M-TF EQUIVALENCES IN THE REAL GROTHENDIECK GROUPS

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Abstract. For an abelian length category $\mathcal{A}$ with only finitely many isoclasses of simple objects, we have the wall-chamber structure and the TF equivalence in the dual real Grothendeick group $K_0(\mathcal{A})^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(K_0(\mathcal{A})_{\mathbb{R}}, \mathbb{R})$, which are defined by semistable subcategories and semistable torsion pairs in $\mathcal{A}$ associated to elements $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. In this paper, we introduce the M-TF equivalence for each object $M \in \mathcal{A}$ as a systematic way to coarsen the TF equivalence. We show that the set $\Sigma(M)$ of the closures of M-TF equivalence classes is a finite complete fan in $K_0(\mathcal{A})^*_{\mathbb{R}}$, and that $\Sigma(M)$ is the normal fan of the Newton polytope $\mathrm{N}(M)$ in $K_0(\mathcal{A})_{\mathbb{R}}$.

Contents

1. Introduction

There are many situations where an abelian length category $\mathcal{A}$ with only finitely many isoclasses $S(1), \dots, S(n)$ of simple objects appears. The objects $S(1), \dots, S(n)$ give a canonical basis $[S(1)], \dots, [S(n)]$ of the *real Grothendieck group* $K_0(\mathcal{A})_{\mathbb{R}}$, so $K_0(\mathcal{A})_{\mathbb{R}} = \bigoplus_{i=1}^n \mathbb{R}[S(i)]$ is naturally identified with the Euclidean space $\mathbb{R}^n$. We also consider the dual real Grothendieck group $K_0(\mathcal{A})^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(K_0(\mathcal{A}), \mathbb{R})$, which we mainly focus on in this paper rather than $K_0(\mathcal{A})_{\mathbb{R}}$. Each element $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ is just an $\mathbb{R}$-linear form $\theta \colon K_0(\mathcal{A})_{\mathbb{R}} \to \mathbb{R}$. Thus for any $M \in \mathcal{A}$, we get a real number $\theta(M) \in \mathbb{R}$. This enables us to define several subcategories of $\mathcal{A}$ and subsets of $K_0(\mathcal{A})^*_{\mathbb{R}}$.

For each $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$, an object $M \in \mathcal{A}$ is said to be *θ-semistable* if $\theta(M) = 0$ and each factor object N of M satisfies the linear inequality $\theta(N) \geq 0$. These *stability conditions* were originally introduced by King [K] to characterize a condition in *geometrical invariant theory* of *quiver representations*. The *θ-semistable subcategory* $\mathcal{W}_\theta \subset \mathcal{A}$ consisting of all θ-semistable objects is an abelian category of finite length [R, HR], so $\mathcal{W}_\theta$ has the Jordan-Hölder property. Thus for any object $M \in \mathcal{W}_\theta$, the set $\mathsf{supp}_\theta M$ of isoclasses of composition factors of M in $\mathcal{W}_\theta$ is well-defined. Though $\mathcal{W}_\theta$ may have infinitely many isoclasses of simple objects, $\mathsf{supp}_\theta M$ is always a finite set.

In a similar way, Baumann-Kamnitzer-Tingley [BKT] associated two *semistable torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ to each $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ so that $\mathcal{T}_\theta \subset \overline{\mathcal{T}}_\theta$, $\mathcal{F}_\theta \subset \overline{\mathcal{F}}_\theta$ and $\mathcal{W}_\theta = \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$, to study the *canonical bases* of *quantum groups* via *preprojective algebras* of affine type.

For each nonzero object $M \in \mathcal{A}$, Bridgeland [B] and Brüstle-Smith-Treffinger [BST] defined the rational polyhedral cone $\Theta_M \subset K_0(\mathcal{A})^*_{\mathbb{R}}$ consisting of all $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ such that M are θ-semistable. By considering the subsets Θ_M for all nonzero objects M as walls, [BST, B] introduced the *wall-chamber structure* (or the *stablity scattering diagram*) in the Euclidean space $K_0(\mathcal{A})^*_{\mathbb{R}}$. Then Bridgeland [B] showed that this wall-chamber structure is strongly related to the *cluster theory* introduced by Fomin-Zelevinsky [FZ].

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To study this wall-chamber structure more, the first named author [A] defined the *TF equivalence* as an equivalence relation on $K_0(\mathcal{A})^*_{\mathbb{R}}$ so that θ and η are TF equivalent if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$.

TF equivalences are strongly related to *silting theory* of finite dimensional algebras. Let A be a finite dimensional algebra A over a field k, and set $\mathcal{A}$ as the category $\operatorname{\mathsf{mod}} A$ of finitely generated A-modules. Then $S(1), \dots, S(n)$ are nothing but the isoclasses of simple A-modules, and we take the projective covers $P(1), \dots, P(n)$ of $S(1), \dots, S(n)$ respectively. Then we have the real Grothendieck group $K_0(\operatorname{\mathsf{proj}} A)_{\mathbb{R}} = \bigoplus_{i=1}^n \mathbb{R}[P(i)] \simeq \mathbb{R}^n$ of the exact category $\operatorname{\mathsf{proj}} A$ of finitely generated projective A-modules, and there is an isomorphism $K_0(\operatorname{\mathsf{proj}} A)_{\mathbb{R}} \simeq K_0(\mathcal{A})^*_{\mathbb{R}}$ induced by the *Euler bilinear form*

$$\begin{aligned}\langle !, ? \rangle \colon K_0(\operatorname{\mathsf{proj}} A)_{\mathbb{R}} \times K_0(\operatorname{\mathsf{mod}} A)_{\mathbb{R}} &\to \mathbb{R}, \\ ([P(i)], [S(j)]) &\mapsto \delta_{i,j} \dim_k \operatorname{End}_A(S(j)).\end{aligned}$$

Then based on the works [BST, Y] of Brüstle-Smith-Treffinger and Yurikusa, [A] showed that the wall-chamber structure and the TF equivalence in $K_0(\operatorname{\mathsf{proj}} A)_{\mathbb{R}}$ contain the *silting fan* of the *silting cones* [DIJ] of basic 2-term presilting complexes U [KV, AI, AIR] in the homotopy category $\mathsf{K}^{\mathrm{b}}(\operatorname{\mathsf{proj}} A)$. More precisely, the silting cone $C^\circ(U)$ of each basic 2-term presilting complex U is a TF equivalence class. He also proved that each chamber in the wall-chamber structure in $K_0(\operatorname{\mathsf{proj}} A)_{\mathbb{R}}$ is the silting cone $C^\circ(T)$ for a unique isoclass of basic 2-term silting complex T.

On the other hand, there are TF equivalence classes which are not silting cones $C^\circ(U)$ if the algebra A is *τ-tilting infinite*, that is, there are infitinely many isoclasses of basic 2-term (pre)silting complexes in $\mathsf{K}^{\mathrm{b}}(\operatorname{\mathsf{proj}} A)_{\mathbb{R}}$; see [ZZ, A]. For τ-tilting infinite algebras, the wall-chamber structure and the TF equivalence in $K_0(\operatorname{\mathsf{proj}} A)_{\mathbb{R}} \simeq K_0(\mathcal{A})^*_{\mathbb{R}}$ often become very complicated, but in many situations, it is enough to get only important information on TF equivalences depending on purposes.

Therefore in this paper, we introduce the *M-TF equivalence* in $K_0(\mathcal{A})^*_{\mathbb{R}}$ for each object $M \in \mathcal{A}$ as a systematic way to coarsen the TF equivalence. For $M \in \mathcal{A}$ and $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$, the semistable torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ give the *canonical sequences*

$$\begin{aligned}&0 \to \overline{\mathrm{t}}_\theta M \to M \to \mathrm{f}_\theta M \to 0 \quad (\overline{\mathrm{t}}_\theta M \in \overline{\mathcal{T}}_\theta,\ \mathrm{f}_\theta M \in \mathcal{F}_\theta), \\ &0 \to \mathrm{t}_\theta M \to M \to \overline{\mathrm{f}}_\theta M \to 0 \quad (\mathrm{t}_\theta M \in \mathcal{T}_\theta,\ \overline{\mathrm{f}}_\theta M \in \overline{\mathcal{F}}_\theta)\end{aligned}$$

with $\mathrm{t}_\theta M \subset \overline{\mathrm{t}}_\theta M \subset M$ subobjects. The subfactor $\mathrm{w}_\theta M := \overline{\mathrm{t}}_\theta M / \mathrm{t}_\theta M$ is in $\mathcal{W}_\theta$, so the set $\operatorname{\mathsf{supp}}_\theta(\mathrm{w}_\theta M)$ of composition factors of $\mathrm{w}_\theta M$ in $\mathcal{W}_\theta$ is well-defined. Then we introduce the M-TF equivalence as follows.

Definition 1.1 (Definition 3.1)**.** Let $M \in \mathcal{A}$.

(a) Let $\theta, \eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. Then we say that θ and η are *M-TF equivalent* if the following conditions are satisfied.
   (i) The conditions $\mathrm{t}_\theta M = \mathrm{t}_\eta M$, $\mathrm{w}_\theta M = \mathrm{w}_\eta M$ and $\mathrm{f}_\theta M = \mathrm{f}_\eta M$ hold, or equivalently, $(\overline{\mathrm{t}}_\theta M, \mathrm{f}_\theta M) = (\overline{\mathrm{t}}_\eta M, \mathrm{f}_\eta M)$ and $(\mathrm{t}_\theta M, \overline{\mathrm{f}}_\theta M) = (\mathrm{t}_\eta M, \overline{\mathrm{f}}_\eta M)$ hold.
   (ii) We have $\operatorname{\mathsf{supp}}_\theta(\mathrm{w}_\theta M) = \operatorname{\mathsf{supp}}_\eta(\mathrm{w}_\eta M)$.

(b) We define $\mathsf{TF}(M)$ as the set of all M-TF equivalence classes, and

$$\Sigma(M) := \{\overline{E} \mid E \in \mathsf{TF}(M)\}$$

as the set of the closures of all $E \in \mathsf{TF}(M)$.

By definition, θ and η are TF equivalent if and only if they are M-TF equivalent for all objects $M \in \mathcal{A}$. Thus the M-TF equivalence for each $M \in \mathcal{A}$ is coarser than the TF equivalence.

The following is the main result of this paper, where σ° denotes the *relative interior* of σ, that is, the interior of σ as a subset of the $\mathbb{R}$-vector space $\mathbb{R}\sigma$ spanned by σ.

Theorem 1.2 (Theorem 3.2 and Corollary 3.16(a))**.** *Let $M \in \mathcal{A}$.*

(a) *Each* $\sigma \in \Sigma(M)$ *is a cone in* $K_0(\mathcal{A})^*_{\mathbb{R}}$. *Moreover we have mutually inverse bijections* $\mathsf{TF}(M) \leftrightarrow \Sigma(M)$ *of finite sets given by* $E \mapsto \overline{E}$ *for any* $E \in \mathsf{TF}(M)$ *and* $\sigma \mapsto \sigma^\circ$ *for any* $\sigma \in \Sigma(M)$.
(b) *The set* $\Sigma(M)$ *is a finite complete fan in* $K_0(\mathcal{A})^*_{\mathbb{R}}$.
(c) *Let* $\sigma \in \Sigma(M)$. *Then the face decomposition*

$$\sigma = \bigsqcup_{\tau \in \mathsf{Face}\,\sigma} \tau^\circ$$

is also the decomposition to M*-TF equivalence classes.*

For each $M \in \mathcal{A}$, the wall Θ_M in the wall-chamber structure and its faces are examples of elements of $\Sigma(M)$; see Example 3.4. Thus we can say that the M-TF equivalence is an extension of the face structure of the cone $\Theta_M \subset K_0(\mathcal{A})^*_{\mathbb{R}}$ to a finite complete fan in $K_0(\mathcal{A})^*_{\mathbb{R}}$.

We remark that the condition (ii) $\mathsf{supp}_\theta M = \mathsf{supp}_\eta M$ in Definition 1.1(a) is necessary to make $\Sigma(M)$ a fan; see Example 3.5.

Theorem 1.2 will be proved in Theorem 3.15 by showing that $\Sigma(M)$ coincides with the *normal fan* $\Sigma(\mathrm{N}(M))$ [AHIKM, BKT, Fe2] of the *Newton polytope* $\mathrm{N}(M)$ in $K_0(\mathcal{A})_{\mathbb{R}}$, which is the convex hull of the set $\{[L] \mid L \text{ is a subobject of } M\}$.

The organization of this paper is as follows. In Section 2, we recall the definitions and basic properties of stability conditions, the wall-chamber structure, semistable torsion pairs and the TF equivalence. Then in Subsection 3.1, we state the definition and the main result on M-TF equivalences, and give some examples. The proof of the main result is done in Subsection 3.2. Subsection 3.3 is devoted to proving useful properties of the fan $\Sigma(M)$.

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2. Preliminary

In this paper, any subcategory is assumed to be a full subcategory.

We first recall some general properties of abelian categories of finite length. Let $\mathcal{C}$ be an abelian category of finite length. Then $\mathcal{C}$ has the Jordan-Hölder property. Thus each $M \in \mathcal{C}$ has a *composition series* $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_\ell = M$, that is, a sequence where each subfactor M_i/M_{i-1} is a simple object of $\mathcal{C}$. The subfactors M_i/M_{i-1} are called the *composition factors* of M in $\mathcal{C}$. Up to reordering and isomorphisms, the composition factors of M in $\mathcal{C}$ and their multiplicities do not depend on the choice of a composition series of M in $\mathcal{C}$. Composition factors play a very important role in this paper.

Throughout this paper, we let $\mathcal{A}$ be an abelian category of finite length with $S(1), \ldots, S(n)$ all the distinct isoclasses of simple objects. Then the Grothendieck group $K_0(\mathcal{A})$ is the free abelian group $\bigoplus_{i=1}^n \mathbb{Z}[S(i)] \simeq \mathbb{Z}^n$, so the real Grothendieck group $K_0(\mathcal{A})_{\mathbb{R}} := K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ is the $\mathbb{R}$-vector space $\bigoplus_{i=1}^n \mathbb{R}[S(i)] \simeq \mathbb{R}^n$.

The main target of this paper is the dual real Grothendieck group $K_0(\mathcal{A})^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(K_0(\mathcal{A}), \mathbb{R})$. We write $[S(1)]^*, \ldots, [S(n)]^* \in K_0(\mathcal{A})^*_{\mathbb{R}}$ for the dual basis of $[S(1)], \ldots, [S(n)] \in K_0(\mathcal{A})_{\mathbb{R}}$, and then we have $K_0(\mathcal{A})^*_{\mathbb{R}} = \bigoplus_{i=1}^n \mathbb{R}[S(i)]^* \simeq \mathbb{R}^n$.

By definition, each element $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ is an $\mathbb{R}$-linear form $\theta\colon K_0(\mathcal{A})_{\mathbb{R}} \to \mathbb{R}$. In particular, for any $M \in \mathcal{A}$, we have a real number $\theta(M) \in \mathbb{R}$, and if $\theta = [S(i)]^*$, then $\theta(M)$ is the multiplicity of $S(i)$ in the composition factors of M. By using this, stability conditions are defined as follows.

Definition 2.1. [K, Definition 1.1] Let $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$.
(a) Let $M \in \mathcal{A}$. Then M is said to be *θ-semistable* if $\theta(M) = 0$ and $\theta(N) \geq 0$ for any factor object N of M. A θ-semistable object M is said to be *θ-stable* if moreover $M \neq 0$ and $\theta(N) > 0$ for any nonzero factor object $N \neq 0$ of M.
(b) We define the *θ-semistable subcategory* $\mathcal{W}_\theta \subset \mathcal{A}$ as the full subcategory consisting of all θ-semistable objects.

Then we have the following basic properties; see [R, HR].

Proposition 2.2. *Let $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$.*

(a) *$\mathcal{W}_\theta$ is a wide subcategory of $\mathcal{A}$, that is, $\mathcal{W}_\theta$ is closed under taking kernels, cokernels and extensions in $\mathcal{A}$.*
(b) *$\mathcal{W}_\theta$ is an abelian category of finite length; hence $\mathcal{W}_\theta$ satisfies the Jordan-Hölder property.*
(c) *For any $M \in \mathcal{W}_\theta$, M is a simple object in $\mathcal{W}_\theta$ if and only if M is θ-stable.*

We write $\mathsf{sim}\,\mathcal{W}_\theta$ for the set of all isoclasses of simple objects of $\mathcal{W}_\theta$. By (b), for each $M \in \mathcal{W}_\theta$, the composition factors of M in $\mathcal{W}_\theta$ is well-defined. Thus we define the following symbol.

Definition 2.3. For any $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ and $M \in \mathcal{W}_\theta$, we set

$$\mathsf{supp}_\theta\, M := \{\text{isoclasses of composition factors of } M \text{ in } \mathcal{W}_\theta\} \subset \mathsf{sim}\,\mathcal{W}_\theta.$$

We remark that $\mathsf{supp}_\theta\, M$ is always a finite set, but $\mathsf{sim}\,\mathcal{W}_\theta$ can be an infinite set. We do not consider the Grothendieck group of $\mathcal{W}_\theta$ in this paper.

The category $\mathsf{Filt}\,\mathsf{supp}_\theta\, M$ is the smallest Serre subcategory of $\mathcal{W}_\theta$ containing M, where $\mathsf{Filt}\,\mathcal{C} \subset \mathcal{A}$ is the full subcategory of all $M \in \mathcal{A}$ admitting a sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ with $M_i/M_{i-1} \in \mathcal{C}$.

Next we recall the wall-chamber structure in $K_0(\mathcal{A})^*_{\mathbb{R}}$ given by stability conditions.

Definition 2.4. [B, Definition 6.1][BST, Definition 3.2] We set the following notions.

(a) Let $M \in \mathcal{A} \setminus \{0\}$. Then we define the *wall* $\Theta_M \subset K_0(\mathcal{A})^*_{\mathbb{R}}$ associated to M by

$$\Theta_M := \{\theta \in K_0(\mathcal{A})^*_{\mathbb{R}} \mid M \in \mathcal{W}_\theta\}.$$

(b) The *wall-chamber structure* in $K_0(\mathcal{A})^*_{\mathbb{R}}$ is defined to be the structure given by all walls Θ_M for $M \in \mathcal{A} \setminus \{0\}$.

To explain properties of Θ_M, we briefly prepare some notions on cones. See [Fu] for the details.

In this paper, a *cone* in the Euclidean space $\mathbb{R}^n$ means a rational polyhedral cone, that is, a nonempty subset $C \subset \mathbb{R}^n$ which admits finitely many elements $v_1, \ldots, v_m \in \mathbb{Z}^n$ such that $C = \sum_{i=1}^m \mathbb{R}_{\geq 0} v_i$. In particular, $\{0\}$ is a cone in $\mathbb{R}^n$ as $m = 0$.

If C is a cone in $\mathbb{R}^n$, a subset $F \subset C$ is called a *face* of C if the following conditions are satisfied:

(a) $F \neq \emptyset$, $\mathbb{R}_{\geq 0} F \subset F$.
(b) If $v, w \in C$ satisfy $v + w \in F$, then $v, w \in F$.

In this case, the face F is a cone again. We set $\mathsf{Face}\, C$ as the set of all faces of the cone C.

For any subset $X \subset \mathbb{R}^n$, the symbol $\mathbb{R}X$ denotes the $\mathbb{R}$-vector subspace spanned by X. Then for a cone C, the *dimension* $\dim_{\mathbb{R}} C$ of a cone is defined as $\dim_{\mathbb{R}} \mathbb{R}C$, and we write C° for its *relative interior*, that is, the interior of C as a subset of $\mathbb{R}C$.

Note also that we have a hyperplane $\mathrm{Ker}\langle ?, \mathsf{supp}_\theta\, M\rangle$ in $K_0(\mathcal{A})^*_{\mathbb{R}}$ for each $M \in \mathcal{A}$ and $\theta \in \Theta_M$. Under this terminology, the face structure of Θ_M is related to $\mathsf{supp}_\theta\, M$ as follows.

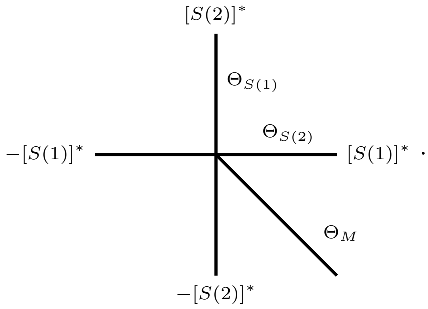
Lemma 2.5. [A, Lemma 2.7] *Let $M \in \mathcal{A} \setminus \{0\}$, and $\theta, \eta \in \Theta_M$. Take the unique faces $\sigma, \tau \in \mathsf{Face}\,\Theta_M$ such that $\theta \in \sigma^\circ$ and $\eta \in \tau^\circ$.*

(a) *The element θ belongs to the interior of $\Theta_M \cap \mathrm{Ker}\langle ?, \mathsf{supp}_\theta\, M\rangle$ in $\mathrm{Ker}\langle ?, \mathsf{supp}_\theta\, M\rangle$.*
(b) *We have $\mathbb{R}\sigma = \mathrm{Ker}\langle ?, \mathsf{supp}_\theta\, M\rangle$, so F° is the interior of F in $\mathrm{Ker}\langle ?, \mathsf{supp}_\theta\, M\rangle$. Thus $\dim_{\mathbb{R}} \sigma = n - \dim_{\mathbb{R}}(\mathbb{R}\,\mathsf{supp}_\theta\, M)$.*
(c) *The inclusion $\sigma \subset \tau$ holds if and only if $\mathsf{supp}_\theta\, M \supset \mathsf{supp}_\eta\, M$.*

We give an easy example of the wall-chamber structure. In examples of this paper, we often let k be a field, A be a finite-dimensional k-algebra, and the abelian category $\mathcal{A}$ be the category $\mathsf{mod}\, A$ of finitely generated right A-modules. In this setting, $S(1), \ldots, S(n)$ are the isoclasses of simple A-modules.

Example 2.6. Let A be the path algebra $k(1 \to 2)$. There are exactly three isoclasses of indecomposable A-modules $S(1), S(2)$ and $M = \frac{1}{2}$. Then the wall-chamber structure of $K_0(\mathsf{mod}\, A)^*_{\mathbb{R}}$

is depicted as follows:



The wall Θ_M is a half-line from the origin 0, so $\mathsf{Face}\,\Theta_M = \{\{0\}, \Theta_M\}$. For each $\theta \in \Theta_M$, $\mathsf{supp}_\theta\, M$ is $\{S(2), S(1)\}$ if $\theta = 0$, and is $\{M\}$ otherwise. Thus we can see that Lemma 2.5 holds.

We remark that Θ_M is not necessarily a simplicial cone by the following example.

Example 2.7. Define an algebra A, and a module $M \in \mathsf{mod}\, A$ by

$$A := k\begin{pmatrix} 1 \longrightarrow 3 \\ \downarrow \qquad \downarrow \\ 2 \longrightarrow 4 \end{pmatrix}, \quad M := \begin{pmatrix} k \longrightarrow k \\ \downarrow \qquad \downarrow\lambda \\ k \longrightarrow k \end{pmatrix} \quad (\lambda \in k^\times).$$

Then A is a tame hereditary algebra, and M is a module in the mouth of some homogeneous tube in the Auslander-Reiten quiver. The wall Θ_M is not simplicial, because direct calculation gives that the one-dimensional faces of Θ_M are

$$\mathbb{R}_{\geq 0}([S(1)]^* - [S(2)]^*), \mathbb{R}_{\geq 0}([S(1)]^* - [S(3)]^*), \mathbb{R}_{\geq 0}([S(2)]^* - [S(4)]^*), \mathbb{R}_{\geq 0}([S(3)]^* - [S(4)]^*).$$

We next recall torsion pairs.

Definition 2.8. Let $\mathcal{T}, \mathcal{F}$ be full subcategories of $\mathcal{A}$.

(a) We say that $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* in $\mathcal{A}$ if $\operatorname{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$ holds and for each $M \in \mathcal{A}$, there exists a short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0 \quad (M_1 \in \mathcal{T},\ M_2 \in \mathcal{F}).$$

(b) The subcategory $\mathcal{T}$ (resp. $\mathcal{F}$) is called a *torsion class* (resp. a *torsion-free class*) in $\mathcal{A}$ if there exists $\mathcal{F} \subset \mathcal{A}$ (resp. $\mathcal{T} \subset \mathcal{A}$) such that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$.

Assume that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$. For each $M \in \mathcal{A}$, a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ such that $M_1 \in \mathcal{T}$ and $M_2 \in \mathcal{F}$ is unique up to isomorphisms, so it is called a *canonical sequence* of M with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. Since $\mathcal{A}$ is of finite length, we can use the following well-known characterizations.

Lemma 2.9. *Let $\mathcal{T}, \mathcal{F}$ be full subcategories of $\mathcal{A}$.*

(a) *The pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ if and only if $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = {}^\perp\mathcal{F}$, where $\mathcal{T}^\perp := \{M \in \mathcal{A} \mid \operatorname{Hom}_A(\mathcal{T}, M) = 0\}$ and ${}^\perp\mathcal{F} := \{M \in \mathcal{A} \mid \operatorname{Hom}_A(M, \mathcal{F}) = 0\}$.*
(b) *The subcategory $\mathcal{T}$ (resp. $\mathcal{F}$) is a torsion class (resp. torsion-free class) in $\mathcal{A}$ if and only if $\mathcal{T}$ (resp. $\mathcal{F}$) is closed under taking factor objects (resp. subobjects) and extensions in $\mathcal{A}$.*
(c) *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair of $M \in \mathcal{A}$. Set $\mathrm{t}M$ as the largest subobject $L \subset M$ such that $L \in \mathcal{T}$, and $\mathrm{f}M := M/\mathrm{t}M$. Then $\mathrm{f}M \in \mathcal{F}$, so we have a canonical sequence*

$$0 \to \mathrm{t}M \to M \to \mathrm{f}M \to 0 \quad (\mathrm{t}M \in \mathcal{T},\ \mathrm{f}M \in \mathcal{F}).$$

In this paper, we are interested only in the following torsion pairs.

Definition 2.10. [BKT, Subsection 3.1] Let $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. We define the following subcategories of $\mathcal{A}$:

$$\begin{aligned}\overline{\mathcal{T}}_\theta &:= \{M \in \mathcal{A} \mid \theta(N) \geq 0 \text{ for any factor object } N \text{ of } M\},\\ \mathcal{F}_\theta &:= \{M \in \mathcal{A} \mid \theta(L) < 0 \text{ for any nonzero subobject } L \neq 0 \text{ of } M\},\\ \mathcal{T}_\theta &:= \{M \in \mathcal{A} \mid \theta(N) > 0 \text{ for any nonzero factor object } N \neq 0 \text{ of } M\},\\ \overline{\mathcal{F}}_\theta &:= \{M \in \mathcal{A} \mid \theta(L) \leq 0 \text{ for any subobject } L \text{ of } M\}.\end{aligned}$$

By definition, we get $\mathcal{W}_\theta = \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$. By Lemma 2.9, it is easy to see that $\mathcal{T}_\theta \subset \overline{\mathcal{T}}_\theta$ are torsion classes in $\mathcal{A}$, and $\mathcal{F}_\theta \subset \overline{\mathcal{F}}_\theta$ are torsion-free classes in $\mathcal{A}$. They form torsion pairs as follows.

Proposition 2.11. [BKT, Proposition 3.1] *Let $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. Then $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ are torsion pairs in $\mathcal{A}$.*

Thus we call $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ the *semistable torsion pairs* associated to each $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. The equality $\mathcal{W}_\theta = \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$ implies that the interval $[\mathcal{T}_\theta, \overline{\mathcal{T}}_\theta]$ in the lattice of torsion classes in $\mathcal{A}$ is a wide interval in the sense of [AP]. By using the semistable torsion pairs, the following equivalence relation is defined.

Definition 2.12. [A, Definition 2.13] Let $\theta, \eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. We say that θ and η are *TF equivalent* if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$.

The relationship with the wall-chamber structure is given as follows. This is a slightly stronger version of [A, Theorem 2.17]. Here, $[\theta, \eta]$ is the line segment in $\mathbb{R}^n$ connecting θ and η, and a *brick* in $\mathcal{A}$ means an object $S \in \mathcal{A}$ such that any nonzero endomorphism on S is isomorphic.

Proposition 2.13. *Let $\theta \neq \eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. Then the following conditions are equivalent.*

(a) *The elements θ and η are TF equivalent.*
(b) *Any $\zeta \in [\theta, \eta]$ is TF equivalent to θ and η.*
(c) *For any $\zeta \in [\theta, \eta]$, the subcategory $\mathcal{W}_\zeta \subset \mathcal{A}$ is constant.*
(d) *For any nonzero object $M \in \mathcal{A} \setminus \{0\}$, the intersection $\Theta_M \cap [\theta, \eta]$ is $[\theta, \eta]$ or $\emptyset$.*
(e) *There exists no brick $S \in \mathcal{A}$ such that* $\operatorname{Ker}\langle ?, S\rangle \cap [\theta, \eta]$ *is one point.*
(f) *There exists no brick $S \in \mathcal{A}$ such that* $\operatorname{Ker}\langle ?, S\rangle \cap [\theta, \eta]$ *has a unique element ζ and that S is ζ-stable.*

Proof. The original proof of [A, Theorem 2.17] works to deduce our (a)⇒(b)⇒(c)⇒(d)⇒(e). Moreover (e)⇒(f) is obvious. It remains to show (f)⇒(a).

We assume that θ and η are not TF equivalent and show the existence of a brick S satisfying the condition in (f).

Since θ and η are not TF equivalent, $\overline{\mathcal{T}}_\theta \cap \mathcal{F}_\eta \neq \{0\}$ or $\overline{\mathcal{F}}_\theta \cap \mathcal{T}_\eta \neq \{0\}$ holds by Lemma [A, Lemma 2.19]. We may assume the former. Then we can take $S \in \overline{\mathcal{T}}_\theta \cap \mathcal{F}_\eta$ such that the length $\ell(S)$ of S in $\mathcal{A}$ satisfies $\ell(S) > 0$ and $\ell(S) \leq \ell(M)$ for any nonzero $M \in (\overline{\mathcal{T}}_\theta \cap \mathcal{F}_\eta) \setminus \{0\}$.

In this case, $\theta(S) \geq 0$ and $\eta(S) < 0$ holds, so $\operatorname{Ker}\langle ?, S\rangle \cap [\theta, \eta]$ is one point; write $\operatorname{Ker}\langle ?, S\rangle \cap [\theta, \eta] = \{\zeta\}$.

It remains to show that S is ζ-stable. Clearly $\zeta(S) = 0$. Let $X \subsetneq S$ be a nonzero proper subobject. Then $\overline{\mathrm{t}}_\theta X \in \overline{\mathcal{T}}_\theta$ is also in $\mathcal{F}_\eta$, because $\overline{\mathrm{t}}_\theta X \subset X \subset S \in \mathcal{F}_\eta$. Thus we have $\overline{\mathrm{t}}_\theta X \in \overline{\mathcal{T}}_\theta \cap \mathcal{F}_\eta$, and $\overline{\mathrm{t}}_\theta X \subset X \subsetneq S$ gives $\ell(\overline{\mathrm{t}}_\theta X) < \ell(S)$. By minimality, we get $\overline{\mathrm{t}}_\theta X = 0$, or equivalently, $X \in \mathcal{F}_\theta$. Since $X \neq 0$, we have $\theta(X) < 0$. On the other hand, $X \subset S \in \mathcal{F}_\eta$ implies $\eta(X) < 0$. Thus $\zeta(X) < 0$ by $\zeta \in [\theta, \eta]$. Therefore S is ζ-stable; hence S is a brick. ☐

By applying Lemma 2.9(c) to semistable torsion pairs, we have the following short exact sequences.

Definition-Proposition 2.14. *Let $M \in \mathcal{A}$ and $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. Define subobjects $\overline{\mathrm{t}}_\theta M$ and $\mathrm{t}_\theta M$ of M as the largest subobjects L of M such that $L \in \overline{\mathcal{T}}_\theta$ and $L \in \mathcal{T}_\theta$ respectively, and set $\mathrm{f}_\theta M :=$*

$M/\overline{\mathrm{t}}_\theta M$ and $\overline{\mathrm{f}}_\theta M := M/\mathrm{t}_\theta M$. Then we have short exact sequences

$$0 \to \overline{\mathrm{t}}_\theta M \to M \to \mathrm{f}_\theta M \to 0 \quad (\overline{\mathrm{t}}_\theta M \in \overline{\mathcal{T}}_\theta,\ \mathrm{f}_\theta M \in \mathcal{F}_\theta),$$
$$0 \to \mathrm{t}_\theta M \to M \to \overline{\mathrm{f}}_\theta M \to 0 \quad (\mathrm{t}_\theta M \in \mathcal{T}_\theta,\ \overline{\mathrm{f}}_\theta M \in \overline{\mathcal{F}}_\theta).$$

Moreover we set $\mathrm{w}_\theta M := \overline{\mathrm{t}}_\theta M/\mathrm{t}_\theta M \in \mathcal{W}_\theta$.

In the rest of this section, we explain the relationship between silting theory and TF equivalences. This is not necessary to define M-TF equivalences in the next section.

Let A be a finite dimensional algebra over a field k, and set $\mathcal{A} := \mathsf{mod}\, A$. Then the simple objects $S(1), \dots, S(n)$ of $\mathcal{A}$ are precisely the simple A-modules.

We also consider the category $\mathsf{proj}\, A$ of finitely generated right projective A-modules, and write $P(1), \dots, P(n)$ for the distinct isoclasses of indecomposable projective A-modules. Without loss of generality, we may assume that $P(i)$ is the projective cover of the simple module $S(i)$. The category $\mathsf{proj}\, A$ is an exact category by the split short exact sequences in $\mathsf{proj}\, A$, and then the Grothendieck group $K_0(\mathsf{proj}\, A)$ as an exact category is the free abelian group $\bigoplus_{i=1}^n \mathbb{Z}[P(i)] \simeq \mathbb{Z}^n$, so the real Grothendieck group $K_0(\mathsf{proj}\, A)^*_{\mathbb{R}} = K_0(\mathsf{proj}\, A) \otimes_{\mathbb{Z}} \mathbb{R}$ is the $\mathbb{R}$-vector space $\bigoplus_{i=1}^n \mathbb{R}[P(i)] \simeq \mathbb{R}^n$.

We define the *Euler bilinear form* as the non-degenerate $\mathbb{R}$-bilinear form

$$\begin{aligned} \langle !, ? \rangle \colon K_0(\mathsf{proj}\, A)_{\mathbb{R}} \times K_0(\mathsf{mod}\, A)_{\mathbb{R}} &\to \mathbb{R}, \\ ([P(i)], [S(j)]) &\mapsto \delta_{i,j} \dim_k \mathrm{End}_A(S(j)). \end{aligned}$$

Then we have an isomorphism $K_0(\mathsf{proj}\, A)_{\mathbb{R}} \to K_0(\mathcal{A})^*_{\mathbb{R}}$, where $[P(i)] \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ is identified with $(\dim_k \mathrm{End}_A(S(i)))[S(i)]^* \in K_0(\mathcal{A})^*_{\mathbb{R}}$.

As in [KV, 5.1], a complex U in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ of bounded complexes over $\mathsf{proj}\, A$ is said to be *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(U, U[>0]) = 0$, and *silting* if U is presilting and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ itself is the *thick subcategory* generated by U in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$, that is, the smallest triangulated subcategory closed under direct summands and containing U. By [AI, Theorem 2.27], a presilting complex U is silting if and only if the number $|U|$ of isoclasses of its direct summands is $n = |A|$.

Moreover a complex $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is said to be *2-term* if U is isomorphic to a complex $U^{-1} \to U^0$ whose terms except the -1st and 0th ones are zero. We write $\text{2-}\mathsf{psilt}\, A$ (resp. $\text{2-}\mathsf{silt}\, A$) for the set of isoclasses of basic 2-term presilting (resp. 2-term silting) complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

For each $U \in \text{2-}\mathsf{psilt}\, A$, decompose U as $U = \bigoplus_{i=1}^m U_i$ with each U_i indecomposable. Then the *silting cones* for U are defined as

$$C^\circ(U) := \sum_{i=1}^m \mathbb{R}_{>0}[U_i], \quad C(U) := \sum_{i=1}^m \mathbb{R}_{\geq 0}[U_i].$$

The silting cone $C(U)$ is an m-dimensional simplicial cone by [AI, Theorem 2.27], so its faces are nothing but $C(V)$ for the direct summands V of U. Moreover [DIJ, Theorem 6.5, Corollary 6.7] imply that, for any $U, U' \in \text{2-}\mathsf{psilt}\, A$, the intersection $C(U) \cap C(U')$ is $C(U'')$ for the largest common direct summand U'' of U and U'. Thus the set $\{C(U) \mid U \in \text{2-}\mathsf{psilt}\, A\}$ is a fan in $K_0(\mathcal{A})^*_{\mathbb{R}}$ (the precise definition of fans is written before Theorem 3.2). This fan is called the *silting fan*. By [A, Theorem 4.7] (see also [ZZ]), the silting fan is complete if and only if $\text{2-}\mathsf{psilt}\, A$ is a finite set, or equivalently, A is τ-tilting finite.

By [A, Proposition 3.11] (see also [BST, Proposition 3.27] and [Y, Proposition 3.3]), $C^\circ(U)$ is a TF equivalence class in $K_0(\mathcal{A})^*_{\mathbb{R}}$. Its closure $C(U)$ is the disjoint union of $C^\circ(V)$ for all direct summands V of U, each of which is a TF equivalence class again. This phenomenon holds also in M-TF equivalences defined in the next section; see Corollary 3.16(c).

A *chamber* in the wall-chamber structure on $K_0(\mathcal{A})^*_{\mathbb{R}} = K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ is defined to be a connected component of the complement

$$K_0(\mathsf{proj}\, A)^*_{\mathbb{R}} \setminus \overline{\bigcup_{M \in (\mathsf{mod}\, A) \setminus \{0\}} \Theta_M}.$$

By [A, Theorem 3.17], there is a bijection from $\mathsf{2\text{-}silt}\, A$ to the set of chambers in $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ given by $T \mapsto C^\circ(T)$. In other words, any chamber is of the form $C^\circ(T)$ for a unique $T \in \mathsf{2\text{-}silt}\, A$.

3. M-TF equivalences

3.1. **The definition and the main result.** In this section, for each $M \in \mathcal{A}$, we define the M-TF equivalence by coarsening the TF equivalence, and construct a complete fan $\Sigma(M)$ in $K_0(\mathcal{A})^*_{\mathbb{R}}$. Recall that we have defined the following short exact sequences and notions for $M \in \mathcal{A}$, $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ and $X \in \mathcal{W}_\theta$:

$$\begin{aligned}
&0 \to \overline{\mathrm{t}}_\theta M \to M \to \mathrm{f}_\theta M \to 0 \quad (\overline{\mathrm{t}}_\theta M \in \overline{\mathcal{T}}_\theta,\ \mathrm{f}_\theta M \in \mathcal{F}_\theta),\\
&0 \to \mathrm{t}_\theta M \to M \to \overline{\mathrm{f}}_\theta M \to 0 \quad (\mathrm{t}_\theta M \in \mathcal{T}_\theta,\ \overline{\mathrm{f}}_\theta M \in \overline{\mathcal{F}}_\theta),\\
&\mathrm{w}_\theta M := \overline{\mathrm{t}}_\theta M / \mathrm{t}_\theta M \in \mathcal{W}_\theta,\\
&\mathsf{supp}_\theta X := \{\text{isoclasses of composition factors of } X \text{ in } \mathcal{W}_\theta\}.
\end{aligned}$$

By using these, we introduce the M-TF equivalence for each $M \in \mathcal{A}$ as follows.

Definition 3.1. Let $M \in \mathcal{A}$.

(a) Let $\theta, \eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$. Then we say that θ and η are *M-TF equivalent* if the following conditions are satisfied.
  - (i) The conditions $\mathrm{t}_\theta M = \mathrm{t}_\eta M$, $\mathrm{w}_\theta M = \mathrm{w}_\eta M$ and $\mathrm{f}_\theta M = \mathrm{f}_\eta M$ hold, or equivalently, $(\overline{\mathrm{t}}_\theta M, \mathrm{f}_\theta M) = (\overline{\mathrm{t}}_\eta M, \mathrm{f}_\eta M)$ and $(\mathrm{t}_\theta M, \overline{\mathrm{f}}_\theta M) = (\mathrm{t}_\eta M, \overline{\mathrm{f}}_\eta M)$ hold.
  - (ii) We have $\mathsf{supp}_\theta(\mathrm{w}_\theta M) = \mathsf{supp}_\eta(\mathrm{w}_\eta M)$.

(b) We write $[\theta]_M$ for the M-TF equivalence class of θ. For each $E = [\theta]_M$, we set

$$\begin{aligned}
&\mathrm{t}_E M := \mathrm{t}_\theta M, \quad \mathrm{w}_E M := \mathrm{w}_\theta M, \quad \mathrm{f}_E M := \mathrm{f}_\theta M, \quad \overline{\mathrm{t}}_E M := \overline{\mathrm{t}}_\theta M, \quad \overline{\mathrm{f}}_E M := \overline{\mathrm{f}}_\theta M,\\
&\mathsf{supp}_E(\mathrm{w}_E M) := \mathsf{supp}_\theta(\mathrm{w}_\theta M).
\end{aligned}$$

(c) We define $\mathsf{TF}(M)$ as the set of all M-TF equivalence classes, and

$$\Sigma(M) := \{\overline{E} \mid E \in \mathsf{TF}(M)\}$$

as the set of the closures of all $E \in \mathsf{TF}(M)$.

Notice that $\mathrm{t}_E X$, $\mathrm{w}_E X$ and $\mathrm{f}_E X$ for other objects $X \in \mathcal{A}$ are not defined.

We will show the following main result of this paper in the next subsection. Here, a *fan* Σ in $\mathbb{R}^n$ means a set of cones in $\mathbb{R}^n$ satisfying (1) $\mathsf{Face}\, \sigma \subset \Sigma$ for each $\sigma \in \Sigma$ and (2) $\sigma \cap \tau \in \mathsf{Face}\, \sigma \cap \mathsf{Face}\, \tau$ for any $\sigma, \tau \in \Sigma$. In particular, we do not assume that each σ in a fan Σ is strongly convex; in other words, we allow that σ contains a nonzero $\mathbb{R}$-vector subspace of $\mathbb{R}^n$. A fan $\Sigma \subset \mathbb{R}^n$ is said to be *finite* if Σ is a finite set, and is said to be *complete* if the union $\bigcup_{\sigma \in \Sigma} \sigma$ is $\mathbb{R}^n$.

Theorem 3.2. *Let $M \in \mathcal{A}$. Then the following statements hold.*

(a) *Each $\sigma \in \Sigma(M)$ is a cone in $K_0(\mathcal{A})^*_{\mathbb{R}}$. Moreover we have mutually inverse bijections $\mathsf{TF}(M) \leftrightarrow \Sigma(M)$ of finite sets given by $E \mapsto \overline{E}$ for any $E \in \mathsf{TF}(M)$ and $\sigma \mapsto \sigma^\circ$ for any $\sigma \in \Sigma(M)$.*

(b) *The set $\Sigma(M)$ is a finite complete fan in $K_0(\mathcal{A})^*_{\mathbb{R}}$.*

Before proving this, we can easily see the following properties. Later we will have its stronger version on bricks in Proposition 3.8.

Lemma 3.3. *Let $\theta, \eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$.*

(a) *Assume $M = M_1 \oplus M_2 \in \mathcal{A}$. Then θ and η are M-TF equivalent if and only if they are M_1-TF equivalent and are M_2-TF equivalent.*

(b) *θ and η are TF equivalent if and only if, for any indecomposable $M \in \mathcal{A}$, θ and η are M-TF equivalent.*

Proof. (a) is clear, because $\mathrm{t}_\theta M = \mathrm{t}_\theta M_1 \oplus \mathrm{t}_\theta M_2$ and similar properties hold.

(b) The "only if" part is obvious.

We show the "if" part. For any indecomposable $M \in \mathcal{A}$, $\mathrm{t}_\theta M = M$ holds if and only if $\mathrm{t}_\eta M = M$, which means $M \in \mathcal{T}_\theta$ is equivalent to $M \in \mathcal{T}_\eta$. Thus we get $\mathcal{T}_\theta = \mathcal{T}_\eta$. The same argument shows $\mathcal{F}_\theta = \mathcal{F}_\eta$. Thus θ and η are TF equivalent. □

We show the following important examples of M-TF equivalence classes.

Example 3.4. Let $M \in \mathcal{A}$.

(a) For each $\sigma \in \mathsf{Face}\,\Theta_M$, we have $\sigma^\circ \in \mathsf{TF}(M)$ and $\sigma \in \Sigma(M)$.
(b) We have $\mathsf{Face}\,\Theta_M \subset \Sigma(M)$ as a subfan.

Proof. (a) Take $\eta \in \sigma^\circ$. It suffices to show $\sigma^\circ = [\eta]_M$. For any $\theta \in K_0(\mathcal{A})^*_\mathbb{R}$, $\theta \in \Theta_M$ if and only if $\mathrm{t}_\theta M = 0$, $\mathrm{w}_\theta M = M$ and $\mathrm{f}_\theta M = 0$, which is equivalent to $\mathrm{t}_\theta M = \mathrm{t}_\eta M$, $\mathrm{w}_\theta M = \mathrm{w}_\eta M$ and $\mathrm{f}_\theta M = \mathrm{f}_\eta M$. Thus we get $[\eta]_M = \{\theta \in \Theta_M \mid \mathsf{supp}_\eta M = \mathsf{supp}_\theta M\}$, which is σ° by Lemma 2.5(c) and $\eta \in \sigma^\circ$. Thus $\sigma^\circ \in \mathsf{TF}(M)$ holds; hence $\sigma = \overline{\sigma^\circ} \in \Sigma(M)$.

(b) follows from (a). □

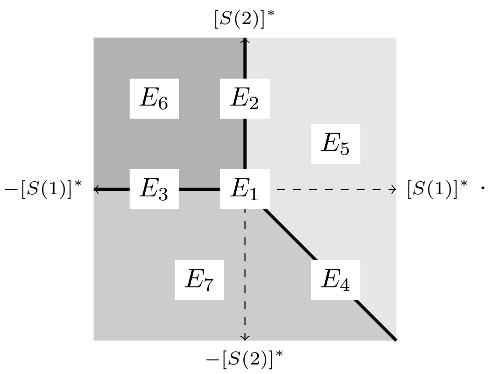
As in Example 2.7, Θ_M may not be simplicial, so $\sigma \in \Sigma(M)$ is not necessarily a simplicial cone.

We give an example of $\Sigma(M)$. It shows that the condition (ii) $\mathsf{supp}_\theta M = \mathsf{supp}_\eta M$ of Definition 3.1(a) is necessary to assure that $\Sigma(M)$ is a fan.

Example 3.5. Let A be the path algebra $k(1 \to 2)$, and $M := \begin{smallmatrix}1\\2\end{smallmatrix}$. Then $\mathsf{TF}(M)$ has exactly seven elements in the following table.

M-TF equivalence classes E	$\mathrm{t}_E M$	$\mathrm{w}_E M$	$\mathrm{f}_E M$	$\mathsf{supp}_E(\mathrm{w}_E M)$
$E_1 = \{0\}$	0	M	0	$\{S(2), S(1)\}$
$E_2 = \mathbb{R}_{>0}[S(2)]^*$	$S(2)$	$S(1)$	0	$\{S(1)\}$
$E_3 = \mathbb{R}_{>0}(-[S(1)]^*)$	0	$S(2)$	$S(1)$	$\{S(2)\}$
$E_4 = \mathbb{R}_{>0}([S(1)]^* - [S(2)]^*)$	0	M	0	$\{M\}$
$E_5 = E_4 + E_2$	M	0	0	$\emptyset$
$E_6 = E_2 + E_3$	$S(2)$	0	$S(1)$	$\emptyset$
$E_7 = E_3 + E_4$	0	0	M	$\emptyset$

They are depicted as follows, and $\Sigma(M) = \{\overline{E_i} \mid i \in \{1, \dots, 7\}\}$ is indeed a complete fan:



Note that $\Theta_M = E_1 \sqcup E_4$ holds. Moreover the two M-TF equivalence classes E_1 and E_4 are distinguished only by $\mathsf{supp}_\theta(\mathrm{w}_\theta M)$. If we consider the equivalence relation on $K_0(\mathcal{A})^*_\mathbb{R}$ defined by the condition (i) in Definition 3.1(a) alone, then the equivalence classes are $E_1 \sqcup E_4 = \overline{E_4}$ and E_2, E_3, E_5, E_6, E_7. The set of the closures of these six sets is $\Sigma(M) \setminus \{\{0\}\}$. This is not closed under taking faces, because $\{0\}$ is a face of $\overline{E_4}$.

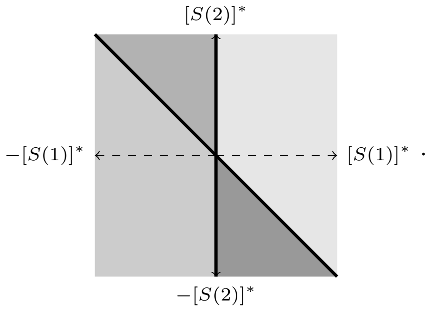
Examples 2.6 and 3.5 show that the fan $\Sigma(M)$ is not necessarily given by a union of walls.

We have another example, which implies that the M-TF equivalence for an object $M \in \mathcal{A}$ which is not a brick can be different from the M'-TF equivalence for any object M' which is a direct sum of bricks. Thus it is meaningful to consider M-TF equivalences for objects $M \in \mathcal{A}$ which are not bricks.

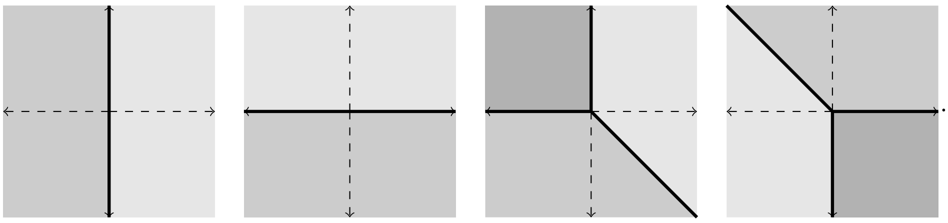
Example 3.6. Define a finite dimensional algebra A and a module $M \in \mathsf{mod}\, A$ by

$$A := k\left(1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \right)/\langle aba, bab\rangle, \quad M := \begin{smallmatrix}1\\2\\1\end{smallmatrix}.$$

Then the M-TF equivalence classes are depicted as follows:



There are exactly four isoclasses of bricks 1, 2, $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}2\\1\end{smallmatrix}$, and the S-TF equivalence classes for each brick S is given as follows:



Thus if M' is a direct sum of bricks, then the M'-TF equivalence never coincides with the M-TF equivalence by Lemma 3.3(a).

3.2. **Proof of Theorem 3.2.** In this subsection, we prove Theorem 3.2. We first describe each M-TF equivalence class E and its closure σ as follows.

Lemma 3.7. *Let $M \in \mathcal{A}$ and $E \in \mathsf{TF}(M)$, $\sigma := \overline{E} \in \sigma(M)$.*

(a) *We have*

$$E = \{\theta \in K_0(\mathcal{A})^*_{\mathbb{R}} \mid \mathrm{t}_E M \in \mathcal{T}_\theta,\ \mathsf{supp}_E(\mathrm{w}_E M) \subset \mathsf{sim}\, \mathcal{W}_\theta,\ \mathrm{f}_E M \in \mathcal{F}_\theta\}.$$

In particular, E is an nonempty open subset of the hyperplane $\mathrm{Ker}\langle ?, \mathsf{supp}_E(\mathrm{w}_E M)\rangle$.

(b) *We have*

$$\sigma = \{\theta \in K_0(\mathcal{A})^*_{\mathbb{R}} \mid \mathrm{t}_E M \in \overline{\mathcal{T}}_\theta,\ \mathsf{supp}_E(\mathrm{w}_E M) \subset \mathcal{W}_\theta,\ \mathrm{f}_E M \in \overline{\mathcal{F}}_\theta\}.$$

In particular, σ is a cone in $K_0(\mathcal{A})^*_{\mathbb{R}}$ such that $\dim_{\mathbb{R}} \sigma = n - \dim_{\mathbb{R}}(\mathbb{R}\, \mathsf{supp}_E(\mathrm{w}_E M))$.

(c) *We have $E = \sigma^\circ$.*

Proof. (a) The "$\subset$" part is clear.

To show the "$\supset$" part, let θ in the right-hand side. Then $\mathrm{t}_E M \in \mathcal{T}_\theta$ implies $\mathrm{t}_E M \subset \mathrm{t}_\theta M$, and $\mathrm{f}_E M \in \mathcal{F}_\theta$ implies $\overline{\mathrm{t}}_\theta M \subset \overline{\mathrm{t}}_E M$, so we have $\mathrm{t}_E M \subset \mathrm{t}_\theta M \subset \overline{\mathrm{t}}_\theta M \subset \overline{\mathrm{t}}_E M$. Thus $\mathrm{t}_\theta M/\mathrm{t}_E M$ is a subobject of $\overline{\mathrm{t}}_E M/\mathrm{t}_E M = \mathrm{w}_E M$, and $\mathrm{w}_E M$ is in $\mathcal{W}_\theta$ by $\mathsf{supp}_E(\mathrm{w}_E M) \subset \mathsf{sim}\, \mathcal{W}_\theta$. Thus we have $\mathrm{t}_\theta M/\mathrm{t}_E M \in \overline{\mathcal{F}}_\theta$. On the other hand, $\mathrm{t}_\theta M/\mathrm{t}_E M$ is a factor object of $\mathrm{t}_\theta M \in \mathcal{T}_\theta$, so $\mathrm{t}_\theta M/\mathrm{t}_E M \in \mathcal{T}_\theta$. Therefore $\mathrm{t}_\theta M/\mathrm{t}_E M = \mathcal{T}_\theta \cap \overline{\mathcal{F}}_\theta = \{0\}$; hence $\mathrm{t}_E M = \mathrm{t}_\theta M$. Similarly, $\mathrm{f}_E M = \mathrm{f}_\theta M$ holds, and we have $\mathrm{w}_E M = \mathrm{w}_\theta M$. Then $\mathsf{supp}_E(\mathrm{w}_E M) \subset \mathsf{sim}\, \mathcal{W}_\theta$ implies $\mathsf{supp}_E(\mathrm{w}_E M) = \mathsf{supp}_\theta(\mathrm{w}_\theta M)$. Thus $\theta \in E$ as desired.

The equality in the statement is proved. In particular, E is the subset of $\mathrm{Ker}\langle ?, \mathsf{supp}_E(\mathrm{w}_E M)\rangle$ given by finitely many strict linear inequalities. Thus the last statement holds.

(b) By (a) and $\sigma = \overline{E}$, σ is contained in the right-hand side.

To show the converse, take some $\eta \in E$. Then if θ is in the right-hand side, for any $\varepsilon \in \mathbb{R}_{>0}$, we can check $\theta' := \theta + \varepsilon\eta$ satisfies $\mathrm{t}_E M \in \overline{\mathcal{T}}_\theta \cap \mathcal{T}_\eta \subset \mathcal{T}_{\theta'}$, $\mathsf{supp}_E(\mathrm{w}_E M) \subset \mathcal{W}_\theta \cap (\mathsf{sim}\,\mathcal{W}_\eta) \subset \mathsf{sim}\,\mathcal{W}_{\theta'}$ and $\mathrm{f}_E M \in \overline{\mathcal{F}}_\theta \cap \mathcal{F}_\eta \subset \mathcal{F}_{\theta'}$. Thus (a) implies $\theta' \in E$ for any $\varepsilon \in \mathbb{R}_{>0}$. Then we have $\theta \in \overline{E} = \sigma$.

Now we have proved the equality in the statement. The right-hand side is a cone, and so is σ. By (a), we have $\dim_\mathbb{R} \sigma = n - \dim_\mathbb{R}(\mathbb{R}\,\mathsf{supp}_E(\mathrm{w}_E M))$.

(c) We show the "$\subset$" part first. By (a)(b), E is an open set of $\mathbb{R}\sigma = \mathrm{Ker}\langle ?, \mathsf{supp}_E(\mathrm{w}_E M)\rangle$ with $E \subset \sigma \subset \mathbb{R}\sigma$. Thus $E \subset \sigma^\circ$.

For the "$\supset$" part, let $\theta \in \sigma^\circ$. Take some $\eta \in E$. Then since $\theta \in \sigma^\circ$ and $\eta \in E \subset \mathbb{R}\sigma$, there exists a sufficiently small $\varepsilon \in \mathbb{R}_{>0}$ such that $\theta' := \theta - \varepsilon\eta \in \sigma$. We get $\theta = \theta' + \varepsilon\eta \in \sigma + E$, and (a)(b) give $\sigma + E \subset E$. Thus $\theta \in E$ as desired. $\square$

Then we can prove Theorem 3.2(a). For any $v, w \in K_0(\mathcal{A})_\mathbb{R}$ with $v \in \sum_{i=1}^n a_i[S(i)]$ and $w \in \sum_{i=1}^n b_i[S(i)]$, we write $v \le w$ if $a_i \le b_i$ for all i, and $v < w$ if $v \le w$ and $v \ne w$.

Proof of Theorem 3.2(a). Each $\sigma \in \Sigma(M)$ is a cone in $K_0(\mathcal{A})^*_\mathbb{R}$ by Lemma 3.7(b).

The map $\Phi\colon \mathsf{TF}(M) \to \Sigma(M)$ with $E \mapsto \overline{E}$ is clearly well-defined and surjective. Thus the map $\Psi\colon \Sigma(M) \to \mathsf{TF}(M)$ with $\sigma \mapsto \sigma^\circ$ is well-defined and $\Psi\Phi$ is identity by Lemma 3.7(c). Thus Φ and Ψ are bijective, and $\Phi = \Psi^{-1}$.

It remains to show that $\mathsf{TF}(M)$ is a finite set. Consider all the linear equalities/inequalities on $\theta \in K_0(\mathcal{A})^*_\mathbb{R}$ of the forms $\langle\theta, v\rangle = 0$, $\langle\theta, v\rangle > 0$, $\langle\theta, v\rangle < 0$ for some $v \in K_0(\mathcal{A})$ with $0 \le v \le [M]$. These are a finite collection of linear equalities/inequalities on $\theta \in K_0(\mathcal{A})^*_\mathbb{R}$. By Lemma 3.7(a), each $E \in \mathsf{TF}(M)$ is determined by some of such linear equalities/inequalities on $\theta \in K_0(\mathcal{A})^*_\mathbb{R}$. Thus $\mathsf{TF}(M)$ is a finite set. $\square$

We also have the following result.

Proposition 3.8. *Let $M \in \mathcal{A}$.*

(a) *Each $E \in \mathsf{TF}(M)$ is convex.*
(b) *Let $\theta, \eta \in K_0(\mathcal{A})^*_\mathbb{R}$. Then θ and η are TF equivalent if and only if, for any brick $S \in \mathcal{A}$, θ and η are S-TF equivalent.*

Proof. (a) This follows from Lemma 3.7(a).

(b) The "only if" part is obvious.

To show the "if" part, assume θ and η are not TF equivalent. Then by Proposition 2.13, we can take a brick $S \in \mathcal{A}$ and $\zeta \in [\theta, \eta]$ such that $\mathrm{Ker}\langle ?, S\rangle \cap [\theta, \eta] = \{\zeta\}$ and S is ζ-stable. Thus there exists $\zeta' \in [\theta, \eta]$ such that $S \in \mathcal{T}_{\zeta'}$ or $S \in \mathcal{F}_{\zeta'}$. In particular, ζ and ζ' are not S-TF equivalent. Thus (a) implies θ and η are not S-TF equivalent. $\square$

The following is a crucial property of elements of $\Sigma(M)$.

Proposition 3.9. *Let $M \in \mathcal{A}$, $E \in \mathsf{TF}(M)$, and $\sigma := \overline{E}$. For any $\theta \in \sigma$, there exists a unique sequence*

$$0 = M_0 \subset M_1 \subset M_2 \subset M_3 \subset M_4 \subset M_5 = M$$

such that $M_1 := \mathrm{t}_\theta M$, $M_2 := \mathrm{t}_E M$, $M_3 := \overline{\mathrm{t}}_E M$ and $M_4 := \overline{\mathrm{t}}_\theta M$. Thus we have

$$\begin{aligned}
&M_1/M_0 = \mathrm{t}_\theta M = \mathrm{t}_\theta(\mathrm{t}_E M), && M_2/M_1 = \mathrm{w}_\theta(\mathrm{t}_E M), && M_2/M_0 = \mathrm{t}_E M,\\
&M_3/M_2 = \mathrm{w}_E M = \mathrm{w}_\theta(\mathrm{w}_E M), && && M_4/M_1 = \mathrm{w}_\theta M,\\
&M_4/M_3 = \mathrm{w}_\theta(\mathrm{f}_E M), && M_5/M_4 = \mathrm{f}_\theta M = \mathrm{f}_\theta(\mathrm{f}_E M), && M_5/M_3 = \mathrm{f}_E M.
\end{aligned}$$

In particular, we have

$$\mathsf{supp}_\theta(\mathrm{w}_\theta M) = \mathsf{supp}_\theta(\mathrm{w}_\theta(\mathrm{t}_E M)) \cup \mathsf{supp}_\theta(\mathrm{w}_E M) \cup \mathsf{supp}_\theta(\mathrm{w}_\theta(\mathrm{f}_E M)).$$

Proof. Since $\theta \in \sigma$, Lemma 3.7(b) gives the following inclusions and canonical surjections:

$$0 \subset \mathrm{t}_\theta(\mathrm{t}_E M) \subset \mathrm{t}_E M \subset \overline{\mathrm{t}}_E M,$$
$$\overline{\mathrm{f}}_E M \twoheadrightarrow \mathrm{f}_E M \twoheadrightarrow \mathrm{f}_\theta(\mathrm{f}_E M) \to 0.$$

Then we obtain a sequence

$$M_0 := 0 \subset M_1 := \mathrm{t}_\theta(\mathrm{t}_E M) \subset M_2 := \mathrm{t}_E M \subset M_3 := \overline{\mathrm{t}}_E M \subset M_4 \subset M_5 := M,$$

where we set M_4 as the kernel of the canonical surjection $M \twoheadrightarrow \mathrm{f}_\theta(\mathrm{f}_E M)$, so M_4/M_3 is the kernel $\overline{\mathrm{t}}_\theta(\mathrm{f}_E M)$ of the canonical surjection $\mathrm{f}_E M \twoheadrightarrow \mathrm{f}_\theta(\mathrm{f}_E M)$. Then $M_2/M_0 = \mathrm{t}_E M$, $M_3/M_2 = \mathrm{w}_E M$ and $M_5/M_3 = \mathrm{f}_E M$ hold.

Clearly $M_2/M_1 = \overline{\mathrm{f}}_\theta(\mathrm{t}_E M)$ is a factor object of $\mathrm{t}_E M$. Since $\theta \in \sigma = \overline{E}$, we have $\mathrm{t}_E M \in \overline{\mathcal{T}}_\theta$ by Lemma 3.7(b). Thus we get $M_2/M_1 \in \mathcal{W}_\theta$, and $M_2/M_1 = \mathrm{w}_\theta(\mathrm{t}_E M)$. Similarly $M_4/M_3 = \overline{\mathrm{t}}_\theta(\mathrm{f}_E M) \in \mathcal{W}_\theta$, and $M_4/M_3 = \mathrm{w}_\theta(\mathrm{f}_E M)$. Finally $M_3/M_2 = \mathrm{w}_E M \in \mathsf{Filt}(\mathsf{supp}_E(\mathrm{w}_E M)) \subset \mathcal{W}_\theta$ by $\theta \in \sigma = \overline{E}$ and Lemma 3.7(b), so $M_3/M_2 = \mathrm{w}_\theta(\mathrm{w}_E M)$.

Since $M_1/M_0 \in \mathcal{T}_\theta$ and $M_2/M_1, M_3/M_2, M_4/M_3 \in \mathcal{W}_\theta$ and $M_5/M_4 \in \mathcal{F}_\theta$, we have $M_1/M_0 = \mathrm{t}_\theta M$ and $M_5/M_4 = \mathrm{f}_\theta M$.

The last statement comes from the sequence $M_1 \subset M_2 \subset M_3 \subset M_4$ whose subfactors are all in $\mathcal{W}_\theta$. □

For $M \in \mathcal{A}$, and $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$, we set

$$\begin{aligned}\mathcal{L}(\theta, M) &:= \{\text{subobjects } L \text{ of } M \text{ such that } \mathrm{t}_\theta M \subset L \text{ with } L/\mathrm{t}_\theta M \in \mathcal{W}_\theta\}\\ &= \{\text{subobjects } L \text{ of } M \text{ such that } L \subset \overline{\mathrm{t}}_\theta M \text{ with } \overline{\mathrm{t}}_\theta M/L \in \mathcal{W}_\theta\},\end{aligned}$$

where the second equality comes from $\overline{\mathrm{t}}_\theta M/\mathrm{t}_\theta M = \mathrm{w}_\theta M \in \mathcal{W}_\theta$. In particular, $\mathrm{t}_\theta M, \overline{\mathrm{t}}_\theta M \in \mathcal{L}(\theta, M)$. We remark that $\mathcal{L}(\theta, M)$ coincides with the one defined in [Fe2, Section 4] by Lemma 3.13 later. By $\mathcal{L}(\theta, M)$, we can characterize elements in $\Sigma(M)$ and $\mathsf{TF}(M)$ as follows.

Lemma 3.10. *Let $M \in \mathcal{A}$, and $\theta, \eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$.*

(a) *We have $\mathrm{t}_\theta M, \overline{\mathrm{t}}_\theta M \in \mathcal{L}(\theta, M)$ and $\mathcal{L}(\theta, M) \subset \overline{\mathcal{T}}_\theta$.*

(b) *The following conditions are equivalent.*
  - (i) $\theta \in \overline{[\eta]_M}$.
  - (ii) $\mathcal{L}(\eta, M) \subset \mathcal{L}(\theta, M)$.

(c) *The following conditions are equivalent.*
  - (i) *θ and η are M-TF equivalent.*
  - (ii) $\mathcal{L}(\eta, M) = \mathcal{L}(\theta, M)$.

Proof. (a) is clear by definition.

(b) Set $E := [\eta]_M$.

(i)⇒(ii) Let $L \in \mathcal{L}(\eta, M)$. We have $\mathrm{t}_E M \subset L$ and $L/\mathrm{t}_E M \in \mathcal{W}_\eta$.

Then by $\theta \in \overline{[\eta]_M} = \overline{E}$ and Proposition 3.9, we have $\mathrm{t}_\theta M \subset \mathrm{t}_E M \subset L$ and $\mathrm{t}_E M/\mathrm{t}_\theta M \in \mathcal{W}_\theta$. To show $L/\mathrm{t}_\theta M \in \mathcal{W}_\theta$, it remains to show $L/\mathrm{t}_E M \in \mathcal{W}_\theta$.

Since $L/\mathrm{t}_E M$ is a subobject of $\mathrm{w}_E M$ in $\mathcal{W}_\eta$, we have $L/\mathrm{t}_E M \in \mathsf{Filt}\,\mathsf{supp}_E(\mathrm{w}_E M)$. Since $\theta \in [\eta]_M$, Lemma 3.7(b) implies $\mathsf{Filt}\,\mathsf{supp}_E(\mathrm{w}_E M) \subset \mathcal{W}_\theta$. Thus $L/\mathrm{t}_E M \in \mathcal{W}_\theta$.

Therefore $\mathrm{t}_\theta M \subset L$ and $L/\mathrm{t}_\theta M \in \mathcal{W}_\theta$ hold. Thus we get $L \in \mathcal{L}(\theta, M)$.

(ii)⇒(i) By Lemma 3.7(b), it suffices to show (1) $\mathrm{t}_E M \in \overline{\mathcal{T}}_\theta$, (2) $\mathsf{supp}_E(\mathrm{w}_E M) \in \mathcal{W}_\theta$ and (3) $\mathrm{f}_E M \in \overline{\mathcal{F}}_\theta$.

(1) Since $\mathrm{t}_E M \in \mathcal{L}(\eta, M)$, (ii) implies $\mathrm{t}_E M \in \mathcal{L}(\theta, M)$. By (a), we have $\mathrm{t}_E M \in \overline{\mathcal{T}}_\theta$.

(2) Considering a composition series of $\mathrm{w}_E M$ in $\mathcal{W}_\eta$, the desired condition $\mathsf{supp}_E(\mathrm{w}_E M) \in \mathcal{W}_\theta$ holds if any subobject $L' \subset \mathrm{w}_E M$ in $\mathcal{W}_\eta$ belongs to $\mathcal{W}_\theta$. Let L' be a subobject of $\mathrm{w}_E M$ in $\mathcal{W}_\eta$. Since L' is a subobject of $\overline{\mathrm{f}}_E M = M/\mathrm{t}_E M$, take a subobject $L \subset M$ such that $\mathrm{t}_E M \subset L$ and $L/\mathrm{t}_E M \simeq L' \in \mathcal{W}_\eta$. Then we have $L \in \mathcal{L}(\eta, M)$, and (ii) implies $L \in \mathcal{L}(\theta, M)$. Thus we get $L/\mathrm{t}_\theta M \in \mathcal{W}_\theta$. By $\mathrm{t}_E M \in \mathcal{L}(\theta, M)$ in the proof of (1), we also obtain $\mathrm{t}_\theta M/\mathrm{t}_E M \in \mathcal{W}_\theta$. Thus $L/\mathrm{t}_E M \in \mathcal{W}_\theta$; hence $L' \in \mathcal{W}_\theta$.

(3) (a) implies $\overline{\mathrm{t}}_E M \in \mathcal{L}(\eta, M)$. Then by (ii), we have $\overline{\mathrm{t}}_E M \in \mathcal{L}(\theta, M)$. Thus $\overline{\mathrm{t}}_\theta M/\overline{\mathrm{t}}_E M \in \mathcal{W}_\theta$ holds. This and $M/\overline{\mathrm{t}}_\theta M = \mathrm{f}_\theta M \in \mathcal{F}_\theta$ give $\mathrm{f}_E M = M/\overline{\mathrm{t}}_E M \in \overline{\mathcal{F}}_\theta$.

Thus the proof is complete.

(b) By (a), it remains to show that, if $\theta \in \overline{[\eta]_M}$ and $\eta \in \overline{[\theta]_M}$, then θ and η are M-TF equivalent. By Proposition 3.9, $\mathrm{t}_\theta M \subset \mathrm{t}_\eta M \subset \mathrm{t}_\theta M$ hold, so we have $\mathrm{t}_\theta M = \mathrm{t}_\eta M$. Similarly $\mathrm{f}_\theta M = \mathrm{f}_\eta M$ holds; hence $\mathrm{w}_\theta M = \mathrm{w}_\eta M$.

By Lemma 3.7(a), it remains to show $\mathsf{supp}_\theta(\mathrm{w}_\theta M) \subset \mathsf{sim}\,\mathcal{W}_\eta$. If $X \in \mathsf{supp}_\theta(\mathrm{w}_\theta M)$, then $\eta \in \overline{[\theta]_M}$ and Lemma 3.7(b) imply $X \in \mathcal{W}_\eta$. Since $X \neq 0$, take a simple subobject Y of X in $\mathcal{W}_\eta$. Then $\eta \in \overline{[\theta]_M}$ and Lemma 3.7(b) imply $Y \in \mathcal{W}_\theta$, so Y is a subobject of X in $\mathcal{W}_\theta$. Since $X \in \mathsf{supp}_\theta(\mathrm{w}_\theta M)$, we have $X = Y \in \mathsf{sim}\,\mathcal{W}_\eta$. Therefore $\mathsf{supp}_\theta(\mathrm{w}_\theta M) \subset \mathsf{sim}\,\mathcal{W}_\eta$ as desired. $\square$

To show that $\Sigma(M)$ is a fan, we explain that the set $\mathcal{L}(\theta, M)$ for each $\theta \in K_0(\mathcal{A})^*_\mathbb{R}$ is strongly related to the Newton polytope $\mathrm{N}(M)$.

We here prepare some symbols and terminology on polytopes.

For any finite set $X \subset \mathbb{R}^n$, we write $\mathsf{conv}\,X$ for the convex hull of X. In this paper, a subset P is called a *polytope* in $\mathbb{R}^n$ if there exist a finite subset $X \subset \mathbb{Z}^n$ such that $X = \mathsf{conv}\,S$.

If P is a polytope in $\mathbb{R}^n$, a subset $F \subset P$ is called a *face* of P if the following are satisfied:

(a) $F \neq \emptyset$, and F is convex.
(b) If $v, w \in P$ satisfy $[v, w] \cap F \neq \emptyset$, then $v, w \in F$.

In this case, F is a polytope again. We set $\mathsf{Face}\,P$ as the set of all faces of the polytope P.

Let P be a polytope, and take $v \in P$. We have an $\mathbb{R}$-vector subspace V_P spanned by $\{w - v \mid w \in P\}$, and then the smallest affine hyperplane containing P is $H_P := v + V_P$. Neither V_P nor H_P depends on the choice of $v \in P$. The *dimension* $\dim_\mathbb{R} P$ of the polytope P is defined as $\dim_\mathbb{R} V_P$.

An element v in a polytope P is called a *vertex* of P if $\{v\} \in \mathsf{Face}\,P$.

Under this terminology, the Newton polytope $\mathrm{N}(M)$ is given as follows.

Definition 3.11. [BKT, Subsection 1.3][Fe1, Definition 2.3] Let $M \in \mathcal{A}$. The *Newton polytope* $\mathrm{N}(M)$ is defined as

$$\mathrm{N}(M) := \mathsf{conv}\{[X] \mid X \text{ is a subobject of } M\} \subset K_0(\mathcal{A})_\mathbb{R}.$$

In [AHIKM, BKT, Fe2] the *normal fan* $\Sigma(\mathrm{N}(M))$ in $K_0(\mathcal{A})^*_\mathbb{R}$ of the Newton polytope $\mathrm{N}(M)$ was defined. We explain the construction of $\Sigma(\mathrm{N}(M))$ following [AHIKM, Subsections 2.1 and 5.3], and see also [BKT, Subsection 1.4] and [Fe2, Section 8]. We will prove Theorem 3.2(b) by showing $\Sigma(\mathrm{N}(M)) = \Sigma(M)$.

Definition-Proposition 3.12. *Fix $M \in \mathcal{A}$.*

(a) [AHIKM, Subsection 5.3, p. 28] *For each $\theta \in K_0(\mathcal{A})^*_\mathbb{R}$, we define $\mathrm{N}(M)_\theta \in \mathsf{Face}\,\mathrm{N}(M)$ by*

$$\mathrm{N}(M)_\theta := \{v \in \mathrm{N}(M) \mid \theta(v) = \max \theta(\mathrm{N}(M))\}.$$

Then $K_0(\mathcal{A})^*_\mathbb{R} \to \mathsf{Face}\,\mathrm{N}(M)$ with $\theta \mapsto \mathrm{N}(M)_\theta$ is surjective, so we have

$$\mathsf{Face}\,\mathrm{N}(M) = \{\mathrm{N}(M)_\theta \mid \theta \in K_0(\mathcal{A})^*_\mathbb{R}\}.$$

(b) [AHIKM, Definition 2.4] *For any face $F \in \mathsf{Face}\,\mathrm{N}(M)$, we set*

$$\sigma_F := \{\theta \in K_0(\mathcal{A})^*_\mathbb{R} \mid F \subset \mathrm{N}(M)_\theta\}.$$

Then we have a finite complete fan $\Sigma(\mathrm{N}(M))$ in $K_0(\mathcal{A})^*_\mathbb{R}$:

$$\Sigma(\mathrm{N}(M)) := \{\sigma_F \mid F \in \mathsf{Face}\,\mathrm{N}(M)\} = \{\sigma_{\mathrm{N}(M)_\theta} \mid \theta \in K_0(\mathcal{A})^*_\mathbb{R}\}.$$

(c) *We have mutually inverse order-reversing bijections $\mathsf{Face}\,\mathrm{N}(M) \leftrightarrow \Sigma(\mathrm{N}(M))$ given by $F \mapsto \sigma_F$ for each $F \in \mathsf{Face}\,\mathrm{N}(M)$ and $\sigma \mapsto \mathrm{N}(M)_\theta$ for each $\sigma \in \Sigma(\mathrm{N}(M))$ taking $\theta \in \sigma^\circ$. Moreover each $F \in \mathsf{Face}\,\mathrm{N}(M)$ satisfies $\dim_\mathbb{R} \sigma_F = n - \dim_\mathbb{R} F$.*

(d) *Let $\theta \in K_0(\mathcal{A})^*_\mathbb{R}$. Then $\sigma_{\mathrm{N}(M)_\theta}$ is the smallest element of the fan $\Sigma(\mathrm{N}(M))$ which contains θ. In particular, $\sigma_{\mathrm{N}(M)_\theta}$ is the unique element $\sigma \in \Sigma(\mathrm{N}(M))$ such that $\theta \in \sigma^\circ$.*

Proof. (c) is general properties of normal fans.

(d) By definition, $\theta \in \sigma_{\mathrm{N}(M)_\theta}$ holds.

Moreover, assume that $\sigma \in \Sigma(\mathrm{N}(M))$ satisfies $\theta \in \sigma$. By (b), take $\eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ such that $\sigma = \sigma_{\mathrm{N}(M)_\eta}$. Since $\theta \in \sigma = \sigma_{\mathrm{N}(M)_\eta}$, we have $\mathrm{N}(M)_\eta \subset \mathrm{N}(M)_\theta$. Thus we get $\sigma_{\mathrm{N}(M)_\eta} \supset \sigma_{\mathrm{N}(M)_\theta}$ by (c), so $\sigma \supset \sigma_{\mathrm{N}(M)_\theta}$ as desired.

Thus $\sigma_{\mathrm{N}(M)_\theta}$ is the smallest element of the fan $\Sigma(\mathrm{N}(M))$ which contains θ.

Then the last statement holds. $\square$

The following lemma is crucial. See also [BKT, Subsection 1.4] and [Fe2, Theorem 4.4].

Lemma 3.13. [AHIKM, Lemmas 5.18, 5.19] *Let $M \in \mathcal{A}$ and $\theta \in K_0(\mathcal{A})^*_{\mathbb{R}}$.*

(a) *For any subobject L of M, $L \in \mathcal{L}(\theta, M)$ is equivalent to $[L] \in \mathrm{N}(X)_\theta$.*

(b) *In $K_0(\mathcal{A})_{\mathbb{R}}$, we have*

$$\mathrm{N}(M)_\theta = [\mathrm{t}_\theta M] + \mathsf{conv}\{[X] \mid X \text{ is a subobject of } \mathrm{w}_\theta M \text{ in } \mathcal{W}_\theta\} = \mathsf{conv}\{[L] \mid L \in \mathcal{L}(\theta, M)\}.$$

Thus we have the following characterization.

Lemma 3.14. *Fix $M \in \mathcal{A}$.*

(a) *For any $\theta, \eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$, the following conditions are equivalent.*
  (i) *θ and η are M-TF equivalent.*
  (ii) *$\mathrm{N}(M)_\theta = \mathrm{N}(M)_\eta$.*

(b) *Let $E \in \mathsf{TF}(M)$. Then $\mathrm{N}(M)_E \in \mathsf{Face}\,\mathrm{N}(M)$ is well-defined. Moreover we have $E = (\sigma_{\mathrm{N}(M)_E})^\circ$ and $\overline{E} = \sigma_{\mathrm{N}(M)_E}$.*

Proof. (a) (i)$\Rightarrow$(ii) By Lemma 3.10, $\mathcal{L}(\theta, M) = \mathcal{L}(\eta, M)$ holds. Then Lemma 3.13(b) gives (ii).

(ii)$\Rightarrow$(i) follows from Lemma 3.13(a).

(b) By (a), $\mathrm{N}(M)_E$ is well-defined.

Then Definition-Proposition 3.12(d) implies that, for any $\theta \in E$, we have $\theta \in (\sigma_{\mathrm{N}(M)_E})^\circ$. Thus we get $E \subset (\sigma_{\mathrm{N}(M)_E})^\circ$.

Conversely, let $\theta \in (\sigma_{\mathrm{N}(M)_E})^\circ$. By the uniqueness in Definition-Proposition 3.12(d), we have $\sigma_{\mathrm{N}(M)_\theta} = \sigma_{\mathrm{N}(M)_E}$. Thus $\mathrm{N}(M)_\theta = \mathrm{N}(M)_E$ holds by Definition-Proposition 3.12(c). Then (a) implies $\theta \in E$. Therefore $(\sigma_{\mathrm{N}(M)_E})^\circ \subset E$ also holds.

Thus $E = (\sigma_{\mathrm{N}(M)_E})^\circ$; hence $\overline{E} = \sigma_{\mathrm{N}(M)_E}$. $\square$

Then we can show the following.

Theorem 3.15. *Let $M \in \mathcal{A}$.*

(a) *We have $\Sigma(M) = \Sigma(\mathrm{N}(M))$. In particular, $\Sigma(M)$ is a finite complete fan.*

(b) *There exist mutually inverse order-reversing bijections $\mathsf{Face}\,\mathrm{N}(M) \leftrightarrow \Sigma(M)$ given by $F \mapsto \sigma_F$ for each $F \in \mathsf{Face}\,\mathrm{N}(M)$ and $\sigma \mapsto \mathrm{N}(M)_E$ for each $\sigma \in \Sigma(M)$ with $E := \sigma^\circ$. Moreover each $F \in \mathsf{Face}\,\mathrm{N}(M)$ satisfies $\dim_{\mathbb{R}} \sigma_F = n - \dim_{\mathbb{R}} F$.*

Proof. (a) By Definition-Proposition 3.12(a)(c), we have a surjection $K_0(\mathcal{A})^*_{\mathbb{R}} \to \Sigma(\mathrm{N}(M))$ given by $\theta \mapsto \sigma_{\mathrm{N}(M)_\theta}$. This induces a bijection $\mathsf{TF}(M) \to \Sigma(\mathrm{N}(M))$ with $E \mapsto \sigma_{\mathrm{N}(M)_E}$ by Lemma 3.14. On the other hand, we also have a bijection $\mathsf{TF}(M) \to \Sigma(M)$ given by $E \mapsto \overline{E}$. These bijections coincide by Lemma 3.14(b), so we have $\Sigma(M) = \Sigma(\mathrm{N}(M))$.

Now $\Sigma(M)$ is a finite complete fan by Definition-Proposition 3.12(b).

(b) follows from (a) and Definition-Proposition 3.12(c). $\square$

Then Theorem 3.2(b) is already proved.

Proof of Theorem 3.2(b). This is in Theorem 3.15(a). $\square$

As a corollary of Theorem 3.2, we have the following properties.

Corollary 3.16. *Let $M \in \mathcal{A}$, $E \in \mathsf{TF}(M)$, $\sigma := \overline{E}$, and $\sigma' \in \mathsf{Face}\,\sigma$.*

(a) *We have $\sigma' \in \Sigma(M)$ and $(\sigma')^\circ \in \mathsf{TF}(M)$. Therefore the decomposition*

$$\sigma = \bigsqcup_{\tau \in \mathsf{Face}\,\sigma} \tau^\circ$$

is also the decomposition to M-TF equivalence classes.

(b) *Set $E' := (\sigma')^\circ$. Then we have*

$$\sigma' = \{\theta \in \sigma \mid \theta(\mathsf{supp}_{E'}(\mathrm{w}_{E'}M)) = 0\}.$$

Proof. (a) Since $\Sigma(M)$ is a fan by Theorem 3.2(b), we have $\sigma' \in \Sigma(M)$. Thus Theorem 3.2(a) gives $(\sigma')^\circ \in \mathsf{TF}(M)$.

(b) Consider the $\mathbb{R}$-vector subspace $V := \mathrm{Ker}\langle ?, \mathsf{supp}_{E'}(\mathrm{w}_{E'}M)\rangle$. The desired equality is $\sigma' = \sigma \cap V$. Since $\sigma' \in \mathsf{Face}\,\sigma$, it suffices to show $\mathbb{R}\sigma' = V$. By (a), $E' \in \mathsf{TF}(M)$ holds. Then Lemma 3.7 gives that E' is open in V, and we have $\mathbb{R}\sigma' = V$, because $\sigma' = \overline{E'}$. ☐

3.3. **More properties of the fan $\Sigma(M)$.** We collect some useful properties of the fan $\Sigma(M)$. Since $\Sigma(M)$ is a finite complete fan, it has the smallest element, which is given as follows.

Lemma 3.17. *Let $M \in \mathcal{A}$. Set*

$$I := \{i \in \{1, \dots, n\} \mid S(i) \text{ is not a composition factor of } M \text{ in } \mathcal{A}\}.$$

Then the vector subspace $\sum_{i \in I} \mathbb{R}P(i) \subset K_0(\mathcal{A})^*_{\mathbb{R}}$ is the smallest element of $\Sigma(M)$, and it is the M-TF equivalence class $[0]_M$. In particular, $\{0\} \in \Sigma(M)$ holds if and only if M is sincere.

Proof. Clearly, $M \in \mathcal{W}_0$. Since $\mathcal{W}_0 = \mathcal{A}$, we have $\mathsf{supp}_0(\mathrm{w}_0 M) = \{S(i) \mid i \notin I\}$. Then Lemma 3.7 gives $[0]_M = \sum_{i \in I} \mathbb{R}P(i)$. This is already closed, so $\sum_{i \in I} \mathbb{R}P(i) \in \Sigma(M)$.

Now let $\sigma \in \Sigma(M)$. Then $0 \in \sigma$ holds, so take the face $\tau \in \mathsf{Face}\,\sigma$ such that $0 \in \tau^\circ$. By Corollary 3.16(a), we get $\tau^\circ \in \mathsf{TF}(M)$. Thus $\tau^\circ = [0]_M$ holds.

We have $\sum_{i \in I} \mathbb{R}P(i) = [0]_M = \tau^\circ \subset \sigma$ for any $\sigma \in \Sigma(M)$. Thus $\sum_{i \in I} \mathbb{R}P(i)$ is the smallest element of $\Sigma(M)$, and it is the M-TF equivalence class $[0]_M$.

In particular, $\{0\} \in \Sigma(M)$ is equivalent to $I = \emptyset$. The condition $I = \emptyset$ is nothing but that M is sincere. ☐

Next we consider maximal elements of $\Sigma(M)$. We set $\mathsf{TF}(M)_n$ and $\Sigma(M)_n$ as the subsets of $\mathsf{TF}(M)$ and $\Sigma(M)$ consisting of full-dimensional elements. They are characterized in $\mathsf{TF}(M)$ and $\Sigma(M)$ as follows.

Lemma 3.18. *Let $M \in \mathcal{A}$, $E \in \mathsf{TF}(M)$ and $\sigma := \overline{E} \in \Sigma(M)$. Then the following conditions are equivalent.*

(a) *$E \in \mathsf{TF}(M)_n$.*
(b) *$\mathrm{w}_E M = 0$.*
(c) *E is open in $K_0(\mathcal{A})^*_{\mathbb{R}}$.*
(d) *$\sigma \in \Sigma(M)_n$.*
(e) *σ is maximal in $\Sigma(M)$.*

In particular, we have a bijection $\mathsf{TF}(M)_n \to \Sigma(M)_n$ given by $E \mapsto \overline{E}$.

Proof. (a), (b) and (c) are equivalent by Lemma 3.7(a).

Since σ is a cone by Lemma 3.7(b), (a) and (d) are equivalent by Lemma 3.7(c).

(d) and (e) are equivalent, because $\Sigma(M)$ is a complete fan by Theorem 3.2(b).

Then the last statement follows from Theorem 3.2(a). ☐

The set $\mathsf{TF}(M)_n$ satisfies the following property.

Lemma 3.19. *Let $M \in \mathcal{A}$ and $\eta \in K_0(\mathcal{A})^*_{\mathbb{R}}$ satisfy $\mathrm{w}_\eta M = 0$. Then the M-TF equivalence class $[\eta]_M$ is in $\mathsf{TF}(M)_n$, and*

$$[\eta]_M = \{\theta \in K_0(\mathcal{A})^*_{\mathbb{R}} \mid \mathrm{t}_\eta M \in \mathcal{T}_\theta,\ \mathrm{f}_\eta M \in \mathcal{F}_\theta\}.$$

Moreover all elements in $\mathsf{TF}(M)_n$ are obtained in this way.

Proof. By Lemma 3.18 and $\mathrm{w}_\eta M = 0$, we have $[\eta]_M \in \mathsf{TF}(M)_n$. By Lemma 3.7 and $\mathrm{w}_\eta M = 0$, we have $[\eta]_M = \{\theta \in K_0(\mathcal{A})^*_{\mathbb{R}} \mid \mathrm{t}_\eta M \in \mathcal{T}_\theta,\ \mathrm{f}_\eta M \in \mathcal{F}_\theta\}$.

For any $E \in \mathsf{TF}(M)_n$, Lemma 3.18 implies $\mathrm{w}_E M = 0$. Thus taking $\eta \in E$, we have $\mathrm{w}_\eta M = 0$ and $[\eta]_M = E$. □

In the case $M = \bigoplus_{i=1}^m M_i$, the maximal elements of $\Sigma(M)$ are obtained in the following way.

Proposition 3.20. *Let $M = \bigoplus_{i=1}^m M_i \in \mathcal{A}$. Then taking intersections give bijections*

$$\left\{(E_i)_{i=1}^m \in \prod_{i=1}^m \mathsf{TF}(M_i)_n \;\middle|\; \bigcap_{i=1}^m E_i \neq \emptyset\right\} \to \mathsf{TF}(M)_n, \qquad (E_i)_{i=1}^m \mapsto \bigcap_{i=1}^m E_i,$$

$$\left\{(\sigma_i)_{i=1}^m \in \prod_{i=1}^m \Sigma(M_i)_n \;\middle|\; \dim_{\mathbb{R}} \bigcap_{i=1}^m \sigma_i = n\right\} \to \Sigma(M)_n, \qquad (\sigma_i)_{i=1}^m \mapsto \bigcap_{i=1}^m \sigma_i.$$

Proof. In general, for any $N \in \mathcal{A}$, Lemma 3.18 implies that $\mathsf{TF}(N)_n$ is the set of open N-TF equivalence classes. Thus the first map denoted by Ψ_1 is well-defined and bijective by Lemma 3.3(a).

To show that the second map denoted by Ψ_2 is well-defined and bijective, we construct bijections Ψ_3 and Ψ_4 which makes the following diagram commutative:

$$\begin{array}{ccc}
\{(E_i)_{i=1}^m \in \prod_{i=1}^m \mathsf{TF}(M_i)_n \mid \bigcap_{i=1}^m E_i \neq \emptyset\} & \xrightarrow{\Psi_1} & \mathsf{TF}(M)_n \\
\Psi_3 \uparrow & & \downarrow \Psi_4 \\
\{(\sigma_i)_{i=1}^m \in \prod_{i=1}^m \Sigma(M_i)_n \mid \dim_{\mathbb{R}} \bigcap_{i=1}^m \sigma_i = n\} & \xrightarrow{\Psi_2} & \Sigma(M)_n
\end{array}$$

By Theorem 3.2(a), there exists a bijection $\prod_{i=1}^m \Sigma(M_i)_n \to \prod_{i=1}^m \mathsf{TF}(M_i)_n$ given by $(\sigma_i)_{i=1}^m \mapsto (\sigma_i^\circ)_{i=1}^m$. This bijection is restricted to a bijection

$$\Psi_3\colon \left\{(\sigma_i)_{i=1}^m \in \prod_{i=1}^m \Sigma(M_i)_n \;\middle|\; \dim_{\mathbb{R}} \bigcap_{i=1}^m \sigma_i = n\right\} \to \left\{(E_i)_{i=1}^m \in \prod_{i=1}^m \mathsf{TF}(M_i)_n \;\middle|\; \bigcap_{i=1}^m E_i \neq \emptyset\right\}.$$

Moreover a bijection $\Psi_4\colon \mathsf{TF}(M)_n \to \Sigma(M)_n$ is just given by taking closures by Lemma 3.18. Then $\Psi_4\Psi_1\Psi_3$ is well-defined and bijective, which is Ψ_2. □

Let $\sigma = \overline{E} \in \Sigma(M)_n$ with $E \in \mathsf{TF}(M)_n$. We consider its boundary $\partial\sigma$ and the subsets

$$\partial^+\sigma := \{\theta \in \sigma \mid \mathrm{t}_E M \notin \mathcal{T}_\theta\}, \quad \partial^-\sigma := \{\theta \in \sigma \mid \mathrm{f}_E M \notin \mathcal{F}_\theta\}.$$

These have the following purity. Here *facets* mean faces whose dimensions are $n-1$.

Theorem 3.21. *Let $M \in \mathcal{A}$ and $\sigma \in \Sigma(M)_n$.*

(a) *We have $\partial\sigma = \partial^+\sigma \cup \partial^-\sigma$.*
(b) *Both $\partial^+\sigma = \{\theta \in \sigma \mid \mathrm{w}_\theta(\mathrm{t}_E M) \neq 0\}$ and $\partial^-\sigma = \{\theta \in \sigma \mid \mathrm{w}_\theta(\mathrm{f}_E M) \neq 0\}$ hold.*
(c) *Both $\partial^+\sigma$ and $\partial^-\sigma$ are union of facets.*
(d) *No facets are contained in $\partial^+\sigma \cap \partial^-\sigma$.*

In the proof of (c), the following properties of the Newton polytope $\mathrm{N}(M)$ is important.

Lemma 3.22. *Let $M \in \mathcal{A}$ and v be a vertex of $\mathrm{N}(M) \subset K_0(\mathcal{A})_{\mathbb{R}}$.*

(a) [Fe2, Proposition 8.6] *Let u be a vertex of $\mathrm{N}(M)$ adjacent to v. Then $v < u$ or $u < v$ holds in $K_0(\mathcal{A})_{\mathbb{R}}$.*
(b) *Assume that a face $F \in \mathsf{Face}\,\mathrm{N}(M)$ satisfies $v \in F$ and has an element $w \in F$ such that $w < v$. Then there exists a vertex $u \in F$ adjacent to v and $u < v$.*

Proof. (b) Take the subset $X \subset K_0(\mathcal{A})_{\mathbb{R}}$ consisting of v and all vertices $u \in F$ adjacent to v. Then as a subset of F, the convex hull $\mathsf{conv}\, X$ is a neighborhood of v, so we may assume $w \in \mathsf{conv}\, X$. By (a), each $u \in X \setminus \{v\}$ satisfies $v < u$ and $u < v$, so $w \in \mathsf{conv}\, X$ and $w < v$ imply the existence of $u \in X \setminus \{v\}$ such that $u < v$. Thus we get the assertion. $\square$

Then Theorem 3.21 can be proved as follows.

Proof of Theorem 3.21. Set $E := \sigma^\circ$. By Lemma 3.18, $E \in \mathsf{TF}(M)_n$.

(a) By Lemma 3.19, we have $E = \{\theta \in K_0(\mathcal{A})^*_{\mathbb{R}} \mid \mathrm{t}_E M \in \mathcal{T}_\theta,\ \mathrm{f}_E M \in \mathcal{F}_\theta\}$. Thus we get $\partial\sigma = \sigma \setminus E = \{\theta \in \sigma \mid \mathrm{t}_E M \notin \mathcal{T}_\theta \text{ or } \mathrm{f}_E M \notin \mathcal{F}_\theta\}$, which is $\partial^+\sigma \cup \partial^-\sigma$.

(b) We show in the case $\varepsilon = +$. Let $\theta \in \sigma$. Then Proposition 3.9 gives a short exact sequence $0 \to \mathrm{t}_\theta M \to \mathrm{t}_E M \to \mathrm{w}_\theta(\mathrm{t}_E M) \to 0$. Thus $\mathrm{t}_E M \notin \mathcal{T}_\theta$ holds if and only if $\mathrm{t}_\theta M \subsetneq \mathrm{t}_E M$, which is also equivalent to $\mathrm{w}_\theta(\mathrm{t}_E M) \neq 0$. Thus we have the assertion for $\partial^+\sigma$.

(c) We show in the case $\varepsilon = +$. By (b), $\partial^+\sigma$ is a union of faces of σ. Let σ' be a maximal face of σ contained in $\partial^+\sigma$. It suffices to show that σ' is a facet.

Since $E \in \mathsf{TF}(M)_n$, Lemma 3.18 implies $\mathrm{w}_E M = 0$, so $\mathrm{N}(M)_E \in \mathsf{Face}\,\mathrm{N}(M)$ is one point $\{v\}$ by Lemma 3.13(b), where $v := [\mathrm{t}_E M] = [\overline{\mathrm{t}}_E M]$. In particular, v is a vertex of $\mathrm{N}(M)$.

Set $E' := (\sigma')^\circ$. Since $\sigma' \subset \partial^+\sigma$, (b) and Proposition 3.9 imply $\mathrm{t}_{E'} M \subsetneq \mathrm{t}_E M$ and $\mathrm{t}_E M/\mathrm{t}_{E'} M = \mathrm{w}_{E'}(\mathrm{t}_E M)$. Thus we have $w := [\mathrm{t}_{E'} M]$ satisfies $w < v$, and by Lemma 3.13(b), we obtain $w, v \in \mathrm{N}(M)_{E'}$. Then Lemma 3.22(b) implies that $\mathrm{N}(M)_{E'}$ has a vertex u adjacent to v with $u < v$.

Then the line segment $[u, v]$ is a one-dimensional face of $\mathrm{N}(M)_{E'}$. By Theorem 3.15(b), $\{v\} \subset [u, v] \subset \mathrm{N}(M)_{E'}$ implies $\sigma \supset \sigma'' := \sigma_{[u,v]} \supset \sigma'$ and $\dim_{\mathbb{R}} \sigma'' = n - \dim_{\mathbb{R}}[u, v] = n - 1$.

Moreover Theorem 3.15(b) implies also that $E'' := (\sigma'')^\circ$ satisfies $\mathrm{N}(M)_{E''} = [u, v]$ with $u < v = [\mathrm{t}_E M]$, so Lemma 3.13(b) gives $u = [\mathrm{t}_{E''} M]$ and $\mathrm{t}_{E''} M \subsetneq \mathrm{t}_E M$.

Thus for any $\theta \in E'' \subset \sigma$, we have $\mathrm{t}_E M \notin \mathcal{T}_\theta$, so $\theta \in \partial^+\sigma$ holds. Thus we get $E'' \subset \partial^+\sigma$; hence $\sigma'' = \overline{E''} \subset \partial^+\sigma$. By the maximality of σ' in $\partial^+\sigma$, we have $\sigma' = \sigma''$, which is a facet.

(d) Let σ' be a facet contained in $\partial^+\sigma \cap \partial^-\sigma$, and set $E' := (\sigma')^\circ$. Then (b) implies $\mathrm{t}_{E'} M \subsetneq \mathrm{t}_E M = \overline{\mathrm{t}}_E M \subsetneq \overline{\mathrm{t}}_{E'} M$. Since σ' is a facet of a full-dimensional element σ, $\mathrm{N}(M)_{E'}$ is a line segment having a vertex $v := [\mathrm{t}_E M]$ by Theorem 3.15(b). Thus Lemma 3.13(b) implies that $\mathrm{N}(M)_{E'} = [u, w]$ has v as a vertex of $\mathrm{N}(M)_{E'}$, where $u := [\mathrm{t}_{E'} M]$ and $w := [\overline{\mathrm{t}}_{E'} M]$. Thus $v = u$ or $v = w$ must hold, but this contradicts $u < v < w$ following from $\mathrm{t}_{E'} M \subsetneq \mathrm{t}_E M = \overline{\mathrm{t}}_E M \subsetneq \overline{\mathrm{t}}_{E'} M$. Therefore $\partial^+\sigma \cap \partial^-\sigma$ contains no facet. $\square$

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