Eigenvalues and graph minors*

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Abstract

Let $spex(n, H_{minor})$ denote the maximum spectral radius of *n*-vertex *H*-minor free graphs. The problem on determining this extremal value can be dated back to the early 1990s. Up to now, it has been solved for *n* sufficiently large and some special minors, such as $\{K_{2,3}, K_4\}$, $\{K_{3,3}, K_5\}$, K_r and $K_{s,t}$. In this paper, we find some unified phenomena on general minors. Every graph G on n vertices with spectral radius $\rho \geq spex(n, H_{minor})$ contains either an H minor or a spanning book $K_{\gamma_H} \nabla(n - \gamma_H) K_1$, where $\gamma_H = |H| - \alpha(H) - 1$. Furthermore, assume that G is H-minor free and $\Gamma_s^*(H)$ is the family of s-vertex irreducible induced subgraphs of H, then G minus its γ_H dominating vertices is $\Gamma^*_{\alpha(H)+1}(H)$ -minor saturate, and it is further edge-maximal if $\Gamma^*_{\alpha(H)+1}(H)$ is a connected family. As applications, we obtain some known results on minors mentioned above. We also determine the extremal values for some other minors, such as flowers, wheels, generalized books and complete multi-partite graphs. Our results extend some conjectures on planar graphs, outer-planar graphs and $K_{s,t}$ -minor free graphs. To obtain the results, we combine stability method, spectral techniques and structural analyses. Especially, we give an exploration of using absorbing method in spectral extremal problems.

Keywords: graph minor; spectral radius; stability method; absorbing method **AMS Classification:** 05C50; 05C35.

1 Introduction

In the last few decades much research has been done on spectra of graphs, especially, the eigenvalues of adjacency matrices of graphs; Alon [2], Bollobás, Lee and Letzter [5], Bollobás and Nikiforov [6], Hoory, Linial and Widgerson [26], Huang [29], Jiang [30],

^{*}Supported by the National Natural Science Foundation of China (Nos. 12171066, 12271162) and the Major Natural Science Research Project of Universities in Anhui Province (No. 2022AH040151).

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Jiang, Tidor, Yao, Zhang and Zhao [31], Lubetzky, Sudakov and Vu [36], Tait [46], Tait and Tobin [47], Wilf [53].

Given a simple graph H, we define H to be a minor of some graph G if H can be obtained from G by means of a sequence of vertex deletions, edge deletions and edge contractions. Given a family of graphs \mathbb{H} , a graph is said to be \mathbb{H} -minor free if it does not have any member of \mathbb{H} as a minor. Minor plays a very important role in graph theory. We refer the reader to a survey by Robertson and Seymour [44]. Wagner [51] proved that a graph is planar if and only if it is $\{K_5, K_{3,3}\}$ -minor free. The study on planar graphs has a rich history, such as, the curvature of planar graphs [1, 19, 22, 27, 28], the automorphism groups of planar graphs [4, 13, 39, 45], the partitions of planar graphs [9, 16, 32], and the eigenvalues of planar graphs [7, 8, 14, 17, 18, 23, 24, 35, 47].

As a generalization of planar graphs, it is interesting to study problems on graphs with no K_r minor or no $K_{s,t}$ minor. Mader [38] showed that $B_{r-2,n-r+2}$ yields the maximum number of edges over all *n*-vertex graphs with no K_r minor when $r \leq 7$, however this is not best possible when r > 7. A famous conjecture was posed by Hadwiger [21] in 1943, which stated that for every integer $r \ge 1$, every graph with no K_r minor is (r-1)colorable. In the 1980s, Kostochka [33, 34] and Thomason [48] independently proved that the maximum number of edges in a K_r minor-free graph G is $\Theta(r\sqrt{logrn})$ for large r and hence G is $O(r\sqrt{\log r})$ -colorable. In 2001, Thomason [49] found the asymptotic value of this edge-extremal function. Very recently, Alon, Krivelevich and Sudakov [3] provided a short and self-contained proof of the celebrated Kostochka-Thomason bound. Simultaneously, Norin, Postle and Song [43] showed that every graph with no K_r minor is $O(r(\log r)^{\beta})$ -colorable for $\beta > 1/4$. From spectral perspective, Tait [46] determined the maximum spectral radius for graphs with no K_r minor. The next one is $K_{s,t}$ minor. Ding, Johnson and Seymour [15] showed that $e(G) \leq {t \choose 2} + n - t$ for every connected graph G with no $K_{1,t}$ minor. Chudnovsky, Reed and Seymour [11] proved that $e(G) \leq C$ $\frac{1}{2}(t+1)(n-1)$ for every graph G with no $K_{2,t}$ minor, which confirms a conjecture of Myers [40]. Zhai and Lin [54] obtained the maximum spectral radius of $K_{s,t}$ -minor free graphs for s = 1 and $s \ge 2$ respectively, which solved a conjecture of Tait [46].

Extensive studies have been conducted on the properties of graphs, and exploring the properties of graphs through minors is a valuable approach, as minor is very useful for characterizing inclusion relations between graphs. In particular, Mader [37] proved that for every given graph H, there exists a constant C_H such that $e(G) \leq C_H |G|$ for any H-minor free G. Let A(G) denote the adjacency matrix of G and $\rho(G)$ be its spectral radius. In this paper, we fundamentally investigate the following extremal problem.

Problem 1.1. Given a graph H or a graph family \mathbb{H} . What is the maximum spectral radius of an H-minor (\mathbb{H} -minor) free graph of order n?

For many specific graphs H, Problem 1.1 has gained great popularity and has attracted the attention of many researchers (see for example, [25, 42, 47, 46, 52, 54]). However, a unified perspective on Problem 1.1 has not yet been found.

A generalized book, denoted by $B_{\gamma,n-\gamma}$, is obtained by joining a γ -clique with an independent set of $n - \gamma$ vertices, in other words, $B_{\gamma,n-\gamma} \cong K_{\gamma} \nabla(n-\gamma) K_1$. For a given graph family \mathbb{H} , we define

$$\gamma_H := |H| - \alpha(H) - 1$$
 and $\gamma_{\mathbb{H}} := \min_{H \in \mathbb{H}} \gamma_H$,

where $\alpha(H)$ (or α_H for simplicity) is the independence number of H. In the following, we use $SPEX(n, \mathbb{H}_{minor})$ to denote the family of graphs with maximum spectral radius over all *n*-vertex \mathbb{H} -minor free graphs. We shall assume that *n* is sufficiently large and \mathbb{H} contains no member isomorphic to a star. Then $\gamma_{\mathbb{H}} \ge 1$. The first main result for Problem 1.1 is as follows.

Theorem 1.1. Every graph in $SPEX(n, \mathbb{H}_{minor})$ contains a spanning subgraph $B_{\gamma_{\mathbb{H}}, n-\gamma_{\mathbb{H}}}$.

We now introduce some new notations and terminologies. Choose an arbitrary graph $H \in \mathbb{H}$. We define a family of induced subgraphs of *H* as follows.

$$\Gamma_s(H) = \{H[S]: S \subseteq V(H) \text{ and } |S| = s\}.$$

A member H[S] in $\Gamma_s(H)$ is said to *irreducible*, if $\Gamma_s(H)$ does not contain any member isomorphic to a proper subgraph of H[S]. Now let $\Gamma_s^*(H)$ be the family of *s*-vertex irreducible induced subgraphs of *H* and

$$\Gamma(\mathbb{H}) = \bigcup_{H \in \mathbb{H}} \Gamma^*_{|H| - \gamma_{\mathbb{H}}}(H).$$

It is obvious that $\Gamma^*_{|H|-\gamma_{\mathbb{H}}}(H) = \Gamma^*_{\alpha_H+1}(H)$ for every $H \in \mathbb{H}$ with $\gamma_H = \gamma_{\mathbb{H}}$.

A graph is said to be \mathbb{H} -*minor saturated*, if it is \mathbb{H} -minor free but adding an edge between a pair of non-adjacent vertices always yields an H minor for some $H \in \mathbb{H}$. Let $SAT(n, \mathbb{H}_{minor})$ be the family of *n*-vertex \mathbb{H} -minor saturated graphs. Choose an arbitrary $G^* \in SPEX(n, \mathbb{H}_{minor})$, and let L be the set of dominating vertices in G^* . Then $|L| = \gamma_{\mathbb{H}}$ by Theorem 1.1. One may expect a characterization of $G^* - L$.

Theorem 1.2. The induced subgraph $G^* - L \in SAT(n - \gamma_{\mathbb{H}}, \Gamma(\mathbb{H})_{minor})$. Particularly, if $\mathbb{H} = \{H\}$, then $G^* - L \in SAT(n - \gamma_H, \Gamma^*_{\alpha_H+1}(H)_{minor})$.

Let $ex(n, \mathbb{H}_{minor})$ denote the maximum number of edges in an *n*-vertex \mathbb{H} -minor free graph, and let $EX(n, \mathbb{H}_{minor})$ be the family of *n*-vertex \mathbb{H} -minor free graphs each of which has $ex(n, \mathbb{H}_{minor})$ edges. If all the members in $\Gamma(\mathbb{H})$ are connected graphs, then we can obtain a stronger result than Theorem 1.2.

Theorem 1.3. If $\Gamma(\mathbb{H})$ is a connected family, then $G^* - L \in EX(n - \gamma_{\mathbb{H}}, \Gamma(\mathbb{H})_{minor})$.

Theorems 1.1, 1.2 and 1.3 can be viewed as three fundamental tools on Problem 1.1. We now show some applications of these theorems.

Example 1. $\mathbb{H} = \{K_t\}$. Then $\gamma_H = |H| - \alpha_H - 1 = t - 2$ and $\Gamma^*_{\alpha_H + 1}(H) = \{K_2\}$ for $H = K_t$. By Theorems 1.1 and 1.2, G^* has t - 2 dominating vertices and $G^* - L$ is $\{K_2\}$ -minor saturated. Thus, $G^* \cong B_{t-2,n-t+2}$.

Example 2. $\mathbb{H} = \{K_{a,3}, K_{a+2}\}$, where $a \in \{2,3\}$. Then $\gamma_{\mathbb{H}} = a - 1$. Consequently, $\Gamma^*_{|H|-\gamma_{\mathbb{H}}}(H) = \Gamma^*_4(H) = \{K_{1,3}, K_{2,2}\}$ for $H = K_{a,3}$ and $\Gamma^*_{|H|-\gamma_{\mathbb{H}}}(H) = \Gamma^*_3(H) = \{K_3\}$ for $H = K_{a+2}$. By Theorem 1.1, G^* has a - 1 dominating vertices, and by Theorem 1.2, $G^* - L$ is $\{K_{1,3}, K_{2,2}, K_3\}$ -minor saturated. Thus, $G^* - L$ is a path, and so $G^* \cong K_{a-1} \nabla P_{n-a+1}$.

To sum up the above discussions, we have the following statements.

Theorem 1.4. *i*) (Tait, [46]) If $r \ge 3$, then $SPEX(n, \{K_r\}_{minor}) = \{B_{r-2,n-r+2}\}$. *ii*) (Tait and Tobin, [47]) If $a \in \{2,3\}$, then $SPEX(n, \{K_{a,3}, K_{a+2}\}_{minor}) = \{K_{a-1}\nabla P_{n-a+1}\}$.

Example 3. $\mathbb{H} = \{K_r - E(H_0)\}$, where $H_0 \subseteq K_r$ and $\delta(H_0) \ge 1$. Then $\alpha_H = \omega(H_0)$ and $\gamma_H = r - \omega(H_0) - 1$ for $H = K_r - E(H_0)$, where $\omega(H_0)$ is the clique number of H_0 . By Theorem 1.1, G^* has $r - \omega(H_0) - 1$ dominating vertices. Specially, if $\omega(H_0) = 2$, then $\Gamma^*_{\alpha_H+1}(H) = \{P_3\}$ for $H_0 = \frac{|H_0|}{2}K_2$ and $\Gamma^*_{\alpha_H+1}(H) = \{K_2 \cup K_1\}$ otherwise.

Recently, Chen, Liu and Zhang [10] characterized the spectral extremal graphs for $(K_r - E(H_0))$ -minor free graphs, where H_0 consists of vertex-disjoint paths. Now we present a result in a slightly stronger form. Let $B_{s,t}^k$ denote the graph obtained from $B_{s,t}$ by adding k isolated edges within its independent set. By Theorems 1.1 and 1.2, we have the following result.

Theorem 1.5. Let $\gamma = r - \omega(H_0) - 1$ and G^* be a graph in SPEX $(n, \{K_r - E(H_0)\}_{minor})$. Then G^* contains $B_{\gamma,n-\gamma}$ as a spanning subgraph. Particularly, if $\omega(H_0) = 2$, then $G^* \cong B_{r-3,n-r+3}^{\lfloor \frac{n-r+3}{2} \rfloor}$ for $H_0 = \frac{|H_0|}{2}K_2$, and $G^* \cong B_{r-3,n-r+3}$ otherwise.

A (k+1)-wheel W_{k+1} , where $k \ge 3$, is defined as $W_{k+1} = K_1 \nabla C_k$. Recently, Cioabă, Desai, Tait [12] obtained a spectral extremal result on W_{2k+1} -free graphs.

Example 4. $\mathbb{H} = \{W_{k+1}\}$. Then $\alpha_H = \lfloor \frac{k}{2} \rfloor$ and $\gamma_H = |H| - \alpha_H - 1 = \lceil \frac{k}{2} \rceil$ for $H = W_{k+1}$. By Theorem 1.1, G^* has $\lceil \frac{k}{2} \rceil$ dominating vertices. Similarly, we have $\Gamma^*_{\alpha_H+1}(H) = \{K_2 \cup (\alpha_H - 1)K_1\}$ for odd k, and $\Gamma^*_{\alpha_H+1}(H) = \{P_3 \cup (\alpha_H - 2)K_1, 2P_2 \cup (\alpha_H - 3)K_1\}$ for even k. By Theorem 1.2, we have the following result on W_{k+1} -minor free graphs.

Theorem 1.6. Let $\gamma = \lceil \frac{k}{2} \rceil$. Then $SPEX(n, \{W_{k+1}\}_{minor}) = \{B_{\gamma,n-\gamma}\}$ for odd k, and $SPEX(n, \{W_{k+1}\}_{minor}) = \{B_{\gamma,n-\gamma}^1\}$ otherwise.

A *t*-flower $F_{s_1,...,s_t}$ is the graph obtained from *t* cycles of lengths $s_1,...,s_t$ respectively by identifying one vertex. A *t*-flower is said to be *odd*, if there exists an odd s_i for $i \in \{1,...,t\}$. If $s_1 = \cdots = s_t = 3$, then it is a friendship graph. Very recently, He, Li and Feng [20] determined $SPEX(n, \{F_{s,...,s}\}_{minor})$ for $s \in \{3,4\}$.

Example 5. $\mathbb{H} = \{F_{s_1,...,s_t}\}$. Then $|H| = \sum_{i=1}^t s_i - (t-1)$, $\alpha_H = \sum_{i=1}^t \lfloor \frac{s_i}{2} \rfloor$ and $\gamma_H = |H| - \alpha_H - 1 = \sum_{i=1}^t \lfloor \frac{s_i}{2} \rfloor - t$. If there exists an odd s_i , then we define a subset $S^* \subseteq V(H)$ by choosing α_H independent vertices and an extra vertex in C_{s_i} . Now $H[S^*] \cong K_2 \cup (\alpha_H - 1)K_1$. Clearly, $\Gamma^*_{\alpha_H+1}(H) = \{H[S^*]\}$. By Theorems 1.1 and 1.2, $G^* \cong B_{\gamma,n-\gamma}$, where

 $\gamma = \gamma_H$. If each s_i is even, then $\alpha_H = \sum_{i=1}^t \frac{s_i}{2}$. Thus, for every $(\alpha_H + 1)$ -subset *S* of V(H), there exists some *i* with $|S \cap C_{s_i}| \ge \frac{s_i}{2} + 1$. Hence, H[S] contains either a P_3 or a copy of $2P_2$, and so $\Gamma^*_{\alpha_H+1}(H) = \{P_3 \cup (\alpha_H - 2)K_1, 2P_2 \cup (\alpha_H - 3)K_1\}$. By Theorem 1.2, $G^* - L$ contains exactly one edge. Thus, $G^* \cong B^1_{\gamma,n-\gamma}$.

Consequently, we have the following result.

Theorem 1.7. Let $\gamma = \sum_{i=1}^{t} \lceil \frac{s_i}{2} \rceil - t$. Then $SPEX(n, \{F_{s_1,...,s_t}\}_{minor}) = \{B_{\gamma,n-\gamma}\}$ for an odd flower $F_{s_1,...,s_t}$, and $SPEX(n, \{F_{s_1,...,s_t}\}_{minor}) = \{B_{\gamma,n-\gamma}\}$ otherwise.

As usual, we denote by \overline{G} the complement of a graph G and $K_{s_1,...,s_r}$ a complete r-partite graph with $\min\{r,s_1\} \ge 2$ and $s_1 \ge \cdots \ge s_r$. Let $H_{s_1,s_2} = (\beta - 1)K_{1,s_2} \cup K_{1,s_2+\beta_0}$, where $\beta(s_2+1) + \beta_0 = s_1 + 1$ and $0 \le \beta_0 \le s_2$. Obviously, H_{s_1,s_2} is a star forest of order $s_1 + 1$. Let $S(\overline{H_{s_1,s_2}})$ denote the graph obtained from $\overline{H_{s_1,s_2}}$ by subdividing an edge uv with minimum degree sum d(u) + d(v). Denote by Pet^* the Petersen graph. Set $\beta = \lfloor \frac{s_1+1}{s_2+1} \rfloor$ and $n - \sum_2^r s_i + 1 := ps_1 + q$ ($1 \le q \le s_1$). Now we introduce a graph G^{\blacktriangle} , which will play an important role in our next theorem.

$$G^{\blacktriangle} = \begin{cases} (p-1)K_{s_1} \cup S(\overline{H_{s_1,s_2}}) & \text{if } (q,\beta) = (2,2); \\ (p-1)K_{s_1} \cup \overline{Pet^{\star}} & \text{if } (q,\beta,s_1) = (2,1,8); \\ (p-q)K_{s_1} \cup q\overline{H_{s_1,s_2}} & \text{if } q \le 2(\beta-1) \text{ and } (q,\beta) \ne (2,2); \\ pK_{s_1} \cup K_q & \text{if } q > 2(\beta-1) \text{ and } (q,\beta,s_1) \ne (2,1,8). \end{cases}$$

Example 6. $\mathbb{H} = \{K_{s_1,...,s_r}\}$. Then $|H| = \sum_{i=1}^r s_i$, $\alpha_H = s_1$ and $\gamma_H = |H| - \alpha_H - 1 = \sum_{i=2}^r s_i - 1$. If $s_2 = 1$, then $\gamma_H = r - 2$. If *H* is not a star, that is, $\sum_{i=2}^r s_i - 1 \ge 1$, then G^* contains a set *L* of γ_H dominating vertices. However, it is not easy to characterize the structure of $G^* - L$. Hence, we present the result here, and give its proof in Section 4.

Theorem 1.8. Let $\min\{r, s_1\} \ge 2$, $s_1 \ge \cdots \ge s_r \ge 1$ and $\gamma = \sum_{i=2}^r s_i - 1 \ge 1$. Then i) if $s_2 \ge 2$ or s_1 is even, then $SPEX(n, \{K_{s_1,...,s_r}\}_{minor}) = \{K_{\gamma}\nabla G^{\blacktriangle}\}$; ii) if $s_2 = 1$ and s_1 is odd, then $SPEX(n, \{K_{s_1,...,s_r}\}_{minor}) = \{K_{\gamma}\nabla G^{\blacktriangledown}\}$, where every component of G^{\blacktriangledown} is a cycle for $s_1 = 3$, and is either K_{s_1} or $\overline{H_{s_1,1}}$ for $s_1 \ge 5$.

The rest of the paper is organized as follows. In Section 2, we provide some preliminary results. Especially, we give a stability theorem, which is a key tool of this paper. For the sake of readability of the article, we shall postpone its proof to the last section. In Section 3, we use the above stability theorem and an absorbing method to prove Theorem 1.1. Based on Theorem 1.1, we further present the proof of Theorems 1.2 and 1.3, which give a more refined description of extremal graphs in $SPEX(n, \mathbb{H}_{minor})$. In Section 4, we will use Theorems 1.1, 1.2 and 1.3 to characterize $SPEX(n, \mathbb{H}_{minor})$ for complete multipartite minors. This extends a conjecture proposed by Tait [46].

2 Preliminary results

Observe that an isolated vertex in a graph H does not work on determining whether a graph G of order large enough is H-minor free or not. Throughout the paper, let \mathbb{H} be a family of graphs in which every member *H* is a finite graph with minimum degree $\delta(H) \ge 1$. We use |G| and e(G) to denote the numbers of vertices and edges in a graph *G*, respectively. For a subset *S* of V(G), let G[S] be the subgraph induced by *S*.

In 1967, Mader [37] proved an elegant result on minors, more precisely, if G is an Hminor free graph of order n then there exists a positive constant C_H such that $e(G) < C_H n$. The following lemma was obtained by Thomason [50].

Lemma 2.1. Every non-empty graph G with $e(G) \ge 2^{s+1}t|G|$ has a proper $K_{s,t}$ minor.

Lemma 2.1 implies a more precise bound for $K_{s,t}$ minor. Using it, we can obtain the following bound for general *H* minor.

Lemma 2.2. Let G be an H-minor free graph on n vertices. Then $e(G) < C_H \cdot n$, where $C_H = 2^{|H|+1}e(H)$.

Proof. Let H' be the graph obtained from H by subdividing every edge once. Clearly, H' is a bipartite graph with |H| + e(H) vertices and it can be embedded in $K_{|H|,e(H)}$. Therefore, $K_{|H|,e(H)}$ contains a subgraph H' and thus an H minor.

Now suppose to the contrary that $e(G) \ge 2^{|H|+1}e(H)n$. Then by Lemma 2.1, *G* has a $K_{|H|,e(H)}$ minor and thus an *H* minor, a contradiction. Hence, the result holds.

The following lemma can be found in [11], and its original version is shown in [15].

Lemma 2.3. ([11, 15]) Let $t \ge 3$ and $n \ge t+2$. If G is an n-vertex connected graph with no $K_{1,t}$ minor, then $e(G) \le {t \choose 2} + n - t$, and for all n, this is the best possible.

Let $S^{\ell}(K_t)$ be the graph obtained from K_t by replacing an edge with a path of length $\ell + 1$. As pointed by Ding, Johnson and Seymour [15], the upper bound in Lemma 2.3 is sharp and $S^{n-t}(K_t)$ is an extremal graph. A natural question is to characterize all the extremal graphs. If t = 3, then $S^{n-t}(K_t)$ is an *n*-cycle and is clearly the unique extremal graph. We now consider the case t = 4, which is useful for our main theorem.

Lemma 2.4. Let $n \ge 5$ and G be a $K_{1,4}$ -minor free connected graph of order n with maximum number of edges. Then $G \cong S^{n-4}(K_4)$.

Proof. Since $S^{n-4}(K_4)$ is $K_{1,4}$ -minor free for $n \ge 5$, we have $e(G) \ge e(S^{n-4}(K_4)) = n+2$. On the other hand, for $n \ge 6$ we obtain $e(G) \le n+2$ by Lemma 2.3, and for n = 5 we can see $e(G) \le \lfloor \frac{3\times 5}{2} \rfloor = 7$ as $\Delta(G) \le 3$. Therefore, e(G) = n+2.

Since *G* is a $K_{1,4}$ -minor free connected graph, we have $\Delta(G) \leq 3$ and $\delta(G) \geq 1$. Now let U_i denote the set of vertices of degree *i* in *G* for $i \in \{1, 2, 3\}$. Then

$$2n - |U_1| + |U_3| = |U_1| + 2|U_2| + 3|U_3| = 2e(G) = 2n + 4,$$

which yields that $|U_3| - |U_1| = 4$.

In the following, we show $G \cong S^{n-4}(K_4)$. The proof is proceeded by induction on *n*. If n = 5, then combining $|U_1| + |U_2| + |U_3| = 5$ and $|U_3| - |U_1| = 4$ gives $2|U_1| + |U_2| = 1$.



Figure 1: The extremal graph G in Case 1.

Thus, $|U_1| = 0$, $|U_2| = 1$ and $|U_3| = 4$. Assume that $U_2 = \{u\}$ and $U_3 = \{v_1, v_2, v_3, v_4\}$, where $N_G(u) = \{v_1, v_2\}$. Then, $N_G(v_3) = \{v_1, v_2, v_4\}$ and $N_G(v_4) = \{v_1, v_2, v_3\}$. It follows that $G \cong S^1(K_4)$, as desired.

Assume now that $n \ge 6$. We first show that U_1 is empty. Suppose to the contrary that $u \in U_1$. Then $e(G - \{u\}) = (n-1)+2$ and $G - \{u\}$ is a $K_{1,4}$ -minor free connected graph. By the induction hypothesis, $G - \{u\} \cong S^{n-5}(K_4)$. Whether u is adjacent to a vertex of degree two or degree three in $G - \{u\}$, G always contains a $K_{1,4}$ minor, a contradiction. Thus, $U_1 = \emptyset$. It follows that $|U_3| = |U_1| + 4 = 4$ and $|U_2| = n - 4 \ge 2$. Assume that $U_3 = \{v_1, v_2, v_3, v_4\}$. In the following, we distinguish two cases.

Case 1. Every vertex in U₂ belongs to a triangle.

Recall that $|U_2| \ge 2$. Choose $u_1, u_2 \in U_2$. If u_1, u_2 contains two common neighbors w_1 and w_2 , then $w_1w_2 \in E(G)$ as u_1, u_2 belong to a triangle respectively. Now $d_G(w_1) = d_G(w_2) = 3$ and $G[\{u_1, u_2, w_1, w_2\}]$ is a component of *G*, contradicting the fact that *G* is a connected graph of order $n \ge 6$.

Assume now that $N_G(u_1) \cap N_G(u_2) = \{w\}$, and $w_i \in N_G(u_i) \setminus \{w\}$ for $i \in \{1,2\}$. Then $ww_1, ww_2 \in E(G)$ and $u_1, u_2, w_1, w_2 \in N_G(w)$, which implies that $u_1u_2 \in E(G)$ (otherwise, G contains a $K_{1,4}$). Thus, $\{u_1, u_2, w\}$ induces a triangle. Since G is connected and $n \ge 6$, we have $d_G(w) = 3$. Let $N_G(w) = \{u_1, u_2, u_3\}$. Then $d_G(u_3) = 2$ (otherwise, $d_G(u_3) \ge 3$, we will get a $K_{1,4}$ minor). Let $N_G(u_3) = \{w, u_4\}$. Then $wu_4 \in E(G)$ as u_3 belongs to a triangle. But now $d_G(w) \ge 4$, a contradiction.

Now we can conclude that $N_G(u_1) \cap N_G(u_2) = \emptyset$. Since u_1 belongs to a triangle, we have $u_1u_2 \notin E(G)$. Let $P = u_1w_1 \dots w_s u_2$ be a shortest path connecting u_1 and u_2 in G. Assume that $N_G(u_1) = \{w_1, u_3\}$ and $N_G(u_2) = \{w_s, u_4\}$. Then $u_3, u_4 \notin V(P)$ and $u_3 \neq u_4$. Furthermore, $w_1u_3, w_su_4 \in E(G)$, as u_1, u_2 belong to triangles. Thus, $u_1, u_3 \in N_G(w_1)$ and $u_2, u_4 \in N_G(w_s)$. Contracting the subpath $w_1 \dots w_s$ as a vertex, we obtain a copy of $K_{1,4}$, a contradiction. The proof of Case 1 is completed.

Case 2. There exists a vertex $u \in U_2$ with two non-adjacent neighbors.

Assume $N_G(u) = \{u_1, u_2\}$ and $u_1u_2 \notin E(G)$. Let G' be the graph obtained from G by contracting the path u_1uu_2 as an edge u_1u_2 (that is, u is absorbed). Then e(G') = e(G) - 1 = (n-1) + 2. Since G is $K_{1,4}$ -minor free, G' is too. By the induction hypothesis, $G' \cong S^{n-5}(K_4)$. Let P be the induced path of length n - 4 in G'. Then both ends are of

degree three in G' and G. We may assume $P = v_1 w_1 \dots w_{n-5} v_2$, where $v_1, v_2 \in U_3$.

If $u_1u_2 \in E(G') \setminus E(P)$, then u_1, u_2 are of degree three in G' and G. Now if $\{u_1, u_2\} = \{v_3, v_4\}$, then $G[\{v_1, v_2, v_3, v_4, w_1, u\}]$ has a $K_{1,4}$ minor (see Fig. 1), a contradiction. Thus $u_1 \in \{v_1, v_2\}$ and $u_2 \in \{v_3, v_4\}$. Assume without loss of generality that $\{u_1, u_2\} = \{v_1, v_3\}$. Then $G[\{v_1, v_2, v_3, v_4, w_1, u\}]$ still has a $K_{1,4}$ minor (see Fig. 1), a contradiction. Hence, $u_1u_2 \in E(P)$, and now $G \cong S^{n-4}(K_4)$, as desired. The proof is completed.

In the following, we introduce some basic notations and terminologies. A *generalized* book, denoted by $B_{\gamma,n-\gamma}$, is obtained by joining a γ -clique with an independent set of $n - \gamma$ vertices. Specially, $B_{2,n-2}$ is called a *book*. Let α_H be the *independence number* of a graph *H*, that is, the cardinality of a maximum independent set in *H*. We now define an important variation on a given family \mathbb{H} which contains no star as a member.

$$\gamma_{\mathbb{H}} := \min_{H \in \mathbb{H}} \gamma_H$$
, where $\gamma_H = |H| - \alpha_H - 1.$ (1)

A graph *H* is said to be *minimal* with respect to \mathbb{H} , if $H \in \mathbb{H}$ such that (i) $\gamma_H = \gamma_{\mathbb{H}}$; (ii) subject to (i), |H| is also minimal. It is obvious that all minimal graphs have the same independence number. Thus, we can set $\alpha_{\mathbb{H}} := \alpha_{H^*}$ for an arbitrary minimal graph H^* . Moreover, we define $C_{\mathbb{H}} := \min C_{H^*}$, where H^* takes over all minimal graphs. Recall that $\gamma_{H^*} = |H^*| - \alpha_{H^*} - 1$ and $C_{H^*} = 2^{|H^*|+1}e(H^*)$. Hence, $\gamma_{\mathbb{H}} + \alpha_{\mathbb{H}} < C_{\mathbb{H}}$.

An *elementary operation* on a graph is one of the following, that is, deleting a vertex, or deleting an edge, or contracting an edge. Clearly, a graph *G* contains an *H* minor if *H* can be obtained from *G* by a sequence of elementary operations. A graph is said to be \mathbb{H} -minor free, if it does not contain an *H* minor for any $H \in \mathbb{H}$. Observe that *H* is a spanning subgraph of B_{γ_H+1,α_H} . Thus we have the following lemma.

Lemma 2.5. B_{γ_H+1,α_H} contains an *H* minor.

Let $spex(n, \mathbb{H}_{minor})$ be the extremal spectral radius of graphs in $SPEX(n, \mathbb{H}_{minor})$.

Lemma 2.6. $B_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}$ is \mathbb{H} -minor free, and $spex(n,\mathbb{H}_{minor}) \geq \sqrt{\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})}$.

Proof. Choose an arbitrary member $H \in \mathbb{H}$. We first claim that $B_{\gamma_H,n-\gamma_H}$ contains no copy of H. Indeed, otherwise, $H \subseteq B_{\gamma_H,n-\gamma_H}$. Let S be the $(n - \gamma_H)$ -vertex independent set and T be the γ_H -clique in $B_{\gamma_H,n-\gamma_H}$. Then,

$$|V(H) \cap S| = |V(H) \setminus T| \ge |H| - \gamma_H = \alpha_H + 1.$$

Furthermore, since $V(H) \cap S$ is also an independent set in H, we have $\alpha_H \ge |V(H) \cap S| \ge \alpha_H + 1$, a contradiction. Therefore, the claim holds.

We now show that $B_{\gamma_H,n-\gamma_H}$ is *H*-minor free. Suppose to the contrary that $B_{\gamma_H,n-\gamma_H}$ has an *H* minor. Then *H* can be obtained by a sequence of elementary operations on $B_{\gamma_H,n-\gamma_H}$. These elementary operations give rise to a graph sequence $H_0, H_1, \dots, H_a (=H)$. From the structure of a generalized book, we know that every elementary operation on a subgraph of $B_{\gamma_H,n-\gamma_H}$ always gives a new subgraph of $B_{\gamma_H,n-\gamma_H}$. This implies that *H* is a subgraph of $B_{\gamma_H,n-\gamma_H}$, contradicting the claim proved above. Thus, $B_{\gamma_H,n-\gamma_H}$ is *H*-minor free. In view of (1), $\gamma_{\mathbb{H}} \leq \gamma_{H}$ and hence $B_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}$ is a subgraph of $B_{\gamma_{H},n-\gamma_{H}}$. Since $B_{\gamma_{H},n-\gamma_{H}}$ is *H*-minor free, $B_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}$ is too. Considering the choice of *H*, we can see that $B_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}$ is \mathbb{H} -minor free, and so $spex(n,\mathbb{H}_{minor}) \geq \rho(B_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}})$. Note that $K_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}} \subseteq B_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}$. Therefore, we have $spex(n,\mathbb{H}_{minor}) \geq \rho(K_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}) = \sqrt{\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})}$, as desired. \Box

We end this section with the following stability result. For the sake of readability of the article, we postpone its proof to the last section.

Theorem 2.1. Let G be a graph of order n large enough. Let X be a non-negative eigenvector corresponding to $\rho(G)$ with $x_{u^*} = \max_{u \in V(G)} x_u$. If $\rho(G) \ge \sqrt{\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})}$, then either G contains an H minor for some $H \in \mathbb{H}$, or G admits a set L of exactly $\gamma_{\mathbb{H}}$ vertices such that $x_u \ge \left(1 - \frac{1}{2(10C_{\mathbb{H}})^2}\right) x_{u^*}$ and $d_G(u) \ge \left(1 - \frac{1}{(10C_{\mathbb{H}})^2}\right) n$ for every $u \in L$.

3 Absorbing method on $SPEX(n, \mathbb{H}_{minor})$

Choose an arbitrary $G^* \in SPEX(n, \mathbb{H}_{minor})$. Let $X^* = (x_1, \ldots, x_n)^T$ be a non-negative unit eigenvector with respect to $\rho(G^*)$, and $u^* \in V(G^*)$ with $x_{u^*} = \max_{u \in V(G^*)} x_u$. Set $\rho^* := \rho(G^*)$. Then, $\rho^* = spex(n, \mathbb{H}_{minor}) \ge \sqrt{\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})}$ by Lemma 2.6. Furthermore, by Theorem 2.1 we have the following proposition.

Proposition 3.1. G^* contains a set L of exactly $\gamma_{\mathbb{H}}$ vertices such that $x_u \ge \left(1 - \frac{1}{2(10C_{\mathbb{H}})^2}\right) x_{u^*}$ and $d_{G^*}(u) \ge \left(1 - \frac{1}{(10C_{\mathbb{H}})^2}\right) n$ for every $u \in L$.

In this section, we partition $V(G^*) \setminus L$ into $L' \cup L''$, where $L'' = \{v : N_{G^*}(v) = L\}$. The key lemma is Lemma 3.3, which will be proved by *absorbing method*. We shall find an *absorbing set* in L'', and then use it to absorb vertices in L'. To this end, we need some more definitions and propositions.

Definition 3.1. A path $P = v_1v_2...v_s$ (where possibly $v_1 = v_s$) is called a **linear path** in *G*, if $P \subseteq G$ and $d_G(v_i) = 2$ for each $i \in \{2,...s-1\}$. A linear path *P* is said to be **maximal**, if there exists no any linear path *P'* such that $P \subseteq P'$ and $P \neq P'$.

By Definition 3.1, every linear path is either an induced path or an induced cycle in *G*. Given a connected graph *G* with $|G| \ge 2$. Definition 3.1 also implies the following two propositions.

Proposition 3.2. Let P be a maximal linear path in G. If P is a path, then $d_G(v) \neq 2$ for each endpoint v; if P is a cycle, then $d_G(v) \neq 2$ for at most one vertex v.

Proposition 3.3. *Every non-trivial connected graph has an edge-decomposition of its maximal linear paths.*

For a graph *H* with $V(H) = \{v_1, ..., v_h\}$, a **model** of *H* in a graph *G* is a collection of vertex-disjoint connected subgraphs $G_{v_1}, ..., G_{v_h}$ such that for any $v_i v_j \in E(H)$, there exists an edge with one end in G_{v_i} and the other end in G_{v_j} . It is not hard to see that *G* has an *H* minor if and only if there is a model of *H* in *G*. Based on this terminology, we can further introduce a definition. **Definition 3.2.** Let G be a graph with an H minor and $\{G_{v_i} : v_i \in V(H)\}$ be a model of H in G. Then, $(V(G_{v_1}), \ldots, V(G_{v_h}))$ is called an H-partition of G.

Note that $\bigcup_{v_i \in V(H)} V(G_{v_i}) \subseteq V(G)$. Hence, an *H*-partition of *G* may not be a partition of V(G) although its members are vertex-disjoint. An *H*-partition is said to be *minimal*, if $\sum_{i=1}^{|H|} |G_{v_i}|$ is minimum over all *H*-partitions of *G*.

By Proposition 3.1, $|L| = \gamma_{\mathbb{H}} = \min_{H \in \mathbb{H}} (|H| - \alpha_H - 1)$. Recall that $V(G^*) \setminus L = L' \cup L''$, where $L'' = \{v : N_{G^*}(v) = L\}$. Also by Proposition 3.1, every vertex in *L* has at most $\frac{n}{(10C_{\mathbb{H}})^2}$ non-neighbors in *L'*. Hence,

$$|L'| \le \frac{n}{(10C_{\mathbb{H}})^2} |L| = \frac{\gamma_{\mathbb{H}} n}{(10C_{\mathbb{H}})^2}.$$
(2)

Before proceeding we need some more notations. Given a graph G with $u \in V(G)$ and $S \subseteq V(G)$, we write $N_S(u) := N_G(u) \cap S$ and $d_S(u) := |N_S(u)|$. Let $G \cup G'$ be the union of two vertex-disjoint graphs G and G'. Specially, we use kG to denote the disjoint union of k copies of G. For two disjoint subsets $S, T \subseteq V(G)$, let G[S,T] be the bipartite subgraph obtained from $G[S \cup T]$ by deleting all its edges within S and within T. We use e(S) and e(S,T) to denote the numbers of edges in G[S] and G[S,T], respectively.

Lemma 3.1. $d_{L''}(v) \leq \alpha_{\mathbb{H}}$ for each $v \in L' \cup L''$ and $G^*[L'']$ is $(K_{1,\alpha_{\mathbb{H}}+1} \cup \gamma_{\mathbb{H}}K_1)$ -minor free. **Proof.** We first show $d_{L''}(v) \leq \alpha_{\mathbb{H}}$ for $v \in L' \cup L''$. Suppose to the contrary that $d_{L''}(v_0) \geq \alpha_{\mathbb{H}} + 1$ for some $v_0 \in L' \cup L''$. Let $L = \{u_1, \ldots, u_{\gamma_{\mathbb{H}}}\}$ and $\{w_0, \ldots, w_{\alpha_{\mathbb{H}}}\} \subseteq N_{L''}(v_0)$.

In view of (2), we have $|L''| = n - |L| - |L'| \ge \gamma_{\mathbb{H}} + \alpha_{\mathbb{H}} + 2$ for *n* large enough. Thus we can choose $\gamma_{\mathbb{H}}$ vertices $v_1, v_2, \ldots, v_{\gamma_{\mathbb{H}}}$ in $L'' \setminus \{v_0, w_0, w_1, \ldots, w_{\alpha_{\mathbb{H}}}\}$. Note that $G^*[L, L''] \cong K_{|L|, |L''|}$. Now let *G* be the graph obtained from G^* by contracting each edge $u_i v_i$ as a new vertex \overline{u}_i for $i \in \{1, \ldots, \gamma_{\mathbb{H}}\}$. Then, $\{\overline{u}_1, \ldots, \overline{u}_{\gamma_{\mathbb{H}}}\}$ is a clique in *G* and $\overline{u}_i \in N_G(w_j)$ for $i \in \{1, \ldots, \gamma_{\mathbb{H}}\}$ and $j \in \{0, \ldots, \alpha_{\mathbb{H}}\}$.

Furthermore, let G' be the graph obtained from G by contracting the edge v_0w_0 as a new vertex \overline{u}_0 . Recall that $\overline{u}_i \in N_G(w_0)$ for $i \in \{1, ..., \gamma_{\mathbb{H}}\}$ and $w_j \in N_G(v_0)$ for $j \in \{1, ..., \alpha_{\mathbb{H}}\}$. Thus, $\overline{u}_i, w_j \in N_{G'}(\overline{u}_0)$ for $i \in \{1, ..., \gamma_{\mathbb{H}}\}$ and $j \in \{1, ..., \alpha_{\mathbb{H}}\}$.

Now, we can see that $G'[\{\overline{u}_i, w_j : 0 \le i \le \gamma_{\mathbb{H}}; 1 \le j \le \alpha_{\mathbb{H}}\}]$ contains $B_{\gamma_{\mathbb{H}}+1,\alpha_{\mathbb{H}}}$ as a spanning subgraph. By Lemma 2.5, G' contains an H minor for some $H \in \mathbb{H}$ and thus G^* too, a contradiction. Therefore, $d_{L''}(v) \le \alpha_{\mathbb{H}}$ for each $v \in L' \cup L''$.

Next suppose that $G^*[L'']$ contains an H_0 -minor, where $H_0 \cong K_{1,\alpha_{\mathbb{H}}+1} \cup \gamma_{\mathbb{H}}K_1$. Let G'' be the graph obtained from G^* by replacing $G^*[L'']$ with a copy of H_0 . Then for G'', $|L''| = |H_0| = \gamma_{\mathbb{H}} + \alpha_{\mathbb{H}} + 2$ and there exists a vertex $v_0 \in L''$ with $d_{L''}(v_0) \ge \alpha_{\mathbb{H}} + 1$. By the above discussion, G'' contains an H minor for some $H \in \mathbb{H}$ and thus G^* too, a contradiction. Therefore, the lemma holds.

Lemma 3.2. $x_v \leq \frac{4x_{u^*}}{100C_{\mathbb{H}}}$ for each $v \in L' \cup L''$.

Proof. Choose an arbitrary $v \in L' \cup L''$. Then $d_L(v) \leq |L| = \gamma_{\mathbb{H}}$, and by Lemma 3.1 $d_{L''}(v) \leq \alpha_{\mathbb{H}}$. Recall that $\gamma_{\mathbb{H}} + \alpha_{\mathbb{H}} < C_{\mathbb{H}}$. Thus

$$d_{G^*}(v) = d_{L \cup L''}(v) + d_{L'}(v) \le (\gamma_{\mathbb{H}} + \alpha_{\mathbb{H}}) + d_{L'}(v) < C_{\mathbb{H}} + d_{L'}(v)$$

Since $\rho^* x_v = \sum_{u \in N_{G^*}(v)} x_u \le d_{G^*}(v) x_{u^*}$, we have

$$\sum_{\nu \in L'} \rho^* x_{\nu} \leq \sum_{\nu \in L'} \big(C_{\mathbb{H}} + d_{L'}(\nu) \big) x_{u^*} = \big(C_{\mathbb{H}} |L'| + 2e(L') \big) x_{u^*},$$

where $e(L') < C_{\mathbb{H}}|L'|$ by Lemma 2.2. Combining (2) gives $\rho^* \sum_{v \in L'} x_v \leq 3C_{\mathbb{H}}|L'|x_{u^*} \leq \frac{3\gamma_{\mathbb{H}}n}{100C_{\mathbb{H}}}x_{u^*}$. Again by $d_{L\cup L''}(v) \leq \gamma_{\mathbb{H}} + \alpha_{\mathbb{H}} < C_{\mathbb{H}}$, we have

$$\rho^* x_v = \sum_{u \in N_{G^*}(v)} x_u = \sum_{u \in N_{L \cup L''}(v)} x_u + \sum_{u \in L'} x_u \le C_{\mathbb{H}} x_{u^*} + \frac{3\gamma_{\mathbb{H}} n}{100C_{\mathbb{H}} \rho^*} x_{u^*}.$$
(3)

Dividing both sides of (3) by ρ^* and combining ${\rho^*}^2 \ge \gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})$, we obtain that $x_v \le \frac{4x_{u^*}}{100C_{\mathbb{H}}}$ for *n* sufficiently large, as desired.

Now we are ready to show a key lemma, which states that L' is empty. We proceed the proof by contradiction and absorbing method. To make the proof readable, we divide it into some claims and present a sketch as follows.

i) Construct a graph G' such that $\rho(G') > \rho(G^*)$ and $G' - L' = G^* - L'$. Then G' admits an H' minor for some $H' \in \mathbb{H}$, and thus an H'-partition \mathscr{V} .

ii) Using the H'-partition \mathscr{V} , we find an absorbing set P^* in G'[L''], which should be a maximal linear path of length sufficiently large.

iii) Based on G', we construct a graph G'' by using P^* to absorb vertices in L'. Now we obtain a new maximal linear path P of order $|P^*| + |L'|$ in $G''[L' \cup L'']$.

iv) Prove $\rho(G'') > \rho(G^*)$. Then G'' contains an H'' minor for some $H'' \in \mathbb{H}$, and thus an H''-partition \mathcal{V}'' . Based on \mathcal{V}'' , we shall construct a graph G''', by contracting P in G'' as a new path of order $r \leq 2|H''| + 1$, such that G''' also contains an H'' minor.

v) Contract P^* in $G^* - L'$ as a new path of order r. The resulting graph is isomorphic to G'''. Hence, G^* contains a G''' minor and thus an H'' minor, a contradiction.

Lemma 3.3. L' is an empty set.

Proof. Suppose to the contrary that $L' \neq \emptyset$. Let G' be the graph obtained from G^* by deleting all edges incident to vertices in L' and joining all edges from L' to L.

Claim 3.1. *Let* $\rho' := \rho(G')$ *. Then* $\rho' > \rho^*$ *.*

Proof. Since $e(L') \leq C_{\mathbb{H}}|L'|$, there exists $v_1 \in L'$ with $d_{L'}(v_1) \leq \frac{2e(L')}{|L'|} \leq 2C_{\mathbb{H}}$. Set $L'_1 := L'$ and $L'_2 := L'_1 \setminus \{v_1\}$. Then we also have $e(L'_2) \leq C_{\mathbb{H}}|L'_2|$, and thus there exists $v_2 \in L'_2$ with $d_{L'_2}(v_2) \leq 2C_{\mathbb{H}}$. Repeating this step, we obtain a sequence $L'_1, \ldots, L'_{|L'|}$ such that $L'_{i+1} = L'_i \setminus \{v_i\}$ and $d_{L'_i}(v_i) \leq 2C_{\mathbb{H}}$ for each *i*. Now we can decompose $E(G^*[L'])$ into |L'|subsets $\{v_iv: v \in N_{L'_i}(v_i)\}$, where $i = 1, \ldots, |L'|$. Thus,

$$\rho' - \rho^* \ge X^{*T} \left(A(G') - A(G^*) \right) X^* \ge \sum_{i=1}^{|L'|} 2x_{v_i} \left(\sum_{u \in L} x_u - \sum_{v \in N_{L \cup L'} \cup L'_i} x_v \right).$$
(4)

Recall that $\gamma_{\mathbb{H}} + \alpha_{\mathbb{H}} < C_{\mathbb{H}}$. Moreover, by Proposition 3.1, we have $|L| = \gamma_{\mathbb{H}}$ and

$$\sum_{u \in L} x_u \ge \gamma_{\mathbb{H}} \left(1 - \frac{1}{2(10C_{\mathbb{H}})^2} \right) x_{u^*} \ge \left(\gamma_{\mathbb{H}} - \frac{1}{10} \right) x_{u^*}.$$
(5)

On the other hand, for each $v_i \in L'$, we have $d_L(v_i) \le |L| - 1 = \gamma_{\mathbb{H}} - 1$ by the definition of L', $d_{L''}(v_i) \le \alpha_{\mathbb{H}}$ by Lemma 3.1, and $d_{L'_i}(v_i) \le 2C_{\mathbb{H}}$ by the choice of v_i . Moreover, Lemma 3.2 gives $x_v \le \frac{4x_{u^*}}{100C_{\mathbb{H}}}$ for $v \in L' \cup L''$. Thus,

$$\sum_{v \in N_{L \cup L'' \cup L'_i}(v_i)} x_v \le d_L(v_i) x_{u^*} + d_{L'' \cup L'_i}(v_i) \frac{4x_{u^*}}{100C_{\mathbb{H}}} \le (\gamma_{\mathbb{H}} - 1) x_{u^*} + (\alpha_{\mathbb{H}} + 2C_{\mathbb{H}}) \frac{4x_{u^*}}{100C_{\mathbb{H}}},$$

which implies that $\sum_{v \in N_{L \cup L'' \cup L'_i}(v_i)} x_v < (\gamma_{\mathbb{H}} - \frac{1}{10}) x_{u^*}$ as $\alpha_{\mathbb{H}} + 2C_{\mathbb{H}} < 3C_{\mathbb{H}}$. Combining with (4) and (5), we obtain $\rho' \ge \rho^*$, with equality if and only if X^* is also an eigenvector corresponding to $\rho(G')$ and $x_{v_i} = 0$ for each $v_i \in L'$. Observe that G' is connected. If X^* is an eigenvector corresponding to $\rho(G')$. then X^* is positive by the Perron-Frobenius theorem. Hence, $\rho' > \rho^*$, as desired.

In view of Claim 3.1 and the choice of G^* , G' must contain an H' minor for some $H' \in \mathbb{H}$. Let $\mathscr{V} = (V_1, \ldots, V_{|H'|})$ be a minimal H'-partition of G'. A set V_i , where $i \in \{1, \ldots, |H'|\}$, is called a *good set* if both $V_i \cap L$ and $V_i \setminus L$ are non-empty. Since $|L| = \gamma_{\mathbb{H}}$ and $V_1, \ldots, V_{|H'|}$ are vertex-disjoint, there are at most $\gamma_{\mathbb{H}}$ good sets in \mathscr{V} . We now give a precise characterization for good sets.

Claim 3.2. Every good set has exactly two vertices.

Proof. By the definition, we have $|V_i| \ge 2$ for each good set V_i . Now, suppose that there exists a good set V_i with $|V_i| \ge 3$. Choose $u \in V_i \cap L$ and $v \in V_i \setminus L$. Note that $G'[L, L' \cup L''] \cong K_{|L'|, |L' \cup L''|}$. Thus $L' \cup L'' \subseteq N_{G'}(u)$ and $L \subseteq N_{G'}(v)$. Thus, if we contract the edge uv as a new vertex w in G', then w is a dominating vertex in the resulting graph. Let $V'_i = \{u, v\}$ and $\mathscr{V}' = (V_1, \ldots, V'_i, \ldots, V_{|H'|})$. Then, \mathscr{V}' is also an H'-partition of G', contradicting the fact that \mathscr{V} is minimal. Therefore, the claim holds.

Claim 3.3. For $i \in \{1, ..., |H'|\}$, every induced subgraph $G'[V_i \cap (L' \cup L'')]$ is connected provided that $V_i \cap (L' \cup L'') \neq \emptyset$.

Proof. Suppose to the contrary that $G'[V_i \cap (L' \cup L'')]$ is not connected for some *i*. Then $|V_i \cap (L' \cup L'')| \ge 2$. However, $G'[V_i]$ is connected by the definition of model. Thus $V_i \cap L \ne \emptyset$. This implies that V_i is a good set and $|V_i| \ge 3$, contradicting Claim 3.2.

In the following, we further assume that $\mathscr{V} = (V_1, \ldots, V_{|H'|})$ is a minimal H'-partition of G' such that $|L' \cap (\bigcup_{i=1}^{|H'|} V_i)|$ is also minimal. Moreover, assume that G'[L''] have c connected components G_1, \ldots, G_c .

Claim 3.4. $L'' \subseteq \bigcup_{i=1}^{|H'|} V_i$ and $c \leq |H'|$.

Proof. Note that $G^*[L \cup L''] = G'[L \cup L'']$. If $\bigcup_{i=1}^{|H'|} V_i \subseteq L \cup L''$, then \mathscr{V} is an H'-partition of G^* , contradicting the fact that G^* is H'-minor free. Hence, $L' \cap (\bigcup_{i=1}^{|H'|} V_i) \neq \emptyset$.

Now suppose to the contrary that there exists a vertex $u \in L'' \setminus (\bigcup_{i=1}^{|H'|} V_i)$. Choose a vertex $v \in L' \cap (\bigcup_{i=1}^{|H'|} V_i)$. We may assume without loss of generality that $v \in L' \cap V_1$. Then, set $V'_1 := (V_1 \setminus \{v\}) \cup \{u\}$ and $\mathscr{V}' := (V'_1, V_2, \dots, V_{|H'|})$. Note that $N_{G'}(v) = L \subseteq N_{G'}(u)$. Then, \mathscr{V}' is also an H'-partition of G', contradicting the assumption that $|L' \cap (\bigcup_{i=1}^{|H'|} V_i)|$ is minimal. Thus, $L'' \subseteq \bigcup_{i=1}^{|H'|} V_i$.

In the following, we show $c \leq |H'|$. On the one hand, $\bigcup_{j=1}^{c} V(G_j) = L'' \subseteq \bigcup_{i=1}^{|H'|} V_i$. On the other hand, by Claim 3.3 we can see that for each V_i (where $1 \leq i \leq |H'|$), there exists at most one G_j with $V(G_j) \cap V_i \neq \emptyset$. Therefore, $c \leq |H'|$.

Claim 3.5. Let $|G_1| = \max_{1 \le j \le c} |G_j|$. Then $e(G_1) \le |G_1| + \frac{1}{6}\sqrt{n}$.

Proof. Recall that $|L| = \gamma_{\mathbb{H}}$. It follows from (2) that $|L''| = n - |L| - |L'| \ge \frac{n}{2}$. Combining Claim 3.4, we have

$$|G_1| \ge \frac{1}{c} |L''| \ge \frac{1}{|H'|} |L''| \ge \frac{n}{2|H'|},\tag{6}$$

which implies that $|G_1| \ge \gamma_{H^*} + \alpha_{H^*} + 3$ (where H^* is minimal with respect to \mathbb{H}). Now using a spanning tree of G_1 , we can find a vertex subset $S \subseteq V(G_1)$ such that $|S| = \gamma_{H^*}$ and $G_1 - S$ is also connected. Observe that $G^*[L''] = G'[L'']$. Set $t := \alpha_{H^*} + 1$. Then by Lemma 3.1, G_1 is $(K_{1,t} \cup \gamma_{H^*}K_1)$ -minor free and thus $G_1 - S$ is $K_{1,t}$ -minor free. Since $G_1 - S$ is a connected graph of order at least t + 2, it contains a $K_{1,2}$. This implies that $t \ge 3$. Now using Lemma 2.3, we obtain $e(G_1 - S) \le {t \choose 2} + |G_1 - S| - t$. Moreover, Lemma 3.1 also gives $d_{L''}(v) \le \alpha_{H^*}$ for each $v \in L''$. Consequently,

$$e(G_1) \le e(G_1 - S) + \sum_{v \in S} d_{L''}(v) \le \frac{1}{2}t(t-1) + |G_1| - |S| - t + |S|\alpha_{H^*}.$$

It follows that $e(G_1) \le |G_1| + \frac{1}{6}\sqrt{n}$ for *n* large enough.

Claim 3.6. Let U_1 be the set of vertices of degree one in G_1 . Then $|U_1| \le |H'|$.

Proof. We first show $|V_i \cap U_1| \le 1$ for each $i \in \{1, ..., |H'|\}$. Suppose to the contrary that $|V_i \cap U_1| \ge 2$ for some *i*. Note that $V_i \cap U_1 \subseteq V_i \setminus L$. Then $|V_i \setminus L| \ge |V_i \cap U_1| \ge 2$. Now if $V_i \cap L \ne \emptyset$, then V_i is a good set, and thus $|V_i| = 2$ by Claim 3.2, a contradiction. Hence, $V_i \cap L = \emptyset$, that is, $V_i \subseteq L' \cup L''$.

Since $G'[V_i]$ is connected and $|V_i| \ge |V_i \cap U_1| \ge 2$, there exist $u_0 \in V_i \cap U_1$ and $w_0 \in N_{V_i}(u_0)$. Recall that $L \subseteq N_{G'}(v)$ for each $v \in L' \cup L''$. Consequently, $L \subseteq N_{G'}(u_0) \cap N_{G'}(w_0)$, as $u_0, w_0 \in V_i \subseteq L' \cup L''$. Since G_1 is a component of $G'[L' \cup L'']$ and u_0 is a vertex of degree one in G_1 , we can see that w_0 is the unique neighbor of u_0 in $L' \cup L''$, and thus $N_{G'}(u_0) \setminus \{w_0\} \subseteq N_{G'}(w_0) \setminus \{u_0\}$. Hence, $(V_1, \ldots, V_i \setminus \{u_0\}, \ldots, V_{|H'|})$ is also an H'-partition of G', contradicting the minimality of \mathscr{V} . Hence, $|V_i \cap U_1| \le 1$ for each i, as desired.

Note that $U_1 \subseteq V(G_1) \subseteq L''$. Moreover, $L'' \subseteq \bigcup_{i=1}^{|H'|} V_i$ by Claim 3.4. Thus, $U_1 \subseteq \bigcup_{i=1}^{|H'|} V_i$. Since $|V_i \cap U_1| \leq 1$ for each $i \in \{1, \dots, |H'|\}$, we obtain $|U_1| \leq |H'|$.

Proof. Let $U_2 = \{v \in V(G_1) : d_{G_1}(v) = 2\}$ and $U_3 = V(G_1) \setminus (U_1 \cup U_2)$. Then, $3|G_1| - 2|U_1| - |U_2| = |U_1| + 2|U_2| + 3|U_3| \le 2e(G_1)$, which yields that $e(G_1) - |U_2| \le 3(e(G_1) - |G_1|) + 2|U_1|$. Combining Claims 3.5 and 3.6 gives that

$$e(G_1) - |U_2| \le \frac{1}{2}\sqrt{n} + 2|H'| \le \sqrt{n}.$$
 (7)

Assume now that there are $\phi(G_1)$ maximal linear paths in G_1 . If G_1 itself is a cycle, then $\phi(G_1) = 1$ by Definition 3.1. If G_1 is not a cycle, then by Proposition 3.2, two ends of every maximal linear path occupies exactly two degrees of vertices in $V(G_1) \setminus U_2$. Thus, $2\phi(G_1) = \sum_{v \in V(G_1) \setminus U_2} d_{G_1}(v)$. Combining (7) gives

$$\phi(G_1) = \frac{1}{2} \sum_{v \in V(G_1) \setminus U_2} d_{G_1}(v) = e(G_1) - |U_2| \le \sqrt{n}.$$

But in view of (6), we have $|G_1| \ge \frac{n}{2|H'|}$. Moreover, by Proposition 3.3, G_1 admits an edge-decomposition of its maximal linear paths. Therefore, there exists a maximal linear path P^* of length at least $\frac{e(G_1)}{\phi(G_1)}$, and thus $|P^*| \ge \frac{e(G_1)}{\phi(G_1)} + 1 \ge \frac{|G_1|}{\phi(G_1)} \ge \frac{\sqrt{n}}{2|H'|}$.

Note that $V(P^*) \subseteq L''$ and L' is an independent set in G'. Now we use P^* to absorb vertices in L'. Assume that $P^* = w_1 w_2 \dots w_a$ and $L' = \{v_1, v_2, \dots, v_b\}$. Let G'' be the graph obtained from G' by replacing the edge $w_1 w_2$ with a path $w_1 v_1 v_2 \dots v_b w_2$.

Claim 3.8. Let $\rho'' := \rho(G'')$. Then $\rho'' > \rho^*$.

Proof. Recall that $\rho(G') = \rho' > \rho^*$ by Claim 3.1. It suffices to show $\rho'' \ge \rho'$. Note that G' is connected. By the Perron-Frobenius theorem, there exists a positive unit eigenvector $Y = (y_1, \ldots, y_n)^T$ corresponding to $\rho(G')$. Set $\sigma_L := \sum_{u \in L} y_u$ and $y_{L''}^* := \max_{w \in L''} y_w$. For every $v_i \in L'$, since $N_{G'}(v_i) = L$, we have $\rho' y_{v_i} = \sigma_L$. Consequently, $y_{v_1} = y_{v_b} = \frac{\sigma_L}{\rho'}$. Moreover, by Lemma 3.1, $d_{L''}(w) \le \alpha_{\mathbb{H}}$ for each $w \in L''$. Thus, $\rho' y_{L''}^* \le \sigma_L + \alpha_{\mathbb{H}} y_{L''}^*$, which yields $y_{L''}^* \le \frac{\sigma_L}{\rho' - \alpha_{\mathbb{H}}}$. By Lemma 2.6, $\rho^* \ge \sqrt{\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})}$, and thus $\rho' > \sqrt{\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})} \ge 2\alpha_{\mathbb{H}}$. Combining these inequalities, we obtain

$$\max\{y_{w_1}, y_{w_2}\} \le y_{L''}^* \le \frac{\sigma_L}{\rho' - \alpha_{\mathbb{H}}} \le \frac{2\sigma_L}{\rho'} = 2y_{v_1} = 2y_{v_b}.$$
(8)

On the other hand, one can see that

$$\rho'' - \rho' \ge 2(y_{w_1}y_{v_1} + y_{w_2}y_{v_b} - y_{w_1}y_{w_2}) = y_{w_1}(2y_{v_1} - y_{w_2}) + y_{w_2}(2y_{v_b} - y_{w_1}).$$

Combining (8), we have $\rho'' \ge \rho'$, and so $\rho'' > \rho^*$.

We are ready to complete the proof of Lemma 3.3. Note that $G''[L' \cup L'']$ has a maximal linear path $P = w_1v_1v_2...v_bw_2...w_{a-1}w_a$. In view of Claim 3.8 and the definition of G^* , G'' contains an H'' minor for some $H'' \in \mathbb{H}$. Let $\mathscr{V}'' = (V_1,...,V_{|H''|})$ be a minimal H''partition of G''. Applying Claim 3.3 on G'' and \mathscr{V}'' , we know that either $V_i \cap (L' \cup L'') = \varnothing$

or $G''[V_i \cap (L' \cup L'')]$ is connected for every $i \in \{1, ..., |H''|\}$. This implies that $V_i \cap V(P) = \emptyset$, $G''[V_i \cap V(P)]$ is a subpath of P, or $G''[V_i \cap V(P)]$ consists of two subpaths P', P'' such that $w_1 \in V(P')$ and $w_a \in V(P'')$. Now let $P(i) = G''[V_i \cap V(P)]$, i = 1, ..., |H''|. Then P(1), ..., P(|H''|) separate P into r subpaths, where $r \leq 2|H''| + 1$. Let G''' be the graph obtained from G'' by contracting each of these r subpaths as a vertex. Then by the definition of model, G''' also has an H'' minor.

On the other hand, recall that $G^*[L''] = G'[L'']$ and $P^* \subseteq G'[L'']$. Hence, P^* is also a maximal linear path in $G^*[L'']$. By Claim 3.7, $|P^*| \ge \frac{\sqrt{n}}{2|H'|} \ge 2|H''| + 1$. One can observe that if we contract the path P^* as a new path of order r in $G^* - L'$, then the resulting graph is isomorphic to G'''. Hence, G^* contains a G''' minor and thus an H'' minor, a contradiction. Therefore, $L' = \emptyset$. This completes the proof of Lemma 3.3.

In the following, we complete the proof of Theorem 1.1.

Proof. Now, we know that $|L| = \gamma_{\mathbb{H}}$ by Proposition 3.1 and $V(G^*) = L \cup L''$ by Lemma 3.3. Thus, G^* contains $K_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}$ as a spanning subgraph. To prove Theorem 1.1, that is, G^* contains $B_{\gamma_{\mathbb{H}},n-\gamma_{\mathbb{H}}}$, it suffices to show $G^*[L] \cong K_{\gamma_{\mathbb{H}}}$.

Suppose to the contrary that *L* is not a clique in G^* . Then we can find $u_1, u_2 \in L$ with $u_1u_2 \notin E(G^*)$. Since $|L''| = n - |L| > \max_{H \in \mathbb{H}}(|H| + 1)$, we can choose a subset $L''' \subset L''$ with $|L'''| = \max_{H \in \mathbb{H}}(|H| + 1)$. Now set $L'''' := L'' \setminus L'''$. By Lemma 3.1, $d_{L''}(w) \leq \alpha_{\mathbb{H}}$ for each $w \in L''$. It follows that

$$e(L'')-e(L'''') \leq \sum_{v \in L'''} d_{L''}(v) \leq \max_{H \in \mathbb{H}} (|H|+1) \alpha_{\mathbb{H}} \leq \sqrt{n}.$$

Moreover, let $x_{L''}^* := \max_{w \in L''} x_w$. Then we have $\rho^* x_{L''}^* \leq \sum_{u \in L} x_u + \alpha_{\mathbb{H}} x_{L''}^*$, Recall that $\rho^* \geq \sqrt{\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})}$. Hence,

$$x_{L''}^* \leq \frac{\sum_{u \in L} x_u}{\rho^* - \alpha_{\mathbb{H}}} \leq \frac{\gamma_{\mathbb{H}} x_{u^*}}{\rho^* - \alpha_{\mathbb{H}}} \leq \sqrt{\frac{2\gamma_{\mathbb{H}}}{n}} x_{u^*}$$

Now let *G* be the graph obtained from G^* by deleting all edges in $E(L'') \setminus E(L''')$ and adding an edge u_1u_2 . By Proposition 3.1, $\min\{x_{u_1}, x_{u_2}\} \ge (1 - \frac{1}{2(10C_{\mathbb{H}})^2})x_{u^*}$, and thus

$$\begin{aligned}
\rho(G) - \rho(G^*) &\geq 2\left(x_{u_1}x_{u_2} - \sum_{w_1w_2 \in E(L'') \setminus E(L''')} x_{w_1}x_{w_2}\right) \\
&\geq 2\left(\left(1 - \frac{1}{2(10C_{\mathbb{H}})^2}\right)^2 - \sqrt{n} \cdot \frac{2\gamma_{\mathbb{H}}}{n}\right) x_{u^*}^2 \\
&> 0,
\end{aligned}$$

which implies that *G* contains an H_0 minor for some $H_0 \in \mathbb{H}$.

Now let $\mathscr{V} = (V_1, \ldots, V_{|H_0|})$ be a minimal H_0 -partition of G. Observe that $N_G(v) = L$ for each vertex $v \in L'''$. Applying Claim 3.3 on G and \mathscr{V} , we can see that $|V_i \cap L'''| \leq 1$ for $i = 1, \ldots, |H_0|$. Since $|L'''| = \max_{H \in \mathbb{H}} (|H| + 1) \geq |H_0| + 1$, there exists a vertex $v \in L''' \setminus \bigcup_{i=1}^{|H_0|} V_i$. Consequently, $\mathscr{V} = (V_1, \ldots, V_{|H_0|})$ is also an H_0 -partition of $G - \{v\}$. In other

words, $G - \{v\}$ contains an H_0 minor. Now let $G_{u_1v}^*$ be the graph obtained from G^* by contracting the edge u_1v as a new vertex \overline{u}_1 . Then $\overline{u}_1u_2 \in E(G_{u_1v}^*)$, and thus $G - \{v\}$ is isomorphic to some subgraph of $G_{u_1v}^*$. This implies that $G_{u_1v}^*$ also contains an H_0 minor. Correspondingly, G^* contains an H_0 minor, a contradiction. Therefore, L is a clique in G^* . This completes the proof of Theorem 1.1.

Having Theorem 1.1, we shall focus on the characterization of $G^* - L$. We now recall some notations and terminologies. For a member $H \in \mathbb{H}$, $\Gamma_s^*(H)$ denotes the family of *s*-vertex irreducible induced subgraphs of *H* and $\Gamma(\mathbb{H}) = \bigcup_{H \in \mathbb{H}} \Gamma_{|H| - \gamma_{\mathbb{H}}}^*(H)$, where $\gamma_{\mathbb{H}} = \min_{H \in \mathbb{H}} \gamma_H$ and $\gamma_H = |H| - \alpha_H - 1$.

Lemma 3.4. Let G be a graph with a set L of $\gamma_{\mathbb{H}}$ dominating vertices. Then, G is \mathbb{H} -minor free if and only if G - L is $\Gamma(\mathbb{H})$ -minor free.

Proof. Firstly, assume that G - L contains an H_0 minor for some $H_0 \in \Gamma(\mathbb{H})$. Then there exists an $H \in \mathbb{H}$ such that H_0 is an $(|H| - \gamma_{\mathbb{H}})$ -vertex induced subgraph of H. Combining this H_0 minor with $\gamma_{\mathbb{H}}$ dominating vertices in L, we obtain an H minor in G.

Conversely, assume that *G* contains an *H* minor for some $H \in \mathbb{H}$. Then by Definition 3.2, *G* has an *H*-partition $\mathscr{V} = (V_1, \ldots, V_{|H|})$. We may assume that \mathscr{V} is a minimal *H*-partition such that $|L \cap (\bigcup_{i=1}^{|H|} V_i)|$ is maximal. Then there exist exactly $\gamma_{\mathbb{H}}$ members of \mathscr{V} , say $V_1, \ldots, V_{\gamma_{\mathbb{H}}}$, such that $|V_i| = |L \cap V_i| = 1$ for $i \in \{1, \ldots, \gamma_{\mathbb{H}}\}$. Consequently, $\bigcup_{i=\gamma_{\mathbb{H}}+1}^{|H|} V_i \subseteq V(G) \setminus L$. This implies that G - L contains an H_0 minor, where $H_0 \in \Gamma_{|H| - \gamma_{\mathbb{H}}}(H)$. It follows that G - L has an H_1 minor with $H_1 \in \Gamma_{|H| - \gamma_{\mathbb{H}}}^*(H)$.

In the following, we give the proof of Theorem 1.2.

Proof. Theorem 1.1 gives that G^* has $\gamma_{\mathbb{H}}$ dominating vertices, which implies that G^* is connected. Hence, adding an arbitrary edge within its independent set increases the spectral radius. Furthermore, by Lemma 3.4, $G^* - L$ is $\Gamma(\mathbb{H})$ -minor free. Thus $G^* - L$ is $\Gamma(\mathbb{H})$ -minor saturated, that is, $G^* - L \in SAT(n - \gamma_{\mathbb{H}}, \Gamma(\mathbb{H})_{minor})$.

Particularly, if $\mathbb{H} = \{H\}$, then $|H| - \gamma_{\mathbb{H}} = \alpha_H + 1$. Hence, $\Gamma(\mathbb{H}) = \Gamma^*_{\alpha_H + 1}(H)$, and thus $G^* - L \in SAT(n - \gamma_H, \Gamma^*_{\alpha_H + 1}(H)_{minor})$. This completes the proof of Theorem 1.2.

A subset *R* of $V(G^*) \setminus L$ is called a *component subset*, if $G^*[R]$ consists of some connected components. Furthermore, *R* is said to be *small*, if $|R| \leq C$ for a constant *C*. To prove Theorem 1.3, we need some more lemmas. We always assume that $\Gamma(\mathbb{H})$ is a family of connected graphs in the following lemmas.

Lemma 3.5. If *R* is a small component subset of $V(G^*) \setminus L$, then $e(R) = ex(|R|, \Gamma(\mathbb{H})_{minor})$.

Proof. Let *H* be a minimal member in \mathbb{H} . Recall that $\gamma_H = \gamma_{\mathbb{H}}$ and $\alpha_H = \alpha_{\mathbb{H}}$. Then $|H| - \gamma_{\mathbb{H}} = \alpha_{\mathbb{H}} + 1$, and thus $\Gamma^*_{\alpha_{\mathbb{H}}+1}(H) \subseteq \Gamma(\mathbb{H})$. By Lemma 3.4, $G^* - L$ is $\Gamma^*_{\alpha_{\mathbb{H}}+1}(H)$ -minor free. Now, choose an $(\alpha_{\mathbb{H}} + 1)$ -subset *S* of V(H) such that it contains $\alpha_{\mathbb{H}}$ independent vertices of *H*. Then $H[S] \subseteq K_{1,\alpha_{\mathbb{H}}}$, and hence $G^* - L$ is $K_{1,\alpha_{\mathbb{H}}}$ -minor free.

Similarly to the proof of Theorem 1.1, we let $L'' = V(G^*) \setminus L$ and $x_{L''}^* = \max_{w \in L''} x_w$. We also set $\sum_{u \in L} x_u := \sigma_L$. Clearly, $x_w \ge \frac{\sigma_L}{\rho^*}$ for each $w \in L''$. Since $G^*[L'']$ is $K_{1,\alpha_{\mathbb{H}}}$ -minor free, we have $\rho^* x_{L''}^* \le \sigma_L + (\alpha_{\mathbb{H}} - 1) x_{L''}^*$, which gives $x_{L''}^* \le \frac{\sigma_L}{\rho^* - \alpha_{\mathbb{H}} + 1}$.

Now suppose to the contrary that there exists a small component subset R of L'' such that $e(R) < ex(|R|, \Gamma(\mathbb{H})_{minor})$. Then we can find a $\Gamma(\mathbb{H})$ -minor free graph G''_i on the vertex set R with at least e(R) + 1 edges. Let G be the graph obtained from G^* by replacing $E(G^*[R])$ with $E(G''_i)$. Since $\Gamma(\mathbb{H})$ is a connected family, G[L''] is still $\Gamma(\mathbb{H})$ -minor free. By Lemma 3.4, G is \mathbb{H} -minor free. Let $\rho = \rho(G)$. Then

$$\frac{1}{2}(\rho - \rho^*) \ge \sum_{uv \in E(G''_i)} x_u x_v - \sum_{uv \in E(G^*[R])} x_u x_v \ge \frac{e(G''_i)\sigma_L^2}{\rho^{*2}} - \frac{e(R)\sigma_L^2}{(\rho^* - \alpha_{\mathbb{H}} + 1)^2}.$$
(9)

Since *R* is small, e(R) is also bounded by a constant. Recall that $e(G''_i) \ge e(R) + 1$ and $\rho^* \ge \sqrt{\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})}$. It is clear that $\rho(G) > \rho^*$ for *n* sufficiently large, a contradiction. The proof is completed.

Let \mathbb{G} be the family of connected $\Gamma(\mathbb{H})$ -minor free graphs on at most $n - \gamma_{\mathbb{H}}$ vertices. In fact, the proof of Lemma 3.5 implies that every member in \mathbb{G} is $K_{1,\alpha_{\mathbb{H}}}$ -minor free. Given a member $G_i \in \mathbb{G}$, we denote by d_i its average degree, and we say that G_i is *small* if $|G_i| \leq c$ for some constant c (in other words, $|G_i|$ is independent of n). Now, let G_0 be a member with $d_0 = \max_{G_i \in \mathbb{G}} d_i$, and let G_1, \ldots, G_s be all the non-isomorphic components in $G^* - L$. We may assume that $m(G_1) \geq \cdots \geq m(G_s)$, where $m(G_i)$ is the number of copies of G_i in $G^* - L$. By Lemma 3.4, $G_i \in \mathbb{G}$ for every $i \in \{1, \ldots, s\}$.

Lemma 3.6. If all the members are small in $\{G_i : i = 0, 1, ..., s\}$, then we have $d_1 = d_0$ and $m(G_i) < |G_1|$ for every G_i with $d_i < d_0$.

Proof. In view of the choice of G_0 , we know that $d_i \leq d_0$ for every $i \in \{0, 1, ..., s\}$. Since $\max_{1 \leq i \leq s} |G_i| = c$ for some constant integer c, we have $s \leq \sum_{k=1}^{c} 2^{\binom{k}{2}}$, that is, s is constant. However, $|G^* - L|$ is sufficiently large, thus $m(G_1)$ is a function on n.

We first prove that $d_1 = d_0$. Otherwise, $d_1 < d_0$, then we define *G* to be the graph obtain from *G*^{*} by replacing $|G_0|$ copies of G_1 with $|G_1|$ copies of G_0 in $G^* - L$. The component subset $V(|G_0|G_1)$ is small and G - L is $\Gamma(\mathbb{H})$ -minor free. However, $|G_0|e(G_1) - |G_1|e(G_0) = \frac{1}{2}|G_0||G_1|(d_1 - d_0) < 0$, contradicting Lemma 3.5. Hence, $d_1 = d_0$.

Now suppose that there exists a component G_i with $d_i < d_0$ but $m(G_i) \ge |G_1|$. Then we define G to be a new graph obtain from G^* by replacing $|G_1|$ copies of G_i with $|G_i|$ copies of G_1 in $G^* - L$. Clearly, $|G_1|e(G_i) - |G_i|e(G_1) = \frac{1}{2}|G_1||G_i|(d_i - d_0) < 0$, and we similarly get a contradiction. Hence, the lemma holds.

Now, let G^* be a graph of order n with a set L of dominating vertices such that $G^* - L \in EX(n - |L|, \Gamma(\mathbb{H})_{minor})$. Moreover, assume that G_{i_1}, \ldots, G_{i_t} are all the non-isomorphic components in $G^* - L$. Clearly, $G_{i_j} \in \mathbb{G}$ for every $j \in \{1, \ldots, t\}$.

Lemma 3.7. If all the members are small in $\{G_i : i = 0, 1, ..., s\}$ and $\{G_{i_j} : j = 1, ..., t\}$, then we have $G^* - L \in EX(n - |L|, \Gamma(\mathbb{H})_{minor})$.

Proof. Suppose to the contrary, then $e(G^* - L) < e(G^* - L)$. Since $\max_{1 \le j \le t} |G_{i_j}|$ is a constant, *t* is too. The proof of Lemma 3.6 also implies that $m(G_{i_j}) < |G_1|$ for every G_{i_j} with $d_{i_j} < d_0$. Recall that $m(G_i)$ denotes the number of copies of G_i in $G^* - L$. We now define $m'(G_{i_j})$ to be the number of copies of G_{i_j} in $G^* - L$. Write $m_i \equiv m(G_i) \pmod{|G_1|}$ for $i \in \{1, \ldots, s\}$ and $m'_{i_j} \equiv m'(G_{i_j}) \pmod{|G_1|}$ for $j \in \{1, \ldots, t\}$. Then, we can see that $m_i = m(G_i)$ if $d_i < d_0$ and $m'_{i_j} = m(G'_{i_j})$ if $d_{i_j} < d_0$.

Now we denote $G'_1 = G^* - L - \bigcup_{i=1}^s V(m_i G_i)$ and $G'_2 = G^* - L - \bigcup_{j=1}^t V(m'_{i_j} G_{i_j})$ for simplicity. Then, both $|G'_1|$ and $|G'_2|$ are divisible by $|G_1|$, which gives that

$$|\cup_{j=1}^{t} V(m_{i_{j}}'G_{i_{j}})| - |\cup_{i=1}^{s} V(m_{i}G_{i})| = |G_{1}'| - |G_{2}'| = r|G_{1}|$$
(10)

for an integer *r*. Moreover, since both $|\bigcup_{j=1}^{t} V(m'_{i_j}G_{i_j})|$ and $|\bigcup_{i=1}^{s} V(m_iG_i)|$ are finite, *r* is too. Note that every component in G'_1 and G'_2 has average degree d_0 . Combining (10) and $d_0 = d_1$, we have $e(G'_1) - e(G'_2) = \frac{1}{2}d_0(|G'_1| - |G'_2|) = re(G_1)$. Consequently,

$$\sum_{j=1}^{t} e(m'_{i_j}G_{i_j}) - \sum_{i=1}^{s} e(m_iG_i) = e(G^* - L) - e(G^* - L) + re(G_1) > re(G_1).$$
(11)

Recall that $m(G_1)$ is a function on n. If $r \ge 0$, we define $R = (\bigcup_{i=1}^{s} V(m_i G_i)) \cup V(rG_1)$. Then R is a small component subset of $V(G^*) \setminus L$, and $|R| = |\bigcup_{j=1}^{t} V(m'_{i_j}G_{i_j})|$ by (10). However, (11) gives $e(R) < \sum_{j=1}^{t} e(m'_{i_j}G_{i_j})$, contradicting Lemma 3.5.

If r < 0, we define $R = \bigcup_{i=1}^{s} V(m_i G_i)$. Then *R* is still a small component subset of $V(G^*) \setminus L$, and by (10) $|R| = |\bigcup_{j=1}^{t} V(m'_{i_j} G_{i_j})| + |V(-rG_1)|$. However, by (11) we have $e(R) < \sum_{j=1}^{t} e(m'_{i_j} G_{i_j}) + (-r)e(G_1)$, also a contradiction. Hence, the lemma holds. \Box

Lemma 3.8. If there exists a member in \mathbb{G} which contains a bicyclic subgraph, then all the members are small in $\{G_i : i = 0, 1, ..., s\}$ and $\{G_{i_j} : j = 1, ..., t\}$.

Proof. Recall that every member in \mathbb{G} is $K_{1,\alpha_{\mathbb{H}}}$ -minor free. If some member in \mathbb{G} contains a bicyclic subgraph, then it has an H_0 minor, where $H_0 \in \{K_1 \nabla 2P_2, K_1 \nabla P_3\}$, and thus a $K_{1,3}$ minor. This implies that $\alpha_{\mathbb{H}} \ge 4$.

Let \mathbb{G}' be the subset of \mathbb{G} in which any member is not small. Suppose to the contrary that \mathbb{G}' is non-empty and $G_{i_0} \in \mathbb{G}'$. Since G_{i_0} is $K_{1,\alpha_{\mathbb{H}}}$ -minor free, by Lemma 2.3 we have $e(G_{i_0}) \leq {\alpha_{\mathbb{H}} \choose 2} + |G_{i_0}| - \alpha_{\mathbb{H}}$. Thus, $d_{i_0} = 2e(G_{i_0})/|G_{i_0}| \to 2$ ($|G_{i_0}| \to \infty$), and hence $d_{i_0} < 2.2$ as $|G_{i_0}|$ is sufficiently large. Now we know that $d_i < 2.2$ for every $G_i \in \mathbb{G}'$.

Since a member in \mathbb{G} contains an H_0 minor, H_0 is $\Gamma(\mathbb{H})$ -minor free. A simple calculation gives that $2e(H_0)/|H_0| \ge 2.4$. By the definition of G_0 , we have $d_0 \ge 2e(H_0)/|H_0| \ge 2.4$. Since $d_i < 2.2$ for every $G_i \in \mathbb{G}'$, it is clear that $G_0 \notin \mathbb{G}'$.

Now suppose that $G_{i_j} \in \mathbb{G}'$ for some $j \in \{1, \ldots, t\}$. that is, $|G_{i_j}|$ is sufficiently large. Then $d_{i_j} < 2.2$ and $|G_{i_j}| = a|H_0| + b$, where $0 \le b < |H_0|$ and a depends on n. Hence, $e(G_{i_j}) < 1.1(a|H_0| + b)$. Now define $G''_{i_j} = aH_0 \cup bK_1$. Then G''_{i_j} is $\Gamma(\mathbb{H})$ -minor free, and $e(G''_{i_j}) = ae(H_0) \ge 1.2a|H_0|$. Thus, $e(G''_{i_j}) > e(G_{i_j})$, contradicting the fact that $G^* - L \in EX(n - |L|, \Gamma(\mathbb{H})_{minor})$. Hence, $G_{i_j} \notin \mathbb{G}'$ for any $j \in \{1, \ldots, t\}$. Finally, suppose $G_i \in \mathbb{G}'$ for some $i \in \{1, ..., s\}$. Then $d_i < 2.2$ and $|G_i| = a'|H_0| + b'$, where $0 \le b' < |H_0|$ and a' depends on n. Thus, $e(G_i) < 1.1(a'|H_0| + b')$. Now define $G''_i = a'H_0 \cup b'K_1$. Then G''_i is $\Gamma(\mathbb{H})$ -minor free, and $e(G''_i) = a'e(H_0) \ge 1.2a'|H_0|$. It is easy to see that $e(G''_i) > 1.05e(G_i)$. Let G be obtained from G^* by replacing $E(G_i)$ with $E(G''_i)$. By Lemma 3.4, G is \mathbb{H} -minor free. Choosing $R = V(G_i)$ in (9), we have

$$\frac{1}{2}(\rho(G) - \rho^*) \ge \sum_{uv \in E(G''_i)} x_u x_v - \sum_{uv \in E(G_i)} x_u x_v \ge \frac{e(G''_i)\sigma_L^2}{\rho^{*2}} - \frac{e(G_i)\sigma_L^2}{(\rho^* - \alpha_{\mathbb{H}} + 1)^2}.$$

Combining $e(G''_i) > 1.05e(G_i)$ yields that $\rho(G) > \rho^*$. This contradicts the fact that $G^* \in SPEX(n, \mathbb{H}_{minor})$. Therefore, $G_i \notin \mathbb{G}'$ for any $i \in \{1, ..., s\}$.

Given a rooted tree *T*. A *branching vertex* in *T* is a vertex of degree at least three. An *edge-switching* on *T* means that we construct a new tree $T' = T - \{u_1v_1\} + \{u_2v_2\}$, where $u_1v_1 \in E(T)$ and u_2v_2 is a non-edge in *T*. A *pruning* on *T* is an edge-switching $T' = T - \{u_1v_1\} + \{v_1v_2\}$, where u_1 is a branching vertex, v_1 is its son and v_2 is a leaf (which is not a descendant of v_1). We end this section by proving Theorem 1.3.

Proof. Combining Lemmas 3.7 and 3.8, we can see that if all the members in \mathbb{G} are small or some member has a bicyclic subgraph, then $G^* - L \in EX(n - |L|, \Gamma(\mathbb{H})_{minor})$ and we are done. Thus we may assume that every member in \mathbb{G} is either a tree or a unicyclic graph, and there exists a member $G_{i_0} \in \mathbb{G}$ such that $|G_{i_0}|$ is sufficiently large.

Let G_i be an arbitrary member in \mathbb{G} and $V(G_i) = \bigcup_{k=1}^3 U_k$, where $U_k = \{v \in V(G_i) : d_{G_i}(v) = k\}$ for $k \in \{1,2\}$ and $U_3 = V(G_i) \setminus (U_1 \cup U_2)$. Recall that every member in \mathbb{G} is $K_{1,\alpha_{\mathbb{H}}}$ -minor free. Then max $\{\Delta(G_i), |U_1|\} < \alpha_{\mathbb{H}}$. Since G_i is a tree or a unicyclic graph, we have $e(G_i) \le |G_i| = \sum_{i=1}^3 |U_k|$. Moreover, $e(G_i) \ge \frac{1}{2}(|U_1| + 2|U_2| + 3|U_3|)$. Combining the above three inequalities, we can deduce that $|U_3| \le |U_1| < \alpha_{\mathbb{H}}$.

On the one hand, the inequality $\max{\{\Delta(G_i), |U_3|\}} < \alpha_{\mathbb{H}}$ implies that every G_i can be transformed to a path or a lollipop graph by at most $\alpha_{\mathbb{H}}^2$ steps of pruning. On the other hand, since $|U_1| + |U_3| < 2\alpha_{\mathbb{H}}$ for G_{i_0} but $|G_{i_0}|$ is sufficiently large, we can see that G_{i_0} contains a path of length large enough. Note that G_{i_0} is $\Gamma(\mathbb{H})$ -minor free. Thus, $P_k \notin \Gamma(\mathbb{H})$ for any positive integer k.

Recall that $G^* - L$ contains *s* non-isomorphic components G_1, \ldots, G_s , each of which is a tree or a unicyclic graph. Suppose that $G^* - L$ contains at least two tree components, say G_1 and G_2 . Then for $i \in \{1,2\}$, G_i can be transformed to a path $P_{|G_i|}$ by at most $\alpha_{\mathbb{H}}^2$ steps of pruning. This implies that $P_{|G_i|} = G_i - E'_i + E''_i$, where $E'_i \subseteq E(G_i)$, E''_i is set of non-edges of G_i and $|E'_i| = |E''_i| \leq \alpha_{\mathbb{H}}^2$. Let $P_{|G_1 \cup G_2|}$ be a path obtained from $P_{|G_1|} \cup P_{|G_2|}$ by adding an edge. Moreover, denote by *G* the graph obtained from G^* by replacing $E(G_1 \cup G_2)$ with $E(P_{|G_1 \cup G_2|})$. Since $P_k \notin \Gamma(\mathbb{H})$ for any positive integer *k*, G - L is still $\Gamma(\mathbb{H})$ -minor free. However, similarly as (9), we have

$$\frac{1}{2}(\rho(G)-\rho^*) \ge \sum_{uv \in E(P)} x_u x_v - \sum_{uv \in E(G_1 \cup G_2)} x_u x_v \ge \frac{(|E_1'' \cup E_2''|+1)\sigma_L^2}{\rho^{*2}} - \frac{|E_1' \cup E_2'|\sigma_L^2}{(\rho^*-\alpha_{\mathbb{H}}+1)^2}, \quad (12)$$

which yields that $\rho(G) > \rho^*$, a contradiction. Therefore, $G^* - L$ contains at most one tree component. If every component in $G^* - L$ is unicyclic, then $e(G^* - L) = |G^* - L|$ and thus $G^* - L \in EX(n - |L|, \Gamma(\mathbb{H})_{minor})$, as desired.

Now, we may assume that $G^* - L$ has exactly one tree component G_1 . We first consider the case that \mathbb{G} admits a unicyclic member G_{j_0} . Then G_{j_0} contains an ℓ_0 -cycle for some $\ell_0 \leq n - \gamma_{\mathbb{H}}$, and thus $C_{\ell} \notin \Gamma(\mathbb{H})$ for any positive integer $\ell \geq \ell_0$. Now if $|G_1| \geq \ell_0$, then we define a new G to be the graph obtained from G^* by replacing $E(G_1)$ with $E(C_{|G_1|})$, where $C_{|G_1|}$ denotes a $|G_1|$ -cycle obtained from $P_{|G_1|}$ by adding an edge. Then G - L is still $\Gamma(\mathbb{H})$ -minor free. But similarly as (12), we have $\rho(G) > \rho^*$, a contradiction. Hence, $|G_1| < \ell_0 \leq n - \gamma_{\mathbb{H}}$, which implies that $G^* - L$ contains another component G_2 and G_2 is a unicyclic graph. Recall that G_1 can be transformed to a path $P_{|G_1|}$ by at most $\alpha_{\mathbb{H}}^2$ steps of pruning, and G_2 can be transformed to a cycle $C_{|G_1 \cup G_2|}$ by adding an edge and switching at most $2\alpha_{\mathbb{H}}^2 + 1$ edges. Let G be the graph obtained from G^* by replacing $E(G_1 \cup G_2)$ with $E(C_{|G_1 \cup G_2|})$. We can similarly see that G - L is $\Gamma(\mathbb{H})$ -minor free and $\rho(G) > \rho^*$, a contradiction. Therefore, every member in \mathbb{G} is a tree. Now $G^* - L = G_1$ and thus $G^* - L \in EX(n - |L|, \Gamma(\mathbb{H})_{minor})$, completing the proof of Theorem 1.3.

4 Complete multi-partite minors

In this section, we will use Theorems 1.1, 1.2 and 1.3 to characterize $SPEX(n, \mathbb{H}_{minor})$ for $\mathbb{H} = \{K_{s_1,...,s_r}\}$. Above all, we shall recall some notations. Let \overline{G} be the complement of a graph G and Pet^* be the Petersen graph. Let $H_{s_1,s_2} = (\beta - 1)K_{1,s_2} \cup K_{1,s_2+\beta_0}$, where $\beta(s_2+1) + \beta_0 = s_1 + 1$ and $0 \le \beta_0 \le s_2$. Obviously, H_{s_1,s_2} is a star forest of order $s_1 + 1$. Moreover, $H_{s_1,1} \cong \frac{s_1+1}{2}K_{1,1}$ for odd s_1 and $H_{s_1,1} \cong \frac{s_1-2}{2}K_{1,1} \cup K_{1,2}$ for even s_1 . Let $S(\overline{H_{s_1,s_2}})$ denote the graph obtained from $\overline{H_{s_1,s_2}}$ by subdividing an edge uv with minimum degree sum d(u) + d(v). Particularly, one can observe that $S(\overline{H_{4,1}}) \cong S^2(K_4)$.

For $(s_1, s_2) \in \{(2, 2), (3, 2), (3, 3)\}$, *SPEX* $(n, \{K_{s_1, s_2}\}_{minor})$ was determined in [41, 42, 52, 55]. In [54], the authors characterized *SPEX* $(n, \{K_{s_1, s_2}\}_{minor})$ for $s_1 \ge 4$ and $s_2 \ge 2$. These results solved a conjecture proposed by Tait [46]. In fact, the above results can be rewritten as a slightly stronger version (see Theorem 4.1).

Theorem 4.1. Assume that $s_1 \ge s_2 \ge 2$, $\beta = \lfloor \frac{s_1+1}{s_2+1} \rfloor$, $\gamma \ge 1$ and $n - \gamma = ps_1 + q$ $(1 \le q \le s_1)$. Let *G* be the join of a copy of K_{γ} and an $(n - \gamma)$ -vertex $\Gamma_{s_1+1}^*(K_{s_1,s_2})$ -minor free graph. Then $\rho(G) \le \rho(K_{\gamma} \nabla G^{\blacktriangle})$, with equality if and only if $G \cong K_{\gamma} \nabla G^{\blacktriangle}$, where

$$G^{\blacktriangle} = \begin{cases} (p-1)K_{s_1} \cup S(\overline{H_{s_1,s_2}}) & \text{if } (q,\beta) = (2,2); \\ (p-1)K_{s_1} \cup \overline{Pet^{\star}} & \text{if } (q,\beta,s_1) = (2,1,8); \\ (p-q)K_{s_1} \cup q\overline{H_{s_1,s_2}} & \text{if } q \le 2(\beta-1) \text{ and } (q,\beta) \ne (2,2); \\ pK_{s_1} \cup K_q & \text{if } q > 2(\beta-1) \text{ and } (q,\beta,s_1) \ne (2,1,8). \end{cases}$$

Now we characterize $SPEX(n, \{H\}_{minor})$ for $H = K_{s_1, \dots, s_r}$, where $s_1 \ge \dots \ge s_r$ and n

is sufficiently large. Clearly, $\alpha_H = s_1$ and $\Gamma^*_{\alpha_H+1}(H) = \Gamma^*_{s_1+1}(H)$. We first consider the case $s_2 \ge 2$. Now we have $\sum_{i=1}^{r} s_i - 1 \ge 1$.

Theorem 4.2. Let $s_2 \ge 2$, $\beta = \lfloor \frac{s_1+1}{s_2+1} \rfloor$, $\gamma = \sum_{i=1}^{r} s_i - 1$ and $n - \gamma = ps_1 + q$ $(1 \le q \le s_1)$. Then $SPEX(n, \{K_{s_1,...,s_r}\}_{minor}) = \{K_{\gamma}\nabla G^{\blacktriangle}\}$, where G^{\blacktriangle} is defined as in Theorem 4.1.

Proof. Let $\bigcup_{i=1}^{r} S_i$ be the *r*-partite partition of V(H), where $|S_i| = s_i$ for $i \in \{1, ..., r\}$. Given an arbitrary $(\alpha_H + 1)$ -subset *S* of V(H). We first claim that there exists an $(\alpha_H + 1)$ -subset *S'* such that $S' \subseteq S_1 \cup S_2$ and H[S'] is isomorphic to a subgraph of H[S].

Let $t_i = |S_i \cap S|$ for $i \in \{1, ..., r\}$. Up to isomorphism of H[S], one can assume that $t_1 \ge \cdots \ge t_r$. If $t_3 = 0$, then we choose S' = S, as required. Suppose now that $t_3 \ge 1$. Then $t_2 \ge 1$. Note that $\sum_{i=1}^r t_i = |S| = s_1 + 1$. Thus, $\sum_{i=3}^r t_i = s_1 + 1 - t_1 - t_2 \le s_1 - t_1 = |S_1 \setminus S|$. Now let S' be obtained from S by replacing $\sum_{i=3}^r t_i$ vertices in $S \setminus (S_1 \cup S_2)$ with $\sum_{i=3}^r t_i$ vertices in $S_1 \setminus S$. Then $S' \subseteq S_1 \cup S_2$, and it is easy to see that H[S'] is isomorphic to a subgraph of H[S]. Hence, the claim holds, which further implies $\Gamma_{\alpha_H+1}^*(H) = \Gamma_{s_1+1}(K_{s_1,s_2})$.

Note that $\gamma_H = |H| - \alpha_H - 1 = \sum_{2}^{r} s_i - 1$. By Theorem 1.1, the extremal graph G^* has a set *L* of dominating vertices, where $|L| = \sum_{2}^{r} s_i - 1$. By Theorem 1.2, $G^* - L$ is $\Gamma^*_{\alpha_H+1}(H)$ -minor free, that is, $\Gamma_{s_1+1}(K_{s_1,s_2})$ -minor free. Now setting $\gamma = \sum_{2}^{r} s_i - 1$ in Theorem 4.1, we obtain that $SPEX(n, \{K_{s_1,\dots,s_r}\}_{minor}) = \{K_{\gamma}\nabla G^{\blacktriangle}\}$ immediately.

Having Theorem 4.2, it remains to characterize $SPEX(n, \{H\}_{minor})$ for a complete *r*-partite graph $H = K_{s_1,1,...,1}$. If $s_1 = 1$, then $H \cong K_r$ and $SPEX(n, \{K_r\}_{minor})$ was determined by Tait (see Theorem 1.4). If r = 2, then $H \cong K_{1,s_1}$ and $SPEX(n, \{K_{1,s_1}\}_{minor})$ was determined in [54]. In the following, we may assume that min $\{r - 1, s_1\} \ge 2$. In fact, we have $H \cong B_{r-1,s_1}$. Moreover, $\alpha_H = s_1$, $\gamma_H = |H| - \alpha_H - 1 = r - 2$ and

$$\Gamma^*_{\alpha_H+1}(H) = \Gamma^*_{s_1+1}(B_{r-1,s_1}) = \{K_{1,s_1}\}.$$
(13)

Assume that $G^* \in SPEX(n, \{B_{r-1,s_1}\}_{minor})$, and $X^* = (x_1, \dots, x_n)^T$ is the positive unit eigenvector corresponding to $\rho^* := \rho(G^*)$. Let *L* be the set of dominating vertices in G^* . Then $|L| = \gamma_H = r - 2$ by Theorem 1.1. In the following, we set $\gamma := |L| = r - 2$.

Lemma 4.1. Assume that $s_1 \ge 2$ and $\gamma \ge 1$. Then we have

$$\rho^{*2} - (s_1 + \gamma - 2)\rho^* \le \gamma(n - \gamma) - (\gamma - 1)(s_1 - 1), \tag{14}$$

with equality if and only if $G^* - L$ is an $(s_1 - 1)$ -regular K_{1,s_1} -minor free graph.

Proof. By Theorem 1.2 and (13), we can see that $G^* - L$ is a K_{1,s_1} -minor saturated graph. Hence, $G^* - L$ is K_{1,s_1} -minor free.

By symmetry, x_u is constant for $u \in L$. Choose $u^* \in L$ and $v^* \in V(G^*) \setminus L$ with $x_{v^*} = \max_{v \in V(G^*) \setminus L} x_v$. Since $G^* - L$ is K_{1,s_1} -minor free, we have $\Delta(G^* - L) \leq s_1 - 1$. Note that $\rho^* x_u = \sum_{v \in N_{G^*}(u)} x_v$ for each $u \in V(G^*)$. Thus,

$$\rho^* x_{\nu^*} \le \gamma x_{u^*} + (s_1 - 1) x_{\nu^*} \text{ and } \rho^* x_{u^*} \le (\gamma - 1) x_{u^*} + (n - \gamma) x_{\nu^*}.$$
 (15)

Combining the two inequalities in (15), we obtain (14) immediately.

Next, we characterize equality case in (14). If the equality holds, then both inequalities in (15) become equalities. Hence, $d_{G^*-L}(v) = s_1 - 1$ for each $v \in V(G^*) \setminus L$, that is, $G^* - L$ is $(s_1 - 1)$ -regular. Conversely, if $G^* - L$ is $(s_1 - 1)$ -regular, then G^* is the join of two regular graphs, which implies that both $X^* \parallel_L$ and $X^* \parallel_{V(G^*) \setminus L}$ are constant vectors. Hence, both inequalities of (15) hold in equality, and thus (14) too.

Given an arbitrary $v \in V(G^*) \setminus L$. Then $\rho^* x_v = \gamma x_{u^*} + \sum_{w \in N_{G^*}(v) \setminus L} x_w$. Clearly, $\rho^* x_v \ge \gamma x_{u^*}$, and by (15) we have $(\rho^* - s_1 + 1)x_v \le \gamma x_{u^*}$. Hence, $\frac{\gamma}{\rho^*} x_{u^*} \le x_v \le \frac{\gamma}{\rho^* - s_1 + 1} x_{u^*}$. Recall that $\Delta(G^* - L) \le s_1 - 1$. We can further deduce that

$$x_u < x_v \tag{16}$$

for any two vertices $u, v \in V(G^*) \setminus L$ with $d_{G^*}(u) < d_{G^*}(v)$.

Lemma 4.2. Assume that $s_1 \ge 4$ and G is a connected K_{1,s_1} -minor free graph. If G is $(s_1 - 1)$ -regular, then either $G \cong K_{s_1}$, or $G \cong \overline{H_{s_1,1}}$ only for odd s_1 .

Proof. We shall first note that both K_{s_1} and $\overline{H_{s_1,1}}$ are K_{1,s_1} -minor free. Indeed, recall that $H_{s_1,1} \cong \frac{s_1+1}{2}K_{1,1}$ for odd s_1 and $H_{s_1,1} \cong \frac{s_1-2}{2}K_{1,1} \cup K_{1,2}$ for even s_1 . Thus, $|H_{s_1,1}| = s_1 + 1$ and $\Delta(\overline{H_{s_1,1}}) = s_1 - 1$. Therefore, $\overline{H_{s_1,1}}$ is K_{1,s_1} -minor free, and K_{s_1} is obviously too.

Since *G* is $(s_1 - 1)$ -regular, we have $|G| \ge s_1$. If $|G| = s_1$, then $G \cong K_{s_1}$. If $|G| = s_1 + 1$, then *G* can only be obtained from K_{s_1+1} by deleting a perfect matching, which implies that s_1 is odd and $\overline{G} \cong \frac{s_1+1}{2}K_{1,1} \cong H_{s_1,1}$. Consequently, $G \cong \overline{H_{s_1,1}}$.

Next assume that $|G| \ge s_1 + 2$. Since *G* is connected and K_{1,s_1} -minor free, by Lemma 2.3 we have $e(G) \le {s_1 \choose 2} + |G| - s_1$, which implies that $e(G) < \frac{1}{2}(s_1 - 1)|G|$ for $s_1 \ge 4$. Thus, *G* is not $(s_1 - 1)$ -regular, a contradiction. Hence, the lemma holds.

The following theorem follows immediately from Lemmas 4.1 and 4.2.

Theorem 4.3. Let $s_1 \ge 3$ be odd and $\gamma \ge 1$. Then $SPEX(n, \{B_{r-1,s_1}\}_{minor}) = \{K_{\gamma}\nabla G^{\blacktriangledown}\}$, where G^{\blacktriangledown} takes over all the $(n - \gamma)$ -vertex $(s_1 - 1)$ -regular K_{1,s_1} -minor free graphs, more precisely, every component of G^{\blacktriangledown} is a cycle for $s_1 = 3$, and is K_{s_1} or $\overline{H_{s_1,1}}$ for $s_1 \ge 5$.

Remark 4.1. In view of Theorem 4.3, $SPEX(n, \{B_{r-1,s_1}\}_{minor})$ is an infinite family for odd $s_1 \ge 3$. Indeed, assume that $s_1 \ge 5$ and $n - \gamma = ps_1 + q$ $(1 \le q \le s_1)$, then G^{\forall} can be constructed as the disjoint union of $p - c - (cs_1 + q)$ copies of K_{s_1} and $cs_1 + q$ copies of $\overline{H_{s_1,1}}$ for an arbitrary non-negative integer c.

Next, we consider the case that s_1 is even. Let \mathbb{G}_i denote the family of *i*-vertex components in $G^* - L$, and $|\mathbb{G}_i|$ be the number of components in \mathbb{G}_i .

Lemma 4.3. Assume that $s_1 \ge 4$ is even and $\gamma \ge 1$. If $\mathbb{G}_i \ne \emptyset$, then we have $s_1 - 1 \le i \le s_1 + 3$, where $i \in \{s_1 + 2, s_1 + 3\}$ only for $s_1 = 4$.

Proof. We first claim that $\mathbb{G}_i = \emptyset$ for each $i \ge 3s_1$. Indeed, otherwise, $G^* - L$ contains a component G_0 with $|G_0| = as_1 + b$, where $a \ge 3$ and $0 \le b < s_1$. Then $e(G_0) \le {\binom{s_1}{2}} + |G_0| - s_1 < {\binom{s_1}{2}} + as_1$ by Lemma 2.3. Now let $G'_0 = aK_{s_1} \cup bK_1$. Clearly, G'_0 is K_{1,s_1} -minor free and $e(G'_0) = a{\binom{s_1}{2}}$. A straightforward calculation gives that $e(G_0) < e(G'_0)$ for $a \ge 3$ and $s_1 \ge 4$, which contradicts Theorem 1.3.

Secondly, we claim that $|\mathbb{G}_i| \le s_1 - 1$ for $i \ne s_1$. Otherwise, there exists some $i_1 \ne s_1$ with $|\mathbb{G}_{i_1}| \ge s_1$. Let G_1 be a union of s_1 components in \mathbb{G}_{i_1} . Then $\Delta(G_1) \le s_1 - 1$ as G_1 is K_{1,s_1} -minor free. Moreover, G_1 is not $(s_1 - 1)$ -regular (otherwise, $G_1 \ge s_1K_{s_1}$ by Lemma 4.2). Thus, $e(G_1) < \frac{1}{2}(s_1 - 1)|G_1| = e(i_1K_{s_1})$, also contradicting Theorem 1.3.

The above claims implies that $\sum_{i \neq s_1} i |\mathbb{G}_i|$ is constant. Thus $s_1 |\mathbb{G}_{s_1}| = n - \gamma - \sum_{i \neq s_1} i |\mathbb{G}_i|$ $\geq \frac{n}{2s_1}$. For an arbitrary $i_2 \in \{1, 2, \dots, 3s_1 - 1\} \setminus \{s_1, s_1 + 1\}$, we set $i_2 = as_1 + b$, where $0 \leq a \leq 2$ and $0 \leq b < s_1$. If $\mathbb{G}_{i_2} \neq \emptyset$, we choose a subgraph G_2 in $G^* - L$, which consists of *b* components in \mathbb{G}_{s_1} and one in \mathbb{G}_{i_2} . Then $e(G_2) \leq b \cdot e(K_{s_1}) + e(K_b)$ for a = 0, and by Lemma 2.3, $e(G_2) \leq b \cdot e(K_{s_1}) + {s_1 \choose 2} + (i_2 - s_1)$ for $a \in \{1, 2\}$. Now let $G'_2 = aK_{s_1} \cup b\overline{H_{s_1,1}}$. Then $|G'_2| = |G_2|$ and $e(G'_2) = a \cdot e(K_{s_1}) + b \cdot \frac{1}{2}(s_1^2 - 2)$. Straightforward calculations give $e(G_2) \leq e(G'_2)$, with equality if and only if $i_2 = s_1 - 1$ or $i_2 \in \{s_1 + 2, s_1 + 3 : s_1 = 4\}$.

On the other hand, recall that both K_{s_1} and $\overline{H_{s_1,1}}$ are K_{1,s_1} -minor free, then G'_2 is too. By Theorem 1.3, $e(G_2) \ge e(G'_2)$. Hence, $e(G_2) = e(G'_2)$. Consequently, $i_2 = s_1 - 1$ or $i_2 \in \{s_1 + 2, s_1 + 3 : s_1 = 4\}$. In view of the choice of i_2 , we completes the proof.

Lemma 4.4. Assume that $s_1 \ge 4$ is even and $\gamma \ge 1$. Then $|\mathbb{G}_{s_1-1}| \le 1$ and $|\mathbb{G}_{s_1+1}| \le s_1-2$. Moreover, if $|\mathbb{G}_{s_1-1}| = 1$, then $\mathbb{G}_i = \emptyset$ unless $i \in \{s_1 - 1, s\}$.

Proof. We first assume that there exists a $G_0 \in \mathbb{G}_{s_1-1}$. By Theorem 1.2, $G^* - L$ is K_{1,s_1} minor saturated. Hence, $G_0 \cong K_{s_1-1}$. Choose another component G_1 arbitrarily in $G^* - L$. We now claim that $|G_1| = s_1$. Indeed, otherwise, $|G_1| \neq s_1$. Then by Lemma 4.2, G_1 is not $(s_1 - 1)$ -regular, and thus there exists $v \in V(G_1)$ with $d_{G_1}(v) \leq s_1 - 2$. Let G be the graph obtained from G^* by replacing $G_0 \cup G_1$ with $K_{s_1} \cup (G_1 - \{v\})$. Since K_{s_1} and $G_1 - \{v\}$ are K_{1,s_1} -minor free, G is B_{r-1,s_1} -minor free by Lemma 3.4. Note that $e(K_{s_1}) - e(G_0) =$ $s_1 - 1$ but $e(G_1) - e(G_1 - \{v\}) \leq s_1 - 2$. Then, $e(G) > e(G^*)$, contradicting Theorem 1.3. Therefore, $|G_1| = s_1$, as claimed.

Now we know that if there exists $G_0 \in \mathbb{G}_{s_1-1}$, then every component in $G^* - L$ other than G_0 can only belong to \mathbb{G}_{s_1} . This implies that $|\mathbb{G}_{s_1-1}| \leq 1$, and if $|\mathbb{G}_{s_1-1}| = 1$, then $\mathbb{G}_i = \emptyset$ unless $i \in \{s_1 - 1, s\}$.

It remains to show $|\mathbb{G}_{s_1+1}| \le s_1 - 2$. Suppose to the contrary that $|\mathbb{G}_{s_1+1}| \ge s_1 - 1$, and let G_2 be a component in \mathbb{G}_{s_1+1} . By Theorem 1.3, $e(G_2) = ex(s_1 + 1, \{K_{1,s_1}\}_{minor})$. Thus, G_2 can only be the complement of $\frac{s_1-2}{2}K_{1,1} \cup K_{1,2}$, that is, $G_2 \cong \overline{H_{s_1,1}}$. Now let Gbe the graph obtained from G^* by replacing $(s_1 - 1)G_2$ with $(s_1 - 1)K_{s_1} \cup K_{s_1-1}$. Then, Gis B_{r-1,s_1} -minor free by Lemma 3.4.

Since $\overline{G_2} \cong \frac{s_1-2}{2}K_2 \cup K_{1,2}$, we may assume that $V(G_2) = \{v_0, v_1, \dots, v_{s_1}\}$ such that $\{v_0v_1, v_0v_2\} \cup \{v_3v_4, v_5v_6, \dots, v_{s_1-1}v_{s_1}\}$ is the set of non-edges in G_2 . Then by symmetry, $x_{v_i} = x_{v_3}$ for each $i \in \{3, \dots, s_1\}$. Furthermore, since $d_{G_2}(v_0) = s_1 - 2$ and $d_{G_2}(v_3) = s_1 - 1$, we have $x_{v_0} < x_{v_3}$ by (16).

Observe that $(s_1 - 1)K_{s_1} \cup K_{s_1-1}$ can be obtained from $(s_1 - 1)G_2$ by replacing the edge set $\{v_0v_i: i = 3, ..., s_1\}$ with $\{v_3v_4, v_5v_6, ..., v_{s_1-1}v_{s_1}\}$ in every copy of G_2 and then forming $s_1 - 1$ copies of v_0 into a copy of K_{s_1-1} . Thus, $e(G) = e(G^*)$ and

$$\sum_{uv \in E(G)} 2x_u x_v - \sum_{uv \in E(G^*)} 2x_u x_v = 2e(K_{s_1-1})x_{v_0}^2 + 2(s-1)\left(\sum_{i=2}^{s_1/2} x_{v_{2i-1}}x_{v_{2i}} - \sum_{i=3}^{s_1} x_{v_0}x_{v_i}\right)$$
$$= (s_1 - 1)(s_1 - 2)(x_{v_0}^2 + x_{v_3}^2 - 2x_{v_0}x_{v_3}).$$

Since $x_{v_0} < x_{v_3}$, we have $\rho(G) > \rho^*$, a contradiction. Hence, $|\mathbb{G}_{s_1+1}| \le s_1 - 2$.

Lemma 4.5. Assume that $s_1 = 4$ and $\gamma \ge 1$. Then $\mathbb{G}_{s_1+3} = \emptyset$ and $\sum_{i \in \{-1,1,2\}} |\mathbb{G}_{s_1+i}| \le 1$.

Proof. Let G_0 be an arbitrary component in $G^* - L$. Then $|G_0| \le s_1 + 3$ by Lemma 4.3. Furthermore, by Theorem 1.3, $e(G_0) = ex(|G_0|, \{K_{1,s_1}\}_{minor})$. Now $s_1 = 4$. By Lemma 2.4, every member in \mathbb{G}_{s_1+i} is isomorphic to $S^i(K_{s_1})$ for $i \in \{1,2,3\}$.

We first show $\mathbb{G}_{s_1+3} = \emptyset$. Suppose to contrary that there exists $G_0 \in \mathbb{G}_{s_1+3}$. Then $G_0 \cong S^3(K_4)$, that is, G_0 is obtained from K_4 by replacing an edge v_1v_2 with a path $v_1w_1w_2w_3v_2$. Now let $G_1 = G_0 - \{v_1w_1, v_2w_3\} + \{v_1v_2, w_1w_3\}$. Then $G_1 \cong K_4 \cup K_3$, and obviously G_1 is $K_{1,4}$ -minor free. Define G to be the graph obtained from G^* by replacing G_0 with G_1 . Then G is B_{r-1,s_1} -minor free by Lemma 3.4. Moreover,

$$\rho(G) - \rho^* \ge 2(x_{v_1}x_{v_2} + x_{w_1}x_{w_3}) - 2(x_{v_1}x_{w_1} + x_{v_2}x_{w_3})$$

By symmetry, we have $x_{v_1} = x_{v_2}$ and $x_{w_1} = x_{w_3}$. Thus $\rho(G) - \rho^* \ge 2(x_{v_1}^2 + x_{w_1}^2 - 2x_{v_1}x_{w_1})$. Note that $d_{G_0}(w_1) = 2$ and $d_{G_0}(v_1) = 3$. By (16) we obtain $x_{w_1} < x_{v_1}$, and hence $\rho(G) > \rho^*$, a contradiction. Thus, $\mathbb{G}_{s_1+3} = \emptyset$.

Secondly, we claim that $|\mathbb{G}_{s_1+i}| \leq 1$ for $i \in \{1,2\}$. Indeed, if $|\mathbb{G}_{s_1+2}| \geq 2$, then we replace two copies of $S^2(K_4)$ in \mathbb{G}_{s_1+2} with three copies of K_4 . Now, $2e(S^2(K_4)) = 16 < 3e(K_4)$, contradicting Theorem 1.3. If $|\mathbb{G}_{s_1+1}| \geq 2$, we choose $G_0, G_1 \in \mathbb{G}_{s_1+1}$. For $j \in \{0,1\}$, $G_j \cong S^1(K_4)$ and thus G_j is obtained from K_4 by replacing an edge $u_j w_j$ with a path $u_j v_j w_j$. Now let $G_2 = (G_0 \cup G_1) - \{u_1 v_1, v_1 w_1, u_0 v_0\} + \{u_1 w_1, u_0 v_1, v_1 v_0\}$. Then $G_2 \cong K_4 \cup S^2(K_4)$, and G_2 is clearly $K_{1,4}$ -minor free. Define G to be the graph obtained from G^* by replacing $G_0 \cup G_1$ with G_2 . Then G is B_{r-1,s_1} -minor free and

$$\rho(G) - \rho^* \ge 2(x_{u_1}x_{w_1} + x_{u_0}x_{v_1} + x_{v_1}x_{v_0}) - 2(x_{u_1}x_{v_1} + x_{v_1}x_{w_1} + x_{u_0}x_{v_0}).$$

By symmetry, $x_{v_0} = x_{v_1}$, $x_{u_0} = x_{u_1} = x_{w_0} = x_{w_1}$, and thus $\rho(G) - \rho^* \ge 2(x_{u_0}^2 + x_{v_0}^2 - 2x_{u_0}x_{v_0})$. Since $d_{G_0}(v_0) = 2$ and $d_{G_0}(u_0) = 3$, By (16) we similarly have $x_{v_0} < x_{u_0}$ and $\rho(G) > \rho^*$, a contradiction. Therefore, the claim holds.

Now we are ready to prove $\sum_{i \in \{-1,1,2\}} |\mathbb{G}_{s_1+i}| \leq 1$. If $\mathbb{G}_{s_1-1} \neq \emptyset$, then we are done by Lemma 4.4. Next, assume that $\mathbb{G}_{s_1-1} = \emptyset$. It suffices to show $\sum_{i=1}^2 |\mathbb{G}_{s_1+i}| \leq 1$. Suppose to the contrary, then by the above claim, there simultaneously exist $G_1 \in \mathbb{G}_{s_1+1}$ and $G_2 \in \mathbb{G}_{s_1+2}$, where $G_i \cong S^i(K_4)$ for $i \in \{1,2\}$. Let $G_3 = (G_1 \cup G_2) - \{u_1u_2, u_2u_3, v_1v_2, v_3v_4\} + \{u_1u_3, v_1v_4, u_2v_2, u_2v_3\}$, where $u_1u_2u_3$ is the induced path of length two in G_1 and $v_1v_2v_3v_4$

is the induced path of length three in G_2 . Then $G_3 \cong 2K_4 \cup K_3$, and thus it is $K_{1,4}$ -minor free. Define *G* to be the graph obtained from G^* by replacing $G_1 \cup G_2$ with G_3 . Then *G* is B_{r-1,s_1} -minor free. By symmetry, $x_{u_1} = x_{u_3}$, $x_{v_1} = x_{v_4}$ and $x_{v_2} = x_{v_3}$. Thus,

$$\begin{aligned}
\rho(G) - \rho^* &\geq 2(x_{u_1}x_{u_3} + x_{v_1}x_{v_4} + x_{u_2}x_{v_2} + x_{u_2}x_{v_3}) - 2(x_{u_1}x_{u_2} + x_{u_2}x_{u_3} + x_{v_1}x_{v_2} + x_{v_3}x_{v_4}) \\
&= 2(x_{u_1}^2 + x_{v_1}^2 + 2x_{u_2}x_{v_2}) - 2(2x_{u_1}x_{u_2} + 2x_{v_1}x_{v_2}) \\
&= 2(x_{u_1} - x_{v_1})^2 + 4(x_{u_1} - x_{v_2})(x_{v_1} - x_{u_2}).
\end{aligned}$$

Note that $d_{G_1}(u_2) = d_{G_2}(v_2) = 2$ and $d_{G_1}(u_1) = d_{G_2}(v_1) = 3$. In view of (16), we get that $\max\{x_{u_2}, x_{v_2}\} < \min\{x_{u_1}, x_{v_1}\}$ and hence $\rho(G) > \rho^*$, a contradiction.

This completes the proof.

We now determine $SPEX(n, \{B_{r-1,s_1}\}_{minor})$ for even s_1 .

Theorem 4.4. Let $s_1 \ge 2$ be even, $\gamma \ge 1$, and $n - \gamma = ps_1 + q$ (where $1 \le q \le s_1$). Then $SPEX(n, \{B_{r-1,s_1}\}_{minor}) = \{K_{\gamma}\nabla G^{\triangle}\}$, where

$$G^{\triangle} = \begin{cases} (p-1)K_{s_1} \cup S(\overline{H_{s_1,1}}) & \text{if } (q,s_1) = (2,4); \\ (p-q)K_{s_1} \cup q\overline{H_{s_1,1}} & \text{if } q \leq s_1 - 2 \text{ and } (q,s_1) \neq (2,4); \\ pK_{s_1} \cup K_q & \text{if } q > s_1 - 2. \end{cases}$$

Proof. By Theorem 1.1, G^* has a set *L* of γ dominating vertices, where $\gamma = r - 2$. Furthermore, by Theorem 1.2 we know that $G^* - L$ is K_{1,s_1} -minor saturated.

Let G_0 be a component in $G^* - L$. By Theorem 1.3, $e(G_0) = ex(|G_0|, \{K_{1,s_1}\}_{minor})$. Therefore, $G_0 \cong K_{|G_0|}$ if $|G_0| \in \{s_1 - 1, s_1\}$, and $G_0 \cong \overline{H_{s_1,1}}$ if $|G_0| = s_1 + 1$. If $|G_0| = s_1 + 2 = 6$, then $G_0 \cong S^2(K_4)$ by Lemma 2.4, and one can further see that $S^2(K_{s_1}) \cong S(\overline{H_{s_1,1}})$ for $s_1 = 4$. In the following, we distinguish the proof into three cases.

If $s_1 = 2$, then $q \in \{1, 2\} = \{s_1 - 1, s_1\}$ and $G^* - L$ is $K_{1,2}$ -minor saturated. It is easy to see that $G^* - L \cong pK_{s_1} \cup K_q$, as desired.

If $s_1 = 4$, then $1 \le q \le 4$, and by Lemmas 4.3 and 4.5, $|G_0| \in \{s_1 + i : -1 \le i \le 2\}$. Lemma 4.5 also gives that $\sum_{i \in \{-1,1,2\}} |\mathbb{G}_{s_1+i}| \le 1$. Consequently, if $q = 1 = s_1 - 3$, then $|G_0| \in \{s_1, s_1 + 1\}$, and $G_0 \in \{K_{s_1}, \overline{H_{s_1,1}}\}$ as stated above. Now $G^* - L \cong (p - 1)K_{s_1} \cup \overline{H_{s_1,1}}$. If $q = 2 = s_1 - 2$, then $|G_0| \in \{s_1, s_1 + 2\}$, and $G_0 \in \{K_{s_1}, S(\overline{H_{s_1,1}})\}$ as discussed above. Hence, $G^* - L \cong (p - 1)K_{s_1} \cup S(\overline{H_{s_1,1}})$. If $q \in \{3, 4\} = \{s_1 - 1, s_1\}$, then $|G_0| \in \{s_1 - 1, s_1\}$, and thus $G_0 \cong K_{|G_0|}$. It follows that $G^* - L \cong pK_{s_1} \cup K_q$.

If $s_1 \ge 6$, then by Lemma 4.3, we have $|G_0| \in \{s_1 + i : -1 \le i \le 1\}$. From Lemma 4.4, we know that $|\mathbb{G}_{s_1-1}| \le 1$, $|\mathbb{G}_{s_1+1}| \le s_1 - 2$, and $\mathbb{G}_{s_1+1} = \emptyset$ provided that $|\mathbb{G}_{s_1-1}| = 1$. Thus, if $q \in \{s_1 - 1, s_1\}$, then $|G_0| \in \{s_1 - 1, s_1\}$ and so $G_0 \cong K_{|G_0|}$, which implies that $G^* - L \cong pK_{s_1} \cup K_q$. If $q \le s_1 - 2$, then $|G_0| \in \{s_1, s_1 + 1\}$ and thus $G_0 \in \{K_{s_1}, \overline{H_{s_1,1}}\}$, which implies that $G^* - L \cong (p - q)K_{s_1} \cup q\overline{H_{s_1,1}}$.

This completes the proof.

We end this section with the proof of Theorem 1.8.

Proof. Let $H = K_{s_1,s_2,...,s_r}$, where $s_1 \ge s_2 \ge \cdots \ge s_r \ge 1$, and $\gamma = \sum_{i=2}^r s_i - 1 \ge 1$. Let $\beta = \lfloor \frac{s_1+1}{s_2+1} \rfloor$. If $s_2 \ge 2$, then Theorem 1.8 holds by Theorem 4.2. If $s_2 = 1$ and s_1 is odd, then Theorem 1.8 holds by Theorem 4.3. If $s_2 = 1$ and s_1 is even, then $\beta = \lfloor \frac{s_1+1}{s_2+1} \rfloor = \frac{s_1}{2}$, and so $2(\beta - 1) = s_1 - 2$. Moreover, the case $(q, \beta, s_1) = (2, 1, 8)$ never occurs. Hence, $G^{\triangle} = G^{\blacktriangle}$, and Theorem 1.8 follows from Theorem 4.4. This completes the proof. \Box

5 **Proof of Theorem 2.1**

Let *G* be a graph of order *n* sufficiently large, and *X* be a non-negative eigenvector corresponding to $\rho(G)$. Choose $u^* \in V(G)$ with $x_{u^*} = \max_{u \in V(G)} x_u$, and assume that *G* is an \mathbb{H} -minor free graph with $\rho(G) \ge \sqrt{\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})}$. To prove Theorem 2.1, it suffices to find a set *L* of exactly $\gamma_{\mathbb{H}}$ vertices in *G* such that $x_u \ge (1 - \frac{1}{2(10C_{\mathbb{H}})^2})x_{u^*}$ and $d_G(u) \ge (1 - \frac{1}{(10C_{\mathbb{H}})^2})n$ for every $u \in L$. To this end, we define a subset of V(G) as follows:

$$L^{\lambda} = \{ u \in V(G) : x_u \ge (10C_{\mathbb{H}})^{-\lambda} x_{u^*} \},$$

where λ is a positive constant. We now establish some lemmas on L^{λ} .

Lemma 5.1. $|L^{\lambda}| < (10C_{\mathbb{H}})^{\lambda - 10} n.$

Proof. Given an arbitrary $u \in L^{\lambda}$. Then $\rho(G)x_u \ge \sqrt{\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})}(10C_{\mathbb{H}})^{-\lambda}x_{u^*}$, and thus $\rho(G)x_u > 2C_{\mathbb{H}}(10C_{\mathbb{H}})^{10-\lambda}x_{u^*}$ for *n* large enough. On the other hand, $\rho(G)x_u = \sum_{v \in N_1(u)} x_v \le |N_1(u)|x_{u^*}$. Combining the above two inequalities gives that $2C_{\mathbb{H}}(10C_{\mathbb{H}})^{10-\lambda} < |N_1(u)|$. Summing this inequality over all vertices $u \in L^{\lambda}$, we obtain

$$2C_{\mathbb{H}}(10C_{\mathbb{H}})^{10-\lambda}|L^{\lambda}| < \sum_{u \in L^{\lambda}} |N_1(u)| \le \sum_{u \in V(G)} |N_1(u)| = 2e(G).$$

Note that $e(G) < C_{\mathbb{H}}n$ by Lemma 2.2. Thus $|L^{\lambda}| < (10C_{\mathbb{H}})^{\lambda-10}n$.

Now we introduce some notations. For a vertex $u \in V(G)$ and a positive integer *i*, let $N_i(u)$ be the set of vertices at distance *i* from *u* in *G*. We will frequently use $N_1(u)$ and $N_2(u)$. Furthermore, we use L_i^{λ} and \overline{L}_i^{λ} to denote $N_i(u) \cap L^{\lambda}$ and $N_i(u) \setminus L^{\lambda}$, respectively. We also denote $L_{i,j}^{\lambda} = L_i^{\lambda} \cup L_j^{\lambda}$ and $\overline{L}_{i,j}^{\lambda} = \overline{L}_i^{\lambda} \cup \overline{L}_j^{\lambda}$ for simplicity.

Lemma 5.2. For every $u \in V(G)$ and every positive constant λ , we have

$$\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})x_{u} \leq |N_{1}(u)|x_{u} + \left(\frac{2C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{10-\lambda}} + \frac{2C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{\lambda}}\right)x_{u^{*}} + \sum_{\nu \in \overline{L}_{1}^{\lambda}, w \in N_{L_{1,2}^{\lambda}}(\nu)} x_{w}.$$
 (17)

Proof. Set $\rho := \rho(G)$. Recall that $\rho \ge \sqrt{\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})}$. Then

$$\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}}) x_{u} \leq \rho^{2} x_{u} = \sum_{v \in N_{1}(u)} \rho x_{v} = |N_{1}(u)| x_{u} + \sum_{v \in N_{1}(u), w \in N_{1}(v) \setminus \{u\}} x_{w}.$$
 (18)

For a given $u \in V(G)$ and each $v \in N_1(u)$, since $N_1(v) \setminus \{u\} \subseteq \bigcup_{i=1}^2 N_i(u)$ and $N_i(u) = L_i^{\lambda} \cup \overline{L}_i^{\lambda}$, we have $N_1(v) \setminus \{u\} = N_{L_{1,2}^{\lambda}}(v) \cup N_{\overline{L}_{1,2}^{\lambda}}(v)$. Moreover, by the definition of L^{λ} , we know that $x_w < (10C_{\mathbb{H}})^{-\lambda} x_{u^*}$ for each $w \in \overline{L}_{1,2}^{\lambda}$.

We now decompose $N_1(u) = L_1^{\lambda} \cup \overline{L}_1^{\lambda}$, and further decompose $\sum_{w \in N_1(v) \setminus \{u\}} x_w$ into two subitems according to $N_1(v) \setminus \{u\} = N_{L_{1,2}^{\lambda}}(v) \cup N_{\overline{L}_{1,2}^{\lambda}}(v)$. Consequently,

$$\sum_{v \in L_{1}^{\lambda}, w \in N_{1}(v) \setminus \{u\}} x_{w} \leq \sum_{v \in L_{1}^{\lambda}, w \in N_{L_{1,2}^{\lambda}}(v)} x_{u^{*}} + \sum_{v \in L_{1}^{\lambda}, w \in N_{\overline{L}_{1,2}^{\lambda}}(v)} (10C_{\mathbb{H}})^{-\lambda} x_{u^{*}}$$
$$\leq (2e(L_{1}^{\lambda}) + e(L_{1}^{\lambda}, L_{2}^{\lambda})) x_{u^{*}} + e(L_{1}^{\lambda}, \overline{L}_{1,2}^{\lambda}) (10C_{\mathbb{H}})^{-\lambda} x_{u^{*}}.$$
(19)

Similarly as above, we have

$$\sum_{v\in\overline{L}_{1}^{\lambda},w\in N_{1}(v)\setminus\{u\}} x_{w} \leq \sum_{v\in\overline{L}_{1}^{\lambda},w\in N_{L_{1,2}^{\lambda}}(v)} x_{w} + \left(2e(\overline{L}_{1}^{\lambda})+e(\overline{L}_{1}^{\lambda},\overline{L}_{2}^{\lambda})\right)(10C_{\mathbb{H}})^{-\lambda}x_{u^{*}}.$$
 (20)

By Lemma 2.2, we obtain $2e(L_1^{\lambda}) + e(L_1^{\lambda}, L_2^{\lambda}) \leq 2e(L^{\lambda}) \leq 2C_{\mathbb{H}}|L^{\lambda}|$ and $e(L_1^{\lambda}, \overline{L}_{1,2}^{\lambda}) + 2e(\overline{L}_1^{\lambda}) + e(\overline{L}_1^{\lambda}, \overline{L}_2^{\lambda}) \leq 2e(G) < 2C_{\mathbb{H}}n$. Furthermore, $|L^{\lambda}| < (10C_{\mathbb{H}})^{\lambda - 10}n$ by Lemma 5.1. Combining (18-20), we get the inequality (17) immediately.

We now choose $\lambda = 4$ to get a better bound of $|L^{\lambda}|$, which only depends on $C_{\mathbb{H}}$.

Lemma 5.3. $|L^4| < (10C_{\mathbb{H}})^6$.

Proof. We first show $|N_1(u)| \ge (10C_{\mathbb{H}})^{-5}n$ for each $u \in L^4$. Suppose to the contrary that there exists $u_0 \in L^4$ with $|N_1(u_0)| < (10C_{\mathbb{H}})^{-5}n$. Set $u = u_0$ and $\lambda = 5$ in (17). Then

$$\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})x_{u_{0}} \leq |N_{1}(u_{0})|x_{u_{0}} + \frac{4C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{5}}x_{u^{*}} + \sum_{\nu \in \overline{L}_{1}^{5}, w \in N_{L_{1}^{5}}(\nu)} x_{w}.$$
(21)

Recall that $\overline{L}_1^5 \subseteq N_1(u_0)$ and $L_{1,2}^5 \subseteq L^5$. By Lemma 2.2, we have $e(\overline{L}_1^5, L_{1,2}^5) \leq e(N_1(u_0) \cup L^5) \leq C_{\mathbb{H}}(|N_1(u_0)| + |L^5|)$, where $|N_1(u_0)| < (10C_{\mathbb{H}})^{-5}n$ by the assumption. By Lemma 5.1, we also have $|L^5| \leq (10C_{\mathbb{H}})^{-5}n$. Hence,

$$|N_1(u_0)|x_{u_0} + \sum_{v \in \overline{L}_1^5, w \in N_{L_{1,2}^5}(v)} x_w \le \left(|N_1(u_0)| + e(\overline{L}_1^5, L_{1,2}^5)\right) x_{u^*} \le \frac{(1 + 2C_{\mathbb{H}})n}{(10C_{\mathbb{H}})^5} x_{u^*}.$$

Combining this inequality and (21) gives that $\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})x_{u_0} \leq (1+6C_{\mathbb{H}})(10C_{\mathbb{H}})^{-5}nx_{u^*}$.

On the other hand, recall that $\gamma_{\mathbb{H}} \ge 1$ and $x_{u_0} \ge (10C_{\mathbb{H}})^{-4}x_{u^*}$ as $u_0 \in L^4$. Thus, $\gamma_{\mathbb{H}}(n - \gamma_{\mathbb{H}})x_{u_0} \ge \frac{7}{10}nx_{u_0} \ge 7C_{\mathbb{H}}(10C_{\mathbb{H}})^{-5}nx_{u^*}$, a contradiction. Therefore, $|N_1(u)| \ge (10C_{\mathbb{H}})^{-5}n$ for each $u \in L^4$.

Summing the above inequality over all vertices $u \in L^4$, we obtain

$$|L^4|(10C_{\mathbb{H}})^{-5}n \le \sum_{u \in L^4} |N_1(u)| \le \sum_{u \in V(G)} |N_1(u)| = 2e(G) < 2C_{\mathbb{H}}n,$$

which yields that $|L^4| < (10C_{\mathbb{H}})^6$.

Lemma 5.4. For each vertex $u \in L^4$, we have $|N_1(u)| \ge (\frac{x_u}{x_{u^*}} - \frac{1}{(10C_{\mathbb{H}})^3})n$.

Proof. Choose an arbitrary vertex $u \in L^4$ and a minimal graph H^* with respect to \mathbb{H} . Then $\gamma_{\mathbb{H}} = \gamma_{H^*} = |H^*| - \alpha_{H^*} - 1$. Recall that $L_i^4 = N_i(u) \cap L^4$, $\overline{L}_i^4 = N_i(u) \setminus L^4$ and $L_{1,2}^4 = L_1^4 \cup L_2^4$. Let L_0 be the subset of \overline{L}_1^4 in which each vertex has at least γ_{H^*} neighbors in $L_{1,2}^4$.

We first claim that $|L_0| \leq \varphi |H^*|$, where $\varphi := \binom{|L_{1,2}^4|}{\gamma_{H^*}}$. Indeed, if $|L_{1,2}^4| \leq \gamma_{H^*} - 1$, then $L_0 = \emptyset$ and we are done. Now consider the case $|L_{1,2}^4| \geq \gamma_{H^*}$. Suppose to the contrary that $|L_0| > \varphi |H^*|$. Since there are only φ options for vertices in L_0 to choose γ_{H^*} neighbors from $L_{1,2}^4$, we can find γ_{H^*} vertices in $L_{1,2}^4$ with at least $|L_0|/\varphi > |H^*|$ common neighbors in L_0 . Furthermore, $u \notin L_{1,2}^4$ and $L_0 \subseteq \overline{L}_1^4 \subseteq N_1(u)$. Hence, u_0 and those γ_{H^*} vertices have $|H^*|$ common neighbors, which implies that G contains a bipartite subgraph G[S,T] isomorphic to $K_{\gamma_{H^*}+1,|H^*|}$. As noted earlier, $|H^*| - (\gamma_{H^*} + 1) = \alpha_{H^*}$. Then, contracting $\gamma_{H^*} + 1$ independent edges in G[S,T], we obtain a new graph isomorphic to $B_{\gamma_{H^*}+1,\alpha_{H^*}}$. By Lemma 2.5, G has an H^* minor, a contradiction. Thus, $|L_0| \leq \varphi |H^*|$.

By Lemma 5.3, we have $|L_{1,2}^4| \leq |L^4| < (10C_{\mathbb{H}})^6$, which implies that φ is constant. Combining $|L_0| \leq \varphi |H^*|$, we obtain $e(L_0, L_{1,2}^4) \leq |L_0| |L_{1,2}^4| \leq (10C_{\mathbb{H}})^{-4}n$ for n sufficiently large. On the other hand, by the choice of L_0 we know that $e(\overline{L}_1^4 \setminus L_0, L_{1,2}^4) \leq |\overline{L}_1^4 \setminus L_0|(\gamma_{\mathbb{H}^*} - 1) \leq |N_1(u)|(\gamma_{\mathbb{H}} - 1)$. Consequently,

$$e(\overline{L}_{1}^{4}, L_{1,2}^{4}) \leq |N_{1}(u)|(\gamma_{\mathbb{H}} - 1) + (10C_{\mathbb{H}})^{-4}n.$$
(22)

Notice that $\sum_{v \in \overline{L}_1^4} \sum_{w \in N_{L_{1,2}^4}(v)} x_w \le e(\overline{L}_1^4, L_{1,2}^4) x_{u^*}$. Now, setting $\lambda = 4$ in (17) and combining (22), we obtain that

$$\begin{split} \gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})x_{u} &\leq \Big(|N_{1}(u)| + \frac{2C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{6}} + \frac{2C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{4}} + |N_{1}(u)|(\gamma_{\mathbb{H}}-1) + \frac{n}{(10C_{\mathbb{H}})^{4}}\Big)x_{u^{*}} \\ &\leq \gamma_{\mathbb{H}}\Big(|N_{1}(u)| + \frac{3C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{4}}\Big)x_{u^{*}}. \end{split}$$

Thus, $|N_1(u)| \ge (n - \gamma_{\mathbb{H}}) \frac{x_u}{x_{u^*}} - \frac{3C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^4}$, where $\gamma_{\mathbb{H}} \frac{x_u}{x_{u^*}} \le \gamma_{\mathbb{H}} \le \frac{7C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^4}$ for *n* large enough. It follows that $|N_1(u)| \ge (\frac{x_u}{x_{u^*}} - \frac{1}{(10C_{\mathbb{H}})^3})n$, as desired.

Now, choose $\lambda = 1$. By the definition of L^{λ} , it is clear that $u^* \in L^1$ and $L^1 \subseteq L^4$.

Lemma 5.5. For every vertex $u \in L^1$, we have $x_u \ge \left(1 - \frac{1}{2(10C_{\mathbb{H}})^2}\right)x_{u^*}$ and $|N_1(u)| \ge \left(1 - \frac{1}{(10C_{\mathbb{H}})^2}\right)n$. Moreover, we have $|L^1| = \gamma_{\mathbb{H}}$.

Proof. We first show the lower bounds of x_u and $|N_1(u)|$. Suppose to the contrary that there exists $u_0 \in L^1$ with $x_{u_0} < (1 - \frac{1}{2(10C_{\mathbb{H}})^2})x_{u^*}$. According to the definition of L^1 , we know that $x_{u_0} \ge \frac{x_{u^*}}{10C_{\mathbb{H}}}$. By Lemma 5.4, we get

$$|N_1(u^*)| \ge (1 - \frac{1}{(10C_{\mathbb{H}})^3})n \text{ and } |N_1(u_0)| \ge (\frac{1}{10C_{\mathbb{H}}} - \frac{1}{(10C_{\mathbb{H}})^3})n.$$

Now let $L_i^4 = N_i(u^*) \cap L^4$ and $\overline{L}_i^4 = N_i(u^*) \setminus L^4$. By Lemma 5.3, we have $|L^4| < (10C_{\mathbb{H}})^6 < \frac{n}{(10C_{\mathbb{H}})^3}$ for *n* large enough. Hence, $|\overline{L}_1^4| \ge |N_1(u^*)| - |L^4| \ge (1 - \frac{2}{(10C_{\mathbb{H}})^3})n$, and thus

$$\left|\overline{L}_{1}^{4} \cap N_{1}(u_{0})\right| \geq \left|\overline{L}_{1}^{4}\right| + \left|N_{1}(u_{0})\right| - n \geq \left(\frac{1}{10C_{\mathbb{H}}} - \frac{3}{(10C_{\mathbb{H}})^{3}}\right)n \geq \frac{9n}{100C_{\mathbb{H}}}.$$
 (23)

In view of (23), u_0 has neighbors in \overline{L}_1^4 . Since $\overline{L}_1^4 \subseteq N_1(u^*)$, we can observe that u_0 is of distance at most two from u^* , that is, $u_0 \in \bigcup_{i=1}^2 N_i(u^*)$. Recall that $u_0 \in L^1 \subseteq L^4$. Then $u_0 \in L_{1,2}^4$, where $L_{1,2}^4 = L_1^4 \cup L_2^4$. Now setting $u = u^*$ and $\lambda = 4$ in (17), we can see that

$$\begin{aligned} &\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})x_{u^{*}} \\ \leq & \left(|N_{1}(u^{*})| + \frac{2C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{6}} + \frac{2C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{4}} + e(\overline{L}_{1}^{4}, L_{1,2}^{4} \setminus \{u_{0}\})\right)x_{u^{*}} + e(\overline{L}_{1}^{4}, \{u_{0}\})x_{u_{0}} \\ \leq & \left(|N_{1}(u^{*})| + \frac{2.5C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{4}} + e(\overline{L}_{1}^{4}, L_{1,2}^{4})\right)x_{u^{*}} + e(\overline{L}_{1}^{4}, \{u_{0}\})(x_{u_{0}} - x_{u^{*}}), \end{aligned}$$

where $x_{u_0} - x_{u^*} < -\frac{x_{u^*}}{2(10C_{\mathbb{H}})^2}$ by the previous assumption.

From (22) we know that $e(\overline{L}_1^4, L_{1,2}^4) \leq (\gamma_{\mathbb{H}} - 1)|N_1(u^*)| + \frac{C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^4}$. Moreover, it is easy to see $\gamma_{\mathbb{H}}^2 \leq \frac{0.5C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^4}$ for *n* large enough. Hence,

$$\gamma_{\mathbb{H}}n \le \gamma_{\mathbb{H}}|N_{1}(u^{*})| + \frac{4C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{4}} - \frac{e(\overline{L}_{1}^{4}, \{u_{0}\})}{2(10C_{\mathbb{H}})^{2}} < \gamma_{\mathbb{H}}n + \frac{4C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{4}} - \frac{e(\overline{L}_{1}^{4}, \{u_{0}\})}{2(10C_{\mathbb{H}})^{4}} - \frac{e(\overline{L}_{1}^{4}, \{u_{0}\})}{2(10C_$$

which yields that $e(\overline{L}_1^4, \{u_0\}) < \frac{8n}{100C_{\mathbb{H}}}$, contradicting (23). Hence, $x_u \ge (1 - \frac{1}{2(10C_{\mathbb{H}})^2})x_{u^*}$ for each $u \in L^1$. Furthermore, it follows from Lemma 5.4 that for each $u \in L^1$,

$$|N_1(u)| \ge \left(1 - \frac{1}{2(10C_{\mathbb{H}})^2} - \frac{1}{(10C_{\mathbb{H}})^3}\right) n \ge \left(1 - \frac{1}{(10C_{\mathbb{H}})^2}\right) n.$$

It remains to show $|L^1| = \gamma_{\mathbb{H}}$. We first suppose that $|L^1| \ge \gamma_{\mathbb{H}} + 1$. Then $|L^1| \ge \gamma_{H^*} + 1$, where H^* is minimal with respect to \mathbb{H} . Notice that every vertex $u \in L^1$ has at most $n/(10C_{\mathbb{H}})^2$ non-neighbors. Hence, every $\gamma_{H^*} + 1$ vertices in L^1 have at least $n - \frac{(\gamma_{H^*}+1)}{(10C_{\mathbb{H}})^2}n \ge |H^*|$ common neighbors, as $C_{\mathbb{H}} > |H^*| > \gamma_{H^*}$. Hence, *G* contains a bipartite subgraph G[S,T] isomorphic to $K_{\gamma_{H^*}+1,|H^*|}$. Note that $|H^*| - (\gamma_{H^*}+1) = \alpha_{H^*}$. Contracting $\gamma_{H^*} + 1$ independent edges in $K_{\gamma_{H^*}+1,|H^*|}$, we get a copy of $B_{\gamma_{H^*}+1,\alpha_{H^*}}$. By Lemma 2.5, *G* contains an H^* minor, a contradiction. Therefore, $|L^1| \le \gamma_{\mathbb{H}}$. Next suppose $|L^1| \leq \gamma_{\mathbb{H}} - 1$. Since $u^* \in L^1 \setminus L_{1,2}^4$, we have $|L^1 \cap L_{1,2}^4| \leq \gamma_{\mathbb{H}} - 2$, and thus $e(\overline{L}_1^4, L_{1,2}^4 \cap L^1) \leq (\gamma_{\mathbb{H}} - 2)n$. On the other hand, by Lemma 2.2 we have $e(\overline{L}_1^4, L_{1,2}^4 \setminus L^1) \leq e(G) < C_{\mathbb{H}}n$, and by the definition of L^1 we know that $x_w < \frac{x_{u^*}}{10C_{\mathbb{H}}}$ for each $w \in L_{1,2}^4 \setminus L^1$. Now, setting $u = u^*$ and $\lambda = 4$ in (17), we can see that

$$\begin{aligned} &\gamma_{\mathbb{H}}(n-\gamma_{\mathbb{H}})x_{u^{*}} \\ &\leq \left(|N_{1}(u^{*})| + \frac{2.5C_{\mathbb{H}}n}{(10C_{\mathbb{H}})^{4}} + e(\overline{L}_{1}^{4}, L_{1,2}^{4} \cap L^{1})\right)x_{u^{*}} + e(\overline{L}_{1}^{4}, L_{1,2}^{4} \setminus L^{1})\frac{x_{u^{*}}}{10C_{\mathbb{H}}} \\ &\leq \left(n + \frac{n}{10} + (\gamma_{\mathbb{H}} - 2)n + \frac{n}{10}\right)x_{u^{*}} \\ &= (\gamma_{\mathbb{H}} - \frac{4}{5})nx_{u^{*}}. \end{aligned}$$

This gives $\gamma_{\mathbb{H}}^2 \geq \frac{4}{5}n$, a contradiction. Therefore, $|L^1| = \gamma_{\mathbb{H}}$. The proof is completed. \Box

Recall that, to prove Theorem 2.1, it suffices to find a set *L* of exactly $\gamma_{\mathbb{H}}$ vertices in *G* such that $x_u \ge (1 - \frac{1}{2(10C_{\mathbb{H}})^2})x_{u^*}$ and $d_G(u) \ge (1 - \frac{1}{(10C_{\mathbb{H}})^2})n$ for every $u \in L$. By Lemma 5.5, we immediately obtain the desired result by choosing $L = L^1$.

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