QUASI-FROBENIUS NOVIKOV ALGEBRAS AND PRE-NOVIKOV BIALGEBRAS

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ABSTRACT. Pre-Novikov algebras and quasi-Frobenius Novikov algebras naturally appear in the theory of Novikov bialgebras. In this paper, we show that there is a natural pre-Novikov algebra structure associated to a quasi-Frobenius Novikov algebra. Then we introduce the definition of double constructions of quasi-Frobenius Novikov algebras associated to two pre-Novikov algebras and show that it is characterized by a pre-Novikov bialgebra. We also introduce the notion of pre-Novikov Yang-Baxter equation, whose symmetric solutions can produce pre-Novikov bialgebras. Moreover, the operator forms of pre-Novikov Yang-Baxter equation are also investigated.

1. Introduction

Novikov algebras appeared in the study of Hamiltonian operators in the formal variational calculus [7, 8] and Poisson brackets of hydrodynamic type [5]. It was also shown in [19] that Novikov algebras correspond to a class of Lie conformal algebras, which describe the singular part of operator product expansion of chiral fields in conformal field theory [13]. Note that Novikov algebras are also an important subclass of pre-Lie algebras (also called left-symmetric algebras), which are closely related to many fields in mathematics and physics such as affine manifolds and affine structures on Lie groups [15], convex homogeneous cones [18], deformation of associative algebras [9] and vertex algebras [1, 4].

The definitions of pre-Novikov algebras and quasi-Frobenius Novikov algebras naturally appeared in the study of Novikov bialgebras [11]. It was shown in [11] that there is a Novikov algebra associated to a pre-Novikov algebra and pre-Novikov algebras can produce skewsymmetric solutions of Novikov Yang-Baxter equation and hence Novikov bialgebras. Moreover, by [20], pre-Novikov algebras correspond to a class of left-symmetric conformal algebras [12] and there are close relationships between pre-Novikov algebras and Zinbiel algebras with a derivation (see [11, 14]). Quasi-Frobenius Novikov algebras are a class of Novikov algebras with a special bilinear form, which are closely related with the skewsymmetric solutions of Novikov Yang-Baxter equation (see [11]). Moreover, it was shown in [11] that quasi-Frobenius Novikov algebras also correspond to a class of quasi-Frobenius infinite-dimensional Lie algebras.

As we know, in the case of associative algebras, there is a natural construction of Frobenius algebras called a double construction of Frobenius algebras (see [2]). It was shown in [2]

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that a double construction of Frobenius algebras is characterized by an antisymmetric infinitesimal bialgebra. Motivated by this, it is natural to consider double constructions of quasi-Frobenius Novikov algebras and the relationships with the bialgebra theory of pre-Novikov algebras. Note that pre-Novikov algebras are a class of L-dendriform algebras [3] and the theory of L-dendriform bialgebras was developed in [3, 16]. Therefore, it is also natural to investigate the bialgebra theory of pre-Novikov algebras.

In this paper, we introduce the definitions of pre-Novikov bialgebras and double constructions of quasi-Frobenius Novikov algebras associated to two pre-Novikov algebras and show that these two definitions are equivalent. We also investigate a special class of pre-Novikov bialgebras, which resembles the coboundary Lie bialgebras [6]. The notion of pre-Novikov Yang-Baxter equation is also introduced, whose symmetric solutions give pre-Novikov bialgebras. We introduce the 0-operator on a pre-Novikov algebra as an operator form of pre-Novikov Yang-Baxter equation. It gives a symmetric solution of pre-Novikov Yang-Baxter equation in a semi-direct product pre-Novikov algebra. It should be pointed out that although pre-Novikov algebras are L-dendriform algebras, their bialgebra theories are totally different, for example, skewsymmetric solutions of LD-equation give L-dendriform bialgebras, whereas symmetric solutions of pre-Novikov Yang-Baxter equation give pre-Novikov bialgebras.

This paper is organized as follows. In Section 2, we recall some basic facts about Novikov algebras, pre-Novikov algebras and quasi-Frobenius Novikov algebras. Moreover, we show that there is a pre-Novikov algebra associated to a quasi-Frobenius Novikov algebra. In Section 3, the definition of double constructions of quasi-Frobenius Novikov algebras associated to two pre-Novikov algebras is introduced and we show that it is equivalent to a special matched pair of Novikov algebras. Moreover, we introduce the definition of pre-Novikov bialgebras and prove that it can also characterize the double construction of quasi-Frobenius Novikov algebras associated to two pre-Novikov algebras. In Section 4, we introduce the definition of pre-Novikov Yang-Baxter equation and show that symmetric solutions of pre-Novikov Yang-Baxter equation can produce pre-Novikov bialgebras. Moreover, we investigate the operator forms of pre-Novikov Yang-Baxter equation. The definition of O-operators on pre-Novikov algebras is introduced and we show that it produces a symmetric solution of pre-Novikov Yang-Baxter equation in a semi-direct product pre-Novikov algebra.

Notations. Throughout this paper, we fix a base field **k** of characteristic 0. All vector spaces and algebras are over **k**. Unless otherwise stated, they are assumed to be finite-dimensional even though many results still hold in the infinite-dimensional cases. The identity map is denoted by id. Let A be a vector space with a binary operation \circ . Define linear maps L_{\circ} , R_{\circ} : $A \to \operatorname{End}_{\mathbf{k}}(A)$ by

$$L_{\circ}(a)b := a \circ b, \ R_{\circ}(a)b := b \circ a, \ a, b \in A.$$

Let

$$\tau: A \otimes A \to A \otimes A$$
, $a \otimes b \mapsto b \otimes a$, $a, b \in A$,

be the flip operator.

2. Preliminaries

In this section, we recall some known facts about Novikov algebras and pre-Novikov algebras and show that there is a natural pre-Novikov algebra structure associated to a quasi-Frobenius Novikov algebra.

Recall that a **Novikov algebra** is a vector space A with a binary operation $\circ: A \otimes A \to A$ satisfying

$$(1) (a \circ b) \circ c - a \circ (b \circ c) = (b \circ a) \circ c - b \circ (a \circ c),$$

$$(a \circ b) \circ c = (a \circ c) \circ b, \quad a, b, c \in A.$$

Denote it by (A, \circ) .

Definition 2.1. [17] A **representation** of a Novikov algebra (A, \circ) is a triple (V, l, r), where V is a vector space and $l, r : A \to \operatorname{End}_{\mathbf{k}}(V)$ are linear maps satisfying

(3)
$$l(a \circ b - b \circ a)v = l(a)l(b)v - l(b)l(a)v,$$

(4)
$$l(a)r(b)v - r(b)l(a)v = r(a \circ b)v - r(b)r(a)v,$$

$$(5) l(a \circ b)v = r(b)l(a)v,$$

(6)
$$r(a)r(b)v = r(b)r(a)v, \quad a, b \in A, \quad v \in V.$$

Remark 2.2. Obviously, $(A, L_{\circ}, R_{\circ})$ is a representation of (A, \circ) , which is called the **adjoint representation** of (A, \circ) .

Let (A, \circ) be a vector space with a binary operation \circ and V be a vector space. For a linear map $\rho: A \to \operatorname{End}_{\mathbf{k}}(V)$, define a linear map $\rho^*: A \to \operatorname{End}_{\mathbf{k}}(V^*)$ by

(7)
$$\langle \rho^*(a)f, v \rangle = -\langle f, \rho(a)v \rangle, \quad a \in A, v \in V, f \in V^*,$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing between V and V^* .

Proposition 2.3. [11, Proposition 3.3] Let (V, l, r) be a representation of a Novikov algebra (A, \circ) . Then $(V^*, l^* + r^*, -r^*)$ is also a representation of (A, \circ) .

Remark 2.4. The adjoint representation (A, L_o, R_o) of a Novikov algebra (A, o) gives the representation $(A^*, L_o^* + R_o^*, -R_o^*)$.

Definition 2.5. [11] Let *A* be a vector space. A **pre-Novikov algebra** is a triple $(A, \triangleleft, \triangleright)$, where $\triangleleft, \triangleright : A \otimes A \to A$ are binary operations satisfying

(8)
$$a \triangleright (b \triangleright c) = (a \circ b) \triangleright c + b \triangleright (a \triangleright c) - (b \circ a) \triangleright c,$$

(9)
$$a \triangleright (b \triangleleft c) = (a \triangleright b) \triangleleft c + b \triangleleft (a \circ c) - (b \triangleleft a) \triangleleft c,$$

$$(10) (a \circ b) \triangleright c = (a \triangleright c) \triangleleft b,$$

$$(11) (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft b, \quad a, b, c \in A,$$

where $a \circ b = a \triangleleft b + a \triangleright b$.

Proposition 2.6. [11, Proposition 3.33] *Let* $(A, \triangleleft, \triangleright)$ *be a pre-Novikov algebra. The binary operation*

$$(12) a \circ b := a \triangleleft b + a \triangleright b, \quad a, b \in A,$$

defines a Novikov algebra, which is called the **associated Novikov algebra** of $(A, \triangleleft, \triangleright)$. Furthermore, $(A, L_{\triangleright}, R_{\triangleleft})$ is a representation of (A, \circ) . Conversely, let A be a vector space equipped with binary operations \triangleleft and \triangleright . If (A, \circ) defined by Eq. (12) is a Novikov algebra and $(A, L_{\triangleright}, R_{\triangleleft})$ is a representation of (A, \circ) , then $(A, \triangleleft, \triangleright)$ is a pre-Novikov algebra.

Definition 2.7. [11] Let (V, l, r) be a representation of a Novikov algebra (A, \circ) . A linear map $T: V \to A$ is called an \emptyset -operator on (A, \circ) associated to (V, l, r) if T satisfies

(13)
$$T(u) \circ T(v) = T(l(T(u))v) + T(r(T(v))u), \quad u, v \in V.$$

Proposition 2.8. [11, Proposition 3.34] Let (V, l, r) be a representation of a Novikov algebra (A, \circ) . If T is an \bigcirc -operator on (A, \circ) associated to (V, l, r), then there is a pre-Novikov algebra structure on V defined by

(14)
$$u \triangleright v := l(T(u))v, \quad u \triangleleft v := r(T(v))u, \quad u, v \in V.$$

Definition 2.9. [11] Let (A, \circ) be a Novikov algebra. If there is a skewsymmetric nondegenerate bilinear form $\omega(\cdot, \cdot)$ on A satisfying

(15)
$$\omega(a \circ b, c) - \omega(a \circ c + c \circ a, b) + \omega(c \circ b, a) = 0, \quad a, b, c \in A,$$

then $(A, \circ, \omega(\cdot, \cdot))$ is called a quasi-Frobenius Novikov algebra.

Next, we show that there is a natural pre-Novikov algebra structure on A, when $(A, \circ, \omega(\cdot, \cdot))$ is a quasi-Frobenius Novikov algebra.

Theorem 2.10. Let $(A, \circ, \omega(\cdot, \cdot))$ be a quasi-Frobenius Novikov algebra. Then there is a compatible pre-Novikov algebra structure on A given by

(16)
$$\omega(a \triangleright b, c) = \omega(a \circ c + c \circ a, b),$$

(17)
$$\omega(a \triangleleft b, c) = \omega(a, c \circ b), \quad a, b, c \in A,$$

such that (A, \circ) is the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. This pre-Novikov algebra is called the **associated pre-Novikov algebra** of $(A, \circ, \omega(\cdot, \cdot))$.

Proof. Obviously, the nondegenerate bilinear form $\omega(\cdot, \cdot)$ on A can induce an invertible linear map $T: A^* \to A$ given by

(18)
$$\omega(T(f), a) = \langle f, a \rangle, \quad f \in A^*, a \in A.$$

Therefore, for all $a, b, c \in A$, we have

$$\begin{aligned} \omega(a \rhd b, c) &= \omega(a \circ c + c \circ a, b) = -\omega(b, a \circ c + c \circ a) \\ &= -\langle T^{-1}(b), a \circ c + c \circ a \rangle = -\langle T^{-1}(b), (L_{\circ} + R_{\circ})(a)c \rangle \\ &= \langle (L_{\circ}^* + R_{\circ}^*)(a)T^{-1}(b), c \rangle = \omega(T((L_{\circ}^* + R_{\circ}^*)(a)T^{-1}(b)), c), \end{aligned}$$

and

$$\omega(a \triangleleft b, c) = \omega(a, c \circ b) = \langle T^{-1}(a), c \circ b \rangle$$
$$= \langle T^{-1}(a), R_{\circ}(b)c \rangle = -\langle R_{\circ}^{*}(b)T^{-1}(a), c \rangle$$
$$= \omega(T((-R_{\circ}^{*})(b)T^{-1}(a)), c).$$

By the nondegenerate property of $\omega(\cdot, \cdot)$, we obtain

(19)
$$a > b = T((L_{\circ}^* + R_{\circ}^*)(a)T^{-1}(b)), \quad a < b = T((-R_{\circ}^*)(b)T^{-1}(a)), \quad a, b \in A.$$

Set $a = T(f), b = T(g)$. Define

$$f \rhd_* g := (L_\circ^* + R_\circ^*)(T(f))g,$$

 $f \vartriangleleft_* g := (-R_\circ^*)(T(g))f, \quad f, g \in A^*.$

Then we have

$$a \triangleright b = T(f) \triangleright T(g) = T(f \triangleright_* g),$$

 $a \triangleleft b = T(f) \triangleleft T(g) = T(f \triangleleft_* g), \quad a, b \in A.$

If we prove that $(A^*, \triangleleft_*, \triangleright_*)$ is a pre-Novikov algebra, then $(A, \triangleleft, \triangleright)$ is a pre-Novikov algebra and T is an isomorphism of pre-Novikov algebras. Therefore, we only need to show that $(A^*, \triangleleft_*, \triangleright_*)$ is a pre-Novikov algebra.

For all f, g, $h \in A^*$, we have

$$\begin{split} \langle h, T(f) \circ T(g) - T((L_{\circ}^* + R_{\circ}^*)(T(f))g - R_{\circ}^*(T(g))f) \rangle \\ &= \omega(T(h), T(f) \circ T(g)) - \omega(T(h), T((L_{\circ}^* + R_{\circ}^*)(T(f))g - R_{\circ}^*(T(g))f)) \\ &= \omega(T(h), T(f) \circ T(g)) + \omega(T((L_{\circ}^* + R_{\circ}^*)(T(f))g - R_{\circ}^*(T(g))f), T(h)) \\ &= \omega(T(h), T(f) \circ T(g)) + \langle (L_{\circ}^* + R_{\circ}^*)(T(f))g - R_{\circ}^*(T(g))f, T(h) \rangle \\ &= \omega(T(h), T(f) \circ T(g)) - \langle g, T(f) \circ T(h) + T(h) \circ T(f) \rangle + \langle f, T(h) \circ T(g) \rangle \\ &= \omega(T(h), T(f) \circ T(g)) - \omega(T(g), T(f) \circ T(h) + T(h) \circ T(f)) + \omega(T(f), T(h) \circ T(g)) \\ &= 0. \end{split}$$

Therefore, we obtain

$$T(f) \circ T(g) - T((L_{\circ}^* + R_{\circ}^*)(T(f))g - R_{\circ}^*(T(g))f) = 0.$$

Hence $T:A^*\to A$ defined by Eq. (18) is an invertible \emptyset -operator on (A,\circ) associated to $(A^*,L_\circ^*+R_\circ^*,-R_\circ^*)$. By Proposition 2.8, $(A^*,\triangleleft_*,\triangleright_*)$ is a pre-Novikov algebra. Moreover,

$$a \triangleleft b + a \triangleright b = T((L_{\circ}^* + R_{\circ}^*)(a)T^{-1}(b)) + T((-R_{\circ}^*)(b)T^{-1}(a))$$

= $T(f) \circ T(g)$
= $a \circ b$.

Therefore, (A, \circ) is the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. The proof is completed.

3. Double constructions of quasi-Frobenius Novikov algebras and pre-Novikov bialgebras

In this section, we introduce the definitions of double constructions of quasi-Frobenius Novikov algebras and pre-Novikov bialgebras, and show that the double construction of quasi-Frobenius Novikov algebras is equivalent to a pre-Novikov bialgebra, which is also characterized by some matched pair of Novikov algebras.

First, we recall matched pairs of Novikov algebras.

Proposition 3.1. [10] Let (A, \circ) and (B, \bullet) be Novikov algebras. If (B, l_A, r_A) is a representation of (A, \circ) , (A, l_B, r_B) is a representation of (B, \bullet) and the following conditions are satisfied:

- $(20) \quad l_B(x)(a \circ b) = -l_B(l_A(a)x r_A(a)x)b + (l_B(x)a r_B(x)a) \circ b + r_B(r_A(b)x)a + a \circ (l_B(x)b),$
- (21) $r_B(x)(a \circ b b \circ a) = r_B(l_A(b)x)a r_B(l_A(a)x)b + a \circ (r_B(x)b) b \circ (r_B(x)a),$
- (22) $l_A(a)(x \bullet y) = -l_A(l_B(x)a r_B(x)a)y + (l_A(a)x r_A(a)x) \bullet y + r_A(r_B(y)a)x + x \bullet (l_A(a)y),$
- $(23) \quad r_A(a)(x \bullet y y \bullet x) = r_A(l_B(y)a)x r_A(l_B(x)a)y + x \bullet (r_A(a)y) y \bullet (r_A(a)x),$
- (24) $(l_B(x)a) \circ b + l_B(r_A(a)x)b = (l_B(x)b) \circ a + l_B(r_A(b)x)a$,
- (25) $(r_R(x)a) \circ b + l_R(l_A(a)x)b = r_R(x)(a \circ b),$
- (26) $l_A(r_B(x)a)y + (l_A(a)x) \bullet y = l_A(r_B(y)a)x + (l_A(a)y) \bullet x$,
- (27) $l_A(l_B(x)a)y + (r_A(a)x) \bullet y = r_A(a)(x \bullet y), \quad a, b \in A, x, y \in B,$

then there is a Novikov algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$(a + x) \cdot (b + y) := (a \circ b + l_B(x)b + r_B(y)a) + (x \bullet y + l_A(a)y + r_A(b)x), \quad a, b \in A, \quad x, y \in B.$$

 $(A, B, l_A, r_A, l_B, r_B)$ satisfying the above conditions is called a **matched pair of Novikov algebras**. Conversely, any Novikov algebra that can be decomposed into a linear direct sum of two Novikov subalgebras is obtained from a matched pair of Novikov algebras.

Next, we introduce the definition of double constructions of quasi-Frobenius Novikov algebras associated to pre-Novikov algebras.

Definition 3.2. Let $(A, \triangleleft, \triangleright)$, $(A^*, \triangleleft_*, \triangleright_*)$ be pre-Novikov algebras and (A, \circ) , (A^*, \circ_*) be their associated Novikov algebras respectively. If a quasi-Frobenius Novikov algebra $(B, \cdot, \omega(\cdot, \cdot))$ satisfies the following conditions:

- (a) B is the direct sum of A and A^* as vector spaces,
- (b) $(A, \triangleleft, \triangleright)$ and $(A^*, \triangleleft_*, \triangleright_*)$ are pre-Novikov subalgebras of $(B, \trianglelefteq, \trianglerighteq)$, which is the associated pre-Novikov algebra of $(B, \cdot, \omega(\cdot, \cdot))$,
- (c) the bilinear form $\omega(\cdot, \cdot)$ on $B = A \oplus A^*$ is given by

(29)
$$\omega(a+f,b+g) = \langle f,b \rangle - \langle g,a \rangle, \quad a,b \in A, \quad f,g \in A^*,$$

then $(B, \cdot, \omega(\cdot, \cdot))$ is called a **double construction of quasi-Frobenius Novikov algebras** associated to $(A, \triangleleft, \triangleright)$ and $(A^*, \triangleleft_*, \triangleright_*)$.

We give a characterization of double constructions of quasi-Frobenius Novikov algebras associated to pre-Novikov algebras by matched pairs of Novikov algebras.

Proposition 3.3. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and (A, \circ) be the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. Suppose that there exists a pre-Novikov algebra structure $(A^*, \triangleleft_*, \triangleright_*)$ on the dual vector space A^* and (A^*, \circ_*) is the associated Novikov algebra of $(A^*, \triangleleft_*, \triangleright_*)$. Then there is a double construction of quasi-Frobenius Novikov algebras associated to $(A, \triangleleft, \triangleright)$ and $(A^*, \triangleleft_*, \triangleright_*)$ if and only if $(A, A^*, L^*_{\triangleright} + R^*_{\triangleleft}, -R^*_{\triangleleft}, L^*_{\triangleright_*} + R^*_{\triangleleft_*}, -R^*_{\triangleleft_*})$ is a matched pair of Novikov algebras.

Proof. Suppose that $(A, A^*, L^*_{\triangleright} + R^*_{\triangleleft}, -R^*_{\triangleleft}, L^*_{\triangleright_*} + R^*_{\triangleleft_*}, -R^*_{\triangleleft_*})$ is a matched pair of Novikov algebras. Then by Proposition 3.1, there is a Novikov algebra structure $(A \oplus A^*, \cdot)$ on the vector space $A \oplus A^*$ given by Eq. (28). By Eq. (29), for all $a, b, c \in A$, and $f, g, h \in A^*$, we have

$$\begin{split} &\omega((a+f)\cdot(b+g),c+h)\\ &=\omega((a\circ b+(L_{\rhd_*}^*+R_{\vartriangleleft_*}^*)(f)b+(-R_{\vartriangleleft_*}^*)(g)a)+(f\circ_*g+(L_{\rhd}^*+R_{\vartriangleleft}^*)(a)g+(-R_{\vartriangleleft}^*)(b)f),c+h)\\ &=\langle f\circ_*g+(L_{\rhd}^*+R_{\vartriangleleft}^*)(a)g+(-R_{\vartriangleleft}^*)(b)f,c\rangle-\langle h,a\circ b+(L_{\rhd_*}^*+R_{\vartriangleleft_*}^*)(f)b+(-R_{\vartriangleleft_*}^*)(g)a\rangle\\ &=\langle f,c\vartriangleleft b\rangle-\langle g,a\rhd c+c\vartriangleleft a\rangle-\langle h,a\circ b\rangle-\langle h\vartriangleleft_*g,a\rangle\\ &+\langle f\rhd_*h+h\vartriangleleft_*f,b\rangle+\langle f\circ_*g,c\rangle. \end{split}$$

Similarly, we obtain

$$\omega((a+f)\cdot(c+h),b+g) \ = \ \langle f,b \lhd c \rangle - \langle g,a \circ c \rangle - \langle h,a \rhd b+b \lhd a \rangle \\ - \langle g \lhd_* h,a \rangle + \langle f \circ_* h,b \rangle + \langle f \rhd_* g+g \lhd_* f,c \rangle, \\ \omega((c+h)\cdot(a+f),b+g) \ = \ - \langle f,c \rhd b+b \lhd c \rangle - \langle g,c \circ a \rangle + \langle h,b \lhd a \rangle \\ + \langle h \rhd_* g+g \lhd_* h,a \rangle + \langle h \circ_* f,b \rangle - \langle g \lhd_* f,c \rangle, \\ \omega((c+h)\cdot(b+g),a+f) \ = \ - \langle f,c \circ b \rangle - \langle g,c \rhd a+a \lhd c \rangle + \langle h,a \lhd b \rangle \\ + \langle h \circ_* g,a \rangle + \langle h \rhd_* f+f \lhd_* h,b \rangle - \langle f \lhd_* g,c \rangle.$$

Hence we get

$$\omega((a+f)\cdot(b+g),c+h) - \omega((a+f)\cdot(c+h) + (c+h)\cdot(a+f),b+g)$$

$$+\omega((c+h)\cdot(b+g),a+f)$$

$$= \langle f,c \triangleleft b-b \triangleleft c+(c \triangleright b+b \triangleleft c)-c \circ b \rangle$$

$$+\langle g,-(a \triangleright c+c \triangleleft a)+a \circ c+c \circ a-(c \triangleright a+a \triangleleft c) \rangle$$

$$+\langle h,-a \circ b+(a \triangleright b+b \triangleleft a)-b \triangleleft a+a \triangleleft b \rangle$$

$$+\langle a,-h \triangleleft_* g+g \triangleleft_* h-(h \triangleright_* g+g \triangleleft_* h)+h \circ_* g \rangle$$

$$+\langle b,(f \triangleright_* h+h \triangleleft_* f)-f \circ_* h-h \circ_* f+(h \triangleright_* f+f \triangleleft_* h) \rangle$$

$$+\langle c,f \circ_* g-(f \triangleright_* g+g \triangleleft_* f)+g \triangleleft_* f-f \triangleleft_* g \rangle$$

$$= 0.$$

Therefore, $(A \oplus A^*, \cdot, \omega(\cdot, \cdot))$ is a quasi-Frobenius Novikov algebra. Then by Theorem 2.10, there is a compatible pre-Novikov algebra structure on $A \oplus A^*$. Denote it by $(A \oplus A^*, \leq, \succeq)$. It suffices to check that (A, \lhd, \rhd) and (A^*, \lhd_*, \rhd_*) are pre-Novikov subalgebras of $(A \oplus A^*, \leq, \succeq)$. For all $a, b, c \in A, h \in A^*$, we have

$$\omega(a \trianglerighteq b, c+h) - \omega(a \trianglerighteq b, c+h) = \omega(a \cdot (c+h), b) + \omega((c+h) \cdot a, b) - \omega(a \trianglerighteq b, c+h)$$
$$= -\langle h, a \trianglerighteq b + b \triangleleft a \rangle + \langle h, b \triangleleft a \rangle + \langle h, a \trianglerighteq b \rangle$$
$$= 0,$$

and

$$\omega(a \le b, c+h) - \omega(a < b, c+h) = \omega(a, (c+h) \cdot b) - \omega(a < b, c+h)$$
$$= -\langle h, a < b \rangle + \langle h, a < b \rangle$$
$$= 0.$$

By the nondegenerate property of $\omega(\cdot, \cdot)$, we obtain $a \ge b = a > b$ and $a \le b = a < b$ for all $a, b \in A$. Hence $(A, \triangleleft, \triangleright)$ is a pre-Novikov subalgebra of $(A \oplus A^*, \triangleleft, \trianglerighteq)$. Similarly, we can prove that $(A^*, \triangleleft_*, \triangleright_*)$ is a pre-Novikov subalgebra of $(A \oplus A^*, \triangleleft, \trianglerighteq)$. Hence $(A \oplus A^*, \cdot, \omega(\cdot, \cdot))$ is a double construction of quasi-Frobenius Novikov algebras associated to $(A, \triangleleft, \triangleright)$ and $(A^*, \triangleleft_*, \triangleright_*)$.

Conversely, suppose that $(A \oplus A^*, \cdot, \omega(\cdot, \cdot))$ is a double construction of quasi-Frobenius Novikov algebras associated to $(A, \triangleleft, \triangleright)$ and $(A^*, \triangleleft_*, \triangleright_*)$. By Proposition 3.1, the Novikov algebra $(A \oplus A^*, \cdot)$ is obtained from a matched pair of Novikov algebras $(A, A^*, l_A, r_A, l_{A^*}, r_{A^*})$, where \cdot is given by Eq. (28). By Theorem 2.10, we have

$$\langle l_A(a)f,b\rangle = \langle a\cdot f,b\rangle = \omega(b,a\cdot f) = \omega(b,a\cdot f+f\cdot a) - \omega(b,f\cdot a)$$
$$= \omega(f,a\triangleright b) + \omega(f,b\triangleleft a)$$
$$= \langle (L_{\triangleright}^* + R_{\triangleleft}^*)(a)f,b\rangle,$$

and

$$\langle r_A(a)f, b \rangle = \langle f \cdot a, b \rangle = \omega(b, f \cdot a)$$

$$= \omega(b \triangleleft a, f) = \langle R_{\triangleleft}(a)b, f \rangle$$

$$= -\langle R_{\triangleleft}^*(a)f, b \rangle, \qquad a, b \in A, f \in A^*.$$

Therefore, we obtain $l_A = L_{\triangleright}^* + R_{\triangleleft}^*$ and $r_A = -R_{\triangleleft}^*$. Similarly, we can prove $l_{A^*} = L_{\triangleright_*}^* + R_{\triangleleft_*}^*$ and $r_{A^*} = -R_{\triangleleft_*}^*$.

The proof is completed.

Next, we introduce the definition of pre-Novikov coalgebras.

Definition 3.4. A **pre-Novikov coalgebra** is a vector space *A* with linear maps $\alpha, \beta : A \to A \otimes A$ satisfying

- $(30) \quad (\alpha \otimes \mathrm{id})\alpha(a) + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \alpha)\beta(a) (\mathrm{id} \otimes (\alpha + \beta))\alpha(a) (\tau \otimes \mathrm{id})(\beta \otimes \mathrm{id})\alpha(a) = 0,$
- (31) $(id \otimes \beta)\beta(a) + (\tau \otimes id)((\alpha + \beta) \otimes id)\beta(a) ((\alpha + \beta) \otimes id)\beta(a) (\tau \otimes id)(id \otimes \beta)\beta(a) = 0$,
- (32) $(id \otimes \tau)(\beta \otimes id)\alpha(a) ((\alpha + \beta) \otimes id)\beta(a) = 0$,
- (33) $(id \otimes \tau)(\alpha \otimes id)\alpha(a) (\alpha \otimes id)\alpha(a) = 0, \quad a \in A.$

Denote it by (A, α, β) .

Let V and W be vector spaces. Suppose that $\phi: V \to W$ is a linear map. Then there is an induced dual linear map $\phi^*: W^* \to V^*$ defined by

(34)
$$\langle \phi^*(f), v \rangle = \langle f, \phi(v) \rangle, \quad v \in V, f \in W^*.$$

It is easy to see that (A, α, β) is a pre-Novikov coalgebra if and only if (A^*, α^*, β^*) is a pre-Novikov algebra.

Proposition 3.5. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and (A, \circ) be the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. Suppose that there is a pre-Novikov algebra $(A^*, \triangleleft_*, \triangleright_*)$ which is induced from a pre-Novikov coalgebra (A, α, β) , whose associated Novikov algebra is denoted by (A^*, \circ_*) . Then $(A, A^*, L^*_{\triangleright} + R^*_{\triangleleft}, -R^*_{\triangleleft}, L^*_{\triangleright_*} + R^*_{\triangleleft_*}, -R^*_{\triangleleft_*})$ is a matched pair of Novikov algebras if and only if the following equations are satisfied:

(35)
$$(\tau \alpha + \beta)(a \circ b) = ((L_{\triangleright} + 2R_{\triangleleft})(a) \otimes \mathrm{id} + \mathrm{id} \otimes L_{\circ}(a))(\tau \alpha + \beta)(b)$$
$$+ (\mathrm{id} \otimes R_{\circ}(b))(2\tau \alpha + \beta)(a) - (R_{\triangleleft}(b) \otimes \mathrm{id})\tau \alpha(a),$$

(36)
$$\tau \alpha(a \circ b - b \circ a) = ((L_{\triangleright} + R_{\triangleleft})(a) \otimes id + id \otimes L_{\circ}(a))\tau \alpha(b)$$
$$- ((L_{\triangleright} + R_{\triangleleft})(b) \otimes id + id \otimes L_{\circ}(b))\tau \alpha(a),$$

(37)

$$(\alpha + \beta)(a \rhd b + b \vartriangleleft a) = (\mathrm{id} \otimes (R_{\rhd} + L_{\vartriangleleft})(b))(2\tau\alpha + \beta)(a) - (L_{\vartriangleleft}(b) \otimes \mathrm{id})\alpha(a) + ((L_{\rhd} + 2R_{\vartriangleleft})(a) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\rhd} + R_{\vartriangleleft})(a))(\alpha + \beta)(b),$$

(38)

$$(\alpha + \beta - \tau \alpha - \tau \beta)(b \triangleleft a) = (\mathrm{id} \otimes L_{\triangleleft}(b))(\tau \alpha + \beta)(a) - (L_{\triangleleft}(b) \otimes \mathrm{id})(\alpha + \tau \beta)(a) + (\mathrm{id} \otimes R_{\triangleleft}(a))(\alpha + \beta)(b) - (R_{\triangleleft}(a) \otimes \mathrm{id})(\tau \alpha + \tau \beta)(b),$$

$$(39) \qquad (\mathrm{id} \otimes R_{\circ}(b) - R_{\lhd}(b) \otimes \mathrm{id})(\tau \alpha + \beta)(a) = (\mathrm{id} \otimes R_{\circ}(a) - R_{\lhd}(a) \otimes \mathrm{id})(\tau \alpha + \beta)(b),$$

$$(40) \tau \alpha(a \circ b) = (\mathrm{id} \otimes R_{\circ}(b)) \tau \alpha(a) + ((L_{\triangleright} + R_{\triangleleft})(a) \otimes \mathrm{id})(\tau \alpha + \beta)(b),$$

(41)

$$(\mathrm{id} \otimes (R_{\triangleright} + L_{\triangleleft})(b))\tau\alpha(a) = ((R_{\triangleright} + L_{\triangleleft})(b) \otimes \mathrm{id})\alpha(a) + (\mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(a))(\tau\alpha + \tau\beta)(b) - ((L_{\triangleright} + R_{\triangleleft})(a) \otimes \mathrm{id})(\alpha + \beta)(b),$$

$$(42) \hspace{1cm} (\alpha+\beta)(b\vartriangleleft a)=(\mathrm{id}\otimes(R_{\rhd}+L_{\vartriangleleft})(b))(\tau\alpha+\beta)(a)+(R_{\vartriangleleft}(a)\otimes\mathrm{id})(\alpha+\beta)(b), \ \ a,b\in A.$$

Proof. By Proposition 3.1, it suffices to show that Eqs. (35)-(42) are equivalent to Eqs. (20)-(27) respectively when $B = A^*$, $l_A = L_{\triangleright}^* + R_{\triangleleft}^*$, $r_A = -R_{\triangleleft}^*$, $l_{A^*} = L_{\triangleright_*}^* + R_{\triangleleft_*}^*$ and $r_{A^*} = -R_{\triangleleft_*}^*$. Let $a \in A$ and $f, g \in A^*$. By Eq. (27), we have

$$-R_{\triangleleft}^{*}(a)(f \circ_{*} g) = (L_{\triangleright}^{*} + R_{\triangleleft}^{*})((L_{\triangleright_{*}}^{*} + R_{\triangleleft_{*}}^{*})(f)a)g - (R_{\triangleleft}^{*}(a)f) \circ_{*} g$$
$$= (L_{\triangleright}^{*} + R_{\triangleleft}^{*})((L_{\triangleright_{*}}^{*} + R_{\triangleleft_{*}}^{*})(f)a)g - (L_{\triangleright_{*}} + L_{\triangleleft_{*}})(R_{\triangleleft}^{*}(a)f)g.$$

Let both sides of the above equation act on an arbitrary element $b \in A$. Then we have

$$-\langle R_{\lhd}^*(a)(f\circ_*g),b\rangle=\langle (L_{\rhd}^*+R_{\lhd}^*)((L_{\rhd_*}^*+R_{\lhd_*}^*)(f)a)g,b\rangle-\langle (L_{\rhd_*}+L_{\lhd_*})(R_{\lhd}^*(a)f)g,b\rangle,$$

which is equivalent to the following equation

$$\langle f \otimes g, (\alpha + \beta)(b \triangleleft a) \rangle = \langle f \otimes g, (\mathrm{id} \otimes (R_{\triangleright} + L_{\triangleleft})(b))(\tau \alpha + \beta)(a) \rangle + \langle f \otimes g, (R_{\triangleleft}(a) \otimes \mathrm{id})(\alpha + \beta)(b) \rangle.$$

Then it is easy to see that Eq. (42) holds if and only if Eq. (27) holds. The others can be proved similarly.

Definition 3.6. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and (A, α, β) be a pre-Novikov coalgebra. If they also satisfy Eqs. (35)-(42), then we call $(A, \triangleleft, \triangleright, \alpha, \beta)$ a **pre-Novikov bialgebra**.

By Propositions 3.3 and 3.5, we obtain the following conclusion.

Theorem 3.7. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and (A, \circ) be the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. Suppose that there is a pre-Novikov algebra $(A^*, \triangleleft_*, \triangleright_*)$ which is induced from a pre-Novikov coalgebra (A, α, β) , whose associated Novikov algebra is denoted by (A^*, \circ_*) . Then the following conditions are equivalent.

- (a) There is a double construction of quasi-Frobenius Novikov algebras associated to $(A, \triangleleft, \triangleright)$ and $(A^*, \triangleleft_*, \triangleright_*)$;
- (b) $(A, A^*, L^*_{\triangleright} + R^*_{\triangleleft}, -R^*_{\triangleleft}, L^*_{\triangleright_*} + R^*_{\triangleleft_*}, -R^*_{\triangleleft_*})$ is a matched pair of Novikov algebras;
- (c) $(A, \triangleleft, \triangleright, \alpha, \beta)$ is a pre-Novikov bialgebra.

Finally, we present an example of pre-Novikov bialgebras and quasi-Frobenius Novikov algebras.

Example 3.8. Let $(A = \mathbf{k}e_1 \oplus \mathbf{k}e_2, \triangleleft, \triangleright)$ be a two-dimensional vector space with binary operations $\triangleleft, \triangleright$ given by

$$\begin{aligned} e_i \rhd e_j &= 0, & i, j \in \{1, 2\}, \\ e_1 \vartriangleleft e_1 &= e_1, & e_1 \vartriangleleft e_2 &= e_2, & e_2 \vartriangleleft e_1 &= e_2, & e_2 \vartriangleleft e_2 &= 0. \end{aligned}$$

One can directly check that $(A, \triangleleft, \triangleright)$ is a pre-Novikov algebra. Define linear maps $\alpha, \beta : A \rightarrow A \otimes A$ by

$$\alpha(e_1) = e_2 \otimes e_2, \quad \alpha(e_2) = 0,$$

 $\beta(e_1) = -e_2 \otimes e_2, \quad \beta(e_2) = 0.$

Then it is easy to check that $(A, \triangleleft, \triangleright, \alpha, \beta)$ is a pre-Novikov bialgebra.

Let $\{e_1^*, e_2^*\}$ be the dual basis of A^* . Then by Theorem 3.7, there is a double construction of quasi-Frobenius Novikov algebras $(A \oplus A^*, \cdot, \omega(\cdot, \cdot))$ which is defined by non-zero products

$$e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_1 \cdot e_1^* = -e_1^*,$$

 $e_1^* \cdot e_1 = e_1^*, \quad e_1 \cdot e_2^* = e_2 - e_2^*, \quad e_2^* \cdot e_1 = e_2^*, \quad e_2 \cdot e_2^* = -e_1^*, \quad e_2^* \cdot e_2 = e_1^*,$

and the nondegenerate bilinear form $\omega(\cdot, \cdot)$ given by

$$\omega(e_1^*, e_1) = \omega(e_2^*, e_2) = -\omega(e_1, e_1^*) = -\omega(e_2, e_2^*) = 1,$$

$$\omega(e_1, e_2^*) = \omega(e_2, e_1^*) = \omega(e_1^*, e_2) = \omega(e_2^*, e_1) = \omega(e_i, e_j) = \omega(e_i^*, e_j^*) = 0, \quad i, j \in \{1, 2\}.$$

4. Pre-Novikov Yang-Baxter equation

In this section, we introduce the definition of pre-Novikov Yang-Baxter equation whose symmetric solutions can be used to construct pre-Novikov bialgebras. Moreover, the operator forms of pre-Novikov Yang-Baxter equation are investigated.

Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and (A, \circ) be the associated Novikov algebra. For convenience, we define two binary operations \odot and \star on A by

$$a \odot b := a \triangleright b + b \triangleleft a$$
, $a \star b := a \circ b + b \circ a$, $a, b \in A$.

Lemma 4.1. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and $r \in A \otimes A$. Suppose that $\alpha, \beta : A \to A \otimes A$ are linear maps defined by

(44)
$$\alpha(a) := (L_{\circ}(a) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(a))\tau r,$$

$$\beta(a) := -(L_{\triangleright}(a) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\circ} + R_{\circ})(a))r, \quad a \in A.$$

Then α, β satisfy Eqs. (35)-(42) if and only if the following conditions are satisfied:

$$(46) \left(((L_{\triangleright} + 2R_{\triangleleft})(a) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(a))(L_{\circ}(b) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b)) \right. \\ \left. - (L_{\triangleleft}(b) \otimes \operatorname{id})(L_{\circ}(a) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(a)) \right. \\ \left. - (L_{\circ}(a \triangleright b + b \triangleleft a) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(a \triangleright b + b \triangleleft a)))(\tau r - r) = 0,$$

$$(47) \left((\mathrm{id} \otimes R_{\triangleleft}(a))(L_{\circ}(b) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b)) + (R_{\triangleleft}(a) \otimes \mathrm{id})((L_{\circ} + R_{\circ})(b) \otimes \mathrm{id} \right)$$

$$+ \mathrm{id} \otimes L_{\triangleright}(b)) - (L_{\triangleleft}(b) \otimes \mathrm{id})(\mathrm{id} \otimes R_{\triangleleft}(a) - R_{\circ}(a) \otimes \mathrm{id})$$

$$(\mathrm{id} \otimes (2L_{\triangleright} + R_{\circ})(b_{\triangleleft} + a) + (2L_{\triangleright} + R_{\circ})(b_{\triangleleft} + a) \otimes \mathrm{id})(\pi r_{\circ} - r_{\circ}) = 0$$

$$-\left(\mathrm{id}\otimes(2L_{\rhd}+R_{\lhd})(b\vartriangleleft a)+(2L_{\circ}+R_{\circ})(b\vartriangleleft a)\otimes\mathrm{id}\right)\right)(\tau r-r)=0,$$

$$(48) \left(((R_{\triangleright} + L_{\triangleleft})(b) \otimes id)(L_{\circ}(a) \otimes id + id \otimes (L_{\triangleright} + R_{\triangleleft})(a)) - (id \otimes (L_{\triangleright} + R_{\triangleleft})(a))(id \otimes L_{\triangleright}(b) + (L_{\circ} + R_{\circ})(b) \otimes id) - ((L_{\triangleright} + R_{\triangleleft})(a) \otimes id)(L_{\circ}(b) \otimes id + id \otimes (L_{\triangleright} + R_{\triangleleft})(b))(\tau r - r) = 0,$$

(49)
$$((R_{\triangleleft}(a) \otimes id)(L_{\circ}(b) \otimes id + id \otimes (L_{\triangleright} + R_{\triangleleft})(b))$$

 $- (L_{\circ}(b \triangleleft a) \otimes id + id \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a))(\tau r - r) = 0, \quad a, b \in A.$

Proof. It is easy to check that α, β satisfy Eqs. (35), (36), (39) and (40) automatically. By Eq. (42), we have

$$(\operatorname{id} \otimes (R_{\triangleright} + L_{\triangleleft})(b))(\tau \alpha + \beta)(a) + (R_{\triangleleft}(a) \otimes \operatorname{id})(\alpha + \beta)(b) - (\alpha + \beta)(b \triangleleft a)$$

$$= (\operatorname{id} \otimes (R_{\triangleright} + L_{\triangleleft})(b))(R_{\triangleleft}(a) \otimes \operatorname{id} - \operatorname{id} \otimes R_{\circ}(a))r$$

$$+ (R_{\triangleleft}(a) \otimes \operatorname{id})(L_{\circ}(b) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b))\tau r$$

$$- (R_{\triangleleft}(a) \otimes \operatorname{id})(L_{\triangleright}(b) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\circ} + R_{\circ})(b))r$$

$$- (L_{\circ}(b \triangleleft a) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a))\tau r$$

$$+ (L_{\triangleright}(b \triangleleft a) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\circ} + R_{\circ})(b \triangleleft a))r$$

$$= ((R_{\triangleleft}(a) \otimes \operatorname{id})(L_{\circ}(b) \otimes \operatorname{id} + \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b))$$

$$- L_{\circ}(b \triangleleft a) \otimes \operatorname{id} - \operatorname{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a))\tau r$$

$$+ (\operatorname{id} \otimes (-(R_{\triangleright} + L_{\triangleleft})(b)R_{\circ}(a) + (L_{\circ} + R_{\circ})(b \triangleleft a))$$

$$+ (-R_{\triangleleft}(a)L_{\triangleright}(b) + L_{\triangleright}(b \triangleleft a)) \otimes \operatorname{id} - R_{\triangleleft}(a) \otimes (L_{\triangleright} + R_{\triangleleft})(b))r.$$

By Definition 2.5, for all $a, b, c \in A$, we have

$$-(c \circ a) \odot b + c \star (b \triangleleft a) = (b \triangleleft a) \odot c$$

and

$$-(b \triangleright c) \triangleleft a + (b \triangleleft a) \triangleright c = -(b \triangleright a) \triangleright c.$$

Hence we obtain

$$-(R_{\triangleright} + L_{\triangleleft})(b)R_{\circ}(a) + (L_{\circ} + R_{\circ})(b \triangleleft a) = (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a),$$

and

$$-R_{\triangleleft}(a)L_{\triangleright}(b) + L_{\triangleright}(b \triangleleft a) = -L_{\triangleright}(b \triangleleft a).$$

Therefore, one gets

$$(\mathrm{id} \otimes (R_{\triangleright} + L_{\triangleleft})(b))(\tau \alpha + \beta)(a) + (R_{\triangleleft}(a) \otimes \mathrm{id})(\alpha + \beta)(b) - (\alpha + \beta)(b \triangleleft a)$$

$$= ((R_{\triangleleft}(a) \otimes \mathrm{id})(L_{\circ}(b) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b)) - L_{\circ}(b \triangleleft a) \otimes \mathrm{id}$$

$$- \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a))\tau r + (\mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a)$$

$$- R_{\triangleleft}(a) \otimes (L_{\triangleright} + R_{\triangleleft})(b) - L_{\triangleright}(b \triangleright a) \otimes \mathrm{id})r$$

$$= ((R_{\triangleleft}(a) \otimes \mathrm{id})(L_{\circ}(b) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b))$$

$$- (L_{\circ}(b \triangleleft a) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a)))(\tau r - r)$$

$$+ (R_{\triangleleft}(a)L_{\circ}(b) \otimes \mathrm{id} + R_{\triangleleft}(a) \otimes (L_{\triangleright} + R_{\triangleleft})(b) - L_{\circ}(b \triangleleft a) \otimes \mathrm{id}$$

$$- \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a) + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a)$$

$$- R_{\triangleleft}(a) \otimes (L_{\triangleright} + R_{\triangleleft})(b) - L_{\triangleright}(b \triangleright a) \otimes \mathrm{id} r$$

$$= (R_{\triangleleft}(a) \otimes \mathrm{id})(L_{\circ}(b) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b))$$

$$- (L_{\circ}(b \triangleleft a) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(b \triangleleft a))(\tau r - r)$$

$$+ (R_{\triangleleft}(a)L_{\circ}(b) \otimes \mathrm{id} - L_{\circ}(b \triangleleft a) \otimes \mathrm{id} - L_{\triangleright}(b \triangleright a) \otimes \mathrm{id} r .$$

By Definition 2.5, for all $a, b, c \in A$, we have

$$(b \circ a) \triangleleft c - (b \triangleleft a) \circ c - (b \triangleright a) \triangleright c$$

$$= (b \triangleright c) \triangleleft a + (b \triangleleft c) \triangleleft a - (b \triangleleft a) \triangleleft c - (b \triangleleft a) \triangleright c - (b \triangleright a) \triangleright c$$

$$= ((b \triangleright c) \triangleleft a - (b \triangleleft a) \triangleright c - (b \triangleright a) \triangleright c) + ((b \triangleleft c) \triangleleft a - (b \triangleleft a) \triangleleft c)$$

$$= 0.$$

Hence we obtain

$$R_{\triangleleft}(a)L_{\triangleright}(b) \otimes \mathrm{id} - L_{\triangleright}(b \triangleleft a) \otimes \mathrm{id} - L_{\triangleright}(b \triangleright a) \otimes \mathrm{id} = 0.$$

Therefore Eq. (42) is equivalent to Eq. (49).

One similarly verifies that Eq. (37) is equivalent to Eq. (46), Eq. (38) is equivalent to Eq. (47), and Eq. (41) is equivalent to Eq. (48). Then the proof is completed.

Let V be a vector space with a binary operation *. Let $r = \sum_i x_i \otimes y_i \in V \otimes V$ and $r' = \sum_i x_i' \otimes y_i' \in V \otimes V$. Set

$$r_{12} * r_{13}' := \sum_{i,j} x_i * x_j' \otimes y_i \otimes y_j', \quad r_{12} * r_{23}' := \sum_{i,j} x_i \otimes y_i * x_j' \otimes y_j', \quad r_{13} * r_{12}' := \sum_{i,j} x_i * x_j' \otimes y_j' \otimes y_i,$$

$$r_{13} * r_{21}' := \sum_{i,j} x_i * y_j' \otimes x_j' \otimes y_i, \quad r_{13} * r_{23}' := \sum_{i,j} x_i \otimes x_j' \otimes y_i * y_j', \quad r_{21} * r_{13}' := \sum_{i,j} y_i * x_j' \otimes x_i \otimes y_j',$$

$$r_{21} * r_{23} := \sum_{i,j} y_i \otimes x_i * x_j' \otimes y_j', \quad r_{21} * r_{31}' := \sum_{i,j} y_i * y_j' \otimes x_i \otimes x_j', \quad r_{21} * r_{32}' := \sum_{i,j} y_i \otimes x_i * y_j' \otimes x_j',$$

$$\begin{split} r_{31} * r_{21}' &:= \sum_{i,j} y_i * y_j' \otimes x_j' \otimes x_i, \ r_{31} * r_{23}' := \sum_{i,j} y_i \otimes x_j' \otimes x_i * y_j', \ r_{31} * r_{32}' := \sum_{i,j} y_i \otimes y_j' \otimes x_i * x_j', \\ r_{23} * r_{12}' &:= \sum_{i,j} x_j' \otimes x_i * y_j' \otimes y_i, \ r_{23} * r_{21}' := \sum_{i,j} y_j' \otimes x_i * x_j' \otimes y_i, \ r_{23} * r_{13}' := \sum_{i,j} x_j' \otimes x_i \otimes y_i * y_j', \\ r_{23} * r_{31}' &:= \sum_{i,j} y_j' \otimes x_i \otimes y_i * x_j', \ r_{32} * r_{21} := \sum_{i,j} y_j' \otimes y_i * x_j' \otimes x_i. \end{split}$$

Lemma 4.2. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and $r = \sum_{i} x_i \otimes y_i \in A \otimes A$. Let $\alpha, \beta : A \rightarrow A \otimes A$ be linear maps defined by Eqs. (44) and (45) respectively. Then (A, α, β) is a pre-Novikov coalgebra if and only if the following conditions are satisfied for all $a \in A$:

$$(50) \ (L_{\circ}(a) \otimes \operatorname{id} \otimes \operatorname{id})R_{11} + (\operatorname{id} \otimes L_{\triangleright}(a) \otimes \operatorname{id})R_{12} + (\operatorname{id} \otimes \operatorname{id} \otimes L_{\circ}(a))R_{13} \\ - \sum_{j} \left((\operatorname{id} \otimes L_{\triangleright}(a \odot x_{j}) \otimes \operatorname{id})(y_{j} \otimes (\tau r - r)) \right) \\ - \sum_{j} \left((\operatorname{id} \otimes \operatorname{id} \otimes L_{\circ}(a \odot x_{i}))(y_{i} \otimes (\tau r - r)) \right) = 0, \\ (51) \ (L_{\triangleright}(a) \otimes \operatorname{id} \otimes \operatorname{id} - \operatorname{id} \otimes L_{\triangleright}(a) \otimes \operatorname{id})R_{21} + (\operatorname{id} \otimes \operatorname{id} \otimes L_{\star}(a))R_{22} \\ + \sum_{j} \left(((2L_{\triangleright} + R_{\triangleleft})(a \triangleright x_{j}) \otimes \operatorname{id} \otimes \operatorname{id})((\tau r - r) \otimes y_{j}) \right) \\ + (\operatorname{id} \otimes (2L_{\triangleright} + R_{\triangleleft})(a \triangleright x_{j}) \otimes \operatorname{id})((\tau r - r) \otimes y_{j}) = 0, \\ (52) \ - (\operatorname{id} \otimes L_{\circ}(a) \otimes \operatorname{id})R_{21} + (\operatorname{id} \otimes \operatorname{id} \otimes L_{\star}(a))R_{31} \\ + \sum_{j} \left((L_{\triangleright}(a \triangleright x_{j}) \otimes \operatorname{id} \otimes \operatorname{id})((\tau r - r) \otimes y_{j}) + (\operatorname{id} \otimes L_{\circ}(a \triangleright x_{j}) \otimes \operatorname{id})((\tau r - r) \otimes y_{j}) \right) = 0, \\ (53) \ - (\operatorname{id} \otimes L_{\circ}(a) \otimes \operatorname{id})R_{12} + (\operatorname{id} \otimes \operatorname{id} \otimes L_{\circ}(a))R_{41} = 0,$$

where

$$R_{11} = r_{21} \circ r_{31} - r_{21} \circ r_{32} - r_{31} \odot r_{32} + r_{21} \triangleright r_{23} + r_{31} \star r_{23},$$

$$R_{12} = -r_{21} \circ r_{31} - r_{23} \odot r_{31} + r_{21} \triangleleft r_{32},$$

$$R_{13} = r_{31} \circ r_{21} + r_{32} \odot r_{21} + r_{23} \triangleright r_{31} - r_{31} \triangleleft r_{23} + r_{32} \triangleright r_{21} + r_{31} \star r_{21},$$

$$R_{21} = r_{21} \triangleright r_{13} + r_{12} \triangleright r_{23} + r_{13} \star r_{23},$$

$$R_{22} = r_{13} \circ r_{21} + r_{23} \odot r_{21} - r_{13} \triangleright r_{12} - r_{23} \star r_{12} - r_{23} \circ r_{12} - r_{13} \odot r_{12} + r_{23} \odot r_{21} + r_{13} \star r_{21} + r_{23} \circ r_{13} - r_{13} \circ r_{23},$$

$$R_{31} = -r_{13} \circ r_{23} + r_{13} \circ r_{21} + r_{23} \odot r_{21} - r_{13} \triangleright r_{12} - r_{23} \star r_{12},$$

$$R_{41} = -r_{31} \circ r_{21} - r_{32} \odot r_{21} + r_{31} \triangleleft r_{23}.$$

Proof. Let $a \in A$. By Eqs. (44) and (45), we have

$$A(a) = (\alpha \otimes \mathrm{id})\alpha(a) + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \alpha)\beta(a) - (\mathrm{id} \otimes (\alpha + \beta))\alpha(a) - (\tau \otimes \mathrm{id})(\beta \otimes \mathrm{id})\alpha(a)$$

$$= \sum_{i,j} ((a \circ y_i) \circ y_j \otimes x_j \otimes x_i + y_j \otimes (a \circ y_i) \odot x_j \otimes x_i + y_i \circ y_j \otimes x_j \otimes a \odot x_i$$

$$+ y_j \otimes y_i \odot x_j \otimes a \odot x_i - y_i \circ y_j \otimes a \rhd x_i \otimes x_j - y_j \otimes a \rhd x_i \otimes y_i \odot x_j$$

$$- (a \star y_i) \circ y_j \otimes x_i \otimes x_j - y_j \otimes x_i \otimes (a \star y_i) \odot x_j - a \circ y_i \otimes x_i \circ y_j \otimes x_j$$

$$- a \circ y_i \otimes y_j \otimes x_i \odot x_j + a \circ y_i \otimes x_i \rhd x_j \otimes y_j + a \circ y_i \otimes x_j \otimes x_i \star y_j$$

$$- y_i \otimes (a \odot x_i) \circ y_j \otimes x_j - y_i \otimes y_j \otimes (a \odot x_i) \odot x_j + y_i \otimes (a \odot x_i) \rhd x_j \otimes y_j$$

$$+ y_i \otimes x_j \otimes (a \odot x_i) \star y_j + y_j \otimes (a \circ y_i) \rhd x_j \otimes x_i + (a \circ y_i) \star y_j \otimes x_j \otimes x_i$$

$$+ y_j \otimes y_i \rhd x_j \otimes a \odot x_i + y_i \star y_j \otimes x_j \otimes a \odot x_i).$$

Then we obtain $A(a) = P_1(a) + P_2(a) + P_3(a)$, where

$$P_{1}(a) = \sum_{i,j} ((a \circ y_{i}) \circ y_{j} \otimes x_{j} \otimes x_{i} - (a \star y_{i}) \circ y_{j} \otimes x_{i} \otimes x_{j}$$

$$- a \circ y_{i} \otimes x_{i} \circ y_{j} \otimes x_{j} - a \circ y_{i} \otimes y_{j} \otimes x_{i} \odot x_{j}$$

$$+ a \circ y_{i} \otimes x_{i} \triangleright x_{j} \otimes y_{j} + a \circ y_{i} \otimes x_{j} \otimes x_{i} \star y_{j} + (a \circ y_{i}) \star y_{j} \otimes x_{j} \otimes x_{i}),$$

$$P_{2}(a) = \sum_{i,j} (y_{j} \otimes (a \circ y_{i}) \odot x_{j} \otimes x_{i} - y_{i} \circ y_{j} \otimes a \triangleright x_{i} \otimes x_{j} - y_{j} \otimes a \triangleright x_{i} \otimes y_{i} \odot x_{j}$$

$$- y_{i} \otimes (a \odot x_{i}) \circ y_{j} \otimes x_{j} + y_{i} \otimes (a \odot x_{i}) \triangleright x_{j} \otimes y_{j} + y_{j} \otimes (a \circ y_{i}) \triangleright x_{j} \otimes x_{i}),$$

$$P_{3}(a) = \sum_{i,j} (y_{i} \circ y_{j} \otimes x_{j} \otimes a \odot x_{i} + y_{j} \otimes y_{i} \odot x_{j} \otimes a \odot x_{i}$$

$$- y_{j} \otimes x_{i} \otimes (a \star y_{i}) \odot x_{j} - y_{i} \otimes y_{j} \otimes (a \odot x_{i}) \odot x_{j}$$

$$+ y_{i} \otimes x_{j} \otimes (a \odot x_{i}) \star y_{j} + y_{j} \otimes y_{i} \triangleright x_{j} \otimes a \odot x_{i} + y_{i} \star y_{j} \otimes x_{j} \otimes a \odot x_{i}).$$

For $P_1(a)$, since

$$\sum_{i,j} ((a \circ y_i) \circ y_j \otimes x_j \otimes x_i - (a \star y_i) \circ y_j \otimes x_i \otimes x_j + (a \circ y_i) \star y_j \otimes x_j \otimes x_i)$$

$$= \sum_{i,j} a \circ (y_j \circ y_i) \otimes x_j \otimes x_i,$$

we have

$$P_{1}(a) = \sum_{i,j} \left((L_{\circ} \otimes \mathrm{id} \otimes \mathrm{id})(y_{j} \circ y_{i} \otimes x_{j} \otimes x_{i} - y_{i} \otimes x_{i} \circ y_{j} \otimes x_{j} - y_{i} \otimes y_{j} \otimes x_{i} \odot x_{j} \right)$$

$$+ y_{i} \otimes x_{i} \rhd x_{j} \otimes y_{j} + y_{i} \otimes x_{j} \otimes x_{i} \star y_{j})$$

$$= (L_{\circ} \otimes \mathrm{id} \otimes \mathrm{id})(r_{21} \circ r_{31} - r_{21} \circ r_{32} - r_{31} \odot r_{32} + r_{21} \rhd r_{23} + r_{31} \star r_{23}).$$

For $P_2(a)$, since

$$\sum_{i,j} (y_j \otimes (a \circ y_i) \odot x_j \otimes x_i - y_i \otimes (a \odot x_i) \circ y_j \otimes x_j + y_i \otimes (a \odot x_i) \triangleright x_j \otimes y_j$$

$$+ y_j \otimes (a \circ y_i) \triangleright x_j \otimes x_i)$$

$$= \sum_{i,j} y_j \otimes a \triangleright (x_j \triangleleft y_i) \otimes x_i - \sum_j (\mathrm{id} \otimes L_{\triangleright}(a \odot x_j) \otimes \mathrm{id})(y_j \otimes (\tau r - r)),$$

we have

$$P_{2}(a) = \sum_{i,j} \left((\operatorname{id} \otimes L_{\triangleright}(a) \otimes \operatorname{id})(-y_{i} \circ y_{j} \otimes x_{i} \otimes x_{j} - y_{j} \otimes x_{i} \otimes y_{i} \odot x_{j} + y_{j} \odot x_{j} \triangleleft y_{i} \otimes x_{i}) \right)$$

$$- \sum_{j} (\operatorname{id} \otimes L_{\triangleright}(a \odot x_{j}) \otimes \operatorname{id})(y_{j} \otimes (\tau r - r))$$

$$= (\operatorname{id} \otimes L_{\triangleright}(a) \otimes \operatorname{id})(-r_{21} \circ r_{31} - r_{23} \odot r_{31} + r_{21} \triangleleft r_{32})$$

$$- \sum_{j} (\operatorname{id} \otimes L_{\triangleright}(a \odot x_{j}) \otimes \operatorname{id})(y_{j} \otimes (\tau r - r)).$$

For $P_3(a)$, since

$$\sum_{i,j} (-y_j \otimes x_i \otimes (a \star y_i) \odot x_j - y_i \otimes y_j \otimes (a \odot x_i) \odot x_j + y_i \otimes x_j \otimes (a \odot x_i) \star y_j)$$

$$= \sum_{i,j} y_i \otimes x_j \otimes a \odot (y_j \rhd x_i - x_i \triangleleft y_j) - \sum_i (\mathrm{id} \otimes \mathrm{id} \otimes L_{\odot}(a \odot x_i))(y_i \otimes (\tau r - r)),$$

we have

$$P_{3}(a) = \sum_{i,j} ((\mathrm{id} \otimes \mathrm{id} \otimes L_{\odot}(a))(y_{i} \circ y_{j} \otimes x_{j} \otimes x_{i} + y_{j} \otimes y_{i} \odot x_{j} \otimes x_{i} + y_{i} \otimes x_{j} \otimes y_{j} \triangleright x_{i} - y_{i} \otimes x_{j} \otimes x_{i} \vee y_{j} + y_{j} \otimes y_{i} \triangleright x_{j} \otimes x_{i} + y_{i} \star y_{j} \otimes x_{j} \otimes x_{i})) - \sum_{i} (\mathrm{id} \otimes \mathrm{id} \otimes L_{\odot}(a \odot x_{i}))((y_{i} \otimes (\tau r - r)),$$

$$= (\mathrm{id} \otimes \mathrm{id} \otimes L_{\odot}(a))(r_{31} \circ r_{21} + r_{32} \odot r_{21} + r_{23} \triangleright r_{31} - r_{31} \vee r_{23} + r_{32} \triangleright r_{21} + r_{31} \star r_{21})$$

$$- \sum_{i} (\mathrm{id} \otimes \mathrm{id} \otimes L_{\odot}(a \odot x_{i}))(y_{i} \otimes (\tau r - r)).$$

Therefore, Eq. (30) is equivalent to Eq. (50).

Similarly, one can prove that Eq. (31) is equivalent to Eq. (51), Eq. (32) is equivalent to Eq. (52), and Eq. (33) is equivalent to Eq. (53). Then the proof is completed.

Next, we introduce the definition of pre-Novikov Yang-Baxter equation.

Definition 4.3. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and $r \in A \otimes A$. The following equation

$$(54) r_{12} \circ r_{13} + r_{23} \odot r_{13} - r_{12} \triangleleft r_{23} = 0$$

is called the **pre-Novikov Yang-Baxter equation** in $(A, \triangleleft, \triangleright)$.

Theorem 4.4. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and $r \in A \otimes A$ be a symmetric solution of pre-Novikov Yang-Baxter equation in $(A, \triangleleft, \triangleright)$. Let $\alpha, \beta : A \to A \otimes A$ be linear maps defined by Eqs. (44) and (45) respectively. Then $(A, \triangleleft, \triangleright, \alpha, \beta)$ is a pre-Novikov bialgebra.

Proof. Clearly, we have $R_{11} = -R_{12} = 0$ since r is symmetric. Suppose that σ is an element in the permutation group S_3 acting on $\{1,2,3\}$. Then there is a linear map induced by σ from $A \otimes A \otimes A$ to $A \otimes A \otimes A$. We still denote this linear map by σ .

Set $\sigma = (23) \in S_3$. Then we have

$$0 = \sigma(R_{12})$$

$$= \sigma(-r_{21} \circ r_{31} - r_{23} \odot r_{31} + r_{21} \triangleleft r_{32})$$

$$= -r_{\sigma(21)} \circ r_{\sigma(31)} - r_{\sigma(23)} \odot r_{\sigma(31)} + r_{\sigma(21)} \triangleleft r_{\sigma(32)}$$

$$= -r_{31} \circ r_{21} - r_{32} \odot r_{21} + r_{31} \triangleleft r_{23}$$

$$= R_{41}.$$

For R_{21} , set $\sigma = (12) \in S_3$. Since r is symmetric, we have

$$R_{21} = r_{12} \triangleright r_{13} + r_{12} \triangleright r_{23} + r_{13} \star r_{23}$$

$$= (r_{12} \circ r_{13} + r_{23} \odot r_{13} - r_{12} \triangleleft r_{23}) + (r_{12} \circ r_{23} + r_{13} \odot r_{23} - r_{12} \triangleleft r_{13})$$

$$= (r_{21} \circ r_{31} + r_{23} \odot r_{31} - r_{21} \triangleleft r_{32}) + (r_{12} \circ r_{32} + r_{13} \odot r_{32} - r_{12} \triangleleft r_{31})$$

$$= -R_{12} - \sigma(R_{12}) = 0.$$

Similarly, one can verify that R_{13} , R_{22} and R_{31} are equal to zero.

Then this theorem follows directly by Lemmas 4.1 and 4.2.

Next, we investigate the operator forms of pre-Novikov Yang-Baxter equation. First, we introduce the definition of representations of pre-Novikov algebras.

Definition 4.5. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and V be a vector space. Let $l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft}$: $A \to \operatorname{End}_{\mathbf{k}}(V)$ be linear maps. $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ is called a **representation** of $(A, \triangleleft, \triangleright)$ if satisfies

- $(55) l_{\triangleright}(a)l_{\triangleright}(b)v l_{\triangleright}(b)l_{\triangleright}(a)v = l_{\triangleright}(a \circ b b \circ a)v,$
- $(56) l_{\triangleright}(a)l_{\triangleleft}(b)v l_{\triangleleft}(b)l_{\triangleright}(a)v = l_{\triangleleft}(a \triangleright b b \triangleleft a)v + l_{\triangleleft}(b)l_{\triangleleft}(a)v,$
- $(57) r_{\triangleright}(a \triangleright b)v = r_{\triangleright}(b)(r_{\triangleright} + r_{\triangleleft})(a)v + l_{\triangleright}(a)r_{\triangleright}(b)v r_{\triangleright}(b)(l_{\triangleleft} + l_{\triangleright})(a)v,$
- $(58) r_{\triangleright}(a \triangleleft b)v = r_{\triangleleft}(b)r_{\triangleright}(a)v + l_{\triangleleft}(a)(r_{\triangleright} + r_{\triangleleft})(b)v r_{\triangleleft}(b)l_{\triangleleft}(a)v,$
- $(59) l_{\triangleright}(a)r_{\triangleleft}(b)v r_{\triangleleft}(b)l_{\triangleright}(a)v = r_{\triangleleft}(a \circ b)v r_{\triangleleft}(b)r_{\triangleleft}(a)v,$
- (60) $r_{\triangleright}(a)(r_{\triangleright} + r_{\triangleleft})(b)v = r_{\triangleleft}(b)r_{\triangleright}(a)v,$
- (61) $l_{\triangleleft}(a \triangleright b)v = r_{\triangleright}(b)(l_{\triangleright} + l_{\triangleleft})(a)v,$
- $(62) l_{\triangleright}(a \circ b)v = r_{\triangleleft}(b)l_{\triangleright}(a)v,$
- (63) $r_{\triangleleft}(a)r_{\triangleleft}(b)v = r_{\triangleleft}(b)r_{\triangleleft}(a)v,$
- $(64) l_{\triangleleft}(a \triangleleft b)v = r_{\triangleleft}(b)l_{\triangleleft}(a)v, a, b \in A, v \in V,$

where $a \circ b = a \triangleright b + a \triangleleft b$.

Note that $(A, L_{\triangleright}, R_{\triangleright}, L_{\triangleleft}, R_{\triangleleft})$ is a representation of $(A, \triangleleft, \triangleright)$.

Proposition 4.6. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra. Let V be a vector space and $l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft}$: $A \to End_{\mathbf{k}}(V)$ be linear maps. Define two binary operation \triangleleft and \triangleright on the direct sum $A \oplus V$ of the underlying vector spaces of A and V by

$$(a + u) \triangleleft (b + v) := a \triangleleft b + l_{\triangleleft}(a)v + r_{\triangleleft}(b)u,$$

 $(a + u) \triangleright (b + v) := a \triangleright b + l_{\triangleright}(a)v + r_{\triangleright}(b)u, \ a, b \in A, u, v \in V.$

Then $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ is a representation of $(A, \triangleleft, \triangleright)$ if and only if $(A \oplus V, \triangleleft, \triangleright)$ is a pre-Novikov algebra, which is called the **semi-direct product** of A and V and denoted by $A \bowtie_{l_{\triangleright}, l_{\triangleright}, l_{\triangleleft}, r_{\triangleleft}} V$.

Proof. It is straightforward.

Proposition 4.7. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra. If $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ is a representation of $(A, \triangleleft, \triangleright)$, then $(V^*, l_{\triangleright}^* + l_{\triangleleft}^* + r_{\triangleright}^* + r_{\triangleleft}^*, r_{\triangleright}^*, -(r_{\triangleright}^* + l_{\triangleleft}^*), -(r_{\triangleright}^* + r_{\triangleleft}^*))$ is also a representation of $(A, \triangleleft, \triangleright)$.

Proof. For all $a, b \in A$, $f \in V^*$ and $v \in V$, we have

$$\langle (l_{\triangleright}^{*} + l_{\triangleleft}^{*} + r_{\triangleright}^{*} + r_{\triangleleft}^{*})(a)(l_{\triangleright}^{*} + l_{\triangleleft}^{*} + r_{\triangleright}^{*} + r_{\triangleleft}^{*})(b)f - (l_{\triangleright}^{*} + l_{\triangleleft}^{*} + r_{\triangleright}^{*} + r_{\triangleleft}^{*})(b)(l_{\triangleright}^{*} + l_{\triangleleft}^{*} + r_{\triangleright}^{*} + r_{\triangleleft}^{*})(a)f - (l_{\triangleright}^{*} + l_{\triangleleft}^{*} + r_{\triangleright}^{*} + r_{\triangleleft}^{*})(a \circ b - b \circ a)f, v \rangle$$

$$= \langle f, (l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(b)((l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(a)v) - (l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(a)((l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(b)v) + (l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(a \circ b - b \circ a)v \rangle.$$

By Eqs. (55) and (62), we obtain

$$l_{\triangleright}(a)l_{\triangleright}(b)v - l_{\triangleright}(b)l_{\triangleright}(a)v = l_{\triangleright}(a \circ b - b \circ a)v = r_{\triangleleft}(b)l_{\triangleright}(a)v - r_{\triangleleft}(a)l_{\triangleright}(b)v.$$

Then we get

(65)
$$(l_{\triangleright} + r_{\triangleleft})(a)l_{\triangleright}(b)v = (l_{\triangleright} + r_{\triangleleft})(b)l_{\triangleright}(a)v.$$

By Eqs. (56), (61) and (64), one obtains

$$\begin{split} l_{\triangleright}(a)l_{\triangleleft}(b)v - l_{\triangleleft}(b)l_{\triangleright}(a)v \\ &= l_{\triangleleft}(a \triangleright b - b \triangleleft a)v + l_{\triangleleft}(b)l_{\triangleleft}(a)v \\ &= r_{\triangleright}(b)(l_{\triangleright} + l_{\triangleleft})(a)v - r_{\triangleleft}(a)l_{\triangleleft}(b)v + l_{\triangleleft}(b)l_{\triangleleft}(a)v. \end{split}$$

Then we have

(66)
$$(r_{\triangleright} + l_{\triangleleft})(b)(l_{\triangleright} + l_{\triangleleft})(a)v = (l_{\triangleright} + r_{\triangleleft})(a)l_{\triangleleft}(b)v.$$

By Eqs. (65) and (66), one gets

$$(l_{\triangleright} + r_{\triangleleft} + r_{\triangleright} + l_{\triangleleft})(b)(l_{\triangleright} + l_{\triangleleft})(a)v = (l_{\triangleright} + r_{\triangleleft} + r_{\triangleright} + l_{\triangleleft})(a)(l_{\triangleright} + l_{\triangleleft})(b)v.$$

By Eqs. (57), (60) and (61), we have

$$r_{\triangleright}(a \triangleright b)v = r_{\triangleright}(b)(r_{\triangleright} + r_{\triangleleft})(a)v + l_{\triangleright}(a)r_{\triangleright}(b)v - r_{\triangleright}(b)(l_{\triangleright} + l_{\triangleleft})(a)v$$
$$= (l_{\triangleright} + r_{\triangleleft})(a)r_{\triangleright}(b)v - l_{\triangleleft}(a \triangleright b)v.$$

Then we obtain

(67)
$$(r_{\triangleright} + l_{\triangleleft})(a \triangleright b)v = (l_{\triangleright} + r_{\triangleleft})(a)r_{\triangleright}(b)v.$$

By Eqs. (58), (60) and (64), one obtains

$$r_{\triangleright}(a \triangleleft b)v = r_{\triangleleft}(b)r_{\triangleright}(a)v + l_{\triangleleft}(a)(r_{\triangleright} + r_{\triangleleft})(b)v - r_{\triangleleft}(b)l_{\triangleleft}(a)v$$
$$= r_{\triangleright}(a)(r_{\triangleright} + r_{\triangleleft})(b)v + l_{\triangleleft}(a)(r_{\triangleright} + r_{\triangleleft})(b)v - l_{\triangleleft}(a \triangleleft b)v.$$

Then we get

$$(68) (r_{\triangleright} + l_{\triangleleft})(a \triangleleft b)v = (r_{\triangleright} + l_{\triangleleft})(a)(r_{\triangleright} + r_{\triangleleft})(b)v.$$

By Eqs. (59), (62) and (63), we have

$$(l_{\triangleright} + r_{\triangleleft})(a \circ b)v = (l_{\triangleright} + r_{\triangleleft})(a)r_{\triangleleft}(b)v,$$

Then by Eqs. (67), (68) and (69), one gets

$$(r_{\triangleright} + l_{\triangleleft} + l_{\triangleright} + r_{\triangleleft})(a \circ b)v = (r_{\triangleright} + l_{\triangleleft} + l_{\triangleright} + r_{\triangleleft})(a)(r_{\triangleright} + r_{\triangleleft})(b)v.$$

Therefore one obtains

$$\begin{split} (l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(b)(l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(a)v \\ &- (l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(a)(l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(b)v \\ &+ (l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(a \circ b - b \circ a)v \\ &= (l_{\triangleright} + r_{\triangleright} + l_{\triangleleft} + r_{\triangleleft})(b)(l_{\triangleright} + l_{\triangleleft})(a)v - (l_{\triangleright} + r_{\triangleright} + l_{\triangleleft} + r_{\triangleleft})(a)(l_{\triangleright} + l_{\triangleleft})(b)v \\ &+ (l_{\triangleright} + r_{\triangleright} + l_{\triangleleft} + r_{\triangleleft})(b)(r_{\triangleright} + r_{\triangleleft})(a)v - (l_{\triangleright} + r_{\triangleright} + l_{\triangleleft} + r_{\triangleleft})(b \circ a)v \\ &+ (l_{\triangleright} + r_{\triangleright} + l_{\triangleleft} + r_{\triangleleft})(a \circ b)v - (l_{\triangleright} + r_{\triangleright} + l_{\triangleleft} + r_{\triangleleft})(a)(r_{\triangleright} + r_{\triangleleft})(b)v \\ &= 0. \end{split}$$

Hence Eq. (55) holds.

Similarly, one can check that Eqs. (56)-(64) hold. Then the proof is completed.

We introduce the definition of O-operators on a pre-Novikov algebra associated to a representation as follows.

Definition 4.8. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ be a representation of $(A, \triangleleft, \triangleright)$. A linear map $T: V \to A$ is called an \circlearrowleft -**operator** on $(A, \triangleleft, \triangleright)$ associated to $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ if T satisfies

(70)
$$T(u) > T(v) = T(l_{\triangleright}(T(u))v) + T(r_{\triangleright}(T(v))u),$$

(71)
$$T(u) \triangleleft T(v) = T(l_{\triangleleft}(T(u))v) + T(r_{\triangleleft}(T(v))u), \quad u, v \in V.$$

For a finite-dimensional vector space A, the natural isomorphism

$$A \otimes A \cong \operatorname{Hom}_{\mathbf{k}}(A^*, A)$$

identifies an $r \in A \otimes A$ with a linear map $T_r : A^* \to A$, which is defined by

(72)
$$\langle f \otimes g, r \rangle = \langle f, T_r(g) \rangle, \quad f, g \in A^*.$$

Proposition 4.9. Let $(A, \lhd, \triangleright)$ be a pre-Novikov algebra, $r \in A \otimes A$ be symmetric and (A, \circ) be the associated Novikov algebra of $(A, \lhd, \triangleright)$. Then r is a solution of pre-Novikov Yang-Baxter equation in $(A, \lhd, \triangleright)$ if and only if T_r is an \circlearrowleft -operator on (A, \circ) associated to $(A^*, L_{\triangleright}^* + R_{\triangleleft}^*, -R_{\triangleleft}^*)$.

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be a basis of A and $\{e_1^*, e_2^*, \dots, e_n^*\}$ be the dual basis of A^* . Assume that $e_i \triangleright e_j = \sum_{k=1}^n b_{ij}^k e_k$, $e_i \triangleleft e_j = \sum_{k=1}^n c_{ij}^k e_k$ and $r = \sum_{i=1}^n \sum_{j=1}^n a_{ij} e_i \otimes e_j$, where b_{ij}^k , c_{ij}^k , $a_{ij} \in \mathbf{k}$ and $a_{ij} = a_{ji}$.

By Eq. (72), we have $T_r(e_i^*) = \sum_{k=1}^n a_{ik} e_k$. Hence we obtain

$$L_{\triangleright}^{*}(T_{r}(e_{i}^{*}))e_{j}^{*} = -\sum_{t=1}^{n}\sum_{l=1}^{n}a_{it}b_{tl}^{j}e_{l}^{*}, \quad L_{\triangleleft}^{*}(T_{r}(e_{i}^{*}))e_{j}^{*} = -\sum_{t=1}^{n}\sum_{l=1}^{n}a_{it}c_{tl}^{j}e_{l}^{*},$$

$$R_{\triangleright}^*(T_r(e_i^*))e_j^* = -\sum_{t=1}^n \sum_{l=1}^n a_{it}b_{lt}^j e_l^*, \quad R_{\triangleleft}^*(T_r(e_i^*))e_j^* = -\sum_{t=1}^n \sum_{l=1}^n a_{it}c_{lt}^j e_l^*.$$

Then the coefficient of e_k in

$$T_r(e_i^*) \circ T_r(e_i^*) - T_r((L_{\triangleright}^* + R_{\triangleleft}^*)(T_r(e_i^*))e_i^* - R_{\triangleleft}^*(T_r(e_i^*))e_i^*)$$

is

$$\sum_{t=1}^{n} \sum_{l=1}^{n} (a_{it}a_{jl}(b_{tl}^{k} + c_{tl}^{k}) + a_{it}a_{lk}(b_{tl}^{j} + c_{lt}^{j}) - a_{jt}a_{lk}c_{lt}^{i}).$$

Hence T_r is an \mathcal{O} -operator on (A, \circ) associated to $(A^*, L^*_{\triangleright} + R^*_{\triangleleft}, -R^*_{\triangleleft})$ if and only if

(73)
$$\sum_{t=1}^{n} \sum_{l=1}^{n} \left(a_{it} a_{jl} (b_{tl}^{k} + c_{tl}^{k}) + a_{it} a_{lk} (b_{tl}^{j} + c_{lt}^{j}) - a_{jt} a_{lk} c_{lt}^{i} \right) = 0$$

holds for all $i, j \in \{1, \dots, n\}$. The left-hand side of Eq. (73) is just the coefficient of $e_k \otimes e_i \otimes e_j$ in $r_{12} \circ r_{13} + r_{23} \odot r_{13} - r_{12} \triangleleft r_{23}$. Therefore this conclusion holds.

Proposition 4.10. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra, $r \in A \otimes A$ be symmetric and (A, \circ) be the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. Then T_r is an \circlearrowleft -operator on $(A, \triangleleft, \triangleright)$ associated to $(A^*, L^*_{\triangleright} + L^*_{\triangleleft} + R^*_{\triangleright} + R^*_{\triangleleft}, R^*_{\triangleright}, -(R^*_{\triangleright} + L^*_{\triangleleft}), -(R^*_{\triangleright} + R^*_{\triangleleft}))$ if and only if the following equations hold:

$$(74) r_{12} \triangleright r_{13} + r_{23} \star r_{13} + r_{12} \triangleright r_{23} = 0,$$

$$(75) r_{12} \triangleleft r_{13} - r_{13} \odot r_{23} - r_{12} \circ r_{23} = 0.$$

Proof. It follows by a similar proof as that in Proposition 4.9.

Theorem 4.11. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra, $r \in A \otimes A$ be symmetric and (A, \circ) be the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. Then the following conditions are equivalent:

- (a) r is a solution of pre-Novikov Yang-Baxter equation in $(A, \triangleleft, \triangleright)$;
- (b) T_r is an \circlearrowleft -operator on (A, \circ) associated to $(A^*, L^*_{\triangleright} + R^*_{\triangleleft}, -R^*_{\triangleleft})$;
- (c) T_r is an \circlearrowleft -operator on $(A, \triangleleft, \triangleright)$ associated to $(A^*, L_{\triangleright}^* + L_{\triangleleft}^* + R_{\triangleright}^* + R_{\triangleleft}^*, R_{\triangleright}^*, -(R_{\triangleright}^* + L_{\triangleleft}^*), -(R_{\triangleright}^* + R_{\triangleleft}^*))$.

Proof. By a similar proof as that in Theorem 4.4, Eq. (75) holds if and only if Eq. (54) holds. Since r is symmetric, Eq. (74) holds if and only if R_{21} holds, where R_{21} is given in Lemma 4.2. Then by Theorem 4.4, Eq. (74) holds if (54) holds. Therefore, by Propositions 4.9 and 4.10, we obtain the conclusion.

Theorem 4.12. Let $(A, \triangleleft, \triangleright)$ be a pre-Novikov algebra and (A, \circ) be the associated Novikov algebra of $(A, \triangleleft, \triangleright)$. Let $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ be a representation of $(A, \triangleleft, \triangleright)$. Let $T: V \to A$ be a linear map, which is identified with $r_T \in A \otimes V^* \subseteq (A \ltimes_{l_{\triangleright}^* + l_{\triangleleft}^* + r_{\triangleright}^* + r_{\triangleleft}^*, r_{\triangleright}^*, -(r_{\triangleright}^* + l_{\triangleleft}^*), -(r_{\triangleright}^* + r_{\triangleleft}^*)} V^*) \otimes (A \ltimes_{l_{\triangleright}^* + l_{\triangleleft}^* + r_{\triangleright}^* + r_{\triangleleft}^*, r_{\triangleright}^*, -(r_{\triangleright}^* + l_{\triangleleft}^*), -(r_{\triangleright}^* + r_{\triangleleft}^*)} V^*)$ through $Hom_{\mathbf{k}}(V, A) \cong A \otimes V^*$. Then $r = r_T + \tau(r_T)$ is a symmetric solution of pre-Novikov Yang-Baxter equation in $A \ltimes_{l_{\triangleright}^* + l_{\triangleleft}^* + r_{\triangleright}^* + r_{\triangleleft}^*, r_{\triangleright}^*, -(r_{\triangleright}^* + l_{\triangleleft}^*), -(r_{\triangleright}^* + r_{\triangleleft}^*)} V^*$ if and only if T is an \emptyset -operator on $(A, \triangleleft, \triangleright)$ associated to $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$.

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be a basis of V and $\{v_1^*, v_2^*, \dots, v_n^*\}$ be the dual basis of V^* . Then we have

$$r_T = \sum_{i=1}^n T(v_i) \otimes v_i^* \in T(V) \otimes V^* \subset (A \oplus V^*) \otimes (A \oplus V^*),$$

$$r = r_T + \tau(r_T) = \sum_{i=1}^n (T(v_i) \otimes v_i^* + v_i^* \otimes T(v_i)).$$

Therefore, we have

$$r_{12} \circ r_{13} = \sum_{i,j=1}^{n} (T(v_i) \circ T(v_j) \otimes v_i^* \otimes v_j^* + (l_{\triangleright}^* + r_{\triangleleft}^*)(T(v_i))v_j^* \otimes v_i^* \otimes T(v_j) - r_{\triangleleft}^*(T(v_j))v_i^* \otimes T(v_i) \otimes v_j^*).$$

By Eq. (7), we have

$$\begin{split} l_{\triangleright}^*(T(v_i))v_j^* &= -\sum_{k=1}^n v_j^*(l_{\triangleright}(T(v_i))v_k)v_k^*, \quad r_{\triangleright}^*(T(v_i))v_j^* = -\sum_{k=1}^n v_j^*(r_{\triangleright}(T(v_i))v_k)v_k^*, \\ l_{\triangleleft}^*(T(v_i))v_j^* &= -\sum_{k=1}^n v_j^*(l_{\triangleleft}(T(v_i))v_k)v_k^*, \quad r_{\triangleleft}^*(T(v_i))v_j^* = -\sum_{k=1}^n v_j^*(r_{\triangleleft}(T(v_i))v_k)v_k^*. \end{split}$$

Hence we obtain

$$\begin{split} \sum_{i=1}^{n} \sum_{j=1}^{n} (l_{\triangleright}^{*} + r_{\triangleleft}^{*})(T(v_{i}))v_{j}^{*} \otimes v_{i}^{*} \otimes T(v_{j}) &= -\sum_{i=1}^{n} \sum_{j=1}^{n} v_{j}^{*} \otimes v_{i}^{*} \otimes T((l_{\triangleright} + r_{\triangleleft})(T(v_{i}))v_{j}), \\ \sum_{i=1}^{n} \sum_{j=1}^{n} r_{\triangleleft}^{*}(T(v_{j}))v_{i}^{*} \otimes T(v_{i}) \otimes v_{j}^{*} &= -\sum_{i=1}^{n} \sum_{j=1}^{n} v_{i}^{*} \otimes T(r_{\triangleleft}(T(v_{j}))v_{i}) \otimes v_{j}^{*}. \end{split}$$

Then one gets

$$r_{12} \circ r_{13} = \sum_{i=1}^{n} \sum_{j=1}^{n} (T(v_i) \circ T(v_j) \otimes v_i^* \otimes v_j^* - v_j^* \otimes v_i^* \otimes T((l_{\triangleright} + r_{\triangleleft})(T(v_i))v_j) + v_i^* \otimes T(r_{\triangleleft}(T(v_j))v_i) \otimes v_j^*).$$

Similarly, we obtain

$$\begin{split} r_{23} & \triangleright r_{13} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(v_{j}^{*} \otimes T(v_{i}) \otimes r_{\triangleright}^{*}(T(v_{j})) v_{i}^{*} + T(v_{j}) \otimes v_{i}^{*} \otimes (l_{\triangleright}^{*} + l_{\triangleleft}^{*} + r_{\triangleright}^{*} + r_{\triangleleft}^{*}) (T(v_{i})) v_{j}^{*} \right. \\ & + v_{j}^{*} \otimes v_{i}^{*} \otimes T(v_{i}) \triangleright T(v_{j}) \big) \\ & = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(-v_{j}^{*} \otimes T(r_{\triangleright}(T(v_{j})) v_{i}) \otimes v_{i}^{*} - T((l_{\triangleright} + l_{\triangleleft} + r_{\triangleright} + r_{\triangleleft})(T(v_{i})) v_{j}) \otimes v_{i}^{*} \otimes v_{j}^{*} \right. \\ & + v_{j}^{*} \otimes v_{i}^{*} \otimes T(v_{i}) \triangleright T(v_{j}) \big), \\ r_{13} & \triangleleft r_{23} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(T(v_{i}) \otimes v_{j}^{*} \otimes (-r_{\triangleright}^{*} - r_{\triangleleft}^{*})(T(v_{j})) v_{i}^{*} + v_{i}^{*} \otimes T(v_{j}) \otimes (-r_{\triangleright}^{*} - l_{\triangleleft}^{*})(T(v_{i})) v_{j}^{*} \right. \\ & + v_{i}^{*} \otimes v_{j}^{*} \otimes T(v_{i}) \triangleleft T(v_{j}) \big) \\ & = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(T((r_{\triangleright} + r_{\triangleleft})(T(v_{j})) v_{i}) \otimes v_{j}^{*} \otimes v_{i}^{*} + v_{i}^{*} \otimes T((r_{\triangleright} + l_{\triangleleft})(T(v_{i})) v_{j}) \otimes v_{j}^{*} \right. \\ & + v_{i}^{*} \otimes v_{j}^{*} \otimes T(v_{i}) \triangleleft T(v_{j}) \big), \\ -r_{12} & \triangleleft r_{23} = -\sum_{i=1}^{n} \sum_{j=1}^{n} \left(T(v_{i}) \otimes (-r_{\triangleright}^{*} - r_{\triangleleft}^{*})(T(v_{j})) v_{i}^{*} \otimes v_{j}^{*} + v_{i}^{*} \otimes T(v_{i}) \triangleleft T(v_{j}) \otimes v_{j}^{*} \right. \\ & + v_{i}^{*} \otimes (-r_{\triangleright}^{*} - l_{\triangleleft}^{*})(T(v_{i})) v_{j}^{*} \otimes T(v_{j}) \big) \\ & = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(T((-r_{\triangleright} - r_{\triangleleft})(T(v_{j})) v_{i}^{*} \otimes v_{j}^{*} - v_{i}^{*} \otimes T(v_{i}) \triangleleft T(v_{j}) \otimes v_{j}^{*} \right. \\ & + v_{i}^{*} \otimes v_{i}^{*} \otimes T((-r_{\triangleright} - l_{\triangleleft})(T(v_{j})) v_{i} \otimes v_{j}^{*} - v_{i}^{*} \otimes T(v_{i}) \triangleleft T(v_{j}) \otimes v_{j}^{*} \right. \\ & + v_{i}^{*} \otimes v_{i}^{*} \otimes T((-r_{\triangleright} - l_{\triangleleft})(T(v_{j})) v_{i} \otimes v_{j}^{*} - v_{i}^{*} \otimes T(v_{i}) \triangleleft T(v_{j}) \otimes v_{j}^{*} \bigg.$$

Therefore, $r = r_T + \tau(r_T)$ is a symmetric solution of pre-Novikov Yang-Baxter equation in $A \ltimes_{l_b^* + l_a^* + r_b^* + r_a^*, r_b^*, -(r_b^* + l_a^*), -(r_b^* + r_a^*)} V^*$ if and only if

(76)
$$T(v_i) \circ T(v_j) = T((l_{\triangleright} + l_{\triangleleft})(T(v_i))v_j) + T((r_{\triangleright} + r_{\triangleleft})(T(v_j))v_i),$$

(77)
$$T(v_i) \triangleleft T(v_j) = T(l_{\triangleleft}(T(v_i))v_j) + T(r_{\triangleleft}(T(v_j))v_i),$$

(78)
$$T(v_i) > T(v_i) + T(v_i) \triangleleft T(v_i) = T((l_{\triangleright} + r_{\triangleleft})(T(v_i))v_i) + T((r_{\triangleright} + l_{\triangleleft})(T(v_i))v_i),$$

hold for all $i, j \in \{1, \dots, n\}$. It is easy to see that Eqs. (76)-(78) hold for all $i, j \in \{1, \dots, n\}$ if and only if T is an \bigcirc -operator on $(A, \triangleleft, \triangleright)$ associated to $(V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$. Then the proof is completed.

Finally, we present an example to construct pre-Novikov bialgebras by O-operators.

Example 4.13. Let $(A = \mathbf{k}e_1 \oplus \mathbf{k}e_2, \triangleleft, \triangleright)$ be the pre-Novikov algebra given in Example 3.8. Let $T : A \to A$ be a linear map given by

$$T(e_1) = e_2, \quad T(e_2) = 0.$$

Then it is easy to check that T is an \bigcirc -operator on $(A, \triangleleft, \triangleright)$ associated to $(A, L_{\triangleright}, R_{\triangleright}, L_{\triangleleft}, R_{\triangleleft})$.

Denote the semi-direct product of $(A, \triangleleft, \triangleright)$ and its representation $(A^*, L^*_{\triangleright} + L^*_{\triangleleft} + R^*_{\triangleright} + R^*_{\triangleleft}, R^*_{\triangleright}, -(R^*_{\triangleright} + L^*_{\triangleleft}), -(R^*_{\triangleright} + R^*_{\triangleleft})$ by $(B, \triangleleft, \triangleright)$. In fact, B is the vector space $\mathbf{k}e_1 \oplus \mathbf{k}e_2 \oplus \mathbf{k}e_1^* \oplus \mathbf{k}e_2^*$ endowed with two binary operations \triangleright , \triangleleft defined by non-zero products:

$$e_1 \triangleleft e_1 = e_1, \quad e_1 \triangleleft e_2 = e_2 \triangleleft e_1 = e_2,$$

 $e_1 \triangleright e_1^* = e_2 \triangleright e_2^* = -2e_1^*, \quad e_1 \triangleright e_2^* = -2e_2^*,$
 $e_1 \triangleleft e_1^* = e_2 \triangleleft e_2^* = e_1^* \triangleleft e_1 = e_2^* \triangleleft e_2 = e_1^*, \quad e_1 \triangleleft e_2^* = e_2^* \triangleleft e_1 = e_2^*.$

By Theorem 4.12, $r = \sum_{i=1}^{2} \sum_{j=1}^{2} (T(e_i) \otimes e_i^* + e_i^* \otimes T(e_i)) = e_2 \otimes e_1^* + e_1^* \otimes e_2$ is a symmetric solution of pre-Novikov Yang-Baxter equation in the pre-Novikov algebra $(B, \triangleleft, \triangleright)$. Let (B, \circ) be the associated Novikov algebra of $(B, \triangleleft, \triangleright)$. Then by Theorem 4.4, $(B, \triangleleft, \triangleright, \alpha, \beta)$ is a pre-Novikov

$$\alpha(a) := (L_{\circ}(a) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\triangleright} + R_{\triangleleft})(a))\tau r$$

$$= a \circ e_{2} \otimes e_{1}^{*} + e_{2} \otimes (a \triangleright e_{1}^{*} + e_{1}^{*} \triangleleft a) + a \circ e_{1}^{*} \otimes e_{2} + e_{1}^{*} \otimes (a \triangleright e_{2} + e_{2} \triangleleft a),$$

$$\beta(a) := -(L_{\triangleright}(a) \otimes \mathrm{id} + \mathrm{id} \otimes (L_{\circ} + R_{\circ})(a))r$$

$$= -a \triangleright e_{2} \otimes e_{1}^{*} - e_{2} \otimes (a \circ e_{1}^{*} + e_{1}^{*} \circ a) - a \triangleright e_{1}^{*} \otimes e_{2} - e_{1}^{*} \otimes (a \circ e_{2} + e_{2} \circ a), \quad a \in B.$$

Explicitly, $\alpha(e_2^*) = 2e_1^* \otimes e_1^*, \beta(e_1) = -e_2 \otimes e_1^*$, and the others are zero.

bialgebra where $\alpha, \beta: B \to B \otimes B$ are linear maps given by

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