\mathrm{T}} \mathrm{Y}))_{(0)}$ *and in* $\underset{\infty \leftarrow m}{lim} \pi_0(\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}} \times_{\mathrm{T}} \mathrm{Y}))_{(0)}$, *where the subscript* (0) *denotes the localization at the prime ideal* (0) *in* $\mathrm{R}(\mathrm{T})$ *and* $i_{\mathrm{Y}} : \mathrm{Y}^{\mathrm{T}} \to \mathrm{Y}$ *denotes the obvious closed immersion.*

(ii) *The above hypotheses are satisfied whenever* Y *is regular and* X *is* T-*equivariantly projective, for example,* X *is a normal quasi-projective scheme.*

(iii) *If* Y *has trivial action by* T, *equality holds in* $\pi_0(\underset{\infty \leftarrow m}{holim} \mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}} \times_{\mathrm{T}} \mathrm{Y})) \cong \underset{\infty \leftarrow \mathrm{m}}{lim} \pi_0(\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}} \times_{\mathrm{T}} \mathrm{Y})) \cong$ $\mathbb{Z}[\mathrm{T}]\widehat{}_{\mathrm{I}_{\mathrm{T}}} \otimes \pi_0(\mathbf{G}(\mathrm{Y}))$.

(iv) *If* Y *is affine and* X^{T} *is affine (for example,* X^{T} *is a finite set of isolated points on* X*), then one obtains* $\Sigma_i(-1)^i \mathrm{H}^i(\mathrm{X}, \mathcal{F}) = \mathrm{H}^0(\mathrm{X}^{\mathrm{T}}, \mathcal{G})$. *This hypothesis is satisfied if* X *is a projective toric variety for a split torus* T, *or a projective spherical variety for a linear algebraic group* G *over an algebraically closed field, with* T *denoting the maximal torus in* G *and where* Y *denotes the base field.*

(v) *If the map* π *is perfect and* $\mathcal{F}$ *denotes a perfect complex of* T-*equivariant* $\mathcal{O}_{\mathrm{X}}$-*modules, then the results corresponding to (i) through (iv) hold for equivariant homotopy K-theory replacing equivariant G-theory.*

Corollary 1.5. (i) *Assume that* X *and* Y *are* T-*quasi-projective and* $\mathcal{F}$ *is as in Theorem 1.4(i). Composing with the Riemann-Roch transformations into a suitable equivariant Borel-Moore homology theory (see Definition 2.6) localized at the prime ideal* (0) *in* $\underset{\infty \leftarrow m}{lim} \Pi_{\mathrm{r,s}} \mathrm{H}^{\mathrm{r,s}}(\mathrm{BT}^{\mathrm{gm,m}}, \Gamma)_{\mathbb{Q}}$, *one also obtains*

$$f_*(\tau_{\mathrm{Z}}(\mathcal{F})) = \tau_{\mathrm{Y}}(R\pi_*(\mathcal{F})) = i_{\mathrm{Y}*} \pi_*^{\mathrm{T}} \tau_{\mathrm{Z}^{\mathrm{T}}}(\mathcal{G}) = \pi_* \tau_{\mathrm{Z}}(i_* \mathcal{G}).$$

as classes in $(\underset{\infty \leftarrow m}{lim} \Pi_{\mathrm{u,v}} \mathrm{H}_{\mathrm{u,v}}(\mathrm{ET}^{\mathrm{gm,m}} \times_{\mathrm{T}} \mathrm{Y}, \Gamma'))_{(0)}$. *(Here* τ_{Z} *(*τ_{Y}, $\tau_{\mathrm{Z}^{\mathrm{T}}}$*) denote the Riemann-Roch transformation associated to* Z*(* Y, Z^{T} *respectively) and a generalized Borel-Moore homology theory.)*

(ii) *If the map* π *is perfect and* $\mathcal{F}$ *denotes a perfect complex of* T-*equivariant* $\mathcal{O}_{\mathrm{X}}$-*modules, then the results corresponding to (i) holds for equivariant homotopy K-theory replacing equivariant G-theory.*

(iii) *(Todd class formula) Under the assumptions of (i),*

$$f_*(\tau_{\mathrm{X}}) = \pi_* \tau_{\mathrm{Z}}(i_*(Li^*(i_{1*}(\mathcal{O}_{\mathrm{X}})) \underset{\mathcal{O}_{\mathrm{Z}^{\mathrm{T}}}}{\otimes} (\lambda_{-1}(\mathcal{N}))^{-1})).$$

Corollary 1.6. *(Demazure character formula): Let* G/B *denote the flag variety associated to a connected reductive group* G *and a Borel subgroup* B *all defined over an algebraically closed field k. Let* X_w *denote a Schubert variety in* G/B, *associated to* $w \in \mathrm{W}$, *which denotes the corresponding Weyl group. Let* T *denote the maximal torus in* G. *Let* $\mathcal{L}_\lambda$ *denote the* T-*equivariant line bundle on* G/B *associated to a character* λ *of* T. *Then in* $\mathrm{R}(\mathrm{T}) = \pi_0(\mathbf{G}(\mathrm{Spec}\, k, \mathrm{T})$, *one obtains:*

$$\Sigma_i(-1)^i \mathrm{H}^i(\mathrm{X}_w, \mathcal{L}_\lambda) = \mathrm{e}^\rho \mathrm{L}_{s_1} \cdots \mathrm{L}_{s_r} \mathrm{e}^{\lambda - \rho}$$

where $w = s_1 \cdots s_r$, $\mathrm{L}_{s_\alpha}(u) = \frac{u - s_\alpha u}{1 - e^\alpha}$, and ρ is half the sum of the positive roots.

As another application of the derived completion theorems in [CJ23]) (see [CJ23, theorems 1.2 and 5.6]), we obtain the following result that shows how to compute the homotopy groups of the equivariant G-theory in terms of equivariant motivic cohomology. Assume that X is a G-quasi-projective scheme, as in 2.4, with G = T a split torus. Then the approximation $\mathrm{E\tilde{G}}^{\mathrm{gm,m}} \underset{\tilde{\mathrm{G}}}{\times} \mathrm{X}$ (for any integer $m > 0$), is always a scheme, and we obtain the following theorem.

Theorem 1.7. *Assume that* X *is a* T-*quasi-projective scheme as in Definition 2.4, with* T *a split torus. Then, for a fixed value of* $-s - t$, *and for m sufficiently large, there exists a strongly convergent spectral sequence with:*

$$E_2^{s,t} = \mathbb{H}_{\mathrm{BM,M}}^{s-t}(\mathrm{ET}^{\mathrm{gm,m}} \underset{\mathrm{T}}{\times} \mathrm{X}, \mathbb{Z}(-\mathrm{t})) \Rightarrow \pi_{-\mathrm{s}-\mathrm{t}}(\mathbf{G}(\mathrm{ET}^{\mathrm{gm,m}} \underset{\mathrm{T}}{\times} \mathrm{X})) \cong \pi_{-\mathrm{s}-\mathrm{t}}(\mathbf{G}(\mathrm{X}, \mathrm{T})\widehat{}_{\rho_{\mathrm{T,m}}}),$$

where $\mathbb{H}_{\mathrm{BM,M}}$ *denotes Borel-Moore motivic cohomology, which identifies with the higher equivariant Chow groups.*

In addition, rationally, the homotopy groups $\pi_{-s-t}(\mathbf{G}(\mathrm{X}, \mathrm{T})\widehat{}_{\rho_{\mathrm{T,m}}})$ *identify with the rational* T-*equivariant higher Chow groups.*

Remark 1.8. For more general groups other than a split torus, there is still a similar spectral sequence, but only for the full $\mathrm{EG}^{\mathrm{gm}} \underset{\mathrm{G}}{\times} \mathrm{X}$. The corresponding spectral sequence converges only conditionally, and computes the homotopy groups of the full derived completion.

We conclude this discussion by providing the following corollaries to clarify the range of applications of the above Theorems.

The results of Sumihiro (see [SumI, Theorem 1] and [SumII, Theorem 2.5]) and the following Corollary enable us to see that the Theorems [CJP23, Theorem 1.4], Theorems 1.2, 1.4 and 1.7 all apply to toric varieties and spherical varieties, irrespective of whether they are regular or projective. Recall a *normal* variety X provided with the action of a split torus T (of a connected split reductive group G) is called *a toric variety* (*an* G-*spherical variety*) if T has only finitely many orbits on X (G and a Borel subgroup of G both have only finitely many orbits on X, respectively).

Corollary 1.9. *(Toric varieties and Spherical varieties) The Theorems* [CJP23, Theorem 1.4]*, 1.2, 1.4, 1.7 and Corollary 1.5 apply to all quasi-projective split toric varieties, with the group* G *denoting the given torus. Theorems* [CJP23, Theorem 1.4] *and 1.2 apply to all quasi-projective* G-*spherical varieties, where* G *is the given connected reductive group. Theorems 1.4 and 1.7 apply to all quasi-projective* G-*spherical varieties for the action of a split sub-torus of the given group* G.

Here is an *outline of the paper.* Section 2 discusses various forms of equivariant Riemann-Roch theorems culminating in those taking values in any equivariant Borel-Moore motivic cohomology theory. Section 3 then discusses the Lefschetz-Riemann-Roch theorem involving the action of a torus and also discuss various applications stemming from this. Section 4 discusses how to compute the completed equivariant G-theory in terms of equivariant motivic cohomology.

2. HIGHER EQUIVARIANT RIEMANN-ROCH THEOREMS

We consider several variants of equivariant Riemann-Roch theorems in this section.

2.1. Riemann-Roch from forms of equivariant K-theory to the corresponding form of K-theory on the Borel construction. The first theorem considers the Riemann-Roch transformation between equivariant **G**-theory and Borel-style equivariant **G**-theory and the second theorem considers the Riemann-Roch transformation between equivariant Homotopy K-theory and Borel-style equivariant homotopy K-theory.

Theorem 2.1. *(Riemann-Roch for Equivariant G-theory) Let* $f : \mathrm{X} \to \mathrm{Y}$ *denote a proper* G-*equivariant map between two normal quasi-projective* G-*schemes of finite type over* $\mathrm{S} = \mathrm{Spec}\,k$. *Let* $\tilde{\mathrm{G}}$ *denote either a*

GL_n *or a finite product of groups of the form* GL_n *containing* G *as a closed subgroup-scheme. Then the squares*

(2.1.1)

$$
\begin{array}{ccccccc}
\mathbf{G}(X,G) & \longrightarrow & \mathbf{G}(X,G)\widehat{}_{\rho_{\tilde{G},\alpha(m)}} & \longrightarrow & \mathbf{G}(E\tilde{G}^{gm,m} \times X, G) & \xrightarrow{\simeq} & \mathbf{G}(E\tilde{G}^{gm,m}\underset{G}{\times}X) \\
\downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\
\mathbf{G}(Y,G) & \longrightarrow & \mathbf{G}(Y,G)\widehat{}_{\rho_{G,\alpha(m)}} & \longrightarrow & \mathbf{G}(E\tilde{G}^{gm,m} \times Y, G) & \xrightarrow{\simeq} & \mathbf{G}(E\tilde{G}^{gm,m}\underset{G}{\times}Y)
\end{array}
$$

homotopy commute, for any fixed integer $m \geq 0$*, with* $\alpha(m)$ *as in* [CJP23, Definition 5.5]*. Moreover, as* m *varies one obtains an inverse system of such homotopy commutative diagrams. On taking the homotopy inverse limits we obtain the commutative diagram:*
(2.1.2)

$$
\begin{array}{ccccccc}
\pi_*\mathbf{G}(X,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{G}(X,G)\widehat{}_{\rho_{\tilde{G},\alpha(m)}}) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{G}(E\tilde{G}^{gm,m} \times X, G)) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{G}(E\tilde{G}^{gm,m}\underset{G}{\times}X)) \\
\downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\
\pi_*\mathbf{G}(Y,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{G}(Y,G)\widehat{}_{\rho_{G,\alpha(m)}}) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{G}(E\tilde{G}^{gm,m} \times Y, G)) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{G}(E\tilde{G}^{gm,m}\underset{G}{\times}Y)).
\end{array}
$$

Theorem 2.2. *(Riemann-Roch for Equivariant Homotopy K-theory) Let* $f : X \to Y$ *denote a proper* G*-equivariant map between two normal quasi-projective* G*-schemes of finite type over* $S = \operatorname{Spec} k$*, that is also perfect (see* [CJP23, Definition 2.1]*). Let* $\tilde{G}$ *denote either a* GL_n *or a finite product of groups of the form* GL_n *containing* G *as a closed subgroup-scheme. Then the squares*

(2.1.3)

$$
\begin{array}{ccccccc}
\mathbf{KH}(X,G) & \longrightarrow & \mathbf{KH}(X,G)\widehat{}_{\rho_{\tilde{G},\alpha(m)}} & \longrightarrow & \mathbf{KH}(E\tilde{G}^{gm,m} \times X, G) & \xrightarrow{\simeq} & \mathbf{KH}(E\tilde{G}^{gm,m}\underset{G}{\times}X) \\
\downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\
\mathbf{KH}(Y,G) & \longrightarrow & \mathbf{KH}(Y,G)\widehat{}_{\rho_{\tilde{G},\alpha(m)}} & \longrightarrow & \mathbf{KH}(E\tilde{G}^{gm,m} \times Y, G) & \xrightarrow{\simeq} & \mathbf{KH}(E\tilde{G}^{gm,m}\underset{G}{\times}Y)
\end{array}
$$

homotopy commute, for any fixed integer $m \geq 0$*, with* $\alpha(m)$ *as in* [CJP23, Definition 5.5]*. Moreover, as* m *varies one obtains an inverse system of such homotopy commutative diagrams. On taking the homotopy inverse limits we obtain the commutative diagram:*
(2.1.4)

$$
\begin{array}{ccccccc}
\pi_*\mathbf{KH}(X,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{KH}(X,G)\widehat{}_{\rho_{\tilde{G},\alpha(m)}}) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{KH}(E\tilde{G}^{gm,m} \times X, G)) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{KH}(E\tilde{G}^{gm,m}\underset{G}{\times}X)) \\
\downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\
\pi_*\mathbf{KH}(Y,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{KH}(Y,G)\widehat{}_{\rho_{G,\alpha(m)}}) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{KH}(E\tilde{G}^{gm,m} \times Y, G)) & \xrightarrow{\cong} & \pi_*(\underset{\infty\leftarrow m}{holim}\mathbf{KH}(E\tilde{G}^{gm,m}\underset{G}{\times}Y)).
\end{array}
$$

Proofs of the above two theorems. The commutativity of the left-most square in both (2.1.1) and (2.1.3) follows from the definition and functoriality of derived completion for equivariant G-theory and equivariant homotopy K-theory as in [CJ23, 3.0.7]: see also [CJP23, (4.0.3)]. The composition of the left two horizontal maps in the top row of (2.1.1) ((2.1.3)) is given by sending a complex of G-equivariant coherent sheaves on X (G-equivariant perfect complex on $X \times \Delta[n]$) to its pull-back on $E\tilde{G}^{gm,m}\times X$ ($E\tilde{G}^{gm,m}\times X \times \Delta[n]$, respectively). The composition of the left two horizontal maps in the bottom row of (2.1.1) ((2.1.3)) has a similar description.

Therefore, the homotopy commutativity of the of the middle square in (2.1.1) ((2.1.3)) follows from [CJ23, Proposition 4.5(ii)] ([CJP23, Proposition 5.4(ii)], respectively). The commutativity of the last squares in both (2.1.1) and (2.1.3) is clear. In order to show that the homotopy commutative diagrams (2.1.1) and (2.1.3) in both theorems are compatible as m varies, resulting in the commutative diagrams in (2.1.2) and (2.1.4), we invoke [CJP23, Proposition 5.7].

$\square$

2.2. **Equivariant cohomology and equivariant Borel-Moore homology theories.** We proceed to combine the above Riemann-Roch theorems with any of the usual Riemann-Roch theorems into various cohomology/homology theories. For this we will adopt the framework as in [Gi]. We start with the base scheme $S = \operatorname{Spec} k$, the spectrum of a field k, and fix either the big Zariski site, the restricted big Zariski site (where the objects are the same as in the big Zariski site, but where morphisms are required to be flat maps), the full subcategory of the big Zariski site consisting of all smooth schemes of finite type over S, or the corresponding sites associated to the big étale site of S.

Definition 2.3. Let $\Gamma = \oplus_r \Gamma(r)$ denote a graded complex of abelian presheaves on one of the above sites. Then the cohomology groups are defined as the corresponding hypercohomology with respect to the above complexes, that is, $H^{i,j}(X, \Gamma(*)) = \mathbb{H}^i(X, \Gamma(j))$. (The index i (j) will be called the degree (the weight, respectively).) We may also additionally require that such bi-graded cohomology theories are contravariantly functorial for all maps between smooth schemes.

In order to be able to define a suitable Riemann-Roch transformation with values in an equivariant Borel-Moore homology theory, we will need to restrict to a nicer class of schemes as considered in the following definition.

Definition 2.4. For a given linear algebraic group G, a convenient class of schemes we consider are what are called G-*quasi-projective schemes*: a quasi-projective scheme X with a given G-action is G-quasi-projective, if it admits a G-equivariant locally closed immersion into a projective space, $\operatorname{Proj}(V)$, for a representation V of G. Then a well-known theorem of Sumihiro, (see [SumII, Theorem 2.5]) shows that any normal quasi-projective scheme over S provided with the action of a connected linear algebraic group G is G-quasi-projective. This may be extended to the case where G is not necessarily connected by invoking the above result for the action of G^o and then considering the closed G-equivariant closed immersion of the given G-scheme into $\Pi_{g\ \epsilon\ G/G^o}\operatorname{Proj}(V)$ followed by a G-equivariant closed immersion of the latter into $\operatorname{Proj}(\otimes_{g\ \epsilon\ G/G^o}V)$. (See for example. [Th88, pp. 629-630].)

Definition 2.5. (Generalized Borel-Moore homology theories) The following assumptions are essentially those that appear in [Gi]. The homology theories we consider will be defined by graded complexes $\Gamma' = \oplus_r \Gamma'(r)$, that come equipped with pairings: $\Gamma'(r) \otimes \Gamma(s) \to \Gamma'(r-s)$, with the homology groups defined by $H_{s,t}(X, \Gamma') = \mathbb{H}^{-s}(X, \Gamma'(t))$. We will assume such homology groups are *covariantly functorial* for proper maps, with $f_* : H_{s,t}(X, \Gamma') \to H_{s,t}(Y, \Gamma')$ denoting the induced map associated to such a proper map $X \to Y$. We will let $H_{*,\bullet}(X, \Gamma')$ denote $\oplus_{s,t} H_{s,t}(X, \Gamma')$. Moreover, $H_{*,\bullet}(X, \Gamma')_\mathbb{Q}$ will denote the corresponding graded homology group, that is made into a graded $\mathbb{Q}$-vector space in a suitable manner, depending on the choice of the complexes as in the examples considered below. We will also impose the following additional requirements:

(i) If $f : X \to Y$ is a flat map of pure relative dimension d, then there is an induced map $f^* : H_{s,t}(Y, \Gamma') \to H_{s+2d,t+d}(X, \Gamma')$.

(ii) For a closed immersion $i : Y \to X$, there exists a long-exact localization sequence:

$$\cdots \to H_{i,j}(Y, \Gamma') \xrightarrow{i_*} H_{i,j}(X, \Gamma') \xrightarrow{j^*} H_{i,j}(X-Y, \Gamma') \to H_{i-1,j}(Y, \Gamma') \to \cdots .$$

(iii) The given pairing $\Gamma'(r) \otimes \Gamma(s) \to \Gamma'(r-s)$ induces a (cap-product) pairing for every closed subscheme Y of X:

$$H_{i,j}(X, \Gamma') \otimes H_Y^{r,s}(X, \Gamma) \to H_{i-r,j-s}(Y, \Gamma').$$

(iv) For a smooth scheme X of pure dimension d, there exists a fundamental class $[X] \epsilon H_{2d,d}(X, \Gamma')$ so that the cap-product pairing with this class induces an isomorphism: $H_Y^{r,s}(X, \Gamma) \to H_{2d-r,d-s}(Y, \Gamma')$, for Y closed and of pure codimension in X.

(v) For a regular closed immersion $i : Y \to X$ of smooth schemes of pure codimension c, one obtains Thom-isomorphism, that is, an isomorphism: $H^{i,j}(Y, \Gamma) \xrightarrow{\cong} H_Y^{i+2c,j+c}(X, \Gamma)$.

(vi) (Riemann-Roch transformation) Given such a cohomology/homology theory, we also require that one has a Riemann-Roch transformation $\tau : \pi_*(\mathbf{G}(X)) \to H_{*,\bullet}(X, \Gamma')_\mathbb{Q}$ defined for all the schemes considered and which is covariantly functorial in X for proper maps.

(vii) Gysin maps for regular closed immersions: if $i : Y \to X$ is a regular closed immersion of pure codimension c, we assume there is a Gysin map: $i^* : H_{s,t}(X, \Gamma') \to H_{s-2c,t-c}(Y, \Gamma')$.

Let X denote a G-quasi-projective scheme, together with a G-equivariant closed immersion into a smooth G-scheme $\tilde{X}$. Then we define a Gysin map $i^* : H_{*,\bullet}(E\tilde{G}^{gm,m+1} \times_G X, \Gamma') \to H_{*-2c,\bullet-c}(E\tilde{G}^{gm,m} \times_G X, \Gamma')$ by making use of the vertical maps in the following diagram which are isomorphisms:

$$(2.2.1) \qquad \begin{array}{ccc} H_{*,\bullet}(E\tilde{G}^{gm,m+1} \times_G X, \Gamma') & \longrightarrow & H_{*-2c,\bullet-c}(E\tilde{G}^{gm,m} \times_G X, \Gamma') \\ \cong \big\uparrow & & \cong \big\uparrow \\ H^{2\tilde{d}-*,\tilde{d}-\bullet}_{E\tilde{G}^{gm,m+1}\times_G X}(E\tilde{G}^{gm,m+1} \times_G \tilde{X}, \Gamma) & \xrightarrow{i^*} & H^{2\tilde{d}-*,\tilde{d}-\bullet}_{E\tilde{G}^{gm,m}\times_G X}(E\tilde{G}^{gm,m} \times_G \tilde{X}, \Gamma) \end{array}$$

where the map in the bottom row is the obvious restriction, $\tilde{d} = dim(E\tilde{G}^{gm,m+1}\times_G\tilde{X})$ and $c = dim(E\tilde{G}^{gm,m+1}\times_G \tilde{X}) - \dim(E\tilde{G}^{gm,m} \times_G \tilde{X})$. In addition to the above hypotheses, we will require the following additional ones to handle the equivariant framework.

Stability for equivariant cohomology and homology.

(i) Let $\tilde{G}$ denote either a GL_n or a finite product of groups of the form GL_n containing the given linear algebraic group G as a closed subgroup-scheme. We will often make the *following additional assumption*: given a scheme X of finite type over k and provided with an action by the linear algebraic group G, and with $E\tilde{G}^{gm,m}$ as in [CJP23, section 3.1], and for each pair of fixed integers i, j, the inverse system $\{H^{i,j}(E\tilde{G}^{gm,m} \times_G X, \Gamma)|m\}$ stabilizes as $m \to \infty$.

(ii) Assume the dimension of $E\tilde{G}^{gm,m}$ is $\tilde{d}_m$ and the dimension of G is g. Then we also require that for each fixed i, j, the inverse system $\{H_{i+2\tilde{d}_m-2g,j+\tilde{d}_m-g}(E\tilde{G}^{gm,m} \times_G X, \Gamma')|m\}$ (where the structure maps of the inverse system are defined by the Gysin maps in (2.2.1)), stabilizes as $m \to \infty$.

Definition 2.6. (Generalized equivariant cohomology/homology theories)

(i) We will let $H^{i,j}_G(X, \Gamma))$ denote the limiting value of the inverse system $\{H^{i,j}(E\tilde{G}^{gm,m}\times_G X, \Gamma)|m\}$ stabilizes as $m \to \infty$.

(ii) We will let $H^G_{i,j}(X, \Gamma))$ denote the corresponding limiting value of the inverse system $\{H_{i+2\tilde{d}_m-2g,j+\tilde{d}_m-g}(E\tilde{G}^{gm,m} \times_G X, \Gamma')|m\}$ as $m \to \infty$.

(iii) An additional assumption we impose is that the equivariant cohomology $H^{i,j}_G(X, \Gamma)$ and equivariant homology $H^G_{s,t}(X, \Gamma')$ for each fixed i, j, s, t are independent of the choice of the ind-scheme $\{E\tilde{G}^{gm,m}\times_G X|m\}$ defining the Borel construction.

Definition 2.7. (Equivariant fundamental class) Assume X is a G-quasi-projective scheme of pure dimension d. In view of the stability of the inverse system $\{H_{2d+2\tilde{d}_m-2g,d+\tilde{d}_m-g}(E\tilde{G}^{gm,m} \times_G X, \Gamma')|m\}$, we define $H^G_{2d,d}(X, \Gamma') = H_{2d+2\tilde{d}_m-2g,d+\tilde{d}_m-g}(E\tilde{G}^{gm,m} \times_G X, \Gamma')$, for m sufficiently large, and define the corresponding *equivariant fundamental class* $[X]_G \in H^G_{2d,d}(X, \Gamma')$ as the image of the classes

$$[X]_{G,m} \in H_{2d+2\tilde{d}_m-2g,d+\tilde{d}_m-g}(E\tilde{G}^{gm,m} \times_G X, \Gamma')$$

for m sufficiently large.

Examples 2.8. Next we list some well-known examples of cohomology/homology theories satisfying the hypotheses in Definitions 2.3 and 2.5.

(i) One takes the big Zariski or etale site of all smooth schemes of finite type over S and let $\Gamma(r)$ denote the motivic complex of weight r.

(ii) One may take the category of all quasi-projective schemes of finite type over S provided with the restricted Zariski topology. Here ones may take $\Gamma'(r)$ to be Bloch's higher cycle complex of weight r: $z^r(\ ,\bullet)$, re-indexed using the dimension of cycles rather than the codimension, that is, if X has pure dimension d, $(z_{d-r}(X,\bullet))_n$ will denote the cycles of dimension $d - r + n$ in $X \times \Delta[n]$ that intersect the faces of $X \times \Delta[n]$ properly. The corresponding Zariski hypercohomology groups identify with Bloch's higher Chow groups, re-indexed by dimension. These are covariantly functorial for proper maps and have localization sequences. Moreover restricting to smooth schemes, these identify with the motivic cohomology groups, indexed by the codimension of cycles.

(iii) Deligne cohomology and homology: these are defined for schemes of finite type over the field of complex or real numbers by complexes satisfying most of the above properties. (See [Jan].)

(iv) Étale cohomology with $\Gamma(r) = \mu_\ell(r)$, ℓ invertible in the base field k. The corresponding (Borel-Moore) homology theory is defined by $\Gamma'(r)$, the corresponding dualizing complex. One may also consider ℓ-adic étale cohomology and homology.

(v) Singular cohomology for varieties over $k = \mathbb{C}$ or $\mathbb{R}$, with $\Gamma(r) = \mathbb{Z}^{\otimes r}$, where $\mathbb{Z}$ denotes the constant sheaf given by the integers. The corresponding homology theory is the familiar Borel-Moore homology theory.

We will skip the verification that most of the above cohomology/homology theories satisfy the properties listed in Definitions 2.3 and 2.5. However, it is important for us to show that the additional hypotheses in Definition 2.6 are indeed satisfied by several of the cohomology/homology theories listed above and extended to the equivariant framework. One may extend any cohomology/homology theory satisfying the hypotheses in Definitions 2.3 and 2.5 to the equivariant framework, that is, to schemes Y that come equipped with the action of a linear algebraic group G, by taking the hypercohomology with respect to the complexes $\Gamma(r)$ and $\Gamma'(s)$ on the Borel construction $EG^{gm,m} \times_G Y$.

Lemma 2.9. *The following cohomology/homology theories extend to the equivariant framework defining cohomology/homology theories satisfying the hypotheses in Definition 2.6.*

(i) *Bloch's higher Chow groups*

(ii) *ℓ-adic étale cohomology and (Borel-Moore) homology, where ℓ denotes a prime that is invertible in the base field k and k has finite ℓ cohomological dimension.*

(iii) *Singular cohomology and (Borel-Moore) homology, when the base field is the complex numbers.*

Proof. The extension of Bloch's higher Chow groups to the equivariant framework (called equivariant higher Chow groups) is carried out in [Tot] and [EG]. The properties in Definition 2.6 are established in [EG].

We will next consider the case of ℓ-adic étale cohomology. Let BG, $EG \times_G X$ denote the simplicial scheme defined by the simplicial Borel construction as in [J93] and [J20, section 5]. (As should be clear from the discussion in [J20, section 5], one could replace the simplicial Borel construction $EG \times_G X$ by the ind-scheme $\{EG^{gm,m} \times_G X | m\}$ or the ind-scheme $\{E\tilde{G}^{gm,m} \times_G X | m\}$ for a larger linear algebraic group $\tilde{G}$ containing G as a closed subgroup-scheme, as they will give equivalent derived categories. Therefore, we will use the simplicial Borel construction $EG \times_G X$ throughout the following discussion.) Let $\pi_X : EG \times_G X \to BG$ denote the obvious map induced by the projection $X \to \mathrm{Spec} k$. Then we let $\mathbb{D}_X^G = R\pi_X^!(\mathbb{Q}_\ell)$ denote the dualizing complex on $EG \times_G X$. We will define

$$(2.2.2) \qquad H_{s,t}^G(X, \mathbb{Q}_\ell) = \mathbb{H}_{s,t}(EG \times_G X, \mathbb{Q}_\ell) = \mathbb{H}^{-s}(EG \times_G X, \mathbb{D}_X^G(-t)).$$

Next we consider the spectral sequence, for any complex of ℓ-adic sheaves K on $EG \times_G X$:

$$(2.2.3) \qquad E_2^{u,v} = H^u(BG, R^v \pi_{X*}(K)) \Rightarrow \mathbb{H}^{u+v}(EG \times_G X, K).$$

Taking $K = \mathbb{D}_X^G$ in the above spectral sequence, one may observe that there is a fundamental class $[X_{\bar{x}}] \in R^{-2d}\pi_{X*}(\mathbb{D}_X^G)_{\bar{x}}(-d)$, for each geometric point $\bar{x}$ on BG. Moreover, in this case, $E_2^{u,v} = 0$ for $v < -2d$, or $v > 0$ and also for $u < 0$. It follows that

$$(2.2.4) \qquad H^0(BG, R\pi_{X*}^{-2d}(\mathbb{D}_X^G)(-d)) = E_2^{0,-2d} \cong E_3^{0,-2d} \cong \cdots E_\infty^{0,-2d} \cong \mathbb{H}^{-2d}(EG \times_G X, \mathbb{D}_X^G(-d))$$

$$= H_{2d,d}(EG \times_G X, \mathbb{Q}_\ell) = H_{2d,d}^G(X, \mathbb{Q}_\ell).$$

We next consider the special case where G is connected. Then one sees that the sheaf $R\pi_{X*}^{-2d}(\mathbb{D}_X^G)$ is constant on BG, and therefore one obtains a class $[X]_G \in H^0(BG, R\pi_{X*}^{-2d}(\mathbb{D}_X^G)(-d))$. Now the identifications in (2.2.4) shows that this is an infinite cycle in the above spectral sequence, and therefore this defines a fundamental class $[X]_G \in H_{2d}(EG \times_G X, \mathbb{Q}_\ell(-d)) = \mathbb{H}_{2d,d}^G(X, \mathbb{Q}_\ell)$.

When G is not connected, let G^o denote the connected component of the identity. Then the finite group G/G^o acts on the stalks of $R\pi_{X*}(\mathbb{D}_X^G)$. Since we are working with $\mathbb{Q}_\ell$-coefficients, one obtains isomorphisms

$$H^0(BG, R\pi_{X*}^{-2d}(\mathbb{D}_X^G)(-d))) \cong H^0(BG^o, R\pi_{X*}^{-2d}(\mathbb{D}_X^G)(-d))^{G/G^o}, \text{ and}$$

$$\mathbb{H}^{-2d}((EG\underset{G}{\times}X, \mathbb{D}_X^G(-d)) \cong \mathbb{H}^{-2d}(EG^o\underset{G^o}{\times}X, \mathbb{D}_X^G(-d))^{G/G^o}.$$

Therefore, the isomorphisms in (2.2.4) extend to this case. It follows that one obtains a fundamental class $[X]_G \in \mathbb{H}^{-2d}(EG\underset{G}{\times}X, \mathbb{D}_X^G(-d)) = H_{2d,d}^G(X, \mathbb{Q}_\ell)$.

Next suppose that $i : X \to \tilde{X}$ is a closed G-equivariant immersion of a given G-quasi-projective scheme into a smooth G-scheme $\tilde{X}$. Let $\mathbb{D}_X^G$ ($\mathbb{D}_{\tilde{X}}^G$) denote the dualizing complexes on $EG\underset{G}{\times}X$ ($EG\underset{G}{\times}\tilde{X}$). (In view of the assumption that $\tilde{X}$ is smooth, one may identify $\mathbb{D}_{\tilde{X}}^G$ with $\mathbb{Q}_\ell[-2\tilde{d}](-\tilde{d})$ where $\tilde{X}$ is assumed to have pure dimension $\tilde{d}$.) Now the pairing $\mathbb{D}_{\tilde{X}}^G \otimes i_*Ri^!(\mathbb{Q}_\ell) \to i_*\mathbb{D}_X^G$ induces a pairing of the corresponding hypercohomology spectral sequences. Since the fundamental class $[\tilde{X}]_G$ is an infinite cycle, cap-product with this class then induces a map of spectral sequences from the spectral sequence in (2.2.3) for $K = i_*Ri^!(\mathbb{Q}_\ell)$ to the corresponding spectral sequence for $K = \mathbb{D}_X^G$. Since this map of spectral sequences is an isomorphism at the E_2-terms and both spectral sequences converge strongly, we obtain an isomorphism at the abutments, that is, an isomorphism

$$[\tilde{X}]_G \cap (\quad) : H_{EG\underset{G}{\times}X}^{s,t}(EG\underset{G}{\times}\tilde{X}, \mathbb{Q}_\ell) \to H_{2\tilde{d}-s, \tilde{d}-t}(EG\underset{G}{\times}X, \mathbb{Q}_\ell).$$

Finally [J20, Theorem 1.6] provides a comparison of the classifying spaces between the two models, one where BG is defined as an ind-scheme $\{EG^{gm,m}\underset{G}{\times}(\text{Spec } k)|m\}$ and the other is defined using the simplicial Borel construction. It shows that the two models have equivalent equivariant derived categories. Therefore, one may define equivariant Borel-Moore homology using either approaches and this verifies all the three required properties in Definition 2.6.

We skip the discussion for singular cohomology, as it follows along similar lines. (In fact one may also consult [J99, section 2].) □

2.3. Riemann-Roch from forms of equivariant K-theory to equivariant Borel-Moore homology.

Theorem 2.10. *Let* $X \to \{H_{s,t}(X, \Gamma')|s, t\}$ *denote a homology theory defined in the framework discussed in Definition 2.5 so that it defines a generalized equivariant homology theory as in Definition 2.6.*

(i) Let $f : X \to Y$ *denote a proper* G-*equivariant map between two normal* G-*quasi-projective schemes. Let* $\tilde{G}$ *denote either a* GL_n *or a finite product of groups of the form* GL_n *containing* G *as a closed subgroup-scheme. Then one obtains the commutative diagram*

$$(2.3.1) \quad \pi_*\mathbf{G}(X, G) \longrightarrow \pi_*\mathbf{G}(E\tilde{G}^{gm,m}\times X, G) \overset{\cong}{\longrightarrow} \pi_*\mathbf{G}(E\tilde{G}^{gm,m}\underset{G}{\times}X) \overset{\tau_{X,m}}{\longrightarrow} H_{*,\bullet}(E\tilde{G}^{gm,m}\underset{G}{\times}X, \Gamma')_\mathbb{Q}$$

with vertical maps Rf_*, Rf_*, Rf_*, f_* down to

$$\pi_*\mathbf{G}(Y, G) \longrightarrow \pi_*\mathbf{G}(E\tilde{G}^{gm,m}\times Y, G) \overset{\cong}{\longrightarrow} \pi_*\mathbf{G}(E\tilde{G}^{gm,m}\underset{G}{\times}Y) \overset{\tau_{Y,m}}{\longrightarrow} H_{*,\bullet}(E\tilde{G}^{gm,m}\underset{G}{\times}Y, \Gamma')_\mathbb{Q}$$

for any fixed integer $m \geq 0$.

(ii) Assume in addition to the hypotheses in (i) that the map f *is also perfect, for example, a map that is a local complete intersection morphism. Then one obtains the commutative diagram*
(2.3.2)

$$\pi_*\mathbf{KH}(X, G) \longrightarrow \pi_*\mathbf{KH}(E\tilde{G}^{gm,m}\times X, G) \overset{\cong}{\longrightarrow} \pi_*\mathbf{KH}(E\tilde{G}^{gm,m}\underset{G}{\times}X) \overset{\tau_{X,m}}{\longrightarrow} H_{*,\bullet}(E\tilde{G}^{gm,m}\underset{G}{\times}X, \Gamma')_\mathbb{Q}$$

with vertical maps Rf_*, Rf_*, Rf_*, f_* down to

$$\pi_*\mathbf{KH}(Y, G) \longrightarrow \pi_*\mathbf{KH}(E\tilde{G}^{gm,m}\times Y, G) \overset{\cong}{\longrightarrow} \pi_*\mathbf{KH}(E\tilde{G}^{gm,m}\underset{G}{\times}Y) \overset{\tau_{Y,m}}{\longrightarrow} H_{*,\bullet}(E\tilde{G}^{gm,m}\underset{G}{\times}Y, \Gamma')_\mathbb{Q}$$

for any fixed integer $m \geq 0$.

Proof. The commutativity of the first squares in both diagrams 2.3.1 and 2.3.2 is clear from Theorems 2.1 and 2.2. The commutativity of the rest of the diagram 2.3.1 is also clear since the induced map $\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X} \to \mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{Y}$ is a proper map. The commutativity of the second square in (ii) is also clear. To prove the commutativity of the last square in (ii), one first observes that sending a perfect complex to itself, but viewed as a pseudo-coherent complex (or a complex of quasi-coherent $\mathcal{O}$-modules with bounded coherent cohomology) defines a natural map $\mathbf{KH}(\ ) \to \mathbf{G}(\ )$ of presheaves of spectra. Under the assumption that the map f is perfect and G-equivariant, one may see that this map commutes with push-forward by f_*, resulting in the homotopy commutative square:

$$
\begin{array}{ccc}
\mathbf{KH}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}) & \xrightarrow{\hspace{1cm}} & \mathbf{G}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}) \\
\downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\
\mathbf{KH}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{Y}) & \xrightarrow{\hspace{1cm}} & \mathbf{G}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{Y})
\end{array}
$$

One may take the homotopy groups of the above diagram and compose the horizontal maps in the resulting square with the corresponding horizontal maps in the third square of the diagram (2.3.1) to obtain the commutative square forming the third square in the diagram (2.3.2). $\qquad\square$

Lemma 2.11. *Assume the hypotheses as in Definitions 2.3, 2.5 and 2.6 on the homology/cohomology theories and that X is a G-quasi-projective normal scheme, for a linear algebraic group. Then the Riemann-Roch transformation* $\tau_{\mathrm{X,m}} : \pi_* \mathbf{G}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}) \to \mathrm{H}_{*,\bullet}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X},\Gamma')_{\mathbb{Q}}$ *extends to a Riemann-Roch transformation* $\tau_{\mathrm{X}} = \underset{\infty\leftarrow m}{lim}\,\tau_{\mathrm{X,m}} : \underset{\infty\leftarrow m}{lim}\,\pi_* \mathbf{G}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}) \to \underset{\infty\leftarrow\mathrm{m}}{lim}\,\mathrm{H}_{*,\bullet}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X},\Gamma')_{\mathbb{Q}}.$

Proof. One first recalls from [Gi], how the Riemann-Roch transformation is defined. Making use of the assumption that X is G-quasi-projective, one may find a smooth scheme $\tilde{\mathrm{X}}$ provided with a G-action and containing X as a G-stable closed subscheme. Therefore, one first defines the (local equivariant) Chern-character

$$ch_m : \pi_* \mathbf{G}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}) \cong \pi_* \mathbf{K}_{\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\tilde{\mathrm{X}}) \xrightarrow{ch_m} \mathrm{H}^{*,\bullet}_{\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\tilde{\mathrm{X}},\Gamma)_{\mathbb{Q}}.$$

These are clearly compatible as m-varies. Therefore, we obtain an induced local Chern-character:

$$ch = \underset{\infty\leftarrow m}{\lim}\,ch_m : \underset{\infty\leftarrow m}{\lim}\,\pi_* \mathbf{G}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}) \cong \underset{\infty\leftarrow\mathrm{m}}{\lim}\,\pi_* \mathbf{K}_{\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\tilde{\mathrm{X}})$$

$$\xrightarrow[\infty\leftarrow m]{\lim\,ch_m} \underset{\infty\leftarrow m}{\lim}\,\mathrm{H}^{*,\bullet}_{\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X}}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\tilde{\mathrm{X}},\Gamma)_{\mathbb{Q}}.$$

In view of the assumption that for each pair of fixed integers s,t, the inverse system

$$\{\mathrm{H}^{\mathrm{s,t}}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}} \times_{\mathrm{G}} \tilde{\mathrm{X}}, \Gamma)_{\mathbb{Q}}|\mathrm{m}\}$$

stabilizes as $m \to \infty$, one may readily define the Todd class of the tangent bundle to the scheme $\tilde{\mathrm{X}}$ with values in the inverse limit $\underset{\infty\leftarrow m}{\lim}\,\Pi_{s,t}\mathrm{H}^{\mathrm{s,t}}(\mathrm{EG}^{\mathrm{gm,m}} \times_{\mathrm{G}} \tilde{\mathrm{X}}, \Gamma)_{\mathbb{Q}}$. In view of the presence of an equivariant fundamental class $[\tilde{\mathrm{X}}]_{\mathrm{G}}$ as in Definition (2.6), and since $\{\mathrm{H}_{\mathrm{i}+2\tilde{\mathrm{d}}_{\mathrm{m}}-2\mathrm{g},\mathrm{j}+\tilde{\mathrm{d}}_{\mathrm{m}}-\mathrm{g}}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}} \times_{\mathrm{G}} \mathrm{X}, \Gamma')|\mathrm{m}\}$ stabilizes as $m \to \infty$ (see Definition 2.6(ii)), one may now define the Riemann-Roch transformation, by cupping the local-Chern-character homomorphism with the above equivariant Todd class (so that the composite map takes values in $\underset{\infty\leftarrow m}{\lim}\,\Pi_{s,t}\mathrm{H}^{\mathrm{s,t}}_{\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\times_{\mathrm{G}}\mathrm{X}}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}} \times_{\mathrm{G}} \tilde{\mathrm{X}}, \Gamma)_{\mathbb{Q}}$), followed by capping with the equivariant fundamental class $[\tilde{\mathrm{X}}]_{\mathrm{G}}$. It is now straight-forward to verify the resulting Riemann-Roch transformation is the inverse limit

$$\underset{\infty\leftarrow m}{\lim}\,\tau_{\mathrm{X,m}} : \underset{\infty\leftarrow m}{\lim}\,\pi_* \mathbf{G}(\mathrm{E}\tilde{\mathrm{G}}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}) \to \underset{\infty\leftarrow\mathrm{m}}{\lim}\,\mathrm{H}_{*,\bullet}(\mathrm{E}\tilde{\mathrm{G}}^{\mathrm{gm,m}}\underset{\mathrm{G}}{\times}\mathrm{X},\Gamma')_{\mathbb{Q}}.$$

$\qquad\square$

Corollary 2.12. *Assume in addition to the hypotheses in Lemma 2.11 that* Y *is a normal* G-*quasi-projective scheme and that* $f : X \to Y$ *is a* G-*equivariant proper map, with* X *also a normal* G-*quasi-projective scheme. Then the following diagram commutes for every choice of* m *sufficiently large, and where the slant maps in the right square are defined using the Gysin maps as in (2.2.1) and where* $c = dim(\tilde{EG}^{gm,m+1}) - dim(\tilde{EG}^{gm,m})$:
(2.3.3)

$$
\begin{array}{ccc}
\pi_* \mathbf{G}(X, G) & \xrightarrow{\quad\tau_X^{m+1}\quad} & \Pi_{u,v} H_{u+2c,v+c}(\tilde{EG}^{gm,m+1} \times_G X, \Gamma')_{\mathbb{Q}} \\
\end{array}
$$

with inner maps id, τ_X^m, Rf_*, f_*, τ_Y^m, id, τ_Y^{m+1}, f_* as shown, and intermediate terms $\pi_* \mathbf{G}(X, G) \xrightarrow{\tau_X^m} \Pi_{u,v} H_{u,v}(\tilde{EG}^{gm,m} \times_G X, \Gamma')_{\mathbb{Q}}$, $\pi_* \mathbf{G}(Y, G) \xrightarrow{\tau_Y^m} \Pi_{u,v} H_{u,v}(\tilde{EG}^{gm,m} \times_G Y, \Gamma')_{\mathbb{Q}}$, and $\pi_* \mathbf{G}(Y, G) \xrightarrow{\tau_Y^{m+1}} \Pi_{u,v} H_{u+2c,v+c}(\tilde{EG}^{gm,m+1} \times_G Y, \Gamma')_{\mathbb{Q}}$.

Proof. The inner and outer squares clearly commute by Theorem 2.10. Lemma 2.11 shows that the Riemann-Roch transformations extend to define a Riemann-Roch transformation with the inverse limits considered there as the targets. This shows the squares in the top and bottom commute. Therefore, it suffices to show the right square commutes. In view of our assumption, one may factor the map f as the composition of a G-equivariant closed immersion $i : X \to Y \times \mathbb{P}^n$ for some $n >> 0$ and the projection $\pi : Y \times \mathbb{P}^n \to Y$. Then the well-known formula for the homology of the projective spaces enables one to readily check that the corresponding diagram commutes when f is replaced by π.

Therefore, it remains to consider the case of a closed immersion $i : X \to Y$. Then, in view of our assumptions on X and Y as G-quasi-projectives schemes, we may find closed G equivariant immersions $X \to \tilde{X}$ and $Y \to \tilde{Y}$, together with a closed G-equivariant immersion $\tilde{X} \to \tilde{Y}$ so that the square

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & \tilde{Y}
\end{array}
$$

commutes. Then the commutativity of the right square in the diagram (2.3.3) amounts to the commutativity of the diagram (where $c' = dim(\tilde{Y}) - dim(\tilde{X})$):

$$
\begin{array}{ccc}
H^{*,\bullet}_{\tilde{EG}^{gm,m} \times_G X}(\tilde{EG}^{gm,m} \times_G \tilde{X}, \Gamma) & \xleftarrow{\; i_X^* \;} & H^{*,\bullet}_{\tilde{EG}^{gm,m+1} \times_G X}(\tilde{EG}^{gm,m+1} \times_G \tilde{X}, \Gamma) \\
\downarrow{\alpha_m} & & \downarrow{\alpha_{m+1}} \\
H^{*+2c',\bullet+c'}_{\tilde{EG}^{gm,m} \times_G X}(\tilde{EG}^{gm,m} \times_G \tilde{Y}, \Gamma) & \xleftarrow{\; i_Y^* \;} & H^{*+2c',\bullet+c'}_{\tilde{EG}^{gm,m+1} \times_G X}(\tilde{EG}^{gm,m+1} \times_G \tilde{Y}, \Gamma) \\
\downarrow{\beta_m} & & \downarrow{\beta_{m+1}} \\
H^{*+2c',\bullet+c'}_{\tilde{EG}^{gm,m} \times_G Y}(\tilde{EG}^{gm,m} \times_G \tilde{Y}, \Gamma) & \xleftarrow{\; i_Y^* \;} & H^{*+2c',\bullet+c'}_{\tilde{EG}^{gm,m+1} \times_G Y}(\tilde{EG}^{gm,m+1} \times_G \tilde{Y}, \Gamma).
\end{array}
$$

Let the normal bundle associated to the closed immersion $\tilde{EG}^{gm,m} \times_G \tilde{X} \to \tilde{EG}^{gm,m} \times_G \tilde{Y}$ be denoted $\mathcal{N}_m$, while we let the normal bundle associated to the closed immersion $\tilde{EG}^{gm,m+1} \times_G \tilde{X} \to \tilde{EG}^{gm,m+1} \times_G \tilde{Y}$ be denoted $\mathcal{N}_{m+1}$. The map denoted α_m (α_{m+1}) is cup-product with the Koszul-Thom class $\lambda_{-1}(\mathcal{N}_m)$ ($\lambda_{-1}(\mathcal{N}_{m+1})$), respectively. Here i_X denotes the closed immersion $\tilde{EG}^{gm,m} \times_G \tilde{X} \to \tilde{EG}^{gm,m+1} \times_G \tilde{X}$ while i_Y denotes the closed immersion $\tilde{EG}^{gm,m} \times_G \tilde{Y} \to \tilde{EG}^{gm,m+1} \times_G \tilde{Y}$, respectively. The commutativity of the top square follows from the observation that $i_Y^*(\lambda_{-1}(\mathcal{N}_{m+1})) = \lambda_{-1}(\mathcal{N}_m)$. The maps denoted β_m and β_{m+1} are the obvious maps, and therefore the commutativity of the bottom square is clear. $\qquad\square$

Proof of Theorem 1.2 Recall that for a tower of fibrations $\{\mathcal{X}_m \leftarrow \mathcal{X}_{m+1}|m\}$, the map $\pi_n(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathcal{X}_m) \to \underset{\infty\leftarrow m}{\lim}\pi_n(\mathcal{X}_n)$ is natural for such towers, and therefore compatible with maps of coherently homotopy commutative diagrams of such towers $\{\mathcal{X}_m \leftarrow \mathcal{X}_{m+1}|m\}$. We apply this to the commutative diagrams in (2.1.2) and (2.1.4) to obtain the commutative diagrams:

(2.3.4)

$$
\begin{array}{ccccccc}
\pi_*\mathbf{G}(X,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{G}(X,G)\widehat{_{\rho_{\tilde{G},\alpha(m)}}}) & \overset{\cong}{\longrightarrow} & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{G}(\mathrm{E}\tilde{G}^{gm,m}\underset{G}{\times}X)) & \longrightarrow & \underset{\infty\leftarrow m}{\lim}\pi_*(\mathbf{G}(\mathrm{E}\tilde{G}^{gm,m}\times X)) \\
\downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle \underset{\infty\leftarrow m}{\lim}Rf_*} \\
\pi_*\mathbf{G}(Y,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{G}(Y,G)\widehat{_{\rho_{G,\alpha(m)}}}) & \overset{\cong}{\longrightarrow} & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{G}(\mathrm{E}\tilde{G}^{gm,m}\times Y,G)) & \longrightarrow & \underset{\infty\leftarrow m}{\lim}\pi_*(\mathbf{G}(\mathrm{E}\tilde{G}^{gm,m}\underset{G}{\times}Y), \text{ and}
\end{array}
$$

(2.3.5)

$$
\begin{array}{ccccccc}
\pi_*\mathbf{KH}(X,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{KH}(X,G)\widehat{_{\rho_{\tilde{G},\alpha(m)}}}) & \overset{\cong}{\longrightarrow} & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{KH}(\mathrm{E}\tilde{G}^{gm,m}\underset{G}{\times}X)) & \longrightarrow & \underset{\infty\leftarrow m}{\lim}\pi_*(\mathbf{KH}(\mathrm{E}\tilde{G}^{gm,m}\times X)) \\
\downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle \underset{\infty\leftarrow m}{\lim}Rf_*} \\
\pi_*\mathbf{KH}(Y,G) & \longrightarrow & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{KH}(Y,G)\widehat{_{\rho_{G,\alpha(m)}}}) & \overset{\cong}{\longrightarrow} & \pi_*(\underset{\infty\leftarrow m}{\mathrm{holim}}\mathbf{KH}(\mathrm{E}\tilde{G}^{gm,m}\times Y,G)) & \longrightarrow & \underset{\infty\leftarrow m}{\lim}\pi_*(\mathbf{KH}(\mathrm{E}\tilde{G}^{gm,m}\underset{G}{\times}Y).
\end{array}
$$

Lemma 2.11 and Corollary (2.12) now show that the commutativity of the diagram 1.0.1 follows from the commutativity of the diagram 2.3.4. These observations complete the proof of the first statement. Then one deduces the commutativity of the diagram 1.0.2 from that of the diagram 2.3.5, just as in statement (i). □

Observe that Corollary 1.3 follows readily by imbedding the finite group as closed subgroup of a GL_n as discussed in the lines before the statement of this Corollary in the introduction.

Remark 2.13. It may be important to point out that, in view of the difficulties with usual Atiyah-Segal type completion, (for example as in [K]), the only equivariant Riemann-Roch theorems known *until now* are only for Grothendieck groups, unless one assumes that both X and Y are projective smooth schemes. The above Riemann-Roch theorems extend the Riemann-Roch theorems of Gillet (as in [Gi]) for higher algebraic K-theory and G-theory to the equivariant framework and to large classes of schemes with group action by linear algebraic groups.

3. Lefschtez-Riemann-Roch and applications

We conclude this discussion by providing a proof of the *Lefschetz-Riemann-Roch and coherent trace formula* discussed in Theorem 1.4, which extends similar formulae of Thomason for equivariant topological G-theory: see [Th86-1, Theorem 6.4]. But first we prove the following localization theorem for actions by diagonalizable groups.

Let T denote a split torus and let M denote its group of characters, with $\mathbb{Z}[M]$ its associated group ring. For each prime p of $\mathbb{Z}[M]$, there is a minimal subgroup M_p of M so that the prime p is the inverse image of a prime in $\mathbb{Z}[M/M_p]$. Let T_p denote the diagonalizable subgroup $D(M/M_p)$. Under this correspondence the (0)-ideal corresponds to the torus T itself.

Theorem 3.1. *Let* X *denote a scheme of finite type over* k *provided with an action by* T. *Let* $i : X^T \to X$ *denote the closed immersion from the corresponding fixed-point scheme into* X. *Then*

$$(\oplus_n\pi_n(\underset{\infty\leftarrow m}{holim}\mathbf{G}(\mathrm{T}^{gm,m}\times_T(X-X^T),T)))_{(0)} \cong \{0\} \ and \ (\oplus_n\underset{\infty\leftarrow m}{lim}\pi_n(\mathbf{G}(\mathrm{T}^{gm,m}\times_T(X-X^T),T)))_{(0)} \cong \{0\}.$$

Proof. We will assume that conclusion holds for all schemes Z on which T acts with no fixed points and where the Krull dimension of Z is less than the Krull dimension of $X-X^T$. Next assume that $X-X^T$ is connected. Let U denote a non-empty open and T-stable subscheme of $X-X^T$. Now consider the localization sequence, where $Z=(X-X^T)-U$:

(3.0.1)
$$\cdots \to (\oplus_n\pi_n(\underset{\infty\leftarrow m}{holim}\mathbf{G}(\mathrm{T}^{gm,m}\times_T Z,T)))_{(0)} \to (\oplus_n\pi_n(\underset{\infty\leftarrow m}{holim}\mathbf{G}(\mathrm{T}^{gm,m}\times_T(X-X^T),T)))_{(0)}$$
$$\to (\oplus_n\pi_n(\underset{\infty\leftarrow m}{holim}\mathbf{G}(\mathrm{T}^{gm,m}\times_T U,T)))_{(0)} \to \cdots.$$

This localization sequence and the inductive assumption shows that it suffices to prove the theorem with $X - X^T$ replaced by any open and T-stable subscheme U of $X - X^T$. At this point one invokes the *Torus generic slice theorem* in [Th86-2, Proposition 4.10] to conclude that the following hold:

(i) there exists a nonempty open and T-stable subscheme U of X and a sub-torus T' of T, with quotient torus $T/T' = T''$ which is non-trivial, and so that T acts on U through its quotient T'',

(ii) the resulting action of T'' on U is free and

(iii) U is isomorphic as a T-scheme to $T'' \times U/T$, with U/T denoting a geometric quotient.

Therefore, $ET^{gm,m} \times_T U \cong ET'' \times BT'^{gm,m} \times U/T$, for all m. It follows that $\underset{\infty \leftarrow m}{\text{holim}}\mathbf{G}(ET^{gm,m} \times_T U, T) \simeq \underset{\infty \leftarrow m}{\text{holim}}\mathbf{G}(BT'^{gm,m} \times U/T)$ and hence $(\oplus_n \pi_n(\underset{\infty \leftarrow m}{\text{holim}}\mathbf{G}(ET^{gm,m} \times_T U, T)))_{(0)} \cong \{0\}$. Therefore, by the inductive assumption on the Krull dimension of X and by the localization sequence (3.0.1),

$$(\oplus_n \pi_n(\underset{\infty \leftarrow m}{\text{holim}}\mathbf{G}(ET^{gm,m} \times_T (X - X^T), T)))_{(0)} \cong \{0\}.$$

In case $X - X^T$ is not connected, an ascending induction on the number of connected components and a similar argument using the same localization sequence will prove the same result in general. One may observe that $\oplus_n \underset{\infty \leftarrow m}{\lim} \pi_n(\mathbf{G}(ET^{gm,m} \times_T (X - X^T), T))$ is a module over $\oplus_n \pi_n(\underset{\infty \leftarrow m}{\text{holim}}\mathbf{G}(ET^{gm,m} \times_T (X - X^T), T)))$ to obtain the second equality. $\qquad\square$

Corollary 3.2. *Let* Z *denote a regular scheme of finite type over k provided with the action of a split torus* T. *Let* $i : Z^T \to Z$ *denote the closed immersion of the corresponding fixed point scheme and let* $\mathcal{N}$ *denote the conormal sheaf associated to the closed immersion i. Then the class* $\lambda_{-1}(\mathcal{N})$ *is a unit in* $\pi_0(\mathbf{K}(Z^T, T))_{(0)}$ *and the localized Gysin map*

$$i_* : \pi_*(\mathbf{G}(Z^T, T))_{(0)} \to \pi_*(\mathbf{G}(Z, T))_{(0)}$$

is injective with a left-inverse given by the map $\mathcal{F} \mapsto Li^*(\mathcal{F}) \cap \lambda_{-1}(\mathcal{N})^{-1}$. *Moreover the localized Gysin map*

$$i_* : \pi_*(\underset{\infty \leftarrow m}{holim}\mathbf{G}(ET^{gm,m} \times_T Z^T, T))_{(0)} \to \pi_*(\underset{\infty \leftarrow m}{holim}\mathbf{G}(ET^{gm,m} \times_T Z, T))_{(0)}$$
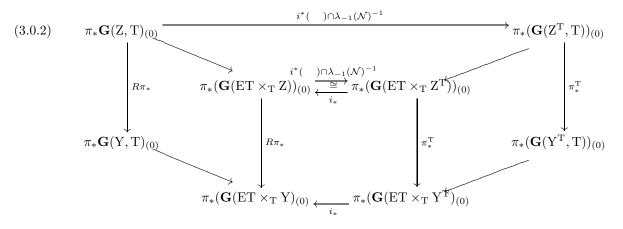
is an isomorphism, with a left-inverse given by the map $\mathcal{F} \mapsto Li^*(\mathcal{F}) \cap \lambda_{-1}(\mathcal{N})^{-1}$.

Proof. It is well-known that the composition of the Gysin map i_* and Li^* is multiplication by the class $\lambda_{-1}(\mathcal{N})$. See [Th86-1, Lemma 6.3] for a proof that $\lambda_{-1}(\mathcal{N})$ is a unit in $\pi_0(\mathbf{K}(Z^T, T))_{(0)}$. The first statement follows from these observations. That the Gysin map in the second statement is an isomorphism follows from Theorem 3.1, using the localization sequence for the open immersion $Z - Z^T \to Z$, with complement Z^T. $\qquad\square$

Proof of Theorem 1.4: The Lefschetz-Riemann-Roch and the Coherent trace formula. We quickly recall the framework of this theorem. Let $f : X \to Y$ denote a T-equivariant proper map between two schemes of finite type over k and provided with actions by split torus T. Assume that f factors T-equivariantly as $\pi \circ i_1$, where $i_1 : X \to Z$ is closed immersion into a regular T-scheme Z followed by a proper T-equivariant map $\pi : Z \to Y$. Let $i : Z^T \to Z$ denote the closed immersion of the fixed point scheme of Z into Z. Let $\mathcal{F}$ denote a T-equivariant coherent sheaf on X and let

$$\mathcal{G} = Li^*(i_{1*}(\mathcal{F})) \underset{\mathcal{O}_{Z^T}}{\otimes} (\lambda_{-1}(\mathcal{N}))^{-1}.$$

One may readily reduce to the case $Z = X$, by replacing $\mathcal{F}$ by $i_{1*}(\mathcal{F})$. Therefore, we will assume $f = \pi$ in what follows. Then the proof of the equality in (i) results from the commutative diagram:

(3.0.2)

where we have let $\mathbf{G}(ET \times_T V) = \underset{\infty \leftarrow m}{\mathrm{holim}}\, \mathbf{G}(ET^{\mathrm{gm.m}} \times_T V)$ for a scheme V with an action by T.

If X is T-equivariantly quasi-projective, one may factor the map f as the composite of the closed immersion $X \to Y \times \mathbb{P}^n$ for some large enough projective space on which the torus T acts and the projection $Y \times \mathbb{P}^n \to Y$. Therefore, when Y is regular, one may take $Z = Y \times \mathbb{P}^n$ in this case. When X is normal, it is T-quasi-projective by [SumII, Theorem 2.5]. This proves (ii).

Next we consider (iii). A key observation now is that, since the T-action on Y is assumed to be trivial, $\pi_0(\mathbf{G}(ET \times_T Y)) = \mathbb{Z}[T]\widehat{_{I_T}} \otimes \pi_0(\mathbf{G}(Y))$. This follows from the observation that $BT^{\mathrm{gm,m}} = (\mathbb{P}^m)^n$, if $T = \mathbb{G}_m^n$, and therefore $\underset{\infty \leftarrow m}{\lim}{}^1 \pi_1(\mathbf{G}(BT^{\mathrm{gm,m}} \times Y)) = 0$. Moreover $\mathbb{Z}[T]\widehat{_{I_T}}$ is a Laurent formal power series ring, and is therefore torsion-free over $\mathbb{Z}[T]$. As the classes that are inverted on taking the localization at the prime ideal (0) are non-zero divisors, equality in (1.0.3) holds in $\pi_0(\underset{\infty \leftarrow m}{\mathrm{holim}}\, \mathbf{G}(ET^{\mathrm{gm,m}} \times_T Y))$, if it holds after localization at the ideal (0). (See [Th86-2, Corollary 6.5], for example.) This completes the proof of (iii). (iv) and the extension to equivariant homotopy K-theory in (v) are now clear. $\square$

Proof of Corollary 1.5 The first two statement should be clear by applying Riemann-Roch transformations into an equivariant Borel-Moore homology theory. The third statement follows simply by taking $\mathcal{F}$ in Corollary 1.5(i) to be $\mathcal{O}_X$.

Proof of Corollary 1.6 (i) This follows readily from Theorem 1.4(iv) by taking X there to be the Schubert variety X_w and $Y = \mathrm{Spec}\, k$. Observe that the T-fixed points on X_w correspond to $\{w' \in W | w' \leq w\}$. Therefore, expanding out $\mathcal{G}$, one gets the term $H^0(X_w, \mathcal{G}) = \Sigma_{w' \leq w} a(w', w) e^{w'\lambda}$, for some coefficients $a(w', w)$. To show that this gives the required expression, one may invoke [Th86-1, Lemma 6.13]. Though the above argument is similar to the one in [Th86-1, pp. 541-542], it needs to be pointed out that as Thomason works there with G-theory with finite coefficients and with the Bott-element inverted, he has to work considerably harder to show the required identity holds in the integral representation ring $R(T)$. Our proof is more direct, as we are working with the $\pi_0(\mathbf{G}(\mathrm{Spec}\, k, T))$. Observe that when w is the longest element in the Weyl group W, $X_w = G/B$: in this case the above formula is just the Weyl character formula. $\square$

4. Computing the homotopy groups of the derived completion in terms of equivariant Borel-Moore motivic cohomology

Proof of Theorem 1.7 Assume that X is a G-scheme as in Definition 2.4, with $G = T$ a split torus. Then the approximation $EG^{\mathrm{gm,m}} \underset{G}{\times} X$ is always a scheme, and it is smooth when X is smooth. We will first assume that X is in fact smooth. Therefore, for a fixed value of $-s - t$, the homotopy groups of its $\mathbf{G}$-theory may be computed using the spectral sequence with m sufficiently large:

$$E_2^{s,t} = \mathbb{H}_{\mathrm{BM,M}}^{s-t}(EG^{\mathrm{gm,m}} \underset{G}{\times} X, \mathbb{Z}(-t)) \Rightarrow \pi_{-s-t}(\mathbf{G}(EG^{\mathrm{gm,m}} \underset{G}{\times} X)) \cong \pi_{-s-t}(\mathbf{G}(X, G)\widehat{_{\rho_{\tilde{G},m}}}).$$

The last isomorphism is from [CJ23, Theorem 5.6(i)] and $\mathbb{H}_{BM,M}$ denotes Borel-Moore motivic cohomology, which identifies with the higher equivariant Chow groups. This identification shows that the $E_2^{s,t} = 0$ for all but finitely many values of s and t, once $-s - t$ is fixed. Therefore, the above spectral sequence converges strongly, and the E_2-terms, and hence the abutment, will be independent of m, if $m >> 0$.

We proceed to show the existence of a spectral sequence as above even when the scheme X is singular, provided it satisfies the assumption that it is G-quasi-projective as in Definition 2.4. In this case, the hypothesis in Definition (2.4) shows that one may find a G-equivariant closed immersion of X into a smooth scheme $\tilde{X}$ with open complement $\tilde{U}$. This provides the commutative diagram (4.0.1), where $s_{\leq q}\mathbf{K}$ denotes the q-th slice of the spectrum $\mathbf{K}$.

$$(4.0.1) \qquad Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{X}_+, \mathbf{H}(\mathbb{Z}(q)[2q])) \longrightarrow Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{U}_+, \mathbf{H}(\mathbb{Z}(q)[2q]))$$

$$Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{X}_+, s_{\leq q}\mathbf{K}) \longrightarrow Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{U}_+, s_{\leq q}\mathbf{K})$$

$$Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{X}_+, s_{\leq q-1}\mathbf{K}) \longrightarrow Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{U}_+, s_{\leq q-1}\mathbf{K})$$

Taking the homotopy fiber of the horizontal maps in each row now provides the fiber sequence:

$$(4.0.2)$$
$$Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{X}_+, \Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{U}_+, \mathbf{H}(\mathbb{Z}(q)[2q])) \to Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{X}_+, \Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{U}_+, s_{\leq q}\mathbf{K}))$$
$$\to Map(\Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{X}_+, \Sigma_{S^1}EG^{gm,m}\underset{G}{\times}\tilde{U}_+, s_{\leq q-1}\mathbf{K}).$$

The exact couple is provided by the above fiber sequence. The same reasoning as above in the case when X is smooth shows the spectral sequence converges strongly and has the required E_2-terms. This completes the proof of the Theorem. $\qquad\square$

Remarks 4.1. For more general groups other than a split torus, there is still a similar spectral sequence (see [CJ23, Theorem 4.7]), but only for the full $EG^{gm}\underset{G}{\times}X$. The exact couple for this spectral sequence is obtained by taking the homotopy inverse limit of the tower of fibrations in (4.0.2). This spectral sequence converges only conditionally, and computes the homotopy groups of the full derived completion. $\qquad\square$

4.1. **Toric varieties, Spherical varieties, Linear varieties and proof of Corollary 1.9.** Here a key observation is that *quasi-projective toric varieties* are defined to be *normal* quasi-projective varieties on which a split torus T acts with finitely many orbits. Therefore, by [SumI, Theorem 1] (and [SumII, Theorem 2.5], such toric varieties are T-quasi-projective. This already includes many familiar varieties: see [Oda].

Given a connected split reductive group G, a *quasi-projective G-spherical variety* is defined to be a normal quasi-projective G-variety on which G, as well as a Borel subgroup of G, have finitely many orbits. (See [Tim].) This class of varieties clearly includes all toric varieties (when the group is a split torus), but also includes many other varieties. The discussion in the last paragraph above, now shows these are also G-quasi-projective varieties. *These observations readily complete the proof of Corollary 1.9.*

Remarks 4.2. The counter-example in [CJ23, 6.1.1] shows that for large classes of toric and spherical varieties, the equivariant algebraic K-theory with finite coefficients away from the characteristic is not isomorphic to the corresponding equivariant K-theory with the Bott element inverted, so that Thomason's theorem does not provide a completion theorem for equivariant K-theory unless the Bott element is inverted. Neither does [K, Theorem 1.2] unless these varieties are projective and smooth (and therefore, stratified by affine spaces).

Finally we recall that *linear varieties* are those varieties that have a stratification where each stratum is a product of an affine space and a split torus. In view of the existence of a localization sequence for equivariant **G**-theory and equivariant homotopy K-theory, it is clear the G-equivariant **G**-theory and homotopy K-theory of such varieties can be readily understood in terms of the corresponding G-equivariant **G**-theory and

homotopy K-theory of the strata. Examples of such varieties are normal quasi-projective varieties on which a connected split solvable group acts with finitely many orbits. (See [Rosen, p. 119].)

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