

# ON MONOIDS OF METRIC PRESERVING FUNCTIONS

VIKTORIIA BILET AND OLEKSIY DOVGOSHEY

**ABSTRACT.** Let  $\mathbf{X}$  be a class of metric spaces and let  $\mathbf{P}_\mathbf{X}$  be the set of all  $f : [0, \infty) \rightarrow [0, \infty)$  preserving  $\mathbf{X}$ ,  $(Y, f \circ \rho) \in \mathbf{X}$  whenever  $(Y, \rho) \in \mathbf{X}$ . For arbitrary subset  $\mathbf{A}$  of the set of all metric preserving functions we show that the equality  $\mathbf{P}_\mathbf{X} = \mathbf{A}$  has a solution iff  $\mathbf{A}$  is a monoid with respect to the operation of function composition. In particular, for the set  $\mathbf{SI}$  of all amenable subadditive increasing functions there is a class  $\mathbf{X}$  of metric spaces such that  $\mathbf{P}_\mathbf{X} = \mathbf{SI}$  holds, which gives a positive answer to the question of paper [1].

## 1. INTRODUCTION

The following is a particular case of the concept introduced by Jacek Jachymski and Filip Turoboś in [2].

**Definition 1.** Let  $\mathbf{A}$  be a class of metric spaces. Let us denote by  $\mathbf{P}_\mathbf{A}$  the set of all functions  $f : [0, \infty) \rightarrow [0, \infty)$  such that the implication

$$((X, d) \in \mathbf{A}) \Rightarrow ((X, f \circ d) \in \mathbf{A})$$

is valid for every metric space  $(X, d)$ .

For mappings  $F : X \rightarrow Y$  and  $\Phi : Y \rightarrow Z$  we use the symbol  $F \circ \Phi$  to denote the mapping

$$X \xrightarrow{F} Y \xrightarrow{\Phi} Z.$$

We also use the following notation:

$\mathbf{F}$ , set of functions  $f : [0, \infty) \rightarrow [0, \infty)$ ;

$\mathbf{F}_0$ , set of functions  $f \in \mathbf{F}$  with  $f(0) = 0$ ;

$\mathbf{Am}$ , set of amenable  $f \in \mathbf{F}$ ;

$\mathbf{SI}$ , set of subadditive increasing  $f \in \mathbf{Am}$ ;

$\mathbf{M}$ , class of metric spaces;

$\mathbf{U}$ , class of ultrametric spaces;

$\mathbf{Dis}$ , class of discrete metric spaces;

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$\mathbf{M}_2$ , class of two-points metric spaces;

$\mathbf{M}_1$ , class of one-point metric spaces.

The main purpose of this paper is to give a solution of the following problems.

**Problem 2.** Let  $\mathbf{A} \subseteq \mathbf{P}_M$ . Find conditions under which the equation

$$(1) \quad \mathbf{P}_X = \mathbf{A}$$

has a solution  $\mathbf{X} \subseteq \mathbf{M}$ .

**Problem 3.** Let  $\mathbf{A} \subseteq \mathbf{P}_U$ . Find conditions under which equation (1) has a solution  $\mathbf{X} \subseteq \mathbf{U}$ .

In addition, we find all solutions to equation (1) for  $\mathbf{A}$  equal to  $\mathbf{F}$ ,  $\mathbf{F}_0$ , or  $\mathbf{Am}$  and answer the following question.

**Question 4.** Is there a subclass  $\mathbf{X}$  of the class  $\mathbf{M}$  such that

$$\mathbf{P}_X = \mathbf{SI}?$$

This question was asked in [1] in a different but equivalent form and it was the original motivation for our research.

The paper is organized as follows. The next section contains some necessary definitions and facts from the theories of metric spaces and metric preserving functions.

In Section 3 we recall some definitions from the semigroup theory and describe solutions to equation (1), for the cases when  $\mathbf{A}$  is  $\mathbf{F}$ ,  $\mathbf{F}_0$  or  $\mathbf{Am}$ . In addition, we show that  $\mathbf{P}_X$  is always a submonoid of  $(\mathbf{F}, \circ)$ . See Theorems 20, 22, 23 and Proposition 26, respectively.

Solutions to Problems 2 and 3 are given, respectively, in Theorems 29 and 31 of Section 4. Theorem 30 gives a positive answer to Question 4.

## 2. PRELIMINARIES ON METRICS AND METRIC PRESERVING FUNCTIONS

Let  $X$  be nonempty set. A function  $d : X \times X \rightarrow [0, \infty)$  is said to be a *metric* on the set  $X$  if for all  $x, y, z \in X$  we have:

- (i)  $d(x, y) \geq 0$  with equality if and only if  $x = y$ , the *positivity property*;
- (ii)  $d(x, y) = d(y, x)$ , the *symmetry property*;
- (iii)  $d(x, y) \leq d(x, z) + d(z, y)$ , the *triangle inequality*.

A metric space  $(X, d)$  is *ultrametric* if the *strong triangle inequality*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

holds for all  $x, y, z \in X$ .

**Example 5.** Let us denote by  $\mathbb{R}_0^+$  the set  $(0, \infty)$ . Then the mapping  $d^+: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow [0, \infty)$ ,

$$d^+(p, q) := \begin{cases} 0 & \text{if } p = q, \\ \max\{p, q\} & \text{otherwise.} \end{cases}$$

is the ultrametric on  $\mathbb{R}_0^+$  introduced by C. Delhomme, C. Laflamme, M. Pouzet, and N. Sauer in [3].

**Definition 6.** Let  $(X, d)$  be a metric space. The metric  $d$  is *discrete* if there is  $k \in (0, \infty)$  such that

$$d(x, y) = k$$

for all distinct  $x, y \in X$ .

In what follows we will say that a metric space  $(X, d)$  is discrete if  $d$  is a discrete metric on  $X$ . We will denote by **Dis** the class of all discrete metric space. In addition, for given nonempty set  $X_1$ , we will denote by **Dis** <sub>$X_1$</sub>  the subclass of **Dis** consisting of all metric spaces  $(X_1, d)$  with discrete  $d$ .

**Example 7.** Let  $\mathbf{M}_k$ , for  $k = 1, 2$ , be the class of all metric spaces  $(X, d)$  satisfying the equality

$$\text{card}(X) = k.$$

Then all metric spaces belonging to  $\mathbf{M}_1 \cup \mathbf{M}_2$  are discrete.

**Proposition 8.** *The following statements are equivalent for each metric space  $(X, d) \in \mathbf{M}$ .*

- (i)  $(X, d)$  is discrete.
- (ii) Every three-point subspace of  $(X, d)$  is discrete.

*Proof.* The implication  $(i) \Rightarrow (ii)$  is evidently valid.

Suppose that  $(ii)$  holds but  $(X, d) \notin \mathbf{Dis}$ . Then there are some different points  $i, j, k, l \in X$  such that

$$(2) \quad d(i, j) \neq d(k, l).$$

Write  $X_1 := \{i, j, k\}$  and  $X_2 := \{j, k, l\}$ . Then the spaces  $(X_1, d|_{X_1 \times X_1})$  and  $(X_2, d|_{X_2 \times X_2})$  are discrete subspaces of  $(X, d)$  by statement  $(ii)$ . Consequently we have

$$(3) \quad d(i, j) = d(j, k)$$

and

$$(4) \quad d(j, k) = d(k, l)$$

by definition of the class **Dis**. Now (3) and (4) give us

$$d(i, j) = d(k, l),$$

which contradicts (2).  $\square$

**Remark 9.** The standard definition of discrete metric can be formulated as: “The metric on  $X$  is discrete if the distance from each point of  $X$  to every other point of  $X$  is one.” (See, for example, [4, p. 14].)

Let  $\mathbf{F}$  be the set of all functions  $f : [0, \infty) \rightarrow [0, \infty)$ .

**Definition 10.** A function  $f \in \mathbf{F}$  is *metric preserving (ultrametric preserving)* iff  $f \in \mathbf{P}_M$  ( $f \in \mathbf{P}_U$ ).

**Remark 11.** The concept of metric preserving functions can be traced back to Wilson [5]. Similar problems were considered by Blumenthal in [6]. The theory of metric preserving functions was developed by Borsík, Doboš, Piotrowski, Vallin and other mathematicians [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. See also lectures by Doboš [20], and the introductory paper by Corazza [21]. The study of ultrametric preserving functions begun by P. Pongsriiam and I. Termwuttipong in 2014 [22] and was continued in [23, 24].

We will say that  $f \in \mathbf{F}$  is *amenable* iff

$$f^{-1}(0) = \{0\}$$

holds and will denote by  $\mathbf{Am}$  the set of all amenable functions from  $\mathbf{F}$ . Let us denote by  $\mathbf{F}_0$  the set of all functions  $f \in \mathbf{F}$  satisfying the equality  $f(0) = 0$ . It follows directly from the definition that  $\mathbf{Am} \subsetneq \mathbf{F}_0 \subsetneq \mathbf{F}$ .

Moreover, a function  $f \in \mathbf{F}$  is *increasing* iff the implication

$$(x \leq y) \Rightarrow (f(x) \leq f(y))$$

is valid for all  $x, y \in [0, \infty)$ .

The following theorem was proved in [22].

**Theorem 12.** A function  $f \in \mathbf{F}$  is ultrametric preserving if and only if  $f$  is increasing and amenable.

**Remark 13.** Theorem 12 was generalized in [25] to the special case of the so-called ultrametric distances. These distances were introduced by S. Priess-Crampe and P. Ribenboim in 1993 [26] and studied in [27, 28, 29, 30].

Recall that a function  $f \in \mathbf{F}$  is said to be *subadditive* if

$$f(x + y) \leq f(x) + f(y)$$

holds for all  $x, y \in [0, \infty)$ . Let us denote by  $\mathbf{SI}$  the set of all subadditive increasing functions  $f \in \mathbf{Am}$ .

Corollary 36 of [1] implies the following result.

**Proposition 14.** *The equality*

$$\mathbf{SI} = \mathbf{P}_U \cap \mathbf{P}_M$$

*holds.*

**Remark 15.** The metric preserving functions can be considered as a special case of metric products (= metric preserving functions of several variables). See, for example, [31, 32, 33, 34, 35, 36]. An important special class of ultrametric preserving functions of two variables was first considered in 2009 [37].

### 3. PRELIMINARIES ON SEMIGROUPS. SOLUTIONS TO $\mathbf{F}_X = \mathbf{A}$ FOR $\mathbf{A} = \mathbf{F}, \mathbf{F}_0, \mathbf{Am}$

Let us recall some basic concepts of semigroup theory, see, for example, “Fundamentals of Semigroup Theory” by John M. Howie [38].

A *semigroup* is a pair  $(S, *)$  consisting of a nonempty set  $S$  and an associative operation  $* : S \times S \rightarrow S$  which is called the *multiplication* on  $S$ . A semigroup  $S = (S, *)$  is a *monoid* if there is  $e \in S$  such that

$$e * s = s * e = s$$

for every  $s \in S$ .

**Definition 16.** Let  $(S, *)$  be a semigroup and  $\emptyset \neq T \subseteq S$ . Then  $T$  is a *subsemigroup* of  $S$  if  $a, b \in T \Rightarrow a * b \in T$ . If  $(S, *)$  is a monoid with the identity  $e$ , then  $T$  is a *submonoid* of  $S$  if  $T$  is a subsemigroup of  $S$  and  $e \in T$ .

**Example 17.** The semigroups  $(\mathbf{F}, \circ)$ ,  $(\mathbf{Am}, \circ)$ ,  $(\mathbf{P}_M, \circ)$  and  $(\mathbf{P}_U, \circ)$  are monoids and the identical mapping  $\text{id} : [0, \infty) \rightarrow [0, \infty)$ ,  $\text{id}(x) = x$  for every  $x \in [0, \infty)$ , is the identity of these monoids.

The following simple lemmas are well known.

**Lemma 18.** *Let  $T$  be a submonoid of a monoid  $(S, *)$  and let  $V \subseteq T$ . Then  $V$  is a submonoid of  $(S, *)$  if and only if  $V$  is a submonoid of  $T$ .*

**Lemma 19.** *Let  $T_1$  and  $T_2$  be submonoids of a monoid  $(S, *)$ . Then the intersection  $T_1 \cap T_2$  also is a submonoid of  $(S, *)$ .*

The next theorem describes all solutions to the equation  $\mathbf{P}_X = \mathbf{F}$ .

**Theorem 20.** *The following statements are equivalent for every  $X \subseteq M$ .*

- (i)  $X$  is the empty subclass of  $M$ .
- (ii) The equality

$$(5) \quad \mathbf{P}_X = \mathbf{F}$$

*holds.*

*Proof.* (i)  $\Rightarrow$  (ii). Let  $\mathbf{X}$  be the empty subclass of  $\mathbf{M}$ . Definition 1 implies the inclusion  $\mathbf{F} \supseteq \mathbf{P}_{\mathbf{X}}$ . Let us consider an arbitrary  $f \in \mathbf{F}$ . To prove equality (5) it suffices to show that  $f \in \mathbf{P}_{\mathbf{X}}$ . Let us do it. Since  $\mathbf{X}$  is empty, the membership relation  $(X, d) \in \mathbf{X}$  is false for every metric space  $(X, d)$ . Consequently, the implication

$$((X, d) \in \mathbf{X}) \Rightarrow ((X, f \circ d) \in \mathbf{X})$$

is valid for every  $(X, d) \in \mathbf{M}$ . It implies  $f \in \mathbf{P}_{\mathbf{X}}$  by Definition 1. Equality (5) follows.

(ii)  $\Rightarrow$  (i). Let (ii) hold. We must show that  $\mathbf{X}$  is empty. Suppose contrary that there is a metric space  $(X, d) \in \mathbf{X}$ . Since, by definition, we have  $X \neq \emptyset$ , there is a point  $x_0 \in X$ . Consequently,  $d(x_0, x_0) = 0$  holds. Let  $c \in (0, \infty)$  and let  $f : [0, \infty) \rightarrow [0, \infty)$  be a constant function,

$$f(t) = c$$

for every  $t \in [0, \infty)$ . In particular, we have

$$(6) \quad f(0) = c > 0.$$

Equality (5) implies that  $f \circ d$  is a metric on  $X$ . Thus, we have

$$0 = f(d(x_0, x_0)) = f(0),$$

which contradicts (6). Statement (i) follows.  $\square$

**Remark 21.** Theorem 20 becomes invalid if we allow the empty metric space to be considered. The equality

$$\mathbf{P}_{\mathbf{X}} = \mathbf{F}$$

holds if the nonempty class  $\mathbf{X}$  contains only the empty metric space.

Let us describe now all possible solutions to  $\mathbf{P}_{\mathbf{X}} = \mathbf{F}_0$ .

**Theorem 22.** *The equality*

$$(7) \quad \mathbf{P}_{\mathbf{X}} = \mathbf{F}_0$$

*holds if and only if  $\mathbf{X}$  is a nonempty subclass of  $\mathbf{M}_1$ .*

*Proof.* Let  $\mathbf{X} \subseteq \mathbf{M}_1$  be nonempty. Equality (7) holds iff

$$(8) \quad \mathbf{P}_{\mathbf{X}} \supseteq \mathbf{F}_0$$

and

$$(9) \quad \mathbf{P}_{\mathbf{X}} \subseteq \mathbf{F}_0.$$

Let us prove the validity of (8). Let  $f \in \mathbf{F}_0$  be arbitrary. Since every  $(X, d) \in \mathbf{X}$  is an one-point metric space, we have  $f \circ d = d$  for all  $(X, d) \in \mathbf{X}$  by positivity property of metric spaces, Inclusion (8) follows.

Let us prove (9). The inclusion  $\mathbf{P}_X \subseteq \mathbf{F}$  follows from Definition 1. Thus, if (9) does not hold, then there is  $f_0 \in \mathbf{F}$  such that  $f_0 \notin \mathbf{P}_X$ ,

$$(10) \quad f_0(0) = k \quad \text{and} \quad k > 0.$$

Since  $\mathbf{X}$  is nonempty, there is  $(X_0, d_0) \in \mathbf{X}$ . Let  $x_0$  be a (unique) point of  $X_0$ . Since  $f_0$  belongs to  $\mathbf{P}_X$ , the function  $f_0 \circ d_0$  is a metric on  $X_0$ . Now, using (10), we obtain

$$f_0(d_0(x_0, x_0)) = f_0(0) = k > 0,$$

which contradicts the positivity property of metric spaces. Inclusion (9) follows.

Let (7) hold. We must show that  $\mathbf{X}$  is a nonempty subclass of  $\mathbf{M}_1$ . If  $\mathbf{X}$  is empty, then

$$(11) \quad \mathbf{P}_X = \mathbf{F}$$

holds by Theorem 20. Equality (11) contradicts equality (7). Hence,  $\mathbf{X}$  is nonempty. To complete the proof we must show that

$$(12) \quad \mathbf{X} \subseteq \mathbf{M}_1.$$

Let us consider the constant function  $f_0 : [0, \infty) \rightarrow [0, \infty)$  such that

$$(13) \quad f_0(t) = 0$$

for every  $t \in [0, \infty)$ . Then  $f_0$  belongs to  $\mathbf{F}_0$ . Hence, for every  $(X, d) \in \mathbf{X}$ , the mapping  $d_0 := f_0 \circ d$  is a metric on  $X$ . Now (13) implies  $d_0(x, y) = 0$  for all  $x, y \in X$  and  $(X, d) \in \mathbf{X}$ . Hence,  $\text{card}(X) = 1$  holds, because the metric space  $(X, d_0)$  is one-point by positivity property. Inclusion (12) follows. The proof is completed.  $\square$

The next theorem gives us all solutions to the equation  $\mathbf{P}_X = \mathbf{Am}$ .

**Theorem 23.** *The following statements are equivalent for every  $\mathbf{X} \subseteq \mathbf{M}$ .*

(i) *The inclusion*

$$(14) \quad \mathbf{X} \subseteq \mathbf{Dis}$$

*holds, and there is  $(Y, \rho) \in \mathbf{X}$  with*

$$(15) \quad \text{card}(Y) \geq 2,$$

*and we have*

$$(16) \quad \mathbf{Dis}_{X_1} \subseteq \mathbf{X}$$

*for every  $(X_1, d_1) \in \mathbf{X}$ .*

(ii) *The equality*

$$(17) \quad \mathbf{P}_X = \mathbf{Am}$$

*holds.*

*Proof.* (i)  $\Rightarrow$  (ii). Let (i) hold. Equality (17) holds iff

$$(18) \quad \mathbf{P}_X \supseteq \mathbf{Dis}$$

and

$$(19) \quad \mathbf{P}_X \subseteq \mathbf{Dis}.$$

Let us prove (18). Inclusion (18) holds iff we have

$$(20) \quad (X_1, f \circ d_1) \in \mathbf{X}$$

for all  $f \in \mathbf{Am}$  and  $(X_1, d_1) \in \mathbf{X}$ . Relation (20) follows from Theorem 22 if  $(X_1, d_1) \in \mathbf{M}_1$ . To see it we only note that  $\mathbf{Am} \subseteq \mathbf{F}_0$ . Let us consider the case when

$$\text{card}(X_1) \geq 2.$$

Since  $(X_1, d_1)$  is discrete by (14), Definition 6 implies that there is  $k_1 \in (0, \infty)$  satisfying

$$d_1(x, y) = k_1$$

for all distinct  $x, y \in X_1$ . Let  $f \in \mathbf{Am}$  be arbitrary. Then  $f(k_1)$  is strictly positive and

$$f(d_1(x, y)) = f(k_1)$$

holds for all distinct  $x, y \in X_1$ . Thus,  $f \circ d_1$  is discrete metric on  $X_1$ , i.e. we have

$$(21) \quad (X_1, f \circ d_1) \in \mathbf{Dis}_{X_1}.$$

Now, (20) follows from (16) and (21).

Let us prove (19). To do it we must show that every  $f \in \mathbf{P}_X$  is amenable.

Suppose contrary that  $f$  belongs to  $\mathbf{P}_X$  but the equality

$$(22) \quad f(t_1) = 0$$

holds with some  $t_1 \in (0, \infty)$ . By statement (i) we can find  $(Y, \rho) \in \mathbf{X}$  such that (15) and

$$\rho(x, y) = t_1$$

hold for all distinct  $x, y \in Y$ . Now  $f \in \mathbf{P}_X$  and  $(Y, \rho) \in \mathbf{X}$  imply that  $f \circ \rho$  is a metric on  $Y$ . Consequently, for all distinct  $x, y \in Y$ , we have

$$f(\rho(x, y)) = f(t_1) > 0,$$

which contradicts (22). The validity of (19) follows.

(ii)  $\Rightarrow$  (i). Let  $\mathbf{X}$  satisfy equality (4). Since  $\mathbf{Am} \neq \mathbf{F}$  holds, the class  $\mathbf{X}$  is nonempty by Theorem 20. Moreover, using Theorem 22 we see that  $\mathbf{X}$  contains a metric space  $(X, d)$  with  $\text{card}(X) \geq 2$ , because  $\mathbf{Am} \neq \mathbf{F}_0$ .

If the inequality

$$\text{card}(Y) \leq 2$$

holds for every  $(Y, \rho) \in \mathbf{X}$ , then all metric spaces belonging to  $\mathbf{X}$  are discrete (see Example 7). Using the definitions of **Dis** and **Am**, it is easy to prove that for each  $(X_1, d_1) \in \mathbf{Dis}$  and every  $(X_1, d) \in \mathbf{Dis}_{X_1}$  there exists  $f \in \mathbf{Am}$  such that  $d = f \circ d_1$ . Hence to complete the proof it suffices to show that every  $(X, d) \in \mathbf{X}$  is discrete when

$$(23) \quad \text{card}(X) \geq 3.$$

Let us consider arbitrary  $(X, d) \in \mathbf{X}$  satisfying (23). Suppose that  $(X, d) \notin \mathbf{Dis}$ . Then by Proposition 8 there are distinct  $a, b, c \in X$  such that

$$(24) \quad d(a, b) \neq d(b, c) \neq d(c, a).$$

Let  $c_1$  and  $c_2$  be points of  $(0, \infty)$  such that

$$(25) \quad c_2 > 2c_1.$$

Now we can define  $f \in \mathbf{Am}$  as

$$(26) \quad f(t) := \begin{cases} 0 & \text{if } t = 0, \\ c_2 & \text{if } t = d(b, c), \\ c_1 & \text{otherwise.} \end{cases}$$

Equality (17) implies that  $f \circ d$  is a metric on  $X$ . Consequently, we have

$$(27) \quad f(d(b, c)) \leq f(d(b, a)) + f(d(b, c))$$

by triangle inequality. Now using (24) and (26) we can rewrite (27) as

$$c_2 \leq c_1 + c_1,$$

which contradicts (15). It implies  $(X, d) \in \mathbf{Dis}$ . The proof is completed.  $\square$

**Corollary 24.** *The equalities*

$$\mathbf{P}_{\mathbf{Dis}} = \mathbf{P}_{\mathbf{M}_2} = \mathbf{Am}$$

*hold.*

**Remark 25.** The equality

$$\mathbf{P}_{\mathbf{M}_2} = \mathbf{Am}$$

is known, see, for example, Remark 1.2 in paper [13]. This paper contains also “constructive” characterizations of the smallest bilateral ideal and the largest subgroup of the monoid  $\mathbf{P}_{\mathbf{M}}$ .

**Proposition 26.** *Let  $\mathbf{X}$  be a subclass of  $\mathbf{M}$ . Then  $\mathbf{P}_{\mathbf{X}}$  is a submonoid of  $(\mathbf{F}, \circ)$ .*

*Proof.* It follows directly from Definition 1 that

$$\mathbf{P}_\mathbf{X} \subseteq \mathbf{F}$$

holds and that the identity mapping  $\text{id} : [0, \infty) \rightarrow [0, \infty)$  belongs to  $\mathbf{P}_\mathbf{X}$ . Hence, by Lemma 18, it is suffices to prove

$$(28) \quad f \circ g \in \mathbf{P}_\mathbf{X}$$

for all  $f, g \in \mathbf{P}_\mathbf{X}$ .

Let us consider arbitrary  $f \in \mathbf{P}_\mathbf{X}$  and  $g \in \mathbf{P}_\mathbf{X}$ . Then, using Definition 1, we see that  $(X, g \circ d)$  belongs to  $\mathbf{X}$  for every  $(X, d) \in \mathbf{X}$ . Consequently,

$$(29) \quad (X, f \circ (g \circ d)) \in \mathbf{X}$$

holds. Since the composition of functions is always associative, the equality

$$(30) \quad (f \circ g) \circ d = f \circ (g \circ d)$$

holds for every  $(X, d) \in \mathbf{X}$ . Now (28) follows from (29) and (30).  $\square$

The above proposition implies the following corollary.

**Corollary 27.** *If the equation*

$$\mathbf{P}_\mathbf{X} = \mathbf{A}$$

*has a solution, then  $\mathbf{A}$  is a submonoid of  $\mathbf{F}$ .*

The following example shows that the converse of Corollary 27 is, generally speaking, false.

**Example 28.** Let us define  $\mathbf{A}_1 \subseteq \mathbf{F}$  as

$$\mathbf{A}_1 = \{f_1, \text{id}\},$$

where  $f_1 \in \mathbf{F}$  is defined such that

$$(31) \quad f_1(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t = 1, \\ t & \text{otherwise} \end{cases}$$

and  $\text{id}$  is the identical mapping of  $[0, \infty)$ . The equalities  $f_1 \circ f_1 = \text{id}$ ,  $f_1 \circ \text{id} = f_1 = \text{id} \circ f_1$  show that  $\mathbf{A}_1$  is a submonoid of  $(\mathbf{F}, \circ)$ . Suppose that there is  $\mathbf{X}_1 \subseteq \mathbf{M}$  satisfying the equality

$$(32) \quad \mathbf{P}_{\mathbf{X}_1} = \mathbf{A}_1.$$

Then using Theorem 20, we see that  $\mathbf{X}_1$  is nonempty because  $\mathbf{A}_1 \neq \mathbf{F}$  holds. Let  $(X_1, d_1)$  be an arbitrary metric space from  $\mathbf{A}_1$ . Since  $X_1$  is

nonempty, we can find  $x_1 \in X_1$ . Then (32) implies that  $f_1 \circ d_1$  is metric on  $X_1$ . Consequently, we have

$$f_1(d_1(x_1, x_1)) = f_1(0) = 0,$$

which contradicts (31).

#### 4. SUBMONOIDS OF MONOIDS $\mathbf{P}_M$ AND $\mathbf{P}_U$

The following theorem gives a solution to Problem 2.

**Theorem 29.** *Let  $\mathbf{A}$  be a nonempty subset of the set  $\mathbf{P}_M$  of all metric preserving functions. Then the following statements are equivalent.*

(i) *The equality*

$$(33) \quad \mathbf{P}_X = \mathbf{A}$$

*has a solution  $\mathbf{X} \subseteq \mathbf{M}$ .*

(ii)  *$\mathbf{A}$  is a submonoid of  $(\mathbf{F}, \circ)$ .*

(iii)  *$\mathbf{A}$  is a submonoid of  $(\mathbf{P}_M, \circ)$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Suppose that there is  $\mathbf{X} \subseteq \mathbf{M}$  such that (33) holds. Then  $\mathbf{A}$  is a submonoid of  $(\mathbf{F}, \circ)$  by Proposition 26.

(ii)  $\Rightarrow$  (iii). Let  $\mathbf{A}$  be a submonoid of  $(\mathbf{F}, \circ)$ . By Proposition 26, the monoid  $(\mathbf{P}_M, \circ)$  also is a submonoid of  $(\mathbf{F}, \circ)$ . Then using the inclusion  $\mathbf{A} \subseteq \mathbf{P}_M$  we obtain that  $\mathbf{A}$  is a submonoid of  $(\mathbf{P}_M, \circ)$  by Lemma 18.

(iii)  $\Rightarrow$  (i). Let  $\mathbf{A}$  be a submonoid of  $(\mathbf{P}_M, \circ)$ . We must prove that (33) has a solution  $\mathbf{X} \subseteq \mathbf{M}$ .

Let  $(X, d)$  be a metric space such that

$$(34) \quad \{d(x, y) : x, y \in X\} = [0, \infty).$$

Write

$$(35) \quad \mathbf{X} := \{(X, f \circ d) : f \in \mathbf{A}\}.$$

We claim that (33) holds if  $\mathbf{X}$  is defined by (35). To prove it we note that (33) holds iff

$$(36) \quad \mathbf{A} \subseteq \mathbf{P}_X$$

and

$$(37) \quad \mathbf{A} \supseteq \mathbf{P}_X.$$

Let us prove (36). This inclusion holds if for every  $f \in \mathbf{A}$  and each  $(Y, \rho) \in \mathbf{X}$  we have  $(Y, f \circ \rho) \in \mathbf{X}$ . Let us consider arbitrary  $(Y, \rho) \in \mathbf{X}$  and  $f \in \mathbf{A}$ . Then, using (35), we can find  $g \in \mathbf{A}$  such that

$$(38) \quad X = Y \quad \text{and} \quad \rho = g \circ d.$$

Since  $\mathbf{A}$  is a monoid, the membership relations  $f \in \mathbf{A}$  and  $g \in \mathbf{A}$  imply  $g \circ f \in \mathbf{A}$ . Hence, we have

$$(39) \quad (X, g \circ f \circ d) \in \mathbf{X}$$

by (35). Now  $(Y, f \circ \rho) \in \mathbf{X}$  follows from (38) and (39).

Let us prove (37). Let  $g_1$  belong to  $\mathbf{P}_\mathbf{X}$  and let  $(X, d)$  be the same as in (35). Then  $(X, g_1 \circ d)$  belongs to  $\mathbf{X}$  and, using (35), we can find  $f_1 \in \mathbf{A}$  such that

$$(40) \quad (X, g_1 \circ d) = (X, f_1 \circ d).$$

The last equality implies

$$(41) \quad g_1(d(x, y)) = f_1(d(x, y))$$

for all  $x, y \in X$ . Consequently,  $g_1(t) = f_1(t)$  holds for every  $t \in [0, \infty)$  by (34). Thus, we have  $g_1 = f_1$ . That implies  $g_1 \in \mathbf{A}$ . Inclusion (37) follows. The proof is completed.  $\square$

Let us turn now to Question 4. Proposition 14 and Lemma 19 give us the following result.

**Theorem 30.** *There is  $\mathbf{X} \subseteq \mathbf{M}$  such that*

$$(42) \quad \mathbf{P}_\mathbf{X} = \mathbf{S}\mathbf{I}.$$

*Proof.* By Proposition 26, the monoids  $(\mathbf{P}_\mathbf{M}, \circ)$  and  $(\mathbf{P}_\mathbf{U}, \circ)$  are submonoids of  $(\mathbf{F}, \circ)$ . The equality

$$(43) \quad \mathbf{S}\mathbf{I} = \mathbf{P}_\mathbf{M} \cap \mathbf{P}_\mathbf{U}$$

holds by Proposition 14. Using (43) and Lemma 19 with  $T_1 = \mathbf{P}_\mathbf{M}$ ,  $T_2 = \mathbf{P}_\mathbf{U}$  and  $\mathbf{S} = \mathbf{F}$  we see that  $\mathbf{S}\mathbf{I}$  also is a submonoid of  $\mathbf{F}$ . Consequently, Theorem 29 with  $\mathbf{A} = \mathbf{S}\mathbf{I}$  implies that there is  $\mathbf{X} \subseteq \mathbf{M}$  such that (42) holds.  $\square$

The next theorem is an ultrametric analog of Theorem 29 and it gives us a solution to Problem 3.

**Theorem 31.** *Let  $\mathbf{A}$  be a nonempty subset of the set  $\mathbf{P}_\mathbf{U}$  of all ultrametric preserving functions. Then the following statements are equivalent.*

- (i) *The equality  $\mathbf{P}_\mathbf{X} = \mathbf{A}$  has a solution  $\mathbf{X} \subseteq \mathbf{U}$ .*
- (ii)  *$\mathbf{A}$  is a submonoid of  $(\mathbf{F}, \circ)$ .*
- (iii)  *$\mathbf{A}$  is a submonoid of  $(\mathbf{P}_\mathbf{U}, \circ)$ .*

A proof of Theorem 31 can be obtained by a simple modification of the proof of Theorem 29. We only note that the ultrametric space defined in Example 5 satisfies the equality (34) with  $X = \mathbb{R}_0^+$  and  $d = d^+$ .

## 5. TWO CONJECTURES

**Conjecture 32.** *The equality*

$$\mathbf{P}_X = \mathbf{A}$$

*has a solution  $\mathbf{X} \subseteq \mathbf{M}$  for every submonoid  $\mathbf{A}$  of the monoid  $\mathbf{Am}$ .*

Example 28 shows that we cannot replace  $\mathbf{Am}$  with  $\mathbf{F}$  in Conjecture 32, but we hope that the following is valid.

**Conjecture 33.** *For every submonoid  $\mathbf{A}$  of the monoid  $\mathbf{F}$  there exists  $\mathbf{X} \subseteq \mathbf{M}$  such that  $\mathbf{P}_X$  and  $\mathbf{A}$  are isomorphic submonoids.*

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**Viktoria Bilet**

DEPARTMENT OF THEORY OF FUNCTIONS  
INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS OF NASU  
DOBROVOLSKOGO STR. 1, SLOVYANSK 84100, UKRAINE  
*Email address:* viktoriabilet@gmail.com

**Oleksiy Dovgoshey**

DEPARTMENT OF THEORY OF FUNCTIONS  
INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS OF NASU  
DOBROVOLSKOGO STR. 1, SLOVYANSK 84100, UKRAINE  
AND  
UNIVERSITY OF TURKU  
FI-200014 TURUN YLIOPISTO, FINLAND  
*Email address:* oleksiy.dovgoshey@gmail.com; oleksiy.dovgoshey@utu.fi