# On the local solvability and stability of the partial inverse problems for the non-self-adjoint Sturm-Liouville operators with a discontinuity 

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#### Abstract

In this work, we study the inverse spectral problems for the Sturm-Liouville operators on $[0,1]$ with complex coefficients and a discontinuity at $x=a \in(0,1)$. Assume that the potential on $(a, 1)$ and some parameters in the discontinuity and boundary conditions are given. We recover the potential on $(0, a)$ and the other parameters from the eigenvalues. This is the so-called partial inverse problem. The local solvability and stability of the partial inverse problems are obtained for $a \in(0,1)$, in which the error caused by the given partial potential is considered. As a by-product, we also obtain two new uniqueness theorems for the partial inverse problem.


Keywords: Non-self-adjoint Sturm-Liouville operator, inverse spectral problem, discontinuity, local solvability, stability
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## 1. INTRODUCTION

Consider the following Sturm-Liouville problem

$$
\begin{equation*}
-y^{\prime \prime}(x)+q(x) y(x)=\lambda y(x), \quad x \in(0, a) \cup(a, 1), \tag{1.1}
\end{equation*}
$$

with the boundary conditions

$$
\begin{equation*}
y^{\prime}(0)-h y(0)=0, \quad y^{\prime}(1)+H y(1)=0, \tag{1.2}
\end{equation*}
$$

and the jump conditions

$$
\begin{equation*}
y(a+0)=a_{1} y(a-0), \quad y^{\prime}(a+0)=a_{1}^{-1} y^{\prime}(a-0)+a_{2} y(a-0) \tag{1.3}
\end{equation*}
$$

where $\lambda$ is spectral parameter, the complex-valued potential $q$ belongs to $L^{2}(0,1), h, H, a_{2} \in$ $\mathbb{C}, a_{1}>0$ and $a \in(0,1)$.

Inverse spectral problems for the Sturm-Liouville operators consist in recovering the coefficients of the operators from their spectral characteristics. The basic results of inverse SturmLiouville problems can be found in, e.g., the monographs [13, 24 26]. The Sturm-Liouville problems with discontinuities inside the interval arise in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology. Such problems are usually connected with discontinuous material properties, for example, the transmission eigenvalue problem with a discontinuous index of refraction [15], the geophysical models for oscillations of the Earth [2, 17] and the electromagnetic and elastic inverse problems for media with discontinuous material properties [23].

[^0]The Sturm-Liouville problem (1.1)-(1.3) has attracted much attention of scholars (see, e.g., [1, 11, 17, 29 33, 36, 37, 40, 41] and the references therein). In order to uniquely recover the potential on $[0,1]$ and all the coefficients, one needs to know two spectra [29, 40, 41]. However, when partial information on the potential and a part of coefficients are known a priori, then only a part of two spectra are needed (see, e.g., [17, 30, 32, 33, 36]). In particular, roughly speaking, if $q(x)$ is known a priori on a half interval, then only one spectrum is sufficient; if $q(x)$ is given on a subinterval more than one half, then only a part of one spectrum is enough. An overview of classical and modern results on partial inverse Sturm-Liouville problems is presented in [6]. Without the jump conditions, the uniqueness for solutions of partial inverse problems was considered in [10, 14, 18 20] and other works.

In this paper, we consider the local solvability and stability the partial inverse problems for the problem (1.1)-(1.3) with complex coefficients, in which, generally speaking, a part of coefficients and $q(x)$ on $(a, 1)$ are known a priori. It is known [37] that the problem (1.1)-(1.3) is equivalent to the following problem $B_{1}=B\left(d_{1}, d_{2}, q_{1}, q_{2}, h, H, a_{1}, a_{2}\right)$ :

$$
\begin{gather*}
-y_{j}^{\prime \prime}(x)+q_{j}(x) y_{j}(x)=\lambda y_{j}(x), \quad 0<x<d_{j}, \quad j=1,2,  \tag{1.4}\\
y_{1}^{\prime}(0)-h y_{1}(0)=0, \quad y_{2}^{\prime}(0)-H y_{2}(0)=0  \tag{1.5}\\
y_{1}\left(d_{1}\right)-a_{1}^{-1} y_{2}\left(d_{2}\right)=0, \quad y_{1}^{\prime}\left(d_{1}\right)+\left[a_{1} y_{2}^{\prime}\left(d_{2}\right)+a_{2} y_{2}\left(d_{2}\right)\right]=0, \tag{1.6}
\end{gather*}
$$

where $d_{1}=a, d_{2}=1-a, q_{1}(x)=q(x)$ for $x \in\left(0, d_{1}\right)$ and $q_{2}(x)=q(1-x)$ for $x \in\left(0, d_{2}\right)$.
Since the coefficients are complex, the problem (1.1)-(1.3) is non-self-adjoint. Thus, there may exist multiple and non-real eigenvalues. The asymptotic behavior of the eigenvalues is the same as that in the self-adjoint case. So, the appearance of non-real eigenvalues cause almost no difficulty in studying the inverse problems. However, the appearance of multiple eigenvalues will cause the main difficulties in the non-self-adjoint cases when we study the inverse problems, especially, for the local solvability and stability (see, e.g., [3-5, 7, 8, 22, 27]). In order to overcome the difficulties caused by multiple eigenvalues, we shall develop the methods and techniques of 3 , $5,7,22,27]$. Let $\left\{\lambda_{1, n}\right\}_{n \geq 0}$ (counted with multiplicities) be the eigenvalues of the problem $B_{1}$.
Inverse Problem 1. Assume that $a \in(0,1 / 2]$, and let $I$ be the subset of $\mathbb{N}_{0}:=\mathbb{N} \cup\{0\}$. Given $\left\{\lambda_{1, n}\right\}_{n \in I}, a_{1}, a_{2}, H, \omega_{1}:=h+\frac{1}{2} \int_{0}^{a} q_{1}(x) d x$ and $q_{2}$, find $q_{1}$ and $h$.

In the self-adjoint case, the local solvability and stability of Inverse Problem 1 are proved in [37], where, in particular, if $a=1 / 2$ then $I=\mathbb{N}_{0}$ and the coefficients $a_{2}, \omega_{1}$ can be recovered from the spectrum. In this paper, we consider the local solvability and stability for the non-self-adjoint case. Moreover, we shall also study the case $a \in(1 / 2,1)$. It is known that if $a \in(1 / 2,1)$ and $q(x)$ on $(a, 1)$ is given, then one spectrum is not sufficient to uniquely determine the potential on the whole interval. One should add some other eigenvalues. Consider the problem $B_{0}=B\left(d_{1}, d_{2}, q_{1}, q_{2}, h, \infty, a_{1}, a_{2}\right)$ which means that $y^{\prime}(1)+H y(1)=0$ is replaced by $y(1)=0$. Let $\left\{\lambda_{0, n}\right\}_{n \geq 0}$ (counted with multiplicities) be the eigenvalues of the problem $B_{0}$. Let us consider the following Inverse Problem 2,
Inverse Problem 2. Let $I_{i}(i=0,1)$ the subsets of $\mathbb{N}_{0}$. Given $\left\{\lambda_{i, n}\right\}_{n \in I_{i}}(i=0,1), a_{1}$, $a_{2}, H, \omega_{1}:=h+\frac{1}{2} \int_{0}^{a} q_{1}(x) d x$ and $q_{2}$, find $q_{1}$ and $h$.

Note that, since $a \in(0,1)$ in Inverse Problem 2, it includes Inverse Problem 11 as a special case. For example, if $a \in(0,1 / 2]$, then we can put $I_{1}=I$ and $I_{0}=\emptyset$. The local solvability
of Inverse Problems 2 depends on the index sets $I_{i}(i=0,1)$. We will give the descriptions of these index sets in the corresponding main theorems. For convenience of formulation, we shall renumber the sequences $\left\{\lambda_{i, n}\right\}_{n \in I_{i}}$ by letting $\left\{\mu_{i, k}\right\}_{k \geq 0}=\left\{\lambda_{i, n}\right\}_{n \in I_{i}}$ with $\left|\mu_{i, k+1}\right| \geq\left|\mu_{i, k}\right|$, $i=0,1$. Our results can also be generalized into the partial inverse problems from parts of $N+1$ spectra of the problems with different boundary conditions at $x=1$, where $N \geq 1$. But the proportion of the needed eigenvalues should remain the same (see Remark (3).

Another motivation of this paper is that, in the local solvability and stability, the error caused by the given potential $q_{2}$ and the parameter $H$ should be considered. In 37], the authors gave the local solvability and stability for Inverse Problem 1 by assuming that only the given subspectrum contains $\varepsilon$-error and the given potential $q_{2}$ and the parameter $H$ contain no error. However, as one of the input data, the given potential $q_{2}$ or the parameter $H$ may also contain $\varepsilon$-error. Therefore, for Inverse Problem 1, it is natural to consider the local solvability and stability with the subspectrum, the potential $q_{2}$ and $H$ containing $\varepsilon$-error. For Inverse Problem 2, since partial eigenvalues of the problem $B_{0}$ are a part of the input data, the given parameter $H$ is assumed to contain no error, and the subspectra and the potential $q_{2}$ contain $\varepsilon$-error.

Let us discuss the essential novelties of our results comparing with the previous studies. First, consider the case of real-valued potential $q_{1}(x)$ and simple eigenvalues $\left\{\lambda_{i, n}\right\}$. In this special case, Inverse Problem 2 can be treated as the problem of Horváth [19] (see Remark 2), which consists in recovering $q_{1}$ and $h$ from the eigenvalues $\left\{\lambda_{n}\right\}_{n \geq 1}$ of the following problems

$$
\begin{equation*}
-y^{\prime \prime}+q_{1}(x) y=\lambda y, \quad y^{\prime}(0)-h y(0)=0, \quad y\left(d_{1}\right) \cos \alpha_{n}+y^{\prime}\left(d_{1}\right) \sin \alpha_{n}=0, \quad n \geq 1 . \tag{1.7}
\end{equation*}
$$

We mean that the eigenvalues are taken from different spectra: $\lambda_{n} \in \sigma\left(q_{1}, h_{1}, \alpha_{n}\right), n \geq 1$, where $\sigma\left(q_{1}, h, \alpha_{n}\right)$ is the spectrum of (1.7). In the case of $h_{1}=\infty$ (i.e. the Dirichlet boundary condition at $x=0$ ), Horváth [19] gave a necessary and sufficient condition for the uniqueness of the inverse problem solution. In the case of $h \in \mathbb{R}$, Horváth has separately obtained a necessary condition and a sufficient one for the unique determination of $q_{1}$ and $h$. The necessary and sufficient conditions of [19] are formulated in terms of the closedness for some exponential systems. Latter on, in the case of $h=\infty$, Horváth and Kiss studied the stability for the self-adjoint case in [21] and for the non-self-adjoint case in [22]. However, in this paper, we investigate the local solvability of the inverse problem, which was not considered by Horváth and Kiss. Moreover, in the case of multiple eigenvalues, our problem statement is different from [19, 21, 22], so it requires a separate investigation. Also, we obtain a necessary and sufficient condition for the uniqueness of solution for Inverse Problem 2 in terms of the completeness for a sequence of two-element vector functions. This condition is different from that in [19].

Second, the boundary value problem (1.4)-(1.6) can be represented in the following form:

$$
\begin{gather*}
-y^{\prime \prime}(x)+q_{1}(x) y(x)=\lambda y(x), \quad 0<x<d_{1}  \tag{1.8}\\
y^{\prime}(0)-h y(0)=0, \quad f_{1}(\lambda) y^{\prime}\left(d_{1}\right)+f_{2}(\lambda) y\left(d_{1}\right)=0, \tag{1.9}
\end{gather*}
$$

where $f_{1}(\lambda)$ and $f_{2}(\lambda)$ are some entire analytic functions, which are constructed by the known data $q_{2}(x), H, a_{1}, a_{2}$ (see Remark 2 for details). Thus, Inverse Problem 1 is reduced to the recovery of $q_{1}$ and $h$ from a subspectrum of (1.8)-(1.9), while the entire functions $f_{1}(\lambda)$ and $f_{2}(\lambda)$ are known a priori. The inverse spectral theory for the problems of form (1.8)-(1.9) has been created in [3-5, 38] and subsequent studies (see the overview [6]). In particular, the case of multiple eigenvalues was considered and local solvability and stability of inverse problems
were proved. However, Inverse Problem 2 cannot be represented as (1.8)-(1.9), since it implies different functions $f_{1}^{i}(\lambda)$ and $f_{2}^{i}(\lambda)$ for the problems $B_{i}$ with $i=0$ and $i=1$. Therefore, the methods of [4, 5, 38] cannot be directly applied here. Moreover, an important feature of our local solvability and stability analysis is that not only eigenvalue perturbations but also perturbations of $q_{2}$ and $H$ are taken into account. In the papers [4, 5, 38] the functions $f_{1}(\lambda)$ and $f_{2}(\lambda)$ remain fixed, so the results of [4, 5, 38] do not give us an opportunity to study perturbations of $q_{2}$ and $H$. It is worth mentioning that, for the case without discontinuity ( $a_{1}=1, a_{2}=0, h=H=\infty, a=1 / 2$ ), the local solvability and stability of the inverse problem for the first time was proved in [3], during the investigation of the inverse transmission eigenvalue problem. In [3], perturbations of the both subspectrum and the known potential were considered in the case of multiple eigenvalues. Nevertheless, the presence of the discontinuity causes additional difficulties and so requires a separate investigation. In this paper, we develop the ideas of the previous studies [3-5, 22, 38], since we cannot directly apply their methods to our problems.

The paper is organized as follows. In Section 2, we derive the main equations and introduce the main results in this paper, including two uniqueness theorems, a reconstruction algorithm, a theorem of the local solvability and stability, and two corollaries. In Section 3, we study the asymptotic behavior of the vector functional sequence in the main equations. In Section 4, we provide the proofs of the uniqueness theorems for Inverse Problem 2. In Section 5, we prove the theorem for the local solvability and stability of Inverse Problems 1 and 2. In particular, we first consider the general case $a \in(0,1)$, namely, Inverse Problem 2, in which the given $q_{2}$ contains $\varepsilon$-error. Then, we consider the case $a \in(0,1 / 2]$, namely, Inverse Problem [1, in which the given $H$ and $q_{2}$ contain $\varepsilon$-error. In Appendix, some auxiliary propositions of complex and functional analyses are provided.

## 2. Main Results

In this section, we first derive the main equations for solving Inverse Problems 1 and 2, and then present the main results of this paper.

Let $\varphi(x, \lambda)$ be the solution of (1.4) for $j=1$ satisfying the initial conditions $\varphi(0, \lambda)=$ $1, \varphi^{\prime}(0, \lambda)=h$. Let $\psi_{0}(x, \lambda)$ and $\psi_{1}(x, \lambda)$ be the solutions of (1.4) for $j=2$ satisfying the initial conditions

$$
\begin{equation*}
\psi_{0}(0, \lambda)=0, \psi_{0}^{\prime}(0, \lambda)=1, \quad \psi_{1}(0, \lambda)=1, \psi_{1}^{\prime}(0, \lambda)=H \tag{2.1}
\end{equation*}
$$

respectively. Then, in view of (1.6), the eigenvalues $\left\{\lambda_{i, n}\right\}_{n \geq 0}$ of the problem $B_{i}(i=0,1)$, respectively, coincide with the zeros of the characteristic functions

$$
\Delta_{i}(\lambda)=\left|\begin{array}{cc}
\varphi(a, \lambda) & -a_{1}^{-1} \psi_{i}\left(d_{2}, \lambda\right)  \tag{2.2}\\
\varphi^{\prime}(a, \lambda) & a_{1} \psi_{i}^{\prime}\left(d_{2}, \lambda\right)+a_{2} \psi_{i}\left(d_{2}, \lambda\right)
\end{array}\right|:=\left|\begin{array}{cc}
\varphi_{0}(\lambda) & g_{i, 0}(\lambda) \\
\varphi_{1}(\lambda) & g_{i, 1}(\lambda)
\end{array}\right| .
$$

Let $\lambda=\rho^{2}$. It is known [25, 26] that $\varphi(x, \lambda)$ has the expression

$$
\begin{equation*}
\varphi(x, \lambda)=\cos \rho x+\int_{0}^{x} K(x, t) \cos \rho t d t, \quad 0 \leq x \leq a \tag{2.3}
\end{equation*}
$$

where the kernel $K(x, t)$ is a function of two variables, which has the first partial derivatives $K_{x}(x, \cdot), K_{t}(x, \cdot) \in L^{2}(0, x)$, and

$$
\begin{equation*}
\omega_{1}=K(a, a)=h+\frac{1}{2} \int_{0}^{a} q_{1}(t) d t . \tag{2.4}
\end{equation*}
$$

The relation (2.3) together with (2.4) yield

$$
\begin{gather*}
\varphi_{0}(\lambda)=\cos \rho a+\omega_{1} \frac{\sin \rho a}{\rho}-\int_{0}^{a} K_{1}(t) \frac{\sin \rho t}{\rho} d t  \tag{2.5}\\
\varphi_{1}(\lambda)=-\rho \sin \rho a+\omega_{1} \cos \rho a+\int_{0}^{a} K_{2}(t) \cos \rho t d t \tag{2.6}
\end{gather*}
$$

where $K_{1}(t):=K_{t}(a, t)$ and $K_{2}(t):=K_{x}(a, t)$. The set $\left\{K_{1}(t), K_{2}(t), \omega_{1}\right\}$ is called the Cauchy data for $q_{1}$ and $h$.

Substituting (2.5) and (2.6) into (2.2), we get

$$
\begin{align*}
-\Delta_{i}(\lambda)= & \rho^{-1} g_{i, 1}(\lambda) \int_{0}^{a} K_{1}(t) \sin \rho t d t+g_{i, 0}(\lambda) \int_{0}^{a} K_{2}(t) \cos \rho t d t \\
& -\left[\cos \rho a+\frac{\omega_{1} \sin \rho a}{\rho}\right] g_{i, 1}(\lambda)+g_{i, 0}(\lambda)\left[\omega_{1} \cos \rho a-\rho \sin \rho a\right] \tag{2.7}
\end{align*}
$$

Introduce the Hilbert space of vector-valued functions $\mathcal{H}:=L^{2}(0, a) \times L^{2}(0, a)$ with the inner product $\langle\cdot, \cdot\rangle$ defined by

$$
\begin{equation*}
\langle\mathbf{h}, \mathbf{p}\rangle=\int_{0}^{a} \overline{h_{1}(x)} p_{1}(x)+\overline{h_{2}(x)} p_{2}(x) d x, \quad \forall \mathbf{h}:=\left(h_{1}, h_{2}\right), \mathbf{p}:=\left(p_{1}, p_{2}\right) \in \mathcal{H} . \tag{2.8}
\end{equation*}
$$

Rewrite (2.7) as

$$
\begin{equation*}
-\Delta_{i}(\lambda)=\left\langle\mathbf{K}(\cdot), \mathbf{U}_{i}(\cdot, \lambda)\right\rangle-f_{i}(\lambda), \quad i=0,1, \tag{2.9}
\end{equation*}
$$

where

$$
\begin{gather*}
f_{i}(\lambda):=\left[\cos \rho a+\omega_{1} \frac{\sin \rho a}{\rho}\right] g_{i, 1}(\lambda)-g_{i, 0}(\lambda)\left[\omega_{1} \cos \rho a-\rho \sin \rho a\right]  \tag{2.10}\\
\mathbf{U}_{i}(t, \lambda):=\left(U_{i, 1}(t, \lambda), U_{i, 2}(t, \lambda)\right), \quad \mathbf{K}(t):=\left(\overline{K_{1}(t)}, \overline{K_{2}(t)}\right)  \tag{2.11}\\
U_{i, 1}(t, \lambda):=g_{i, 1}(\lambda) s(t, \lambda), \quad s(t, \lambda):=\frac{\sin \rho t}{\rho}, \quad U_{i, 2}(t, \lambda):=g_{i, 0}(\lambda) c(t, \lambda), \quad c(t, \lambda):=\cos \rho t \tag{2.12}
\end{gather*}
$$

Recall $\left\{\mu_{i, k}\right\}_{k \geq 0}=\left\{\lambda_{i, n}\right\}_{n \in I_{i}}, i=0,1$. Let $m_{n}^{i}$ be the multiplicity of the value $\mu_{i, n}$ in the sequence $\left\{\mu_{i, n}\right\}_{n \geq 0}$. Without loss of generality, assume $\mu_{i, n}=\mu_{i, n+1}=\cdots=\mu_{i, n+m_{n}^{i}-1}$. Note that $m_{n}^{i}=1$ for $n \geq n_{i}$ for some large $n_{i}$. Consider the set

$$
\mathcal{S}_{i}:=\left\{n \in \mathbb{N}: \mu_{i, n} \neq \mu_{i, n-1}, n \geq 1\right\} \cup\{0\}, \quad i=0,1
$$

It is obvious that the sequence $\left\{\mu_{i, n}\right\}_{n \in \mathcal{S}_{i}}$ consists of elements of $\left\{\mu_{i, n}\right\}_{n \geq 0}$ being taken only once. Denote

$$
f^{\langle\nu\rangle}(\lambda):=\frac{1}{\nu!} \frac{d^{\nu} f(\lambda)}{d \lambda^{\nu}}, \quad P^{\langle\nu\rangle}(t, \lambda):=\frac{1}{\nu!} \frac{\partial^{\nu} P(t, \lambda)}{\partial \lambda^{\nu}} .
$$

Define

$$
\begin{equation*}
\mathbf{U}_{n+\nu}^{i}(t):=\mathbf{U}_{i}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right), \quad n \in \mathcal{S}_{i}, \quad \nu=\overline{0, m_{n}^{i}-1}, \quad i=0,1, \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau_{n+\nu}^{i}:=f_{i}^{\langle\nu\rangle}\left(\mu_{i, n}\right), \quad n \in \mathcal{S}_{i}, \quad \nu=\overline{0, m_{n}^{i}-1}, \quad i=0,1 \tag{2.14}
\end{equation*}
$$

Together with (2.9) $-(2.14)$, we get the main equations of Inverse Problem 2

$$
\begin{equation*}
\left\langle\mathbf{K}(\cdot), \mathbf{U}_{n}^{i}(\cdot)\right\rangle=\tau_{n}^{i}, \quad n \geq 0, \quad i=0,1 \tag{2.15}
\end{equation*}
$$

Let us formulate the uniqueness results for the solution of Inverse Problem 2. Firstly, using the given data $a_{1}, a_{2}, H$ and $q_{2}$, we can uniquely recover the functions $\mathbf{U}_{i}(t, \lambda)(i=0,1)$ with the help of (2.2), (2.11), and (2.12). Then, using the given eigenvalues $\left\{\mu_{i, n}\right\}_{n \geq 0, i=0,1}$, we can construct the system of functions $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ by (2.13).

Theorem 1 (Uniqueness 1). Assume that Inverse Problem 2 is solvable. Then the solution of Inverse Problem 2 is unique if and only if the system $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.13) is complete in $\mathcal{H}$.

In Theorem 1, the condition depends on the system of functions $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ which, visually, relies on not only the subspectra but also the data $a_{1}, a_{2}, H$ and $q_{2}$. In the following uniqueness theorem, the condition, visually, only depends on the subspectra. Denote

$$
\begin{equation*}
c_{n+\nu}^{i}(t):=c^{\langle\nu\rangle}\left(t, \mu_{i, n}\right)=\left.\frac{1}{\nu!} \frac{\partial^{\nu} \cos \rho t}{\partial \lambda^{\nu}}\right|_{\lambda=\mu_{i, n}} \quad, \quad n \in \mathcal{S}_{i}, \quad \nu=\overline{0, m_{n}^{i}-1}, \quad i=0,1 \tag{2.16}
\end{equation*}
$$

Theorem 2 (Uniqueness 2). Assume that Inverse Problem 2 is solvable. The solution of Inverse Problem 2 is unique if the system $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.16) is complete in $L^{2}(0,2 a)$.

Remark 1. Theorem 1 gives the necessary and sufficient condition for the uniqueness of the solution of Inverse Problem 2. Theorem 2 only gives the sufficient condition. From Lemmas 1 and 2 in Section 4, we know that the condition in Theorem 2 implies the condition in Theorem 1. However, the condition in Theorem 2 seems easier to verify in some special cases (see the proof of Corollary (2). In general, if $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is complete in $\mathcal{H}$, it not clear whether $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is complete in $L^{2}(0,2 a)$.
Remark 2. The eigenvalues $\left\{\lambda_{n}\right\}_{n \geq 1}$ and coefficients $\cos \alpha_{n}$ and $\sin \alpha_{n}$ in (1.7), and the entire functions $f_{1}(\lambda)$ and $f_{2}(\lambda)$ in (1.9) can be constructed from the given data in Inverse Problems 2 and 1, respectively. Indeed, in (1.7), $\left\{\lambda_{n}\right\}_{n \geq 1}=\left\{\mu_{0, k}, \mu_{1, k}\right\}_{k \geq 0}$ and $\left\{\alpha_{n}\right\}_{n \geq 1}=$ $\left\{\alpha_{0, k}, \alpha_{1, k}\right\}_{k \geq 0}$, where $\alpha_{i, k} \quad(i=0,1)$ satisfy

$$
\cos \alpha_{i, k}=\frac{g_{i, 1}\left(\mu_{i, k}\right)}{\sqrt{g_{i, 1}\left(\mu_{i, k}\right)^{2}+g_{i, 0}\left(\mu_{i, k}\right)^{2}}}, \quad \sin \alpha_{i, k}=\frac{-g_{i, 0}\left(\mu_{i, k}\right)}{\sqrt{g_{i, 1}\left(\mu_{i, k}\right)^{2}+g_{i, 0}\left(\mu_{i, k}\right)^{2}}}
$$

and in (1.9),

$$
f_{1}(\lambda)=g_{1,1}(\lambda), \quad f_{2}(\lambda)=-g_{1,0}(\lambda)
$$

where the entire functions $g_{i, j}(\lambda)$ are defined in (2.2). In view of this observation, for the realvalued potential $q_{1}$ and simple eigenvalues, Theorem 2 can be viewed as a variant of Theorem 1.2 in [19]. Furthermore, for Inverse Problem 1, Theorem 2 can be obtained from Theorem 2.1 in [38]. However, for Inverse Problem 2 in the non-self-adjoint case with possible eigenvalue multiplicities, Theorem 2 does not follow from previously known results.

Remark 3. The above results can be easily generalized to the partial inverse problems from parts of $N+1$ subspectra, where $N \geq 1$. Indeed, consider the problems $B_{i}=B\left(d_{1}, d_{2}, q_{1}, q_{2}, h, H_{i}, a_{1}, a_{2}\right)$ $(i=\overline{0, N})$ with $H_{0}=\infty, H_{1}=H$ and $H_{l} \neq H_{j}$ for $l \neq j$. For $i=\overline{0, N}$, let $\left\{\mu_{i, n}\right\}_{n \geq 0}$ be a subspectrum of the corresponding problem $B_{i}$. Similarly to the definitions of $\mathbf{U}_{n}^{i}(t)$ and $c_{n}^{i}(t)$ for $i=0,1$, we can also define $\mathbf{U}_{n}^{i}(t)$ and $c_{n}^{i}(t)$ for $i=\overline{2, N}$ by $\left\{\mu_{i, n}\right\}_{n \geq 0, i=\overline{2, N}}$. Then we have the generalized result: $q_{1}$ and $h$ are uniquely determined by $a_{1}, a_{2}, q_{2}, \omega_{1}, H_{i}$ and $\left\{\mu_{i, n}\right\}_{n \geq 0}, i=\overline{0, N}$, if and only if $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=\overline{0, N}}$ is complete in $\mathcal{H}$ (or if $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=\overline{0, N}}$ is complete in $\left.L^{2}(0,2 a)\right)$.

The next result is the algorithm for recovering the solution of Inverse Problem 2, It is known that, in the self-adjoint case, one can use the Cauchy data $\left\{K_{1}(t), K_{2}(t), \omega_{1}\right\}$ to recover $q_{1}$ and $h$ directly (see [28]). In the non-self-adjoint case, one can first use the Cauchy data to recover the functions $\varphi_{0}(\lambda)$ and $\varphi_{1}(\lambda)$ defined in (2.5) and (2.6), and then recover the complex-valued potential $q_{1}$ and $h$ by the method of spectral mapping (see, e.g., [8, 12]). Therefore, we only need to recover the Cauchy data. The solution of Inverse Problem 2 can be found by the following algorithm under the assumption that $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.13) is a basis in $\mathcal{H}$.

Algorithm 1. Let $\left\{\mu_{i, n}\right\}_{n \geq 0}(i=0,1)$ be the given subspectra of the corresponding problems $B_{i}$. We have to find $q_{1}$ and $h$ from $a_{1}, a_{2}, H, \omega_{1}$, and $q_{2}$.
(1) Find the solutions $\psi_{i}(x, \lambda)(i=0,1)$ of equation (1.4) with $j=2$ under the initial conditions (2.1) and then determine the functions $g_{i, j}$ by (2.2) from $q_{2}, H, a_{1}$, and $a_{2}$, where $i=0,1$ and $j=0,1$.
(2) Construct the basis $\left\{\mathbf{U}_{n}^{i}\right\}_{n \geq 0}(i=0,1)$ by (2.11), (2.12), and (2.13).
(3) Construct $\left\{\tau_{n}^{i}\right\}_{n \geq 0}(i=0,1)$ by (2.10) and (2.14).
(4) Determine $\mathbf{K}=\left(K_{1}, K_{2}\right) \in \mathcal{H}$ by the following formula

$$
\mathbf{K}(t)=\sum_{n \geq 0}^{\overline{\tau_{n}^{0}}} \mathbf{U}_{n}^{0 *}(t)+\sum_{n \geq 0} \overline{\tau_{n}^{1}} \mathbf{U}_{n}^{1 *}(t),
$$

where $\left\{\mathbf{U}_{n}^{0 *}(t)\right\}_{n \geq 0} \cup\left\{\mathbf{U}_{n}^{1 *}(t)\right\}_{n \geq 0}$ is the basis, biorthonormal to $\left\{\mathbf{U}_{n}^{0}(t)\right\}_{n \geq 0} \cup\left\{\mathbf{U}_{n}^{1}(t)\right\}_{n \geq 0}$.
(5) Construct the functions $\varphi_{0}(\lambda)$ and $\varphi_{1}(\lambda)$ by (2.5) and (2.6).
(6) Recover the potential $q_{1}$ and $h$ from $\varphi_{0}(\lambda)$ and $\varphi_{1}(\lambda)$ by using the method of spectral mappings (see [8, 12]).

The next result is a theorem on the local solvability and stability of Inverse Problem 2. This theorem takes the error caused by the given partial potential into account.

Theorem 3. For $i=0,1$, let $\left\{\lambda_{i, n}\right\}_{n \in I_{i}}\left(=\left\{\mu_{i, k}\right\}_{k \geq 0}\right)$ be a subspectrum of the corresponding problem $B_{i}=B\left(d_{1}, d_{2}, q_{1}, q_{2}, h, H_{i}, a_{1}, a_{2}\right)$ with the complex-valued potentials $q_{j} \in L^{2}\left(0, d_{j}\right), h, H_{1}$, $a_{2} \in \mathbb{C}, a_{1}>0$ and $H_{0}=\infty$. Suppose that the system of functions $\left\{\sqrt{\left(\left|\mu_{0, n}\right|+1\right)} \mathbf{U}_{n}^{0}(t)\right\}_{n \geq 0} \cup$ $\left\{\mathbf{U}_{n}^{1}(t)\right\}_{n \geq 0}$ constructed in (2.13) is a Riesz basis in $\mathcal{H}$. Then there exists $\varepsilon>0$ (depending on the problem $B_{1}$ ) such that, for arbitrary sequences $\left\{\tilde{\lambda}_{i, n}\right\}_{n \in I_{i}}$ and any function $\tilde{q}_{2} \in L^{2}\left(0, d_{2}\right)$ satisfying

$$
\begin{equation*}
\Lambda:=\sqrt{\sum_{n \in I_{0}}\left|\lambda_{0, n}-\tilde{\lambda}_{0, n}\right|^{2}+\sum_{n \in I_{1}}\left|\lambda_{1, n}-\tilde{\lambda}_{1, n}\right|^{2}} \leq \varepsilon \tag{2.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{d_{2}} \tilde{q}_{2}(t) d t=\int_{0}^{d_{2}} q_{2}(t) d t, \quad Q:=\left\|\tilde{q}_{2}-q_{2}\right\|_{L^{2}\left(0, d_{2}\right)} \leq \varepsilon \tag{2.18}
\end{equation*}
$$

respectively, there exist unique $\tilde{q}_{1} \in L^{2}\left(0, d_{1}\right)$ and $\tilde{h} \in \mathbb{C}$ such that $\tilde{h}+\frac{1}{2} \int_{0}^{d_{1}} \tilde{q}_{1}(x) d x=\omega_{1}$ and, for $i=0,1,\left\{\tilde{\lambda}_{i, n}\right\}_{n \in I_{i}}$ is a subspectrum of the problem $\tilde{B}_{i}:=B\left(d_{1}, d_{2}, \tilde{q}_{1}, \tilde{q}_{2}, \tilde{h}, H_{i}, a_{1}, a_{2}\right)$. Moreover,

$$
\begin{equation*}
\left\|\tilde{q}_{1}-q_{1}\right\|_{L^{2}\left(0, d_{1}\right)} \leq C(\Lambda+Q), \quad|\tilde{h}-h| \leq C(\Lambda+Q) \tag{2.19}
\end{equation*}
$$

where $C>0$ depends only on the problem $B_{1}$.
Remark 4. From Lemma 3 in Section 4, we know that, in Theorem 3, the condition that the system of functions $\left\{\sqrt{\left(\left|\mu_{0, n}\right|+1\right)} \mathbf{U}_{n}^{0}(t)\right\}_{n \geq 0} \cup\left\{\mathbf{U}_{n}^{1}(t)\right\}_{n \geq 0}$ constructed in (2.13) is a Riesz basis in $\mathcal{H}$, can be replaced by the stronger but easier to verify condition that the system $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.16) is a Riesz basis in $L^{2}(0,2 a)$.

As a corollary, we also give the local solvability and stability result for Inverse Problem 1, in which the error caused by $q_{2}$ and $H$ is taken into account.
Corollary 1. Assume that $a \in(0,1 / 2]$. Let $\left\{\lambda_{1, n}\right\}_{n \in I}\left(=\left\{\mu_{1, k}\right\}_{k \geq 0}\right)$ be a subspectrum of the problem $B_{1}=B\left(d_{1}, d_{2}, q_{1}, q_{2}, h, H, a_{1}, a_{2}\right)$ with complex-valued potentials $q_{j} \in L^{2}\left(0, d_{j}\right), h, H$, $a_{2} \in \mathbb{C}$ and $a_{1}>0$. Suppose that the system of functions $\left\{\mathbf{U}_{n}^{1}(t)\right\}_{n \geq 0}$ constructed in (2.13) is a Riesz basis in $\mathcal{H}$. Then, there exists $\varepsilon>0$ (depending on the problem $B_{1}$ ) such that, for arbitrary sequence $\left\{\tilde{\lambda}_{1, n}\right\}_{n \in I}$, any $\tilde{q}_{2} \in L^{2}\left(0, d_{2}\right)$ and $\tilde{H} \in \mathbb{C}$ satisfying

$$
\begin{equation*}
\Lambda_{1}:=\sqrt{\sum_{n \in I}\left|\lambda_{1, n}-\tilde{\lambda}_{1, n}\right|^{2}} \leq \varepsilon \tag{2.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{H}+\frac{1}{2} \int_{0}^{d_{2}} \tilde{q}_{2}(t) d t=H+\frac{1}{2} \int_{0}^{d_{2}} q_{2}(t) d t, \quad Q_{1}:=|\tilde{H}-H|+\left\|\tilde{q}_{2}-q_{2}\right\|_{L^{2}\left(0, d_{2}\right)} \leq \varepsilon \tag{2.21}
\end{equation*}
$$

respectively, there exist unique $\tilde{q}_{1} \in L^{2}\left(0, d_{1}\right)$ and $\tilde{h} \in \mathbb{C}$ such that $\tilde{h}+\frac{1}{2} \int_{0}^{d_{1}} \tilde{q}_{1}(x) d x=\omega_{1}$, and $\left\{\tilde{\lambda}_{1, n}\right\}_{n \in I}$ is a subspectrum of the problem $\tilde{B}_{1}:=B\left(d_{1}, d_{2}, \tilde{q}_{1}, \tilde{q}_{2}, \tilde{h}, \tilde{H}, a_{1}, a_{2}\right)$. Moreover,

$$
\begin{equation*}
\left\|\tilde{q}_{1}-q_{1}\right\|_{L^{2}\left(0, d_{1}\right)} \leq C\left(\Lambda_{1}+Q_{1}\right), \quad|\tilde{h}-h| \leq C\left(\Lambda_{1}+Q_{1}\right) \tag{2.22}
\end{equation*}
$$

where $C>0$ depends only on the problem $B_{1}$.
Furthermore, in the case of the half inverse problem (i.e., $d=1 / 2$ ), we do not need to require the Riesz-basis property of $\left\{\mathbf{U}_{n}^{1}(t)\right\}_{n \geq 0}$ and so obtain the following result.
Corollary 2. Assume that $a=1 / 2$. Let $\left\{\lambda_{1, n}\right\}_{n \geq 0}$ be the spectrum of the problem $B_{1}=$ $B\left(\frac{1}{2}, \frac{1}{2}, q_{1}, q_{2}, h, H, a_{1}, a_{2}\right)$ with complex-valued potentials $q_{j} \in L^{2}(0,1 / 2), h, H, a_{2} \in \mathbb{C}$ and $a_{1}>$ 0 . Then, there exists $\varepsilon>0$ (depending on the problem $B_{1}$ ) such that, for an arbitrary sequence $\left\{\tilde{\lambda}_{1, n}\right\}_{n \geq 0}$, any $\tilde{q}_{2} \in L^{2}(0,1 / 2)$ and $\tilde{H} \in \mathbb{C}$ satisfying

$$
\Lambda_{1}:=\sqrt{\sum_{n \geq 0}\left|\lambda_{1, n}-\tilde{\lambda}_{1, n}\right|^{2}} \leq \varepsilon
$$

and

$$
\tilde{H}+\frac{1}{2} \int_{0}^{\frac{1}{2}} \tilde{q}_{2}(t) d t=H+\frac{1}{2} \int_{0}^{\frac{1}{2}} q_{2}(t) d t, \quad Q_{1}:=|\tilde{H}-H|+\left\|\tilde{q}_{2}-q_{2}\right\|_{L^{2}(0,1 / 2)} \leq \varepsilon
$$

respectively, there exist unique $\tilde{q}_{1} \in L^{2}\left(0, \frac{1}{2}\right)$ and $\tilde{h} \in \mathbb{C}$ such that $\tilde{h}+\frac{1}{2} \int_{0}^{\frac{1}{2}} \tilde{q}_{1}(x) d x=\omega_{1}$, and $\left\{\tilde{\lambda}_{1, n}\right\}_{n \geq 0}$ is the spectrum of the problem $\tilde{B}_{1}:=B\left(\frac{1}{2}, \frac{1}{2}, \tilde{q}_{1}, \tilde{q}_{2}, \tilde{h}, \tilde{H}, a_{1}, a_{2}\right)$. Moreover,

$$
\left\|\tilde{q}_{1}-q_{1}\right\|_{L^{2}(0,1 / 2)} \leq C\left(\Lambda_{1}+Q_{1}\right), \quad|\tilde{h}-h| \leq C\left(\Lambda_{1}+Q_{1}\right),
$$

where $C>0$ depends only on the problem $B_{1}$.
Remark 5. Under the requirement $Q_{1}=0$, Corollaries 1 and 2 can be obtained by the method of [38].

## 3. Some asymptotic estimates

In this section, we investigate the asymptotic behavior of the functions $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.13).

Note that $\psi_{1}\left(d_{2}, \lambda\right)$ and $\psi_{1}^{\prime}\left(d_{2}, \lambda\right)$ have expressions similar to (2.5) and (2.6) with $a$ and $\omega_{1}$ replaced by $d_{2}$ and $\omega_{2}:=H+\frac{1}{2} \int_{0}^{d_{2}} q_{2}(t) d t$, respectively. Then, we obtain

$$
\begin{gather*}
g_{1,0}(\lambda)=-a_{1}^{-1} \cos \rho d_{2}-a_{1}^{-1} \omega_{2} \frac{\sin \rho d_{2}}{\rho}+\frac{1}{\rho} \int_{-d_{2}}^{d_{2}} P_{1,0}(t) e^{\mathrm{i} \rho t} d t  \tag{3.1}\\
g_{1,1}(\lambda)=-a_{1} \rho \sin \rho d_{2}+\left(a_{1} \omega_{2}+a_{2}\right) \cos \rho d_{2}+\int_{-d_{2}}^{d_{2}} P_{1,1}(t) e^{\mathrm{i} \rho t} d t \tag{3.2}
\end{gather*}
$$

where $P_{1, j}(\cdot) \in L^{2}\left(-d_{2}, d_{2}\right)$. Using the transformation operator expression for $\psi_{0}(x, \lambda)$, we also have

$$
\begin{align*}
& g_{0,0}(\lambda)=-a_{1}^{-1} \frac{\sin \rho d_{2}}{\rho}+a_{1}^{-1} \omega_{0} \frac{\cos \rho d_{2}}{\rho^{2}}+\frac{1}{\rho^{2}} \int_{-d_{2}}^{d_{2}} P_{0,0}(t) e^{\mathrm{i} \rho t} d t  \tag{3.3}\\
& g_{0,1}(\lambda)=a_{1} \cos \rho d_{2}+\left(a_{1} \omega_{0}+a_{2}\right) \frac{\sin \rho d_{2}}{\rho}+\frac{1}{\rho} \int_{-d_{2}}^{d_{2}} P_{0,1}(t) e^{\mathrm{i} \rho t} d t \tag{3.4}
\end{align*}
$$

where $P_{0, j} \in L^{2}\left(-d_{2}, d_{2}\right)$ and $\omega_{0}=\frac{1}{2} \int_{0}^{d_{2}} q_{2}(t) d t$. Substituting (2.5)-(2.6) and (3.1)-(3.4) into (2.2) and using the Paley-Wiener Theorem (see, e.g., [39, p.101]), we calculate

$$
\begin{align*}
& \Delta_{1}(\lambda)=\Delta_{1}^{0}(\lambda)+\eta_{+} \cos \rho+\eta_{-} \cos \rho(2 a-1)+\int_{-1}^{1} P_{1}(t) e^{\mathrm{i} \rho t} d t  \tag{3.5}\\
& \Delta_{0}(\lambda)=\Delta_{0}^{0}(\lambda)+\zeta_{+} \frac{\sin \rho}{\rho}+\zeta_{-} \frac{\sin \rho(2 a-1)}{\rho}+\frac{1}{\rho} \int_{-1}^{1} P_{0}(t) e^{\mathrm{i} \rho t} d t \tag{3.6}
\end{align*}
$$

where $P_{i} \in L^{2}(-1,1), i=0,1$, and

$$
\begin{gathered}
\Delta_{1}^{0}(\lambda):=-\rho\left[b_{+} \sin \rho-b_{-} \sin \rho(2 a-1)\right], \quad \Delta_{0}^{0}(\lambda):=b_{+} \cos \rho+b_{-} \cos \rho(2 a-1), \\
b_{ \pm}=\frac{a_{1} \pm a_{1}^{-1}}{2}, \quad \eta_{ \pm}=b_{ \pm}\left(\omega_{2} \pm \omega_{1}\right) \pm \frac{a_{2}}{2}, \quad \zeta_{ \pm}=b_{ \pm}\left(\omega_{1} \pm \omega_{0}\right) \pm \frac{a_{2}}{2} .
\end{gathered}
$$

For $i=0,1$, let $\left\{\lambda_{i, n}^{0}\right\}_{n \geq 0}$ be the zeros of $\Delta_{i}^{0}(\lambda)$, which are real and simple, since $a_{1}>0$. Denote $\rho_{i, n}^{0}:=\sqrt{\lambda_{i, n}^{0}}$. Using a similar method to Lemma 1 in [1], it is easy to get that for
each fixed $i=0,1$, the sequence $\left\{\rho_{i, n}^{0}\right\}$ is separated, namely, $\left|\rho_{i, n}^{0}-\rho_{i, m}^{0}\right| \geq c_{0}>0$ whenever $n \neq m$. Using the standard method involving the Rouchè Theorem together with (3.5) and (3.6), one can prove the following proposition (see, e.g., [1, 13, 40]).

Proposition 1. The eigenvalues $\left\{\lambda_{i, n}\right\}_{n \geq 0}$ of the problem $B_{i}$ have the asymptotic behavior

$$
\begin{equation*}
\rho_{i, n}:=\sqrt{\lambda_{i, n}}=\rho_{i, n}^{0}+\frac{\theta_{i, n}}{\rho_{i, n}^{0}}+\frac{\kappa_{i, n}}{\rho_{i, n}^{0}}, \quad\left\{\kappa_{i, n}\right\} \in l^{2}, \quad i=0,1, \tag{3.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\theta_{1, n}=\frac{\eta_{+} \cos \rho_{1, n}^{0}+\eta_{-} \cos \rho_{1, n}^{0}(2 a-1)}{2 \dot{\Delta}_{1}^{0}\left(\lambda_{1, n}^{0}\right)}, \quad \theta_{0, n}=-\frac{\zeta_{+} \sin \rho_{0, n}^{0}+\zeta_{-} \sin \rho_{0, n}^{0}(2 a-1)}{2 \rho_{0, n}^{0} \dot{\Delta}_{0}^{0}\left(\lambda_{0, n}^{0}\right)} \tag{3.8}
\end{equation*}
$$

here $\dot{\Delta}_{i}^{0}(\lambda):=\frac{d \Delta_{i}^{0}(\lambda)}{d \lambda}$. Moreover, for each $i=0,1$, there is $n_{i} \in \mathbb{N}_{0}$ such that the sequence $\left\{\rho_{i, n}\right\}_{n \geq n_{i}}$ is separated.

Using (3.1), (3.2) in (2.12) and (3.3), (3.4) in (2.10), we obtain the following relations for large $|\rho|$ :

$$
\begin{align*}
& U_{1,1}(t, \lambda)=u_{1,1}(t, \lambda)+O\left(\frac{e^{|\operatorname{Im} \rho|\left(d_{2}+t\right)}}{|\rho|}\right), U_{1,2}(t, \lambda)=u_{1,2}(t, \lambda)+O\left(\frac{e^{|\operatorname{Im} \rho|\left(d_{2}+t\right)}}{|\rho|}\right),  \tag{3.9}\\
& U_{0,1}(t, \lambda)=u_{0,1}(t, \lambda)+O\left(\frac{e^{|\operatorname{Im} \rho|\left(d_{2}+t\right)}}{|\rho|^{2}}\right), U_{0,2}(t, \lambda)=u_{0,2}(t, \lambda)+O\left(\frac{e^{|\operatorname{Im} \rho|\left(d_{2}+t\right)}}{|\rho|^{2}}\right) \tag{3.10}
\end{align*}
$$

where

$$
\begin{array}{ll}
u_{1,1}(t, \lambda)=\frac{a_{1}}{2}\left[\cos \rho\left(d_{2}+t\right)-\cos \rho\left(d_{2}-t\right)\right], & u_{1,2}(t, \lambda)=\frac{-1}{2 a_{1}}\left[\cos \rho\left(d_{2}-t\right)+\cos \rho\left(d_{2}+t\right)\right] \\
u_{0,1}(t, \lambda)=\frac{a_{1}}{2 \rho}\left[\sin \rho\left(d_{2}+t\right)-\sin \rho\left(d_{2}-t\right)\right], & u_{0,2}(t, \lambda)=\frac{-1}{2 a_{1} \rho}\left[\sin \rho\left(d_{2}-t\right)+\sin \rho\left(d_{2}+t\right)\right] \tag{3.12}
\end{array}
$$

Denote $\mathbf{u}_{i}(t, \lambda)=\left(u_{i, 1}(t, \lambda), u_{i, 2}(t, \lambda)\right), i=0,1$. From (3.11) and (3.12), and recalling $d_{2}=1-a$, we have

$$
\begin{align*}
\left\langle\mathbf{u}_{1}, \mathbf{u}_{1}\right\rangle= & \frac{a_{1}^{2}}{4} \int_{0}^{a}\left[\cos \bar{\rho}\left(d_{2}-t\right)-\cos \bar{\rho}\left(d_{2}+t\right)\right]\left[\cos \rho\left(d_{2}-t\right)-\cos \rho\left(d_{2}+t\right)\right] d t \\
& +\frac{1}{4 a_{1}^{2}} \int_{0}^{a}\left[\cos \bar{\rho}\left(d_{2}-t\right)+\cos \bar{\rho}\left(d_{2}+t\right)\right]\left[\cos \rho\left(d_{2}-t\right)+\cos \rho\left(d_{2}+t\right)\right] d t \\
= & \frac{a_{1}^{2}+a_{1}^{-2}}{4} \int_{1-2 a}^{1} \cos (\bar{\rho} t) \cos (\rho t) d t+\frac{a_{1}^{-2}-a_{1}^{2}}{4} \int_{1-2 a}^{1} \cos \left(\rho\left(2 d_{2}-t\right)\right) \cos (\bar{\rho} t) d t \\
= & \frac{a_{1}^{2}+a_{1}^{-2}}{8} \int_{1-2 a}^{1}[\cos (\bar{\rho}-\rho) t+\cos (\bar{\rho}+\rho) t] d t \\
& +\frac{a_{1}^{-2}-a_{1}^{2}}{8} \int_{1-2 a}^{1}\left[\cos \left(2 d_{2} \rho-(\rho+\bar{\rho}) t\right)+\cos \left(2 d_{2} \rho-(\rho-\bar{\rho}) t\right)\right] d t \tag{3.13}
\end{align*}
$$

$$
\begin{align*}
\left\langle\mathbf{u}_{0}, \mathbf{u}_{0}\right\rangle= & \frac{a_{1}^{2}}{4|\lambda|} \int_{0}^{a}\left[\sin \bar{\rho}\left(d_{2}-t\right)-\sin \bar{\rho}\left(d_{2}+t\right)\right]\left[\sin \rho\left(d_{2}-t\right)-\sin \rho\left(d_{2}+t\right)\right] d t \\
& +\frac{1}{4 a_{1}^{2}|\lambda|} \int_{0}^{a}\left[\sin \bar{\rho}\left(d_{2}-t\right)+\sin \bar{\rho}\left(d_{2}+t\right)\right]\left[\sin \rho\left(d_{2}-t\right)+\sin \rho\left(d_{2}+t\right)\right] d t \\
= & \frac{a_{1}^{2}+a_{1}^{-2}}{4|\lambda|} \int_{1-2 a}^{1} \sin (\bar{\rho} t) \sin (\rho t) d t+\frac{a_{1}^{-2}-a_{1}^{2}}{4} \int_{1-2 a}^{1} \sin (\bar{\rho} t) \sin \left(\rho\left(2 d_{2}-t\right)\right) d t \\
= & \frac{a_{1}^{2}+a_{1}^{-2}}{8|\lambda|} \int_{1-2 a}^{1}[\cos (\bar{\rho}-\rho) t-\cos (\bar{\rho}+\rho) t] d t \\
& +\frac{a_{1}^{-2}-a_{1}^{2}}{8|\lambda|} \int_{1-2 a}^{1}\left[\cos \left(2 d_{2} \rho-(\rho+\bar{\rho}) t\right)-\cos \left(2 d_{2} \rho-(\rho-\bar{\rho}) t\right)\right] d t \tag{3.14}
\end{align*}
$$

By Proposition 1, we know that $\operatorname{Im} \sqrt{\lambda_{i, n}} \rightarrow 0$ and $\operatorname{Re} \sqrt{\lambda_{i, n}} \rightarrow \infty$ as $n \rightarrow \infty$. Using (3.9) and (3.10) in (2.13), together with (3.13) and (3.14), and with the help of the mean value theorem, we obtain

$$
\begin{align*}
\left\|\mathbf{U}_{n}^{1}\right\|_{\mathcal{H}}^{2} & =\frac{\left(a_{1}^{2}+a_{1}^{-2}\right) a}{4}[1+o(1)]+\frac{\left(a_{1}^{-2}-a_{1}^{2}\right) a}{4}\left[\cos \left(2 \sqrt{\mu_{1, n}}(1-a)\right)+o(1)\right], n \rightarrow \infty  \tag{3.15}\\
\left\|\mathbf{U}_{n}^{0}\right\|_{\mathcal{H}}^{2} & =\frac{\left(a_{1}^{2}+a_{1}^{-2}\right) a}{4\left|\mu_{0, n}\right|}[1+o(1)]-\frac{\left(a_{1}^{-2}-a_{1}^{2}\right) a}{4\left|\mu_{0, n}\right|}\left[\cos \left(2 \sqrt{\mu_{0, n}}(1-a)\right)+o(1)\right], n \rightarrow \infty . \tag{3.16}
\end{align*}
$$

By virtue of the eigenvalue asymptotics, we know that $\left|\cos \left(2 \sqrt{\mu_{i, n}}(1-a)\right)\right| \leq 1, i=0,1$, for sufficiently large $n$. It follows from (3.15) and (3.16) that

$$
\begin{equation*}
\left\|\mathbf{U}_{n}^{i}\right\|_{\mathcal{H}}^{2}=\frac{\left(a_{1}^{2}+a_{1}^{-2}\right) a}{4\left|\mu_{0, n}\right|^{1-i}}\left[1+c_{i, n}+o(1)\right], \quad i=0,1, \quad n \rightarrow \infty \tag{3.17}
\end{equation*}
$$

where $\left\{c_{i, n}\right\}$ is a sequence bounded by a constant less than 1 .

## 4. Proofs of the uniqueness theorems

In this section, we give the proofs of Theorems 1 and 2. In order to prove the necessity in Theorem 1, we need the following proposition on the local solvability and stability of the inverse problem by the Cauchy data.

Proposition 2 (See 34]). Let $q_{1}(x)$ be a fixed complex-valued function from $L^{2}(0, a)$, and let $h \in \mathbb{C}$ be a fixed number. Denote by $\left\{K_{1}, K_{2}, \omega_{1}\right\}$ the corresponding Cauchy data. Then there exists $\varepsilon>0$ (depending only on $q_{1}$ and $h$ ) such that, for any functions $\left\{\tilde{K}_{1}, \tilde{K}_{2}\right\}$ satisfying

$$
\begin{equation*}
\Xi:=\max \left\{\left\|\tilde{K}_{1}-K_{1}\right\|_{L^{2}(0, a)},\left\|\tilde{K}_{2}-K_{2}\right\|_{L^{2}(0, a)}\right\} \leq \varepsilon \tag{4.1}
\end{equation*}
$$

there exists a unique function $\tilde{q}_{1} \in L^{2}(0, a)$ such that $\left\{\tilde{K}_{1}, \tilde{K}_{2}, \omega_{1}\right\}$ are the Cauchy data for $\tilde{q}_{1}$ and $\tilde{h}=\omega_{1}-\frac{1}{2} \int_{0}^{a} \tilde{q}_{1}(x) d x$. Moreover,

$$
\begin{equation*}
\left\|\tilde{q}_{1}-q_{1}\right\|_{L^{2}(0, a)} \leq C \Xi, \quad|\tilde{h}-h| \leq C \Xi \tag{4.2}
\end{equation*}
$$

where $C$ depends only on $q_{1}$ and $h$.

Proof of Theorem 1. Firstly, note that the given data in Inverse Problem 2 implies the unique determination of $U(t, \lambda), f(\lambda),\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ and $\left\{\tau_{n}^{i}\right\}_{n \geq 0, i=0,1}$. Due to the condition that Inverse problem 2 is solvable, assume $\left(q_{1}, h\right)$ is the solution such that $\left\{\mu_{i, n}\right\}_{n \geq 0}$ are the subspectra of the corresponding problems $B\left(d_{1}, d_{2}, q_{1}, q_{2}, h, H_{i}, a_{1}, a_{2}\right)(i=0,1)$, where $H_{0}=\infty$ and $H_{1}=H$. Let $\left\{K_{1}, K_{2}, \omega_{1}\right\}$ be the Cauchy data for $q_{1}$ and $h$.
(Sufficiency). If the system $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.13) is complete in $\mathcal{H}$, then there is at most one $\mathbf{K}(t)$ satisfying the main equations (2.15). It is obvious that $\mathbf{K}(t)=$ $\left(\overline{K_{1}(t)}, \overline{K_{2}(t)}\right)$. Note that there is at most one pair $\left(q_{1}, h\right)$ corresponding to the Cauchy data $\left\{K_{1}, K_{2}, \omega_{1}\right\}$. Hence, the sufficiency is valid.
(Necessity). If the system $\left\{\underline{\mathbf{U}}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.13) is not complete in $\mathcal{H}$, then there exists a solution $\hat{\mathbf{K}}(t)=\left(\frac{n}{\hat{K}_{1}(t)}, \frac{\hat{K}_{2}(t)}{}\right) \quad\left(\|\hat{\mathbf{K}}\|_{\mathcal{H}} \neq 0\right)$ satisfying

$$
\begin{equation*}
\left\langle\hat{\mathbf{K}}, \mathbf{U}_{n}^{i}\right\rangle=0, \quad n \geq 0, \quad i=0,1 \tag{4.3}
\end{equation*}
$$

Due to the linearity of (4.3) for $\hat{\mathbf{K}}$, we can choose $\hat{\mathbf{K}}$ such that $\|\hat{\mathbf{K}}\|_{\mathcal{H}} \leq \varepsilon$ for $\varepsilon$ from Proposition 2. Define

$$
\tilde{\mathbf{K}}:=\left(\overline{\tilde{K}_{1}}, \overline{\tilde{K}_{2}}\right), \quad \tilde{K}_{1}:=K_{1}+\hat{K}_{1}, \quad \tilde{K}_{2}:=K_{2}+\hat{K}_{2} .
$$

Then using Proposition 2, we know that there is a unique $\tilde{q}_{1} \in L^{2}(0,1)$ such that $\left\{\tilde{K}_{1}, \tilde{K}_{2}, \omega_{1}\right\}$ are the Cauchy data for $\tilde{q}_{1}$ and $\tilde{h}=\omega_{1}-\frac{1}{2} \int_{0}^{a} \tilde{q}_{1}(x) d x$. Define the functions

$$
\begin{gather*}
\tilde{\varphi}_{0}(\lambda):=\cos \rho a+\omega_{1} \frac{\sin \rho a}{\rho}-\int_{0}^{a} \tilde{K}_{1}(t) \frac{\sin \rho t}{\rho} d t \\
\tilde{\varphi}_{1}(\lambda):=-\rho \sin \rho a+\omega_{1} \cos \rho a+\int_{0}^{a} \tilde{K}_{2}(t) \cos \rho t d t \\
\tilde{\Delta}_{i}(\lambda)=\left\langle\tilde{\mathbf{K}}(\cdot), \mathbf{U}_{i}(\cdot, \lambda)\right\rangle-f_{i}(\lambda), \quad i=0,1 . \tag{4.4}
\end{gather*}
$$

It is easy to get that functions $\tilde{\Delta}_{i}(\lambda)$ defined in (4.4) have the expressions $\tilde{\Delta}_{i}(\lambda)=\tilde{\varphi}_{0}(\lambda) g_{i, 1}(\lambda)-$ $\tilde{\varphi}_{1}(\lambda) g_{i, 0}(\lambda), \quad i=0,1$. Thus, $\tilde{\Delta}_{i}(\lambda)$ are the characteristic functions of the corresponding problems $B\left(d_{1}, d_{2}, \tilde{q}_{1}, q_{2}, \tilde{h}, H_{i}, a_{1}, a_{2}\right), i=0,1$. Due to (2.15), (4.3) and (4.4), we get that $\mu_{i, n}\left(n \in \mathcal{S}_{i}\right)$ are zeros of $\tilde{\Delta}_{i}(\lambda), i=0,1$, with the corresponding multiplicities $m_{n}^{i}$. Thus, $\left\{\mu_{i, n}\right\}_{n \geq 0}$ are the subspectra of the problems $B\left(d_{1}, d_{2}, \tilde{q}_{1}, q_{2}, \tilde{h}, H_{i}, a_{1}, a_{2}\right)$, respectively, $i=0,1$. We have proved that the solution of Inverse Problem2 is not unique if $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ defined in (2.13) is not complete in $\mathcal{H}$. The proof of necessity is complete.

Now, let us prove Theorem 2 by proving two lemmas. Similar to (2.12) and (2.13), define

$$
\begin{equation*}
\mathbf{V}(t, \lambda):=\left(V_{1}(t, \lambda), V_{2}(t, \lambda)\right), \quad V_{1}(t, \lambda):=\varphi_{1}(\lambda) s(t, \lambda), \quad V_{2}(t, \lambda):=\varphi_{0}(\lambda) c(t, \lambda) \tag{4.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{V}_{n+\nu}^{i}(t):=V^{\langle\nu\rangle}\left(t, \mu_{i, n}\right), \quad n \in \mathcal{S}_{i}, \quad \nu=\overline{0, m_{n}^{i}-1}, \quad i=0,1 . \tag{4.6}
\end{equation*}
$$

Lemma 1. The system $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is complete in $\mathcal{H}$ if and only if $\left\{\mathbf{V}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is complete in $\mathcal{H}$.

Proof. Let $\mathbf{h}=\left(\overline{h_{1}}, \overline{h_{2}}\right) \in \mathcal{H}$. It is sufficient to show that

$$
\begin{equation*}
\left\langle\mathbf{h}, \mathbf{U}_{n}^{i}\right\rangle=0, \quad n \geq 0, \quad i=0,1 \tag{4.7}
\end{equation*}
$$

is equivalent to

$$
\begin{equation*}
\left\langle\mathbf{h}, \mathbf{V}_{n}^{i}\right\rangle=0, \quad n \geq 0, \quad i=0,1 \tag{4.8}
\end{equation*}
$$

Consider the functions

$$
\begin{gather*}
G_{i}(\lambda):=\left\langle\mathbf{h}(\cdot), \mathbf{U}_{i}(\cdot, \lambda)\right\rangle=\int_{0}^{a}\left(h_{1}(t) g_{i, 1}(\lambda) s(t, \lambda)+h_{2}(t) g_{i, 0}(\lambda) c(t, \lambda)\right) d t, \quad i=0,1  \tag{4.9}\\
F(\lambda):=\langle\mathbf{h}(\cdot), \mathbf{V}(\cdot, \lambda)\rangle=\int_{0}^{a}\left(h_{1}(t) \varphi_{1}(\lambda) s(t, \lambda)+h_{2}(t) \varphi_{0}(\lambda) c(t, \lambda)\right) d t \tag{4.10}
\end{gather*}
$$

By the definitions of $\mathbf{U}_{n}^{i}(t)$ and $\mathbf{V}_{n}^{i}(t)$, we know that (4.7) and (4.8) are equivalent to

$$
\begin{equation*}
G_{i}^{\langle\nu\rangle}\left(\mu_{i, n}\right)=0, \quad n \in \mathcal{S}_{i}, \quad \nu=\overline{0, m_{n}^{i}-1}, \quad i=0,1 \tag{4.11}
\end{equation*}
$$

and

$$
\begin{equation*}
F^{\langle\nu\rangle}\left(\mu_{i, n}\right)=0, \quad n \in \mathcal{S}_{i}, \quad \nu=\overline{0, m_{n}^{i}-1}, \quad i=0,1, \tag{4.12}
\end{equation*}
$$

respectively. Therefore, it is sufficient to show that (4.11) is equivalent to (4.12). Let us first prove that (4.11) implies (4.12). Note that $\varphi_{0}(\lambda)$ and $\varphi_{1}(\lambda)$ have no common zero, and so do $g_{i, 0}(\lambda)$ and $g_{i, 1}(\lambda), i=0,1$. From (2.2) and Proposition A.1, we have

$$
\begin{equation*}
\varphi_{j}^{\langle\nu\rangle}\left(\mu_{i, n}\right)=\sum_{k=0}^{\nu} M_{n, k}^{i} g_{i, j}^{\langle\nu-k\rangle}\left(\mu_{i, n}\right), \quad n \in \mathcal{S}_{i}, \quad \nu=\overline{0, m_{n}^{i}-1}, \quad j=0,1 \tag{4.13}
\end{equation*}
$$

where $M_{n, \nu}^{i}$ are constants. Using (4.9) and (4.13), we obtain

$$
\begin{aligned}
& F^{\langle\nu\rangle}\left(\mu_{i, n}\right)=\sum_{j=0}^{\nu} \int_{0}^{a}\left[h_{1}(t) \varphi_{1}^{\langle j\rangle}\left(\mu_{i, n}\right) s^{\langle\nu-j\rangle}\left(t, \mu_{i, n}\right)+h_{2}(t) \varphi_{0}^{\langle j\rangle}\left(\mu_{i, n}\right) c^{\langle\nu-j\rangle}\left(t, \mu_{i, n}\right)\right] d t \\
& =\sum_{j=0}^{\nu} \sum_{l=0}^{j} M_{n, l}^{i} \int_{0}^{a}\left[h_{1}(t) g_{i, 1}^{\langle j-l\rangle}\left(\mu_{i, n}\right) s^{\langle\nu-j\rangle}\left(t, \mu_{i, n}\right)+h_{2}(t) g_{i, 0}^{\langle j-l\rangle}\left(\mu_{i, n}\right) c^{\langle\nu-j\rangle}\left(t, \mu_{i, n}\right)\right] d t \\
& =\sum_{l=0}^{\nu} M_{n, l}^{i} \sum_{j=0}^{\nu-l} \int_{0}^{a}\left[h_{1}(t) g_{i, 1}^{\langle j\rangle}\left(\mu_{i, n}\right) s^{\langle\nu-l-j\rangle}\left(t, \mu_{i, n}\right)+h_{2}(t) g_{i, 0}^{\langle j\rangle}\left(\mu_{i, n}\right) c^{\langle\nu-l-j\rangle}\left(t, \mu_{i, n}\right)\right] d t \\
& =\sum_{l=0}^{\nu} M_{n, l}^{i} G_{i}^{\langle\nu-l\rangle}\left(\mu_{i, n}\right)=0 .
\end{aligned}
$$

Similarly, one can also prove that (4.12) implies (4.11).
Lemma 2. If the system $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is complete in $L^{2}(0,2 a)$, then $\left\{\mathbf{V}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is complete in $\mathcal{H}$.

Proof. Substituting (2.5) and (2.6) into (4.5), we get

$$
\begin{equation*}
V_{1}(t, \lambda)=v_{1}(t, \lambda)+O\left(\frac{e^{2 a|\operatorname{Im} \rho|}}{|\rho|}\right), \quad V_{2}(t, \lambda)=v_{2}(t, \lambda)+O\left(\frac{e^{2 a|\operatorname{Im} \rho|}}{|\rho|}\right), \quad|\rho| \rightarrow \infty \tag{4.14}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{1}(t, \lambda)=\frac{1}{2}[\cos \rho(a+t)-\cos \rho(a-t)], \quad v_{2}(t, \lambda)=\frac{1}{2}[\cos \rho(a-t)+\cos \rho(a+t)] . \tag{4.15}
\end{equation*}
$$

Substituting (2.5) and (2.6) into (4.10) and taking (4.14) and (4.15) into account, we obtain

$$
\begin{equation*}
F(\lambda)=\int_{0}^{2 a} b_{0}(t) \cos \rho t d t+F_{1}(\rho) \tag{4.16}
\end{equation*}
$$

where

$$
b_{0}(t)= \begin{cases}\frac{h_{1}(a-t)+h_{2}(a-t)}{2}, & 0<t<a \\ \frac{h_{2}(t-a)-h_{1}(t-a)}{2}, & a<t<2 a\end{cases}
$$

and the function $F_{1}(\rho)$ is an even and entire function of exponential type $\leq 2 a$, moreover,

$$
F_{1}(\rho)=O\left(|\rho|^{-1}\right), \quad|\rho| \rightarrow \infty, \quad \rho \in \mathbb{R}
$$

It follows from the Paley-Wiener Theorem and the evenness of $F_{1}(\rho)$ that there exists $b_{1}(t)$ in $L^{2}(0,2 a)$ such that

$$
\begin{equation*}
F(\lambda)=\int_{0}^{2 a}\left[b_{0}(t)+b_{1}(t)\right] \cos \rho t d t \tag{4.17}
\end{equation*}
$$

Let $\mathbf{h}=\left(\overline{h_{1}}, \overline{h_{2}}\right) \in \mathcal{H}$ such that (4.8) holds, or equivalently, (4.12) holds. From (4.17), (2.16) and the completeness of $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$, we get that $b_{0}(t)+b_{1}(t)=0$ in $L^{2}(0,2 a)$. Therefore, $F(\lambda) \equiv 0$. Using (4.10) and Proposition A.2 in Appendix, we conclude that $h_{2}(t)=0$ and $h_{1}(t)=0$ in $L^{2}(0, a)$.
Proof of Theorem 2. Using Lemmas 1 and 2, together with Theorem 1, we immediately finish the proof of Theorem 2 .

In the end of this section, let us prove the following lemma, which indicates that the Rieszbasicity $\mathcal{H}$ for the functional sequence $\left\{\sqrt{\left(\left|\mu_{0, n}\right|+1\right)} \mathbf{U}_{n}^{0}(t)\right\}_{n \geq 0} \cup\left\{\mathbf{U}_{n}^{1}(t)\right\}_{n \geq 0}$, constructed in (2.13), is reasonable. This lemma is also useful in the proof of Corollary 2.

Lemma 3. If the system $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is a Riesz basis in $L^{2}(0,2 a)$ then: (i) $\left\{\mathbf{V}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is a Riesz basis in $\mathcal{H}$; (ii) $\left\{\sqrt{\left(\left|\mu_{0, n}\right|+1\right)} \mathbf{U}_{n}^{0}(t)\right\}_{n \geq 0} \cup\left\{\mathbf{U}_{n}^{1}(t)\right\}_{n \geq 0}$ is a Riesz basis in $\mathcal{H}$.
Proof. (i) Note that $m_{n}^{i}=1$ for $n \geq n_{i}$ for some large $n_{i}$. Let $\left\{\alpha_{n}\right\}_{n \geq 0}$ be the numbers such that $\alpha_{k} \neq \alpha_{l}$ and $\alpha_{k} \neq \overline{\alpha_{l}}$ for all $k \neq l$ and

$$
\left\{\alpha_{n}\right\}_{n \geq n_{0}+n_{1}}=\left\{\sqrt{\mu_{0, n}}\right\}_{n \geq n_{0}} \cup\left\{\sqrt{\mu_{1, n}}\right\}_{n \geq n_{1}}
$$

Since $\left\{c_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is a Riesz basis in $L^{2}(0,2 a)$, then $\left\{\cos \alpha_{n} t\right\}_{n \geq 0}$ is also a Riesz basis in $L^{2}(0,2 a)$. From Proposition A. 3 in Appendix, we have that $\left\{\left(v_{1}\left(t, \alpha_{n}^{2}\right), v_{2}\left(t, \alpha_{n}^{2}\right)\right)\right\}_{n \geq 0}$ is a Riesz basis in $\mathcal{H}$, where $v_{j}(t, \lambda)(j=1,2)$ are defined in (4.15). By (4.14) and (4.6), we know that $\left\{\mathbf{V}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is quadratically close to $\left\{\left(v_{1}\left(t, \alpha_{n}^{2}\right), v_{2}\left(t, \alpha_{n}^{2}\right)\right)\right\}_{n \geq 0}$. Using Proposition 1.8.5 in [13], together with Lemma 2, we conclude that $\left\{\mathbf{V}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is a Riesz basis in $\mathcal{H}$.
(ii) Using Proposition A.1, together with the definitions of $\mathbf{U}_{n}^{i}$ and $\mathbf{V}_{n}^{i}$, we obtain that $\mathbf{U}_{n}^{i}(t)=C_{n}^{i} \mathbf{V}_{n}^{i}(t)$ for $n \geq n_{i}$, where $C_{n}^{i}$ are nonzero constants. Note that $\left\{\mathbf{U}_{n}^{i}(t)\right\}_{n \geq 0, i=0,1}$ is complete in $\mathcal{H}$ by (i) and Lemma 1. In view of (3.17), we get the conclusion (ii).

## 5. Proofs of the local solvability and stability

In this section, we prove Theorem 3 as well as Corollaries 1 and 2, For $i=0,1$, let $\left\{\mu_{i, n}\right\}_{n \geq 0}$ be a fixed subspectrum of the problem $B_{i}$. We shall use the data $\left\{\tilde{\mu}_{i, n}\right\}_{n \geq 0}, \tilde{q}_{2}, \tilde{H}_{1}, a_{1}, a_{2}$ and $\omega_{1}$ to construct $\tilde{q}_{1}$ and $\tilde{h}$ by Algorithm $\mathbb{1}$. We agree that, if a certain symbol $\delta_{i}$ denotes an object related to the problem $B_{i}$, then $\tilde{\delta}_{i}$ will denote an analogous object related to the sequence $\left\{\tilde{\mu}_{i, n}\right\}_{n \geq 0}, \tilde{q}_{2}$ and $\tilde{H}_{1}$. In Theorem 3, we assume $H_{1}=\tilde{H}_{1}$. The notation $C$ may stand for different positive constants.

Let $\tilde{\psi}_{i}(x, \lambda)$ be the solution of the equation $-y^{\prime \prime}+\tilde{q}_{2}(x) y=\lambda y$ under the initial conditions

$$
\tilde{\psi}_{0}(0, \lambda)=0, \tilde{\psi}_{0}^{\prime}(0, \lambda)=1, \quad \tilde{\psi}_{1}(0, \lambda)=1, \tilde{\psi}_{1}^{\prime}(0, \lambda)=\tilde{H}_{1}
$$

Lemma 4. (i) If (2.18) is fulfilled for some $\varepsilon \leq 1$, then there exist functions $W_{0}^{i} \in L^{2}\left(0, d_{2}\right)(i=$ $0,1)$ such that

$$
\begin{align*}
& \tilde{\psi}_{0}\left(d_{2}, \lambda\right)-\psi_{0}\left(d_{2}, \lambda\right)=\frac{1}{\rho^{2}} \int_{0}^{d_{2}} W_{0}^{0}(t) \cos \rho t d t  \tag{5.1}\\
& \tilde{\psi}_{0}^{\prime}\left(d_{2}, \lambda\right)-\psi_{0}^{\prime}\left(d_{2}, \lambda\right)=\frac{1}{\rho} \int_{0}^{d_{2}} W_{0}^{1}(t) \sin \rho t d t \tag{5.2}
\end{align*}
$$

Moreover, $\left\|W_{0}^{i}\right\|_{L^{2}\left(0, d_{2}\right)} \leq C Q, i=0,1$, where $Q$ is defined in (2.18) and the constant $C>0$ depends only on $\left\|q_{2}\right\|_{L^{2}\left(0, d_{2}\right)}$.
(ii) If (2.21) is fulfilled for some $\varepsilon \leq 1$, then there exist functions $W_{1}^{i} \in L^{2}\left(0, d_{2}\right)(i=0,1)$ such that

$$
\begin{align*}
& \tilde{\psi}_{1}\left(d_{2}, \lambda\right)-\psi_{1}\left(d_{2}, \lambda\right)=\frac{1}{\rho} \int_{0}^{d_{2}} W_{1}^{0}(t) \sin \rho t d t  \tag{5.3}\\
& \tilde{\psi}_{1}^{\prime}\left(d_{2}, \lambda\right)-\psi_{1}^{\prime}\left(d_{2}, \lambda\right)=\int_{0}^{d_{2}} W_{1}^{1}(t) \cos \rho t d t \tag{5.4}
\end{align*}
$$

Moreover, $\left\|W_{1}^{i}\right\|_{L^{2}\left(0, d_{2}\right)} \leq C Q_{1}, \quad i=0,1$, where $Q_{1}$ is defined in (2.21) and the constant $C>0$ depends only on $\left\|q_{2}\right\|_{L^{2}\left(0, d_{2}\right)}$ and $\left|H_{1}\right|$.

Proof. According to the theory of transformation operators [25, 35], there are functions $M^{0}(x, t)$ and $M^{1}(x, t)$, having the first partial derivatives, such that

$$
\begin{equation*}
\tilde{\psi}_{0}(x, \lambda)=\psi_{0}(x, \lambda)+\int_{0}^{x} M^{0}(x, t) \psi_{0}(t, \lambda) d t, \quad \psi_{0}(x, \lambda)=\frac{\sin \rho x}{\rho}+\int_{0}^{x} M^{1}(x, t) \frac{\sin \rho t}{\rho} d t \tag{5.5}
\end{equation*}
$$

Moreover, $M^{i}(x, t)(i=0,1)$ satisfy that $M_{x}^{i}(x, \cdot), M_{t}^{i}(x, \cdot) \in L^{2}(0, x)$ for each fixed $x \in\left[0, d_{2}\right]$ and

$$
\begin{equation*}
M^{0}(x, x)=\frac{1}{2} \int_{0}^{x}\left[\tilde{q}_{2}(t)-q_{2}(t)\right] d t, M^{1}(x, x)=\frac{1}{2} \int_{0}^{x} q_{2}(t) d t, M^{i}(x, 0)=0 \tag{5.6}
\end{equation*}
$$

Substituting the second equation in (5.5) into the first one, integrating (5.5) by parts, and taking (5.6) and the first equation in (2.18) into account, we obtain

$$
\begin{align*}
\tilde{\psi}_{0}\left(d_{2}, \lambda\right)-\psi_{0}\left(d_{2}, \lambda\right)= & \frac{1}{\rho^{2}} \int_{0}^{d_{2}} M^{0}\left(d_{2}, x\right)\left[-M^{1}(x, x) \cos \rho x+\int_{0}^{x} M_{t}^{1}(x, t) \cos \rho t d t\right] d x \\
& +\frac{1}{\rho^{2}} \int_{0}^{d_{2}} M_{t}^{0}\left(d_{2}, t\right) \cos \rho t d t \\
= & \frac{1}{\rho^{2}} \int_{0}^{d_{2}} W_{0}^{0}(t) \cos \rho t d t \tag{5.7}
\end{align*}
$$

with

$$
\begin{equation*}
W_{0}^{0}(t)=M_{t}^{0}\left(d_{2}, t\right)-M^{0}\left(d_{2}, t\right) M^{1}(t, t)+\int_{t}^{d_{2}} M^{0}\left(d_{2}, s\right) M_{t}^{1}(s, t) d s \tag{5.8}
\end{equation*}
$$

Using the estimates for $M^{i}(x, t)$ (see, e.g., Proposition 2.1 in [35]), we have that, for each fixed $x \in\left[0, d_{2}\right]$, there hold

$$
\begin{equation*}
\max _{x \geq t \geq 0}\left|M^{0}(x, t)\right| \leq C Q,\left\|M_{x}^{0}(x, \cdot)\right\|_{L^{2}(0, x)} \leq C Q,\left\|M_{t}^{0}(x, \cdot)\right\|_{L^{2}(0, x)} \leq C Q \tag{5.9}
\end{equation*}
$$

where the positive constant $C$ depends only on the sum of $\left\|q_{2}\right\|_{L^{2}\left(0, d_{2}\right)}$ and $\left\|\tilde{q}_{2}\right\|_{L^{2}\left(0, d_{2}\right)}$. Note that $\left\|\tilde{q}_{2}\right\|_{L^{2}\left(0, d_{2}\right)} \leq\left\|q_{2}\right\|_{L^{2}\left(0, d_{2}\right)}+Q$ and, by (2.18), $Q \leq \varepsilon \leq 1$. Therefore, we have that the constant $C$ depends only on $\left\|q_{2}\right\|_{L^{2}\left(0, d_{2}\right)}$. Using (5.9) in (5.8), we get that $\left\|W_{0}^{0}\right\|_{L^{2}\left(0, d_{2}\right)} \leq C Q$. Similarly, using (5.5) and (5.9), we arrive at (5.2) with

$$
\begin{equation*}
W_{0}^{1}(t)=M_{x}^{0}\left(d_{2}, t\right)+\int_{t}^{d_{2}} M_{x}^{0}\left(d_{2}, s\right) M^{1}(s, t) d s, \quad\left\|W_{0}^{1}\right\|_{L^{2}\left(0, d_{2}\right)} \leq C Q \tag{5.10}
\end{equation*}
$$

Note that there are also functions $N^{0}(x, t)$ and $N^{1}(x, t)$, having the first partial derivatives, such that

$$
\begin{equation*}
\tilde{\psi}_{1}(x, \lambda)=\psi_{1}(x, \lambda)+\int_{0}^{x} N^{0}(x, t) \psi_{1}(t, \lambda) d t, \quad \psi_{1}(x, \lambda)=\cos \rho x+\int_{0}^{x} N^{1}(x, t) \cos \rho t d t \tag{5.11}
\end{equation*}
$$

Moreover, $N^{i}(x, t)(i=0,1)$ satisfy $N_{x}^{i}(x, \cdot), N_{t}^{i}(x, \cdot) \in L^{2}(0, x)$ for each fixed $x \in\left[0, d_{2}\right]$ and

$$
\begin{equation*}
N^{0}(x, x)=\tilde{H}_{1}-H_{1}+\frac{1}{2} \int_{0}^{x}\left[\tilde{q}_{2}(t)-q_{2}(t)\right] d t, N^{1}(x, x)=H_{1}+\frac{1}{2} \int_{0}^{x} q_{2}(t) d t \tag{5.12}
\end{equation*}
$$

Similarly, using the estimates for $N^{0}(x, t)$ from Proposition 2.1 in [16], we get that, for each fixed $x \in\left[0, d_{2}\right]$, there hold

$$
\begin{equation*}
\max _{x \geq t \geq 0}\left|N^{0}(x, t)\right| \leq C Q_{1},\left\|N_{x}^{0}(x, \cdot)\right\|_{L^{2}(0, x)} \leq C Q_{1},\left\|N_{t}^{0}(x, \cdot)\right\|_{L^{2}(0, x)} \leq C Q_{1} \tag{5.13}
\end{equation*}
$$

where the positive constant $C$ depends only on the sum of $\left\|q_{2}\right\|_{L^{2}\left(0, d_{2}\right)}$ and $\left|H_{1}\right|$, since $Q_{1} \leq$ $\varepsilon \leq 1$ by (2.21). Using (5.11), integrating by parts, and taking (5.12), (5.13) and the first equation in (2.21) into account, we can get (5.3) and (5.4) with

$$
\begin{equation*}
W_{1}^{0}(t)=N^{0}\left(d_{2}, t\right) N^{1}(t, t)-N_{t}^{0}\left(d_{2}, t\right)-\int_{t}^{d_{2}} N^{0}\left(d_{2}, s\right) N_{t}^{1}(s, t) d s, \quad\left\|W_{1}^{0}\right\|_{L^{2}\left(0, d_{2}\right)} \leq C Q_{1} \tag{5.14}
\end{equation*}
$$

and

$$
\begin{equation*}
W_{1}^{1}(t)=N_{x}^{0}\left(d_{2}, t\right)+\int_{t}^{d_{2}} N_{x}^{0}\left(d_{2}, s\right) N^{1}(s, t) d s, \quad\left\|W_{1}^{1}\right\|_{L^{2}\left(0, d_{2}\right)} \leq C Q_{1} \tag{5.15}
\end{equation*}
$$

respectively. The proof is complete.
Denote

$$
\begin{equation*}
\tilde{g}_{i, 0}(\lambda):=-a_{1}^{-1} \tilde{\psi}_{i}\left(d_{2}, \lambda\right), \quad \tilde{g}_{i, 1}(\lambda):=a_{1} \tilde{\psi}_{i}^{\prime}\left(d_{2}, \lambda\right)+a_{2} \tilde{\psi}_{i}\left(d_{2}, \lambda\right) \tag{5.16}
\end{equation*}
$$

Similarly to (2.10) $-(2.12)$, define

$$
\begin{align*}
\tilde{\mathbf{U}}_{i}(t, \lambda) & :=\left(\tilde{U}_{i, 1}(t, \lambda), \tilde{U}_{i, 2}(t, \lambda)\right), \quad \tilde{U}_{i, 1}(t, \lambda):=\tilde{g}_{i, 1}(\lambda) s(t, \lambda), \quad \tilde{U}_{i, 2}(t, \lambda):=\tilde{g}_{i, 0}(\lambda) c(t, \lambda) \\
\tilde{f}_{i}(\lambda) & :=\left[\cos \rho a+\omega_{1} \frac{\sin \rho a}{\rho}\right] \tilde{g}_{i, 1}(\lambda)-\tilde{g}_{i, 0}(\lambda)\left[\omega_{1} \cos \rho a-\rho \sin \rho a\right], \quad i=0,1 \tag{5.17}
\end{align*}
$$

Lemma 5. If the condition (2.18) is fulfilled for some $\varepsilon \leq 1$, then

$$
\begin{gather*}
\sum_{n \in \mathcal{S}_{0}}\left(\left|\mu_{0, n}\right|+1\right) \sum_{\nu=0}^{m_{n}^{0}}\left\|\left(\tilde{\mathbf{U}}_{0}^{\langle\nu\rangle}-\mathbf{U}_{0}^{\langle\nu\rangle}\right)\left(t, \mu_{0, n}\right)\right\|_{\mathcal{H}}^{2}+\sum_{n \in \mathcal{S}_{1}} \sum_{\nu=0}^{m_{n}^{1}}\left\|\left(\tilde{\mathbf{U}}_{1}^{\langle\nu\rangle}-\mathbf{U}_{1}^{\langle\nu\rangle}\right)\left(t, \mu_{1, n}\right)\right\|_{\mathcal{H}}^{2} \leq C Q^{2}  \tag{5.19}\\
\sum_{n \in \mathcal{S}_{0}}\left(\left|\mu_{0, n}\right|+1\right) \sum_{\nu=0}^{m_{n}^{0}}\left|\left(\tilde{f}_{0}^{\langle\nu\rangle}-f_{0}^{\langle\nu\rangle}\right)\left(\mu_{0, n}\right)\right|^{2}+\sum_{n \in \mathcal{S}_{1}} \sum_{\nu=0}^{m_{n}^{1}}\left|\left(\tilde{f}_{1}^{\langle\nu\rangle}-f_{1}^{\langle\nu\rangle}\right)\left(\mu_{1, n}\right)\right|^{2} \leq C Q^{2} \tag{5.20}
\end{gather*}
$$

where $C$ depends only on the problem $B_{1}$.
Proof. Using Lemma 4 with $\tilde{H}_{1}=H_{1}$, together with definitions of $U_{i, j}(t, \lambda), \tilde{U}_{i, j}(t, \lambda), f_{i}(\lambda)$ and $\tilde{f}_{i}(\lambda)$, we have that for $\nu \geq 0$ and $i=0,1$ there hold

$$
\begin{array}{r}
\left|\tilde{U}_{i, j}^{\langle\nu\rangle}(t, \lambda)-U_{i, j}^{\langle\nu\rangle}(t, \lambda)\right| \leq C Q \frac{e^{|\operatorname{Im} \rho|}}{(|\rho|+1)^{\nu+2-i}}, \quad j=1,2 \\
\left|\tilde{f}_{i}^{\langle\nu\rangle}(\lambda)-f_{i}^{\langle\nu\rangle}(\lambda)\right| \leq C \frac{e^{|\operatorname{Im} \rho| a}}{|\rho|^{\mid \nu+1-i}}\left|\int_{-d_{2}}^{d_{2}} W_{i}(t) e^{\mathrm{i} \rho t} d t\right|, \quad\left\|W_{i}\right\|_{L^{2}\left(-d_{2}, d_{2}\right)} \leq C Q \tag{5.22}
\end{array}
$$

where $C$ depends only on $\|q\|_{L^{2}\left(0, d_{2}\right)}$. Note that $m_{n}^{i}=1$ for $n \geq n_{i}, i=0,1$. In particular, $\left|\sqrt{\mu_{i, m}}-\sqrt{\mu_{i, n}}\right| \geq c_{0}>0$ for $m, n \geq n_{i}$ and $m \neq n$, and $\left|\operatorname{Im} \sqrt{\mu_{i, n}}\right| \leq c_{1}<\infty$. Using the estimate (5.21) in (2.11) and (5.17), we obtain (5.19). Using (5.22), together with Proposition A.4, we get (5.20).

Note that multiplicities of $\mu_{i, n}$ and $\tilde{\mu}_{i, n}$ may be distinct. However, by virtue of (2.17), we have that $\mathcal{S}_{i} \subseteq \tilde{\mathcal{S}}_{i}$ for sufficiently small $\varepsilon>0$. In particular, $\tilde{m}_{n}^{i}=1$ for $n \geq n_{i}$. Denote

$$
\begin{equation*}
\tilde{\mathbf{U}}_{n+\nu}^{i}(t):=\tilde{\mathbf{U}}_{i}^{\langle\nu\rangle}\left(t, \tilde{\mu}_{i, n}\right), \quad \tilde{\tau}_{n+\nu}^{i}:=\tilde{f}_{i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right), \quad n \in \tilde{\mathcal{S}}_{i}, \quad \nu=\overline{0, \tilde{m}_{n}^{i}-1}, \quad i=0,1 \tag{5.23}
\end{equation*}
$$

Consider the system of equations

$$
\begin{equation*}
\left\langle\tilde{\mathbf{K}}(\cdot), \tilde{\mathbf{U}}_{n}^{i}(\cdot)\right\rangle=\tilde{\tau}_{n}^{i}, \quad n \geq 0, \quad i=0,1, \tag{5.24}
\end{equation*}
$$

where $\tilde{\mathbf{K}}=\left(\tilde{K}_{1}, \tilde{K}_{2}\right)$ is the unknown element in $\mathcal{H}$.
For each $i=0,1$, fix $k \in\left[0, n_{i}\right) \cap \mathcal{S}_{i}$, and assume that the value $\mu_{i, k}$ with the multiplicity $m_{k}^{i}$ corresponds to the numbers $\left\{\tilde{\mu}_{i, n}\right\}_{n \in M_{k}^{i}}$, where $M_{k}^{i}:=\left\{k, k+1, \cdots, k+m_{k}^{i}-1\right\}$. Define $\tilde{\mathcal{S}}_{k}^{i}:=\tilde{\mathcal{S}}_{i} \cap M_{k}^{i}$. Then the relation (5.24) for $n \in \tilde{\mathcal{S}}_{k}^{i}$ can be rewritten as

$$
\begin{equation*}
\left\langle\tilde{\mathbf{K}}(\cdot), \tilde{\mathbf{U}}_{i}^{\langle\nu\rangle}\left(\cdot, \tilde{\mu}_{i, n}\right)\right\rangle=\tilde{f}_{i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right), \quad n \in \tilde{\mathcal{S}}_{k}^{i}, \quad \nu=\overline{0, \tilde{m}_{n}^{i}-1}, \quad i=0,1 \tag{5.25}
\end{equation*}
$$

For each fixed $t \in[0, a]$ and $i=0,1$, let $\tilde{E}_{k, i, j}(t, \lambda), \tilde{F}_{k, i}(\lambda)$ be the unique polynomials of degree at most $m_{k}^{i}-1$, respectively, interpolating $\tilde{U}_{i, j}(t, \lambda)(j=1,2)$ and $\tilde{f}_{i}(\lambda)$ and their derivatives in the usual way at the points $\left\{\tilde{\mu}_{i, n}\right\}_{n \in M_{k}^{i}}$. Namely,

$$
\begin{equation*}
\tilde{E}_{k, i, j}^{\langle\nu\rangle}\left(t, \tilde{\mu}_{i, n}\right)=\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \tilde{\mu}_{i, n}\right), \quad \tilde{F}_{k, i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right)=\tilde{f}_{i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right), \quad n \in \tilde{\mathcal{S}}_{k}^{i}, \quad \nu=\overline{0, \tilde{m}_{n}^{i}-1} \tag{5.26}
\end{equation*}
$$

Denote $\tilde{\mathbf{E}}_{k, i}(t, \lambda):=\left(\tilde{E}_{k, i, 1}(t, \lambda), \tilde{E}_{k, i, 2}(t, \lambda)\right)$. Eqs. (5.25) and (5.26) imply that

$$
\begin{equation*}
\left\langle\tilde{\mathbf{K}}(\cdot), \tilde{\mathbf{E}}_{k, i}^{\langle\nu\rangle}\left(\cdot, \tilde{\mu}_{i, n}\right)\right\rangle=\tilde{F}_{k, i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right), \quad n \in \tilde{\mathcal{S}}_{k}^{i}, \quad \nu=\overline{0, \tilde{m}_{n}^{i}-1}, \quad i=0,1 . \tag{5.27}
\end{equation*}
$$

Since $\tilde{E}_{k, i, j}(t, \lambda), \tilde{F}_{k, i}(\lambda)$ are the polynomials of degree at most $m_{k}^{i}-1$, we have

$$
\begin{equation*}
\left\langle\tilde{\mathbf{K}}(\cdot), \tilde{\mathbf{E}}_{k, i}(\cdot, \lambda)\right\rangle=\tilde{F}_{k, i}(\lambda), \quad \lambda \in \mathbb{C}, \quad n \in \tilde{\mathcal{S}}_{k}^{i}, \quad \nu=\overline{0, \tilde{m}_{n}^{i}-1}, \quad i=0,1 \tag{5.28}
\end{equation*}
$$

In particular, we have

$$
\begin{equation*}
\left\langle\tilde{\mathbf{K}}(\cdot), \tilde{\mathbf{E}}_{k, i}^{\langle\nu\rangle}\left(\cdot, \mu_{i, k}\right)\right\rangle=\tilde{F}_{k, i}^{\langle\nu\rangle}\left(\mu_{i, k}\right), \quad \nu=\overline{0, m_{k}^{i}-1}, \quad i=0,1 \tag{5.29}
\end{equation*}
$$

Define the sequences $\left\{\tilde{\tilde{\mathbf{U}}}_{n}^{i}\right\}_{n \geq 0}$ and $\left\{\tilde{\tilde{\tau}}_{n}^{i}\right\}_{n \geq 0}$ for $i=0,1$ as follows

$$
\left\{\begin{array}{l}
\tilde{\tilde{\mathbf{U}}}_{n+\nu}^{i}(t)=\tilde{\mathbf{E}}_{n, i}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right), \quad \tilde{\tilde{\tau}}_{n+\nu}^{i}=\tilde{F}_{n, i}^{\langle\nu\rangle}\left(\mu_{i, n}\right), n \in \mathcal{S}_{i} \cap\left[0, n_{i}\right), \nu=\overline{0, m_{n}^{i}-1}  \tag{5.30}\\
\tilde{\tilde{\mathbf{U}}}_{n}^{i}(t)=\tilde{\mathbf{U}}_{n}^{i}(t), \quad \tilde{\tilde{\tau}}_{n}^{i}=\tilde{\tau}_{n}^{i}, \quad n \geq n_{i}
\end{array}\right.
$$

Then the system (5.24) is equivalent to

$$
\begin{equation*}
\left\langle\tilde{\mathbf{K}}(\cdot), \tilde{\mathbf{U}}_{n}^{i}(\cdot)\right\rangle=\tilde{\tilde{\tau}}_{n}^{i}, \quad n \geq 0, \quad i=0,1 \tag{5.31}
\end{equation*}
$$

Lemma 6. There exists $\varepsilon>0$ such that, if (2.17) and (2.18) are fulfilled, then the following estimates hold

$$
\begin{gather*}
\sqrt{\sum_{n \geq 0}\left(\left(\left|\mu_{0, n}\right|+1\right)\left\|\mathbf{U}_{n}^{0}-\tilde{\tilde{\mathbf{U}}}_{n}^{0}\right\|_{\mathcal{H}}^{2}+\left\|\mathbf{U}_{n}^{1}-\tilde{\tilde{\mathbf{U}}}_{n}^{1}\right\|_{\mathcal{H}}^{2}\right)}<C(\Lambda+Q),  \tag{5.32}\\
\sqrt{\sum_{n \geq 0}\left(\left(\left|\mu_{0, n}\right|+1\right)\left|\tau_{n}^{0}-\tilde{\tilde{\tau}}_{n}^{0}\right|^{2}+\left|\tau_{n}^{1}-\tilde{\tilde{\tau}}_{n}^{1}\right|^{2}\right)}<C(\Lambda+Q) . \tag{5.33}
\end{gather*}
$$

Proof. From the theory of the transformation operators [25], we know that

$$
\begin{equation*}
\left|\psi_{i}^{\langle\nu\rangle}(x, \lambda)\right| \leq C \frac{e^{|\operatorname{Im} \rho| x}}{(|\rho|+1)^{\nu+1-i}}, \quad\left|\psi_{i}^{\prime\langle\nu}(x, \lambda)\right| \leq C \frac{e^{|\operatorname{Im} \rho| x}}{(|\rho|+1)^{\nu-i}}, \quad \nu \geq 0 \tag{5.34}
\end{equation*}
$$

Using (5.34) in (2.12) and (2.10), together with the definitions of $g_{i, j}(\lambda)$, we obtain the estimates

$$
\begin{gather*}
\left|U_{i, j}^{\langle\nu\rangle}(t, \lambda)\right| \leq C \frac{e^{|\operatorname{Im} \rho|}}{(|\rho|+1)^{\nu+1-i}}, \quad \nu \geq 0, \quad i=0,1, \quad j=1,2  \tag{5.35}\\
\left|f_{i}^{\langle\nu\rangle}(\lambda)\right| \leq C \frac{e^{|\operatorname{Im} \rho|}}{(|\rho|+1)^{\nu-i}}, \quad \nu \geq 0, \quad i=0,1 \tag{5.36}
\end{gather*}
$$

In view of (5.21) and (5.22), and noting $Q \leq \varepsilon \leq 1$, we also have

$$
\begin{gather*}
\left|\tilde{U}_{i, j}^{\langle\nu\rangle}(t, \lambda)\right| \leq C \frac{e^{|\operatorname{Im} \rho|}}{(|\rho|+1)^{\nu+1-i}}, \quad \nu \geq 0, \quad i=0,1, \quad j=1,2  \tag{5.37}\\
\left|\tilde{f}_{i}^{\langle\nu\rangle}(\lambda)\right| \leq C \frac{e^{|\operatorname{Im} \rho|}}{(|\rho|+1)^{\nu-i}}, \quad \nu \geq 0, \quad i=0,1 \tag{5.38}
\end{gather*}
$$

Since

$$
\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right)-\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \tilde{\mu}_{i, n}\right)=(\nu+1) \int_{\tilde{\mu}_{i, n}}^{\mu_{i, n}} \tilde{U}_{i, j}^{\langle\nu+1\rangle}(t, \mu) d \mu, \quad \nu \geq 0, \quad i=0,1, \quad j=1,2
$$

it follows from (5.37) that

$$
\begin{equation*}
\left|\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right)-\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \tilde{\mu}_{i, n}\right)\right| \leq C \frac{\left|\mu_{i, n}-\tilde{\mu}_{i, n}\right|}{\left(\sqrt{\left|\mu_{i, n}\right|}+1\right)^{\nu+2-i}}, \quad \nu \geq 0, \quad i=0,1, \quad j=1,2 \tag{5.39}
\end{equation*}
$$

Similarly, we get

$$
\begin{equation*}
\left|\tilde{f}_{i}^{\langle\nu\rangle}\left(\mu_{i, n}\right)-\tilde{f}_{i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right)\right| \leq C \frac{\left|\mu_{i, n}-\tilde{\mu}_{i, n}\right|}{\left(\sqrt{\left|\mu_{i, n}\right|}+1\right)^{\nu+1-i}}, \quad \nu \geq 0, \quad i=0,1 . \tag{5.40}
\end{equation*}
$$

Note that

$$
\begin{gathered}
\left|\tilde{U}_{i, j}^{\langle \rangle}\left(t, \tilde{\mu}_{i, n}\right)-U_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right)\right| \leq\left|\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \tilde{\mu}_{i, n}\right)-\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right)\right|+\left|\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right)-U_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, n}\right)\right|, \\
\left|\tilde{f}_{i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right)-f_{i}^{\langle\nu\rangle}\left(\mu_{i, n}\right)\right| \leq\left|\tilde{f}_{i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right)-\tilde{f}_{i}^{\langle\nu\rangle}\left(\mu_{i, n}\right)\right|+\left|\tilde{f}_{i}^{\langle\nu\rangle}\left(\mu_{i, n}\right)-f_{i}^{\langle\nu\rangle}\left(\mu_{i, n}\right)\right| .
\end{gathered}
$$

Using (5.39), (5.40) and Lemma 5, and noting $m_{n}^{i}=1$ for $n \geq n_{i}, i=0,1$, we obtain

$$
\begin{gather*}
\sum_{n \geq n_{0}}\left(\left|\mu_{0, n}\right|+1\right)\left\|\mathbf{U}_{n}^{0}-\tilde{\tilde{\mathbf{U}}}_{n}^{0}\right\|_{\mathcal{H}}^{2}+\sum_{n \geq n_{1}}\left\|\mathbf{U}_{n}^{1}-\tilde{\tilde{\mathbf{U}}}_{n}^{1}\right\|_{\mathcal{H}}^{2}<C(\Lambda+Q)^{2}  \tag{5.41}\\
\sum_{n \geq n_{0}}\left(\left|\mu_{0, n}\right|+1\right)\left|\tau_{n}^{0}-\tilde{\tau}_{n}^{0}\right|^{2}+\sum_{n \geq n_{1}}\left|\tau_{n}^{1}-\tilde{\tau}_{n}^{1}\right|^{2}<C(\Lambda+Q)^{2} \tag{5.42}
\end{gather*}
$$

Now let us consider $n \in\left[0, n_{i}\right), i=0,1$. Fix $i=0,1$. By the definitions of $\tilde{E}_{n, i, j}(t, \lambda)$ and $\tilde{F}_{n, i}(\lambda)$, using Proposition A.6, we have that for each fixed $k \in\left[0, n_{i}\right) \cap \mathcal{S}_{i}$,

$$
\begin{gather*}
\left|\tilde{E}_{k, i, j}^{\langle\nu\rangle}\left(t, \mu_{i, k}\right)-\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, k}\right)\right| \leq C \max _{n \in \tilde{S}_{k}^{i}}\left|\tilde{\mu}_{i, n}-\mu_{i, k}\right|, \quad \nu=\overline{0, m_{k}^{i}-1},  \tag{5.43}\\
\left|\tilde{F}_{k, i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)-\tilde{f}_{i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)\right| \leq C \max _{n \in \tilde{\mathcal{S}}_{k}^{i}}\left|\tilde{\mu}_{i, n}-\mu_{i, k}\right|, \quad \nu=\overline{0, m_{k}^{i}-1}, \tag{5.44}
\end{gather*}
$$

for sufficient small $\varepsilon>0$. Note that

$$
\begin{gathered}
\left|\tilde{E}_{k, i, j}^{\langle\nu\rangle}\left(t, \mu_{i, k}\right)-U_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, k}\right)\right| \leq\left|\tilde{E}_{k, i, j}^{\langle \rangle}\left(t, \mu_{i, k}\right)-\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, k}\right)\right|+\left|\tilde{U}_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, k}\right)-U_{i, j}^{\langle\nu\rangle}\left(t, \mu_{i, k}\right)\right|, \\
\left|\tilde{F}_{k, i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)-f_{i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)\right| \leq\left|\tilde{F}_{k, i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)-\tilde{f}_{i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)\right|+\left|\tilde{f}_{i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)-f_{i}^{\langle\nu\rangle}\left(\mu_{i, k}\right)\right|
\end{gathered}
$$

Using Lemma 5 and (5.43), (5.44), we obtain

$$
\sum_{n \in M_{k}^{i}}\left\|\tilde{\mathbf{U}}_{n}^{i}-\mathbf{U}_{n}^{i}\right\|_{\mathcal{H}} \leq C\left(Q+\max _{n \in \tilde{\mathcal{S}}_{k}^{i}}\left|\tilde{\mu}_{i, n}-\mu_{i, k}\right|\right), \quad k \in \mathcal{S}_{i} \cap\left[0, n_{i}\right), \quad i=0,1
$$

$$
\sum_{n \in M_{k}^{i}}\left|\tilde{\tilde{\tau}}_{n}^{i}-\tau_{n}^{i}\right| \leq C\left(Q+\max _{n \in \tilde{\mathcal{S}}_{k}^{i}}\left|\tilde{\mu}_{i, n}-\mu_{i, k}\right|\right), \quad k \in \mathcal{S}_{i} \cap\left[0, n_{i}\right), \quad i=0,1
$$

Hence

$$
\begin{gather*}
\sum_{n=0}^{n_{0}-1}\left(\left|\mu_{0, n}\right|+1\right)\left\|\mathbf{U}_{n}^{0}-\tilde{\tilde{\mathbf{U}}}_{n}^{0}\right\|_{\mathcal{H}}^{2}+\sum_{n=0}^{n_{1}-1}\left\|\mathbf{U}_{n}^{1}-\tilde{\tilde{\mathbf{U}}}_{n}^{1}\right\|_{\mathcal{H}}^{2}<C(\Lambda+Q)^{2}  \tag{5.45}\\
\sum_{n=0}^{n_{0}-1}\left(\left|\mu_{0, n}\right|+1\right)\left|\tau_{n}^{0}-\tilde{\tilde{\tau}}_{n}^{0}\right|^{2}+\sum_{n=0}^{n_{1}-1}\left|\tau_{n}^{1}-\tilde{\tilde{\tau}}_{n}^{1}\right|^{2}<C(\Lambda+Q)^{2} \tag{5.46}
\end{gather*}
$$

Using (5.41), (5.42), (5.45), and (5.46), we arrive at (5.32) and (5.33).
Proof of Theorem 3. Using Lemma 6 and Proposition A.5, we get that, for sufficiently small $\varepsilon>0$, there is a unique $\tilde{\mathbf{K}}=\left(\tilde{K}_{1}, \tilde{K}_{2}\right) \in \mathcal{H}$ satisfying (5.31) that is equivalent to (5.24). Recall the definitions $\left(\underline{(5.16)}-(\sqrt{(5.18)})\right.$ of $\tilde{\mathbf{U}}_{i}(t, \lambda)$ and $\tilde{f}_{i}(\lambda), i=0,1$. Define the functions $\tilde{\Delta}_{i}(\lambda)$ $(i=0,1)$ with $\tilde{\mathbf{K}}(t), \tilde{\mathbf{U}}_{i}(t, \lambda)$ and $\tilde{f}_{i}(\lambda)$

$$
\begin{equation*}
\tilde{\Delta}_{i}(\lambda)=-\left\langle\tilde{\mathbf{K}}(\cdot), \tilde{\mathbf{U}}_{i}(\cdot, \lambda)\right\rangle+\tilde{f}_{i}(\lambda), \quad i=0,1 \tag{5.47}
\end{equation*}
$$

Then Eq.(5.24) together with (5.23) imply

$$
\tilde{\Delta}_{i}^{\langle\nu\rangle}\left(\tilde{\mu}_{i, n}\right)=0, \quad n \in \tilde{\mathcal{S}}_{i}, \quad \nu=\overline{0, \tilde{m}_{n}^{i}-1}, \quad i=0,1
$$

Note that Proposition A.5 also implies $\|\tilde{\mathbf{K}}-\mathbf{K}\|_{\mathcal{H}} \leq C(Q+\Lambda)$. Thus, using Proposition 2, we conclude that there exists a unique $\tilde{q}_{1} \in L^{2}(0, a)$ such that $\left\{\tilde{K}_{1}, \tilde{K}_{2}, \omega_{1}\right\}$ are the Cauchy data for $\tilde{q}_{1}$ and $\tilde{h}=\omega_{1}-\frac{1}{2} \int_{0}^{a} \tilde{q}_{1}(x) d x$, and the estimate (2.19) is valid. Define the functions $\tilde{\varphi}_{0}(\lambda)$ and $\tilde{\varphi}_{1}(\lambda)$ by the Cauchy data $\left\{\tilde{K}_{1}, \tilde{K}_{2}, \omega_{1}\right\}$

$$
\begin{gather*}
\tilde{\varphi}_{0}(\lambda):=\cos \rho a+\omega_{1} \frac{\sin \rho a}{\rho}-\int_{0}^{a} \tilde{K}_{1}(t) \frac{\sin \rho t}{\rho} d t  \tag{5.48}\\
\tilde{\varphi}_{1}(\lambda):=-\rho \sin \rho a+\omega_{1} \cos \rho a+\int_{0}^{a} \tilde{K}_{2}(t) \cos \rho t d t \tag{5.49}
\end{gather*}
$$

Then Eqs.(5.47), (5.48) and (5.49), together with (5.17) and (5.18), imply that the functions $\tilde{\Delta}_{i}(\lambda)(i=0,1)$ defined in (5.47) have the expressions

$$
\begin{equation*}
\tilde{\Delta}_{i}(\lambda):=\tilde{\varphi}_{0}(\lambda) \tilde{g}_{i, 1}(\lambda)-\tilde{\varphi}_{1}(\lambda) \tilde{g}_{i, 0}(\lambda) \tag{5.50}
\end{equation*}
$$

The proof is complete.
Proof of Corollary 1. The proof of Corollary 1 is similar to the proof of Theorem 3. Letting $\left\{\mu_{0, n}\right\}_{n \geq 0}=\emptyset$ and using only (ii) of Lemma 4 in the proof of Theorem 3, we complete the proof of Corollary 1 .

Proof of Corollary 2. If $a=1 / 2$, then $\lambda_{1, n}$ has the following asymptotics (cf.(3.7))

$$
\begin{equation*}
\sqrt{\lambda_{1, n}}=n \pi+\frac{\eta_{+}+\eta_{-}(-1)^{n}}{n \pi b_{+}}+\frac{\kappa_{1, n}}{n}, \quad\left\{\kappa_{1, n}\right\} \in l^{2} . \tag{5.51}
\end{equation*}
$$

Using Proposition 1 in [7] with (5.51), we get that $\left\{c^{\langle\nu\rangle}\left(t, \lambda_{1, n}\right)\right\}_{n \in \mathcal{S}_{1}, \nu=\overline{0, m_{n}^{1}-1}}$ is a Riesz basis in $L^{2}(0,1)$, where we have assumed $\mu_{1, n}=\lambda_{1, n}$ for all $n \geq 0$, and $c(t, \lambda)=\cos \rho t$. Then, using Lemma 3 and Corollary 1, we get Corollary 2 ,

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## Appendix

In Appendix, we provide a few auxiliary propositions. One can also find partially similar results in [3, 4]. For convenience of readers, we summarize them in Appendix.

In Propositions A.1 and A.2, we consider a sequence of complex numbers $\left\{z_{n}\right\}_{n \geq 0}$ with finite multiplicities. Let $m_{n}$ denote the multiplicity of the value $z_{n}$ in the sequence $\left\{z_{n}\right\}_{n \geq 0}$. Without loss of generality, assume that $z_{n}=z_{n+1}=\cdots=z_{n+m_{n}-1}$ and denote

$$
\mathcal{S}:=\left\{n \in \mathbb{N}: z_{n} \neq z_{n-1}, n \geq 1\right\} \cup\{0\}
$$

Proposition A.1. Assume that $\left\{z_{n}\right\}_{n \geq 0}$ (counted with multiplicities) are the zeros of the function

$$
D(z):=\phi_{0}(z) g_{1}(z)-\phi_{1}(z) g_{0}(z)
$$

where the functions $\phi_{j}(z)$ and $g_{j}(z)$ are analytic at $z_{n}(n \geq 0), j=0,1$. If the vectors $\left(g_{0}\left(z_{n}\right), g_{1}\left(z_{n}\right)\right) \neq(0,0) \neq\left(\phi_{0}\left(z_{n}\right), \phi_{1}\left(z_{n}\right)\right)$ for $n \in \mathcal{S}$, then there exist constants $C_{n, \nu}, M_{n, \nu}$, $n \in \mathcal{S}, \nu=\overline{0, m_{n}-1}$ such that $C_{n, 0} \neq 0, M_{n, 0} \neq 0$ and

$$
\begin{gather*}
g_{j}^{\langle\nu\rangle}\left(z_{n}\right):=\left.\frac{1}{\nu!} \frac{d^{\nu} g_{j}(z)}{d z^{\nu}}\right|_{z=z_{n}}=\sum_{k=0}^{\nu} C_{n, k} \phi_{j}^{\langle\nu-k\rangle}\left(z_{n}\right), \quad n \in \mathcal{S}, \quad \nu=\overline{0, m_{n}-1}, \quad j=0,1 .  \tag{A.1}\\
\phi_{j}^{\langle\nu\rangle}\left(z_{n}\right)=\sum_{k=0}^{\nu} M_{n, k} g_{j}^{\langle\nu-k\rangle}\left(z_{n}\right), \quad n \in \mathcal{S}, \quad \nu=\overline{0, m_{n}-1}, \quad j=0,1 \tag{A.2}
\end{gather*}
$$

The proof of Proposition A. 1 repeats the proof of Lemma 1 in [3], so we omit it.
Proposition A.2. Let $\phi_{0}(z)$ and $\phi_{1}(z)$ be nontrivial entire functions and

$$
\begin{equation*}
\int_{0}^{a}\left(h_{1}(t) \phi_{1}(z) \frac{\sin \sqrt{z} t}{\sqrt{z}}+h_{2}(t) \phi_{0}(z) \cos \sqrt{z} t\right) d t \equiv 0, \quad h_{1}, h_{2} \in L^{2}(0, a) \tag{A.3}
\end{equation*}
$$

Assume that $\phi_{1}(z)$ has the zeros $\left\{z_{n}\right\}_{n \geq 0}$ (counted with multiplicities) satisfying the asymptotics

$$
\sqrt{z_{n}}=\frac{n \pi}{a}+\kappa_{n}, \quad\left\{\kappa_{n}\right\} \in l^{2} .
$$

If $\phi_{0}\left(z_{n}\right) \neq 0$ for all $n$, then $h_{1}=0$ and $h_{2}=0$ in $L^{2}(0, a)$.
Proof. Define

$$
c^{\langle\nu\rangle}\left(t, z_{n}\right)=\left.\frac{1}{\nu!} \frac{\partial^{\nu} \cos \sqrt{z} t}{\partial z^{\nu}}\right|_{z=z_{n}}, \quad \nu=\overline{0, m_{n}-1}, \quad n \in \mathcal{S} .
$$

It follows from (A.3) that

$$
\sum_{k=0}^{\nu} \phi_{0}^{\langle k\rangle}\left(z_{n}\right) \int_{0}^{a} h_{2}(t) c^{\langle\nu-k\rangle}\left(t, z_{n}\right) d t=0, \quad \nu=\overline{0, m_{n}-1}, \quad n \in \mathcal{S}
$$

From the asymptotics of $z_{n}$, there is only a finite number of multiple values in $\left\{z_{n}\right\}_{n \geq 0}$. Without loss of generality, assume $m_{n} \geq 2$ for $0 \leq n \leq n_{0}-1$ for some $n_{0} \in \mathbb{N}$, and $m_{n}=1$ for $n \geq n_{0}$. Since $\phi_{0}\left(z_{n}\right) \neq 0$ for $n \in \mathcal{S}$, we have

$$
\begin{gather*}
\int_{0}^{a} h_{2}(t) c\left(t, z_{n}\right) d t=0, \quad n \in \mathcal{S},  \tag{A.4}\\
\int_{0}^{a} h_{2}(t)\left[c^{\langle\nu\rangle}\left(t, z_{n}\right)+\sum_{k=1}^{\nu} \frac{\phi_{0}^{\langle k\rangle}\left(z_{n}\right)}{\phi_{0}\left(z_{n}\right)} c^{\langle\nu-k\rangle}\left(t, z_{n}\right)\right] d t=0, \quad \nu=\overline{1, m_{n}-1}, \quad n \in \mathcal{S} \cap\left[0, n_{0}\right) . \tag{A.5}
\end{gather*}
$$

It is known that $\left\{c^{\langle\nu\rangle}\left(t, z_{n}\right)\right\}_{n \in \mathcal{S}, \nu=\overline{0, m_{n}-1}}$ is a Riesz basis in $L^{2}(0, a)$ (see Appendix in [7]). Hence, by replacing the finite functions $\left\{c^{\langle\nu\rangle}\left(t, z_{n}\right)\right\}_{n \in \mathcal{S} \cap\left[0, n_{0}\right), \nu=\overline{1, m_{n}-1}}$ with the functions

$$
\begin{equation*}
c^{\langle\nu\rangle}\left(t, z_{n}\right)+\sum_{k=1}^{\nu} \frac{\phi_{0}^{\langle k\rangle}\left(z_{n}\right)}{\phi_{0}\left(z_{n}\right)} c^{\langle\nu-k\rangle}\left(t, z_{n}\right), \quad \nu=\overline{1, m_{n}-1}, \quad n \in \mathcal{S} \cap\left[0, n_{0}\right), \tag{A.6}
\end{equation*}
$$

we know that the system of the functions in (A.6) and $\left\{c\left(t, z_{n}\right)\right\}_{n \in \mathcal{S}}$ is also complete in $L^{2}(0, a)$. It follows from (А.4) and (A.5) that $h_{2}(t)=0$ in $L^{2}(0, a)$. Returning to (A.3), we have

$$
\phi_{1}(z) \int_{0}^{a} h_{1}(t) \frac{\sin \sqrt{z} t}{\sqrt{z}} d t=0, \quad z \in \mathbb{C}
$$

which implies $h_{1}(t)=0$ in $L^{2}(0, a)$ since $\phi_{1}(z)$ is a nontrivial entire function.
Proposition A.3. Assume that $\left\{\alpha_{n}\right\}_{n \geq 1}$ are complex numbers satisfying $\alpha_{k} \neq \alpha_{l}$ and $\alpha_{k} \neq \overline{\alpha_{l}}$ for all $k \neq l$. Denote $\mathbf{v}_{n}(t)=\left(v_{1}\left(t, \alpha_{n}^{2}\right), v_{2}\left(t, \alpha_{n}^{2}\right)\right)$, where $v_{j}(t, \lambda)(j=1,2)$ are defined in (4.15), where $\lambda=\rho^{2}$. Then the following assertions are equivalent:
(i) $\left\{\cos \alpha_{n} t\right\}_{n \geq 0}$ is a Riesz basis in $L^{2}(0,2 a)$;
(ii) $\left\{\mathbf{v}_{n}(t)\right\}_{n \geq 0}$ is a Riesz basis in $\mathcal{H}:=L^{2}(0, a) \times L^{2}(0, a)$, the inner product of which is defined in $(2.8)$.

Proof. By Theorem 9 in [39, p.32], we know that a system of functions $\left\{f_{n}(t)\right\}_{n \geq 0}$ is a Riesz basis in some Hilbert space $\mathbb{H}$ if and only if it is complete and satisfies the two side inequality

$$
C_{1} \sum_{n=0}^{N}\left|\beta_{n}\right|^{2} \leq\left\|\sum_{n=0}^{N} \beta_{n} f_{n}\right\|_{\mathbb{H}} \leq C_{2} \sum_{n=0}^{N}\left|\beta_{n}\right|^{2},
$$

where $\left\{\beta_{n}\right\}$ is an arbitrary sequence, and $N \geq 0$ is an arbitrary integer, $C_{1}$ and $C_{2}$ are some fixed constants. In view of (4.15), we have that

$$
\begin{aligned}
\left\langle\mathbf{v}_{j}, \mathbf{v}_{k}\right\rangle= & \frac{1}{4} \int_{0}^{a}\left[\cos \overline{\alpha_{j}}(a-t)-\cos \overline{\alpha_{j}}(a+t)\right]\left[\cos \alpha_{k}(a-t)-\cos \alpha_{k}(a+t)\right] d t \\
& +\frac{1}{4} \int_{0}^{a}\left[\cos \overline{\alpha_{j}}(a-t)+\cos \overline{\alpha_{j}}(a+t)\right]\left[\cos \alpha_{k}(a-t)+\cos \alpha_{k}(a+t)\right] d t \\
= & \frac{1}{2} \int_{0}^{a} \cos \left(\overline{\alpha_{j}} t\right) \cos \left(\alpha_{k} t\right) d t+\frac{1}{2} \int_{a}^{2 a} \cos \left(\overline{\alpha_{j}} t\right) \cos \left(\alpha_{k} t\right) d t \\
= & \frac{1}{2} \int_{0}^{2 a} \cos \left(\overline{\alpha_{j}} t\right) \cos \left(\alpha_{k} t\right) d t .
\end{aligned}
$$

Hence, we have

$$
\begin{equation*}
\left\|\sum_{n=0}^{N} \beta_{n} \mathbf{v}_{n}\right\|_{\mathcal{H}}^{2}=\sum_{j=0}^{N} \sum_{k=0}^{N} \overline{\beta_{j}} \beta_{k}\left\langle\mathbf{v}_{j}, \mathbf{v}_{k}\right\rangle=\frac{1}{2}\left\|\sum_{n=0}^{N} \beta_{n} \cos \left(\alpha_{n} t\right)\right\|_{L^{2}(0,2 a)}^{2} \tag{A.7}
\end{equation*}
$$

In view of (A.7), it only remains to show that $\left\{\cos \alpha_{n} t\right\}_{n \geq 0}$ is complete in $L^{2}(0,2 a)$ if and only if $\left\{\left(\mathbf{v}_{n}(t)\right\}_{n \geq 0}\right.$ is complete in $\mathcal{H}$.

Assume that $\left\{\cos \alpha_{n} t\right\}_{n \geq 0}$ is complete in $L^{2}(0,2 a)$. Let $\mathbf{h}=\left(\overline{h_{1}}, \overline{h_{2}}\right) \in \mathcal{H}$ such that $\left\langle\mathbf{h}, \mathbf{v}_{n}\right\rangle=0$ for all $n \geq 0$. Then the function

$$
\begin{equation*}
F_{0}(\lambda):=\int_{0}^{a}\left(h_{1}(t) v_{1}(t, \lambda)+h_{2}(t) v_{2}(t, \lambda)\right) d t \tag{A.8}
\end{equation*}
$$

has zeros $\left\{\alpha_{n}\right\}_{n \geq 0}$. By the definition of $v_{j}(t, \lambda)$ (cf.(4.15)), we get

$$
F_{0}(\lambda)=\int_{0}^{2 a} b_{0}(t) \cos \rho t d t, \quad b_{0}(t)= \begin{cases}\frac{h_{1}(a-t)+h_{2}(a-t)}{2}, & 0<t<a  \tag{A.9}\\ \frac{h_{2}(t-a)-h_{1}(t-a)}{2}, & a<t<2 a\end{cases}
$$

Since $\left\{\cos \alpha_{n} t\right\}_{n \geq 0}$ is complete in $L^{2}(0,2 a)$, we have $b_{0}(t)=0$ in $L^{2}(0,2 a)$. Hence $h_{1}(t)=$ $h_{2}(t)=0$ in $L^{2}(0, a)$.

Assume that $\left\{\mathbf{v}_{n}(t)\right\}_{n \geq 0}$ is complete in $\mathcal{H}$. Let $b \in L^{2}(0,2 a)$ such that $\int_{0}^{2 a} b(t) \cos \alpha_{n} t d t=0$ for all $n \geq 0$. Namely, the function $F_{2}(\lambda)=\int_{0}^{2 a} b(t) \cos \rho t d t$ has zeros $\left\{\alpha_{n}\right\}_{n \geq 0}$. Note that the function $F_{2}(\lambda)$ also has the form of (A.8) with

$$
h_{1}(t)=b(a-t)-b(a+t), \quad h_{2}(t)=b(a-t)+b(a+t), \quad t \in(0, a)
$$

Since $\left\{\left(\mathbf{v}_{n}(t)\right\}_{n \geq 0}\right.$ is complete in $\mathcal{H}$, then $h_{1}(t)=h_{2}(t)=0$ in $L^{2}(0, a)$ and so $b(t)=0$ in $L^{2}(0,2 a)$.

Proposition A.4. Let $\left\{\rho_{n}\right\}$ be separated complex numbers with bounded imaginary parts, namely, $\left|\rho_{n}-\rho_{m}\right| \geq c_{0}>0$ whenever $n \neq m$ and $\sup _{n}\left|\operatorname{Im} \rho_{n}\right| \leq c_{1}<\infty$. If $W \in L^{2}(-b, b)$, then

$$
\begin{equation*}
\sum_{n}\left|\int_{-b}^{b} W(t) e^{i \rho_{n} t} d t\right|^{2} \leq C_{0}\|W\|_{L^{2}(-b, b)}^{2} \tag{A.10}
\end{equation*}
$$

where $C_{0}>0$ depends only on $b, c_{0}$ and $c_{1}$.

Proof. Define $f(\rho)=\int_{-b}^{b} W(t) e^{i \rho t} d t$. Then $f(\rho)$ is an entire function of exponential type $\leq b$. By the theory of the Fourier transform, we know that $\|f\|_{L^{2}(-\infty, \infty)}^{2}=2 \pi\|W\|_{L^{2}(-b, b)}^{2}$. Since $\left\{\rho_{n}\right\}$ are separated complex numbers with bounded imaginary parts, from Theorem 17 and its Remark in [39, p.96-98], we have

$$
\sum_{n}\left|f\left(\rho_{n}\right)\right|^{2} \leq C_{1}\|f\|_{L^{2}(-\infty, \infty)}^{2}=2 \pi C_{1}\|W\|_{L^{2}(-b, b)}^{2}
$$

which implies (A.10). Another proof of Proposition A. 4 can be obtained from Lemma 1 in [9].
Proposition A.5 (See, e.g., [38]). Let $\left\{v_{n}\right\}$ be a Riesz basis in a Hilbert space $\mathbb{H}$. Then there exists $\varepsilon>0$, such that every sequence $\left\{\tilde{v}_{n}\right\}$, satisfying

$$
\mathcal{V}:=\left(\sum_{n}\left\|v_{n}-\tilde{v}_{n}\right\|_{\mathbb{H}}^{2}\right)^{1 / 2} \leq \varepsilon
$$

is also a Riesz basis in $\mathbb{H}$. Furthermore, for some $\tau \in \mathbb{H}$, denote $\tau_{n}:=\left(\tau, v_{n}\right)_{\mathbb{H}}$, where $(\cdot, \cdot)_{\mathbb{H}}$ is the inner product in $\mathbb{H}$, then for any sequence $\left\{\tilde{\tau}_{n}\right\}$ satisfying

$$
\Omega:=\left(\sum_{n}\left|\tau_{n}-\tilde{\tau}_{n}\right|^{2}\right)^{1 / 2} \leq \varepsilon
$$

there exists a unique $\tilde{\tau} \in \mathbb{H}$, such that $\tilde{\tau}_{n}=\left(\tilde{\tau}, \tilde{v}_{n}\right)_{\mathbb{H}}$ for all $n$, moreover,

$$
\|\tau-\tilde{\tau}\|_{\mathbb{H}} \leq C(\mathcal{V}+\Omega),
$$

where the constant $C$ depends only on $\left\{v_{n}\right\}$ and $\tau$.
To deal with the multiple eigenvalues in the local solvability and stability, we need the following proposition.
Proposition A. 6 (See [27]). Assume that $f(z)$ is an entire function, and $z_{1}, \ldots, z_{m}$ (not necessarily distinct) are in the disc $\left\{z:\left|z-z_{0}\right| \leq r<1 / 2\right\}$. Let $p(z)$ be the unique polynomial of degree at most $m-1$ which interpolates $f(z)$ and its derivatives in the usual way at the points $z_{j}, j=\overline{1, m}$ : namely, if $z_{j}$ appears $m_{j}$ times, then $p^{(n)}\left(z_{j}\right)=f^{(n)}\left(z_{j}\right)$ for $n=\overline{0, m_{j}-1}$. Then for each $j=\overline{0, m-1}$,

$$
\begin{equation*}
\left|f^{(j)}\left(z_{0}\right)-p^{(j)}\left(z_{0}\right)\right| \leq C r^{m-j} \sup _{\left|z-z_{0}\right|=1}|f(z)| \tag{A.11}
\end{equation*}
$$

here the constant $C$ depends only on $m$.

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