# Eigenvalues and graph minors* 

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#### Abstract

Let spex $\left(n, H_{\text {minor }}\right)$ denote the maximum spectral radius of $n$-vertex $H$-minor free graphs. The problem on determining this extremal value can be dated back to the early 1990s. Up to now, it has been solved for $n$ sufficiently large and some special minors, such as $\left\{K_{2,3}, K_{4}\right\},\left\{K_{3,3}, K_{5}\right\}, K_{r}$ and $K_{s, t}$. In this paper, we find some unified phenomena on general minors. Every graph $G$ on $n$ vertices with spectral radius $\rho \geq \operatorname{spex}\left(n, H_{\text {minor }}\right)$ contains either an $H$ minor or a spanning book $K_{\gamma_{H}} \nabla\left(n-\gamma_{H}\right) K_{1}$, where $\gamma_{H}=|H|-\alpha(H)-1$. Furthermore, assume that $G$ is $H$-minor free and $\Gamma_{s}^{*}(H)$ is the family of $s$-vertex irreducible induced subgraphs of $H$, then $G$ minus its $\gamma_{H}$ dominating vertices is $\Gamma_{\alpha(H)+1}^{*}(H)$-minor saturate, and it is further edge-maximal if $\Gamma_{\alpha(H)+1}^{*}(H)$ is a connected family. As applications, we obtain some known results on minors mentioned above. We also determine the extremal values for some other minors, such as flowers, wheels, generalized books and complete multi-partite graphs. Our results extend some conjectures on planar graphs, outer-planar graphs and $K_{s, t}$-minor free graphs. To obtain the results, we combine stability method, spectral techniques and structural analyses. Especially, we give an exploration of using absorbing method in spectral extremal problems.


Keywords: graph minor; spectral radius; stability method; absorbing method AMS Classification: 05C50; 05C35.

## 1 Introduction

In the last few decades much research has been done on spectra of graphs, especially, the eigenvalues of adjacency matrices of graphs; Alon [2], Bollobás, Lee and Letzter [5], Bollobás and Nikiforov [6], Hoory, Linial and Widgerson [26], Huang [29], Jiang [30],

[^0]Jiang, Tidor, Yao, Zhang and Zhao [31], Lubetzky, Sudakov and Vu [36], Tait [46], Tait and Tobin [47], Wilf [53].

Given a simple graph $H$, we define $H$ to be a minor of some graph $G$ if $H$ can be obtained from $G$ by means of a sequence of vertex deletions, edge deletions and edge contractions. Given a family of graphs $\mathbb{H}$, a graph is said to be $\mathbb{H}$-minor free if it does not have any member of $\mathbb{H}$ as a minor. Minor plays a very important role in graph theory. We refer the reader to a survey by Robertson and Seymour [44]. Wagner [51] proved that a graph is planar if and only if it is $\left\{K_{5}, K_{3,3}\right\}$-minor free. The study on planar graphs has a rich history, such as, the curvature of planar graphs [1, 19, 22, 27, 28], the automorphism groups of planar graphs [ $4,13,39,45$ ], the partitions of planar graphs [ $9,16,32$ ], and the eigenvalues of planar graphs $[7,8,14,17,18,23,24,35,47]$.

As a generalization of planar graphs, it is interesting to study problems on graphs with no $K_{r}$ minor or no $K_{s, t}$ minor. Mader [38] showed that $B_{r-2, n-r+2}$ yields the maximum number of edges over all $n$-vertex graphs with no $K_{r}$ minor when $r \leq 7$, however this is not best possible when $r>7$. A famous conjecture was posed by Hadwiger [21] in 1943, which stated that for every integer $r \geq 1$, every graph with no $K_{r}$ minor is $(r-1)$ colorable. In the 1980s, Kostochka [33, 34] and Thomason [48] independently proved that the maximum number of edges in a $K_{r}$ minor-free graph $G$ is $\Theta(r \sqrt{\log r} n)$ for large $r$ and hence $G$ is $O(r \sqrt{\log r})$-colorable. In 2001, Thomason [49] found the asymptotic value of this edge-extremal function. Very recently, Alon, Krivelevich and Sudakov [3] provided a short and self-contained proof of the celebrated Kostochka-Thomason bound. Simultaneously, Norin, Postle and Song [43] showed that every graph with no $K_{r}$ minor is $O\left(r(\log r)^{\beta}\right)$-colorable for $\beta>1 / 4$. From spectral perspective, Tait [46] determined the maximum spectral radius for graphs with no $K_{r}$ minor. The next one is $K_{s, t}$ minor. Ding, Johnson and Seymour [15] showed that $e(G) \leq\binom{ t}{2}+n-t$ for every connected graph $G$ with no $K_{1, t}$ minor. Chudnovsky, Reed and Seymour [11] proved that $e(G) \leq$ $\frac{1}{2}(t+1)(n-1)$ for every graph $G$ with no $K_{2, t}$ minor, which confirms a conjecture of Myers [40]. Zhai and Lin [54] obtained the maximum spectral radius of $K_{s, t}$-minor free graphs for $s=1$ and $s \geq 2$ respectively, which solved a conjecture of Tait [46].

Extensive studies have been conducted on the properties of graphs, and exploring the properties of graphs through minors is a valuable approach, as minor is very useful for characterizing inclusion relations between graphs. In particular, Mader [37] proved that for every given graph $H$, there exists a constant $C_{H}$ such that $e(G) \leq C_{H}|G|$ for any $H$ minor free $G$. Let $A(G)$ denote the adjacency matrix of $G$ and $\rho(G)$ be its spectral radius. In this paper, we fundamentally investigate the following extremal problem.

Problem 1.1. Given a graph $H$ or a graph family $\mathbb{H}$. What is the maximum spectral radius of an $H$-minor ( $\mathbb{H}$-minor) free graph of order $n$ ?

For many specific graphs $H$, Problem 1.1 has gained great popularity and has attracted the attention of many researchers (see for example, [25, 42, 47, 46, 52, 54]). However, a unified perspective on Problem 1.1 has not yet been found.

A generalized book, denoted by $B_{\gamma, n-\gamma}$, is obtained by joining a $\gamma$-clique with an independent set of $n-\gamma$ vertices, in other words, $B_{\gamma, n-\gamma} \cong K_{\gamma} \nabla(n-\gamma) K_{1}$. For a given graph family $\mathbb{H}$, we define

$$
\gamma_{H}:=|H|-\alpha(H)-1 \quad \text { and } \quad \gamma_{H}:=\min _{H \in \mathbb{H}} \gamma_{H},
$$

where $\alpha(H)$ (or $\alpha_{H}$ for simplicity) is the independence number of $H$. In the following, we use $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$ to denote the family of graphs with maximum spectral radius over all $n$-vertex $\mathbb{H}$-minor free graphs. We shall assume that $n$ is sufficiently large and $\mathbb{H}$ contains no member isomorphic to a star. Then $\gamma_{\mathbb{H}} \geq 1$. The first main result for Problem 1.1 is as follows.

Theorem 1.1. Every graph in $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$ contains a spanning subgraph $B_{\gamma_{\Perp}, n-\gamma_{\mathbb{H}}}$.
We now introduce some new notations and terminologies. Choose an arbitrary graph $H \in \mathbb{H}$. We define a family of induced subgraphs of $H$ as follows.

$$
\Gamma_{s}(H)=\{H[S]: S \subseteq V(H) \text { and }|S|=s\} .
$$

A member $H[S]$ in $\Gamma_{s}(H)$ is said to irreducible, if $\Gamma_{s}(H)$ does not contain any member isomorphic to a proper subgraph of $H[S]$. Now let $\Gamma_{s}^{*}(H)$ be the family of $s$-vertex irreducible induced subgraphs of $H$ and

$$
\Gamma(\mathbb{H})=\bigcup_{H \in \mathbb{H}} \Gamma_{|H|-\gamma_{\mathbb{H}}}^{*}(H) .
$$

It is obvious that $\Gamma_{|H|-\gamma_{H}}^{*}(H)=\Gamma_{\alpha_{H}+1}^{*}(H)$ for every $H \in \mathbb{H}$ with $\gamma_{H}=\gamma_{\mathbb{H}}$.
A graph is said to be $\mathbb{H}$-minor saturated, if it is $\mathbb{H}$-minor free but adding an edge between a pair of non-adjacent vertices always yields an $H$ minor for some $H \in \mathbb{H}$. Let $\operatorname{SAT}\left(n, \mathbb{H}_{\text {minor }}\right)$ be the family of $n$-vertex $\mathbb{H}$-minor saturated graphs. Choose an arbitrary $G^{*} \in \operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$, and let $L$ be the set of dominating vertices in $G^{*}$. Then $|L|=\gamma_{\mathbb{H}}$ by Theorem 1.1. One may expect a characterization of $G^{*}-L$.

Theorem 1.2. The induced subgraph $G^{*}-L \in \operatorname{SAT}\left(n-\gamma_{\mathbb{H}}, \Gamma(\mathbb{H})_{\text {minor }}\right)$. Particularly, if $\mathbb{H}=\{H\}$, then $G^{*}-L \in \operatorname{SAT}\left(n-\gamma_{H}, \Gamma_{\alpha_{H}+1}^{*}(H)_{\text {minor }}\right)$.

Let ex $\left(n, \mathbb{H}_{\text {minor }}\right)$ denote the maximum number of edges in an $n$-vertex $\mathbb{H}$-minor free graph, and let $E X\left(n, \mathbb{H}_{\text {minor }}\right)$ be the family of $n$-vertex $\mathbb{H}$-minor free graphs each of which has ex $\left(n, \mathbb{H}_{\text {minor }}\right)$ edges. If all the members in $\Gamma(\mathbb{H})$ are connected graphs, then we can obtain a stronger result than Theorem 1.2.

Theorem 1.3. If $\Gamma(\mathbb{H})$ is a connected family, then $G^{*}-L \in E X\left(n-\gamma_{\mathbb{H}}, \Gamma(\mathbb{H})_{\text {minor }}\right)$.
Theorems 1.1, 1.2 and 1.3 can be viewed as three fundamental tools on Problem 1.1. We now show some applications of these theorems.

Example 1. $\mathbb{H}=\left\{K_{t}\right\}$. Then $\gamma_{H}=|H|-\alpha_{H}-1=t-2$ and $\Gamma_{\alpha_{H}+1}^{*}(H)=\left\{K_{2}\right\}$ for $H=K_{t}$. By Theorems 1.1 and $1.2, G^{*}$ has $t-2$ dominating vertices and $G^{*}-L$ is $\left\{K_{2}\right\}$-minor saturated. Thus, $G^{*} \cong B_{t-2, n-t+2}$.

Example 2. $\mathbb{H}=\left\{K_{a, 3}, K_{a+2}\right\}$, where $a \in\{2,3\}$. Then $\gamma_{\mathbb{H}}=a-1$. Consequently, $\Gamma_{|H|-\gamma_{H I}}^{*}(H)=\Gamma_{4}^{*}(H)=\left\{K_{1,3}, K_{2,2}\right\}$ for $H=K_{a, 3}$ and $\Gamma_{|H|-\gamma_{H I I}}^{*}(H)=\Gamma_{3}^{*}(H)=\left\{K_{3}\right\}$ for $H=K_{a+2}$. By Theorem 1.1, $G^{*}$ has $a-1$ dominating vertices, and by Theorem $1.2, G^{*}-L$ is $\left\{K_{1,3}, K_{2,2}, K_{3}\right\}$-minor saturated. Thus, $G^{*}-L$ is a path, and so $G^{*} \cong K_{a-1} \nabla P_{n-a+1}$.

To sum up the above discussions, we have the following statements.
Theorem 1.4. $i$ ) (Tait, [46]) If $r \geq 3$, then $\operatorname{SPEX}\left(n,\left\{K_{r}\right\}_{\text {minor }}\right)=\left\{B_{r-2, n-r+2}\right\}$.
ii) (Tait and Tobin, [47]) If $a \in\{2,3\}$, then $\operatorname{SPEX}\left(n,\left\{K_{a, 3}, K_{a+2}\right\}_{\text {minor }}\right)=\left\{K_{a-1} \nabla P_{n-a+1}\right\}$.

Example 3. $\mathbb{H}=\left\{K_{r}-E\left(H_{0}\right)\right\}$, where $H_{0} \subseteq K_{r}$ and $\delta\left(H_{0}\right) \geq 1$. Then $\alpha_{H}=\omega\left(H_{0}\right)$ and $\gamma_{H}=r-\omega\left(H_{0}\right)-1$ for $H=K_{r}-E\left(H_{0}\right)$, where $\omega\left(H_{0}\right)$ is the clique number of $H_{0}$. By Theorem 1.1, $G^{*}$ has $r-\omega\left(H_{0}\right)-1$ dominating vertices. Specially, if $\omega\left(H_{0}\right)=2$, then $\Gamma_{\alpha_{H}+1}^{*}(H)=\left\{P_{3}\right\}$ for $H_{0}=\frac{\left|H_{0}\right|}{2} K_{2}$ and $\Gamma_{\alpha_{H}+1}^{*}(H)=\left\{K_{2} \cup K_{1}\right\}$ otherwise.

Recently, Chen, Liu and Zhang [10] characterized the spectral extremal graphs for $\left(K_{r}-E\left(H_{0}\right)\right)$-minor free graphs, where $H_{0}$ consists of vertex-disjoint paths. Now we present a result in a slightly stronger form. Let $B_{s, t}^{k}$ denote the graph obtained from $B_{s, t}$ by adding $k$ isolated edges within its independent set. By Theorems 1.1 and 1.2, we have the following result.

Theorem 1.5. Let $\gamma=r-\omega\left(H_{0}\right)-1$ and $G^{*}$ be a graph in $\operatorname{SPEX}\left(n,\left\{K_{r}-E\left(H_{0}\right)\right\}_{\text {minor }}\right)$. Then $G^{*}$ contains $B_{\gamma, n-\gamma}$ as a spanning subgraph. Particularly, if $\omega\left(H_{0}\right)=2$, then $G^{*} \cong$ $B_{r-3, n-r+3}^{\left\lfloor\frac{n-r+3}{2}\right\rfloor}$ for $H_{0}=\frac{\left|H_{0}\right|}{2} K_{2}$, and $G^{*} \cong B_{r-3, n-r+3}$ otherwise.

A $(k+1)$-wheel $W_{k+1}$, where $k \geq 3$, is defined as $W_{k+1}=K_{1} \nabla C_{k}$. Recently, Cioabă, Desai, Tait [12] obtained a spectral extremal result on $W_{2 k+1}$-free graphs.

Example 4. $\mathbb{H}=\left\{W_{k+1}\right\}$. Then $\alpha_{H}=\left\lfloor\frac{k}{2}\right\rfloor$ and $\gamma_{H}=|H|-\alpha_{H}-1=\left\lceil\frac{k}{2}\right\rceil$ for $H=W_{k+1}$. By Theorem 1.1, $G^{*}$ has $\left\lceil\frac{k}{2}\right\rceil$ dominating vertices. Similarly, we have $\Gamma_{\alpha_{H}+1}^{*}(H)=\left\{K_{2} \cup\right.$ $\left.\left(\alpha_{H}-1\right) K_{1}\right\}$ for odd $k$, and $\Gamma_{\alpha_{H}+1}^{*}(H)=\left\{P_{3} \cup\left(\alpha_{H}-2\right) K_{1}, 2 P_{2} \cup\left(\alpha_{H}-3\right) K_{1}\right\}$ for even $k$. By Theorem 1.2, we have the following result on $W_{k+1}$-minor free graphs.

Theorem 1.6. Let $\gamma=\left\lceil\frac{k}{2}\right\rceil$. Then $\operatorname{SPEX}\left(n,\left\{W_{k+1}\right\}_{\text {minor }}\right)=\left\{B_{\gamma, n-\gamma}\right\}$ for odd $k$, and $\operatorname{SPEX}\left(n,\left\{W_{k+1}\right\}_{\text {minor }}\right)=\left\{B_{\gamma, n-\gamma}^{1}\right\}$ otherwise.

A $t$-flower $F_{s_{1}, \ldots, s_{t}}$ is the graph obtained from $t$ cycles of lengths $s_{1}, \ldots, s_{t}$ respectively by identifying one vertex. A $t$-flower is said to be odd, if there exists an odd $s_{i}$ for $i \in$ $\{1, \ldots, t\}$. If $s_{1}=\cdots=s_{t}=3$, then it is a friendship graph. Very recently, $\mathrm{He}, \mathrm{Li}$ and Feng [20] determined $\operatorname{SPEX}\left(n,\left\{F_{s, \ldots, s}\right\}_{\text {minor }}\right)$ for $s \in\{3,4\}$.

Example 5. $\mathbb{H}=\left\{F_{s_{1}, \ldots, s_{t}}\right\}$. Then $|H|=\sum_{i=1}^{t} s_{i}-(t-1), \alpha_{H}=\sum_{i=1}^{t}\left\lfloor\frac{s_{i}}{2}\right\rfloor$ and $\gamma_{H}=$ $|H|-\alpha_{H}-1=\sum_{i=1}^{t}\left\lceil\frac{s_{i}}{2}\right\rceil-t$. If there exists an odd $s_{i}$, then we define a subset $S^{*} \subseteq V(H)$ by choosing $\alpha_{H}$ independent vertices and an extra vertex in $C_{s_{i}}$. Now $H\left[S^{*}\right] \cong K_{2} \cup\left(\alpha_{H}-\right.$ 1) $K_{1}$. Clearly, $\Gamma_{\alpha_{H}+1}^{*}(H)=\left\{H\left[S^{*}\right]\right\}$. By Theorems 1.1 and $1.2, G^{*} \cong B_{\gamma, n-\gamma}$, where
$\gamma=\gamma_{H}$. If each $s_{i}$ is even, then $\alpha_{H}=\sum_{i=1}^{t} \frac{s_{i}}{2}$. Thus, for every $\left(\alpha_{H}+1\right)$-subset $S$ of $V(H)$, there exists some $i$ with $\left|S \cap C_{s_{i}}\right| \geq \frac{s_{i}}{2}+1$. Hence, $H[S]$ contains either a $P_{3}$ or a copy of $2 P_{2}$, and so $\Gamma_{\alpha_{H}+1}^{*}(H)=\left\{P_{3} \cup\left(\alpha_{H}-2\right) K_{1}, 2 P_{2} \cup\left(\alpha_{H}-3\right) K_{1}\right\}$. By Theorem 1.2, $G^{*}-L$ contains exactly one edge. Thus, $G^{*} \cong B_{\gamma, n-\gamma}^{1}$.

Consequently, we have the following result.
Theorem 1.7. Let $\gamma=\sum_{i=1}^{t}\left\lceil\frac{S_{i}}{2}\right\rceil-t$. Then SPEX $\left(n,\left\{F_{s_{1}, \ldots, s_{t}}\right\}_{\text {minor }}\right)=\left\{B_{\gamma, n-\gamma}\right\}$ for an odd flower $F_{s_{1}, \ldots, s_{t}}$, and $\operatorname{SPEX}\left(n,\left\{F_{s_{1}, \ldots, s_{t}}\right\}_{\text {minor }}\right)=\left\{B_{\gamma, n-\gamma}^{1}\right\}$ otherwise.

As usual, we denote by $\bar{G}$ the complement of a graph $G$ and $K_{s_{1}, \ldots, s_{r}}$ a complete $r$ partite graph with $\min \left\{r, s_{1}\right\} \geq 2$ and $s_{1} \geq \cdots \geq s_{r}$. Let $H_{s_{1}, s_{2}}=(\beta-1) K_{1, s_{2}} \cup K_{1, s_{2}+\beta_{0}}$, where $\beta\left(s_{2}+1\right)+\beta_{0}=s_{1}+1$ and $0 \leq \beta_{0} \leq s_{2}$. Obviously, $H_{s_{1}, s_{2}}$ is a star forest of order $s_{1}+1$. Let $S\left(\overline{H_{s_{1}, s_{2}}}\right)$ denote the graph obtained from $\overline{H_{s_{1}, s_{2}}}$ by subdividing an edge $u v$ with minimum degree sum $d(u)+d(v)$. Denote by Pet ${ }^{\star}$ the Petersen graph. Set $\beta=\left\lfloor\frac{s_{1}+1}{s_{2}+1}\right\rfloor$ and $n-\sum_{2}^{r} s_{i}+1:=p s_{1}+q\left(1 \leq q \leq s_{1}\right)$. Now we introduce a graph $G^{\mathbf{\Delta}}$, which will play an important role in our next theorem.

$$
G^{\mathbf{\Delta}}= \begin{cases}(p-1) K_{s_{1}} \cup S\left(\overline{H_{s_{1}, s_{2}}}\right) & \text { if }(q, \beta)=(2,2) ; \\ (p-1) K_{s_{1}} \cup \overline{\text { Pet }^{\star}} & \text { if }\left(q, \beta, s_{1}\right)=(2,1,8) ; \\ (p-q) K_{s_{1}} \cup q \overline{H_{s_{1}, s_{2}}} & \text { if } q \leq 2(\beta-1) \text { and }(q, \beta) \neq(2,2) ; \\ p K_{s_{1}} \cup K_{q} & \text { if } q>2(\beta-1) \text { and }\left(q, \beta, s_{1}\right) \neq(2,1,8) .\end{cases}
$$

Example 6. $\mathbb{H}=\left\{K_{s_{1}, \ldots, s_{r}}\right\}$. Then $|H|=\sum_{i=1}^{r} s_{i}, \alpha_{H}=s_{1}$ and $\gamma_{H}=|H|-\alpha_{H}-1=$ $\sum_{i=2}^{r} s_{i}-1$. If $s_{2}=1$, then $\gamma_{H}=r-2$. If $H$ is not a star, that is, $\sum_{i=2}^{r} s_{i}-1 \geq 1$, then $G^{*}$ contains a set $L$ of $\gamma_{H}$ dominating vertices. However, it is not easy to characterize the structure of $G^{*}-L$. Hence, we present the result here, and give its proof in Section 4.

Theorem 1.8. Let $\min \left\{r, s_{1}\right\} \geq 2, s_{1} \geq \cdots \geq s_{r} \geq 1$ and $\gamma=\sum_{i=2}^{r} s_{i}-1 \geq 1$. Then i) if $s_{2} \geq 2$ or $s_{1}$ is even, then $\operatorname{SPEX}\left(n,\left\{K_{s_{1}, \ldots, s_{r}}\right\}_{\text {minor }}\right)=\left\{K_{\gamma} \nabla G^{\mathbf{\Delta}}\right\}$;
ii) if $s_{2}=1$ and $s_{1}$ is odd, then $\operatorname{SPEX}\left(n,\left\{K_{s_{1}, \ldots, s_{r}}\right\}_{\text {minor }}\right)=\left\{K_{\gamma} \nabla G^{\mathbf{V}}\right\}$, where every component of $G^{\mathbf{V}}$ is a cycle for $s_{1}=3$, and is either $K_{s_{1}}$ or $\overline{H_{s_{1}, 1}}$ for $s_{1} \geq 5$.

The rest of the paper is organized as follows. In Section 2, we provide some preliminary results. Especially, we give a stability theorem, which is a key tool of this paper. For the sake of readability of the article, we shall postpone its proof to the last section. In Section 3, we use the above stability theorem and an absorbing method to prove Theorem 1.1. Based on Theorem 1.1, we further present the proof of Theorems 1.2 and 1.3, which give a more refined description of extremal graphs in $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$. In Section 4, we will use Theorems 1.1, 1.2 and 1.3 to characterize $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$ for complete multipartite minors. This extends a conjecture proposed by Tait [46].

## 2 Preliminary results

Observe that an isolated vertex in a graph $H$ does not work on determining whether a graph $G$ of order large enough is $H$-minor free or not. Throughout the paper, let $\mathbb{H}$
be a family of graphs in which every member $H$ is a finite graph with minimum degree $\delta(H) \geq 1$. We use $|G|$ and $e(G)$ to denote the numbers of vertices and edges in a graph $G$, respectively. For a subset $S$ of $V(G)$, let $G[S]$ be the subgraph induced by $S$.

In 1967, Mader [37] proved an elegant result on minors, more precisely, if $G$ is an $H$ minor free graph of order $n$ then there exists a positive constant $C_{H}$ such that $e(G)<C_{H} n$. The following lemma was obtained by Thomason [50].

Lemma 2.1. Every non-empty graph $G$ with $e(G) \geq 2^{s+1} t|G|$ has a proper $K_{s, t}$ minor.
Lemma 2.1 implies a more precise bound for $K_{s, t}$ minor. Using it, we can obtain the following bound for general $H$ minor.

Lemma 2.2. Let $G$ be an $H$-minor free graph on $n$ vertices. Then $e(G)<C_{H} \cdot n$, where $C_{H}=2^{|H|+1} e(H)$.

Proof. Let $H^{\prime}$ be the graph obtained from $H$ by subdividing every edge once. Clearly, $H^{\prime}$ is a bipartite graph with $|H|+e(H)$ vertices and it can be embedded in $K_{|H|, e(H)}$. Therefore, $K_{|H|, e(H)}$ contains a subgraph $H^{\prime}$ and thus an $H$ minor.

Now suppose to the contrary that $e(G) \geq 2^{|H|+1} e(H) n$. Then by Lemma 2.1, $G$ has a $K_{|H|, e(H)}$ minor and thus an $H$ minor, a contradiction. Hence, the result holds.

The following lemma can be found in [11], and its original version is shown in [15].
Lemma 2.3. ([11, 15]) Let $t \geq 3$ and $n \geq t+2$. If $G$ is an $n$-vertex connected graph with no $K_{1, t}$ minor, then $e(G) \leq\binom{ t}{2}+n-t$, and for all $n$, this is the best possible.

Let $S^{\ell}\left(K_{t}\right)$ be the graph obtained from $K_{t}$ by replacing an edge with a path of length $\ell+1$. As pointed by Ding, Johnson and Seymour [15], the upper bound in Lemma 2.3 is sharp and $S^{n-t}\left(K_{t}\right)$ is an extremal graph. A natural question is to characterize all the extremal graphs. If $t=3$, then $S^{n-t}\left(K_{t}\right)$ is an $n$-cycle and is clearly the unique extremal graph. We now consider the case $t=4$, which is useful for our main theorem.

Lemma 2.4. Let $n \geq 5$ and $G$ be a $K_{1,4}$-minor free connected graph of order $n$ with maximum number of edges. Then $G \cong S^{n-4}\left(K_{4}\right)$.

Proof. Since $S^{n-4}\left(K_{4}\right)$ is $K_{1,4}$-minor free for $n \geq 5$, we have $e(G) \geq e\left(S^{n-4}\left(K_{4}\right)\right)=n+2$. On the other hand, for $n \geq 6$ we obtain $e(G) \leq n+2$ by Lemma 2.3, and for $n=5$ we can see $e(G) \leq\left\lfloor\frac{3 \times 5}{2}\right\rfloor=7$ as $\Delta(G) \leq 3$. Therefore, $e(G)=n+2$.

Since $G$ is a $K_{1,4}$-minor free connected graph, we have $\Delta(G) \leq 3$ and $\delta(G) \geq 1$. Now let $U_{i}$ denote the set of vertices of degree $i$ in $G$ for $i \in\{1,2,3\}$. Then

$$
2 n-\left|U_{1}\right|+\left|U_{3}\right|=\left|U_{1}\right|+2\left|U_{2}\right|+3\left|U_{3}\right|=2 e(G)=2 n+4,
$$

which yields that $\left|U_{3}\right|-\left|U_{1}\right|=4$.
In the following, we show $G \cong S^{n-4}\left(K_{4}\right)$. The proof is proceeded by induction on $n$. If $n=5$, then combining $\left|U_{1}\right|+\left|U_{2}\right|+\left|U_{3}\right|=5$ and $\left|U_{3}\right|-\left|U_{1}\right|=4$ gives $2\left|U_{1}\right|+\left|U_{2}\right|=1$.


Figure 1: The extremal graph $G$ in Case 1.

Thus, $\left|U_{1}\right|=0,\left|U_{2}\right|=1$ and $\left|U_{3}\right|=4$. Assume that $U_{2}=\{u\}$ and $U_{3}=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$, where $N_{G}(u)=\left\{v_{1}, v_{2}\right\}$. Then, $N_{G}\left(v_{3}\right)=\left\{v_{1}, v_{2}, v_{4}\right\}$ and $N_{G}\left(v_{4}\right)=\left\{v_{1}, v_{2}, v_{3}\right\}$. It follows that $G \cong S^{1}\left(K_{4}\right)$, as desired.

Assume now that $n \geq 6$. We first show that $U_{1}$ is empty. Suppose to the contrary that $u \in U_{1}$. Then $e(G-\{u\})=(n-1)+2$ and $G-\{u\}$ is a $K_{1,4}$-minor free connected graph. By the induction hypothesis, $G-\{u\} \cong S^{n-5}\left(K_{4}\right)$. Whether $u$ is adjacent to a vertex of degree two or degree three in $G-\{u\}, G$ always contains a $K_{1,4}$ minor, a contradiction. Thus, $U_{1}=\varnothing$. It follows that $\left|U_{3}\right|=\left|U_{1}\right|+4=4$ and $\left|U_{2}\right|=n-4 \geq 2$. Assume that $U_{3}=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$. In the following, we distinguish two cases.

## Case 1. Every vertex in $U_{2}$ belongs to a triangle.

Recall that $\left|U_{2}\right| \geq 2$. Choose $u_{1}, u_{2} \in U_{2}$. If $u_{1}, u_{2}$ contains two common neighbors $w_{1}$ and $w_{2}$, then $w_{1} w_{2} \in E(G)$ as $u_{1}, u_{2}$ belong to a triangle respectively. Now $d_{G}\left(w_{1}\right)=$ $d_{G}\left(w_{2}\right)=3$ and $G\left[\left\{u_{1}, u_{2}, w_{1}, w_{2}\right\}\right]$ is a component of $G$, contradicting the fact that $G$ is a connected graph of order $n \geq 6$.

Assume now that $N_{G}\left(u_{1}\right) \cap N_{G}\left(u_{2}\right)=\{w\}$, and $w_{i} \in N_{G}\left(u_{i}\right) \backslash\{w\}$ for $i \in\{1,2\}$. Then $w w_{1}, w w_{2} \in E(G)$ and $u_{1}, u_{2}, w_{1}, w_{2} \in N_{G}(w)$, which implies that $u_{1} u_{2} \in E(G)$ (otherwise, $G$ contains a $K_{1,4}$ ). Thus, $\left\{u_{1}, u_{2}, w\right\}$ induces a triangle. Since $G$ is connected and $n \geq 6$, we have $d_{G}(w)=3$. Let $N_{G}(w)=\left\{u_{1}, u_{2}, u_{3}\right\}$. Then $d_{G}\left(u_{3}\right)=2$ (otherwise, $d_{G}\left(u_{3}\right) \geq 3$, we will get a $K_{1,4}$ minor). Let $N_{G}\left(u_{3}\right)=\left\{w, u_{4}\right\}$. Then $w u_{4} \in E(G)$ as $u_{3}$ belongs to a triangle. But now $d_{G}(w) \geq 4$, a contradiction.

Now we can conclude that $N_{G}\left(u_{1}\right) \cap N_{G}\left(u_{2}\right)=\varnothing$. Since $u_{1}$ belongs to a triangle, we have $u_{1} u_{2} \notin E(G)$. Let $P=u_{1} w_{1} \ldots w_{s} u_{2}$ be a shortest path connecting $u_{1}$ and $u_{2}$ in $G$. Assume that $N_{G}\left(u_{1}\right)=\left\{w_{1}, u_{3}\right\}$ and $N_{G}\left(u_{2}\right)=\left\{w_{s}, u_{4}\right\}$. Then $u_{3}, u_{4} \notin V(P)$ and $u_{3} \neq u_{4}$. Furthermore, $w_{1} u_{3}, w_{s} u_{4} \in E(G)$, as $u_{1}, u_{2}$ belong to triangles. Thus, $u_{1}, u_{3} \in N_{G}\left(w_{1}\right)$ and $u_{2}, u_{4} \in N_{G}\left(w_{s}\right)$. Contracting the subpath $w_{1} \ldots w_{s}$ as a vertex, we obtain a copy of $K_{1,4}$, a contradiction. The proof of Case 1 is completed.

## Case 2. There exists a vertex $u \in U_{2}$ with two non-adjacent neighbors.

Assume $N_{G}(u)=\left\{u_{1}, u_{2}\right\}$ and $u_{1} u_{2} \notin E(G)$. Let $G^{\prime}$ be the graph obtained from $G$ by contracting the path $u_{1} u u_{2}$ as an edge $u_{1} u_{2}$ (that is, $u$ is absorbed). Then $e\left(G^{\prime}\right)=$ $e(G)-1=(n-1)+2$. Since $G$ is $K_{1,4}$-minor free, $G^{\prime}$ is too. By the induction hypothesis, $G^{\prime} \cong S^{n-5}\left(K_{4}\right)$. Let $P$ be the induced path of length $n-4$ in $G^{\prime}$. Then both ends are of
degree three in $G^{\prime}$ and $G$. We may assume $P=v_{1} w_{1} \ldots w_{n-5} v_{2}$, where $v_{1}, v_{2} \in U_{3}$.
If $u_{1} u_{2} \in E\left(G^{\prime}\right) \backslash E(P)$, then $u_{1}, u_{2}$ are of degree three in $G^{\prime}$ and $G$. Now if $\left\{u_{1}, u_{2}\right\}=$ $\left\{v_{3}, v_{4}\right\}$, then $G\left[\left\{v_{1}, v_{2}, v_{3}, v_{4}, w_{1}, u\right\}\right]$ has a $K_{1,4}$ minor (see Fig. 1), a contradiction. Thus $u_{1} \in\left\{v_{1}, v_{2}\right\}$ and $u_{2} \in\left\{v_{3}, v_{4}\right\}$. Assume without loss of generality that $\left\{u_{1}, u_{2}\right\}=\left\{v_{1}, v_{3}\right\}$. Then $G\left[\left\{v_{1}, v_{2}, v_{3}, v_{4}, w_{1}, u\right\}\right]$ still has a $K_{1,4}$ minor (see Fig. 1), a contradiction. Hence, $u_{1} u_{2} \in E(P)$, and now $G \cong S^{n-4}\left(K_{4}\right)$, as desired. The proof is completed.

In the following, we introduce some basic notations and terminologies. A generalized book, denoted by $B_{\gamma, n-\gamma}$, is obtained by joining a $\gamma$-clique with an independent set of $n-\gamma$ vertices. Specially, $B_{2, n-2}$ is called a book. Let $\alpha_{H}$ be the independence number of a graph $H$, that is, the cardinality of a maximum independent set in $H$. We now define an important variation on a given family $\mathbb{H}$ which contains no star as a member.

$$
\begin{equation*}
\gamma_{\mathbb{H}}:=\min _{H \in \mathbb{H}} \gamma_{H}, \text { where } \gamma_{H}=|H|-\alpha_{H}-1 . \tag{1}
\end{equation*}
$$

A graph $H$ is said to be minimal with respect to $\mathbb{H}$, if $H \in \mathbb{H}$ such that (i) $\gamma_{H}=\gamma_{\mathbb{H}}$; (ii) subject to (i), $|H|$ is also minimal. It is obvious that all minimal graphs have the same independence number. Thus, we can set $\alpha_{\mathbb{H}}:=\alpha_{H^{*}}$ for an arbitrary minimal graph $H^{*}$. Moreover, we define $C_{\mathbb{H}}:=\min C_{H^{*}}$, where $H^{*}$ takes over all minimal graphs. Recall that $\gamma_{H^{*}}=\left|H^{*}\right|-\alpha_{H^{*}}-1$ and $C_{H^{*}}=2^{\left|H^{*}\right|+1} e\left(H^{*}\right)$. Hence, $\gamma_{\mathbb{H}}+\alpha_{\mathbb{H}}<C_{\mathbb{H}}$.

An elementary operation on a graph is one of the following, that is, deleting a vertex, or deleting an edge, or contracting an edge. Clearly, a graph $G$ contains an $H$ minor if $H$ can be obtained from $G$ by a sequence of elementary operations. A graph is said to be $\mathbb{H}$-minor free, if it does not contain an $H$ minor for any $H \in \mathbb{H}$. Observe that $H$ is a spanning subgraph of $B_{\gamma_{H}+1, \alpha_{H}}$. Thus we have the following lemma.

Lemma 2.5. $B_{\gamma_{H}+1, \alpha_{H}}$ contains an $H$ minor.
Let $\operatorname{spex}\left(n, \mathbb{H}_{\text {minor }}\right)$ be the extremal spectral radius of graphs in $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$.
Lemma 2.6. $B_{\gamma_{\mathbb{H}}, n-\gamma_{\mathbb{H}}}$ is $\mathbb{H}$-minor free, and spex $\left(n, \mathbb{H}_{\text {minor }}\right) \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}$.
Proof. Choose an arbitrary member $H \in \mathbb{H}$. We first claim that $B_{\gamma_{H}, n-\gamma_{H}}$ contains no copy of $H$. Indeed, otherwise, $H \subseteq B_{\gamma_{H}, n-\gamma_{H}}$. Let $S$ be the $\left(n-\gamma_{H}\right)$-vertex independent set and $T$ be the $\gamma_{H}$-clique in $B_{\gamma_{H}, n-\gamma_{H}}$. Then,

$$
|V(H) \cap S|=|V(H) \backslash T| \geq|H|-\gamma_{H}=\alpha_{H}+1 .
$$

Furthermore, since $V(H) \cap S$ is also an independent set in $H$, we have $\alpha_{H} \geq|V(H) \cap S| \geq$ $\alpha_{H}+1$, a contradiction. Therefore, the claim holds.

We now show that $B_{\gamma_{H}, n-\gamma_{H}}$ is $H$-minor free. Suppose to the contrary that $B_{\gamma_{H}, n-\gamma_{H}}$ has an $H$ minor. Then $H$ can be obtained by a sequence of elementary operations on $B_{\gamma_{H}, n-\gamma_{H}}$. These elementary operations give rise to a graph sequence $H_{0}, H_{1}, \cdots, H_{a}(=H)$. From the structure of a generalized book, we know that every elementary operation on a subgraph of $B_{\gamma_{H}, n-\gamma_{H}}$ always gives a new subgraph of $B_{\gamma_{H}, n-\gamma_{H}}$. This implies that $H$ is a subgraph of $B_{\gamma_{H}, n-\gamma_{H}}$, contradicting the claim proved above. Thus, $B_{\gamma_{H}, n-\gamma_{H}}$ is $H$-minor free.

In view of (1), $\gamma_{\mathbb{H}} \leq \gamma_{H}$ and hence $B_{\gamma_{\Pi}, n-\gamma_{H}}$ is a subgraph of $B_{\gamma_{H}, n-\gamma_{H}}$. Since $B_{\gamma_{H}, n-\gamma_{H}}$ is $H$-minor free, $B_{\gamma_{\Pi}, n-\gamma_{\Pi}}$ is too. Considering the choice of $H$, we can see that $B_{\gamma_{\Pi}, n-\gamma_{\Pi}}$ is $\mathbb{H}$-minor free, and so $\operatorname{spex}\left(n, \mathbb{H}_{\text {minor }}\right) \geq \rho\left(B_{\gamma_{\mathbb{H}}, n-\gamma_{\Perp}}\right)$. Note that $K_{\gamma_{\Perp}, n-\gamma_{\Perp}} \subseteq B_{\gamma_{\Perp}, n-\gamma_{\mathbb{H}}}$. Therefore, we have $\operatorname{spex}\left(n, \mathbb{H}_{\text {minor }}\right) \geq \rho\left(K_{\gamma_{\Pi}, n-\gamma_{\Pi}}\right)=\sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}$, as desired.

We end this section with the following stability result. For the sake of readability of the article, we postpone its proof to the last section.

Theorem 2.1. Let $G$ be a graph of order $n$ large enough. Let $X$ be a non-negative eigenvector corresponding to $\rho(G)$ with $x_{u^{*}}=\max _{u \in V(G)} x_{u}$. If $\rho(G) \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}$, then either $G$ contains an $H$ minor for some $H \in \mathbb{H}$, or $G$ admits a set $L$ of exactly $\gamma_{\mathbb{H}}$ vertices such that $x_{u} \geq\left(1-\frac{1}{2\left(10 C_{H}\right)^{2}}\right) x_{u^{*}}$ and $d_{G}(u) \geq\left(1-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{2}}\right)$ nfor every $u \in L$.

## 3 Absorbing method on $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$

Choose an arbitrary $G^{*} \in \operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$. Let $X^{*}=\left(x_{1}, \ldots, x_{n}\right)^{T}$ be a non-negative unit eigenvector with respect to $\rho\left(G^{*}\right)$, and $u^{*} \in V\left(G^{*}\right)$ with $x_{u^{*}}=\max _{u \in V\left(G^{*}\right)} x_{u}$. Set $\rho^{*}:=\rho\left(G^{*}\right)$. Then, $\rho^{*}=\operatorname{spex}\left(n, \mathbb{H}_{\text {minor }}\right) \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}$ by Lemma 2.6. Furthermore, by Theorem 2.1 we have the following proposition.
Proposition 3.1. $G^{*}$ contains a set $L$ of exactly $\gamma_{H \mathcal{H}}$ vertices such that $x_{u} \geq\left(1-\frac{1}{2\left(10 C_{H I}\right)^{2}}\right) x_{u^{*}}$ and $d_{G^{*}}(u) \geq\left(1-\frac{1}{\left(10 C_{H}\right)^{2}}\right) n$ for every $u \in L$.

In this section, we partition $V\left(G^{*}\right) \backslash L$ into $L^{\prime} \cup L^{\prime \prime}$, where $L^{\prime \prime}=\left\{v: N_{G^{*}}(v)=L\right\}$. The key lemma is Lemma 3.3, which will be proved by absorbing method. We shall find an absorbing set in $L^{\prime \prime}$, and then use it to absorb vertices in $L^{\prime}$. To this end, we need some more definitions and propositions.
Definition 3.1. A path $P=v_{1} v_{2} \ldots v_{s}$ (where possibly $v_{1}=v_{s}$ ) is called a linear path in $G$, if $P \subseteq G$ and $d_{G}\left(v_{i}\right)=2$ for each $i \in\{2, \ldots s-1\}$. A linear path $P$ is said to be maximal, if there exists no any linear path $P^{\prime}$ such that $P \subseteq P^{\prime}$ and $P \neq P^{\prime}$.

By Definition 3.1, every linear path is either an induced path or an induced cycle in $G$. Given a connected graph $G$ with $|G| \geq 2$. Definition 3.1 also implies the following two propositions.

Proposition 3.2. Let $P$ be a maximal linear path in $G$. If $P$ is a path, then $d_{G}(v) \neq 2$ for each endpoint $v$; if $P$ is a cycle, then $d_{G}(v) \neq 2$ for at most one vertex $v$.

Proposition 3.3. Every non-trivial connected graph has an edge-decomposition of its maximal linear paths.

For a graph $H$ with $V(H)=\left\{v_{1}, \ldots, v_{h}\right\}$, a model of $H$ in a graph $G$ is a collection of vertex-disjoint connected subgraphs $G_{v_{1}}, \ldots, G_{v_{h}}$ such that for any $v_{i} v_{j} \in E(H)$, there exists an edge with one end in $G_{v_{i}}$ and the other end in $G_{v_{j}}$. It is not hard to see that $G$ has an $H$ minor if and only if there is a model of $H$ in $G$. Based on this terminology, we can further introduce a definition.

Definition 3.2. Let $G$ be a graph with an $H$ minor and $\left\{G_{v_{i}}: v_{i} \in V(H)\right\}$ be a model of $H$ in $G$. Then, $\left(V\left(G_{v_{1}}\right), \ldots, V\left(G_{v_{h}}\right)\right)$ is called an $H$-partition of $G$.

Note that $\cup_{v_{i} \in V(H)} V\left(G_{v_{i}}\right) \subseteq V(G)$. Hence, an $H$-partition of $G$ may not be a partition of $V(G)$ although its members are vertex-disjoint. An $H$-partition is said to be minimal, if $\sum_{i=1}^{|H|}\left|G_{v_{i}}\right|$ is minimum over all $H$-partitions of $G$.

By Proposition 3.1, $|L|=\gamma_{\text {HI }}=\min _{H \in \mathbb{H}}\left(|H|-\alpha_{H}-1\right)$. Recall that $V\left(G^{*}\right) \backslash L=L^{\prime} \cup$ $L^{\prime \prime}$, where $L^{\prime \prime}=\left\{v: N_{G^{*}}(v)=L\right\}$. Also by Proposition 3.1, every vertex in $L$ has at most $\frac{n}{\left(10 C_{\text {III }}\right)^{2}}$ non-neighbors in $L^{\prime}$. Hence,

$$
\begin{equation*}
\left|L^{\prime}\right| \leq \frac{n}{\left(10 C_{\mathbb{H}}\right)^{2}}|L|=\frac{\gamma_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{2}} \tag{2}
\end{equation*}
$$

Before proceeding we need some more notations. Given a graph $G$ with $u \in V(G)$ and $S \subseteq V(G)$, we write $N_{S}(u):=N_{G}(u) \cap S$ and $d_{S}(u):=\left|N_{S}(u)\right|$. Let $G \cup G^{\prime}$ be the union of two vertex-disjoint graphs $G$ and $G^{\prime}$. Specially, we use $k G$ to denote the disjoint union of $k$ copies of $G$. For two disjoint subsets $S, T \subseteq V(G)$, let $G[S, T]$ be the bipartite subgraph obtained from $G[S \cup T]$ by deleting all its edges within $S$ and within $T$. We use $e(S)$ and $e(S, T)$ to denote the numbers of edges in $G[S]$ and $G[S, T]$, respectively.
Lemma 3.1. $d_{L^{\prime \prime}}(v) \leq \alpha_{\mathbb{H}}$ for each $v \in L^{\prime} \cup L^{\prime \prime}$ and $G^{*}\left[L^{\prime \prime}\right]$ is $\left(K_{1, \alpha_{\mathbb{H}}+1} \cup \gamma_{\mathbb{H}} K_{1}\right)$-minor free.
Proof. We first show $d_{L^{\prime \prime}}(v) \leq \alpha_{\mathbb{H}}$ for $v \in L^{\prime} \cup L^{\prime \prime}$. Suppose to the contrary that $d_{L^{\prime \prime}}\left(v_{0}\right) \geq$ $\alpha_{\mathbb{H}}+1$ for some $v_{0} \in L^{\prime} \cup L^{\prime \prime}$. Let $L=\left\{u_{1}, \ldots, u_{\gamma_{\mathbb{H}}}\right\}$ and $\left\{w_{0}, \ldots, w_{\alpha_{\mathbb{H}}}\right\} \subseteq N_{L^{\prime \prime}}\left(v_{0}\right)$.

In view of (2), we have $\left|L^{\prime \prime}\right|=n-|L|-\left|L^{\prime}\right| \geq \gamma_{\mathbb{H}}+\alpha_{\mathbb{H}}+2$ for $n$ large enough. Thus we can choose $\gamma_{\mathbb{H}}$ vertices $v_{1}, v_{2}, \ldots, v_{\gamma_{\Pi \mathbb{H}}}$ in $L^{\prime \prime} \backslash\left\{v_{0}, w_{0}, w_{1}, \ldots, w_{\alpha_{\mathbb{H}}}\right\}$. Note that $G^{*}\left[L, L^{\prime \prime}\right] \cong$ $K_{|L|,\left|L^{\prime \prime}\right|}$. Now let $G$ be the graph obtained from $G^{*}$ by contracting each edge $u_{i} v_{i}$ as a new vertex $\bar{u}_{i}$ for $i \in\left\{1, \ldots, \gamma_{\mathbb{H}}\right\}$. Then, $\left\{\bar{u}_{1}, \ldots, \bar{u}_{\gamma_{\mathbb{H}}}\right\}$ is a clique in $G$ and $\bar{u}_{i} \in N_{G}\left(w_{j}\right)$ for $i \in\left\{1, \ldots, \gamma_{H}\right\}$ and $j \in\left\{0, \ldots, \alpha_{\mathbb{H}}\right\}$.

Furthermore, let $G^{\prime}$ be the graph obtained from $G$ by contracting the edge $v_{0} w_{0}$ as a new vertex $\bar{u}_{0}$. Recall that $\bar{u}_{i} \in N_{G}\left(w_{0}\right)$ for $i \in\left\{1, \ldots, \gamma_{\mathbb{H}}\right\}$ and $w_{j} \in N_{G}\left(v_{0}\right)$ for $j \in$ $\left\{1, \ldots, \alpha_{\mathbb{H}}\right\}$. Thus, $\bar{u}_{i}, w_{j} \in N_{G^{\prime}}\left(\bar{u}_{0}\right)$ for $i \in\left\{1, \ldots, \gamma_{\mathbb{H}}\right\}$ and $j \in\left\{1, \ldots, \alpha_{\mathbb{H}}\right\}$.

Now, we can see that $G^{\prime}\left[\left\{\bar{u}_{i}, w_{j}: 0 \leq i \leq \gamma_{\mathbb{H}} ; 1 \leq j \leq \alpha_{\mathbb{H}}\right\}\right]$ contains $B_{\gamma_{\mathbb{H}}+1, \alpha_{\mathbb{H}}}$ as a spanning subgraph. By Lemma $2.5, G^{\prime}$ contains an $H$ minor for some $H \in \mathbb{H}$ and thus $G^{*}$ too, a contradiction. Therefore, $d_{L^{\prime \prime}}(v) \leq \alpha_{\mathbb{H}}$ for each $v \in L^{\prime} \cup L^{\prime \prime}$.

Next suppose that $G^{*}\left[L^{\prime \prime}\right]$ contains an $H_{0}$-minor, where $H_{0} \cong K_{1, \alpha_{\mathbb{H}}+1} \cup \gamma_{\mathbb{H}} K_{1}$. Let $G^{\prime \prime}$ be the graph obtained from $G^{*}$ by replacing $G^{*}\left[L^{\prime \prime}\right]$ with a copy of $H_{0}$. Then for $G^{\prime \prime},\left|L^{\prime \prime}\right|=\left|H_{0}\right|=\gamma_{\mathbb{H}}+\alpha_{\mathbb{H}}+2$ and there exists a vertex $v_{0} \in L^{\prime \prime}$ with $d_{L^{\prime \prime}}\left(v_{0}\right) \geq \alpha_{\mathbb{H}}+1$. By the above discussion, $G^{\prime \prime}$ contains an $H$ minor for some $H \in \mathbb{H}$ and thus $G^{*}$ too, a contradiction. Therefore, the lemma holds.
Lemma 3.2. $x_{v} \leq \frac{4 x_{u^{*}}}{100 C_{\mathbb{H}}}$ for each $v \in L^{\prime} \cup L^{\prime \prime}$.
Proof. Choose an arbitrary $v \in L^{\prime} \cup L^{\prime \prime}$. Then $d_{L}(v) \leq|L|=\gamma_{\mathbb{H}}$, and by Lemma 3.1 $d_{L^{\prime \prime}}(v) \leq \alpha_{\mathbb{H}}$. Recall that $\gamma_{\mathbb{H}}+\alpha_{\mathbb{H}}<C_{\mathbb{H}}$. Thus

$$
d_{G^{*}}(v)=d_{L \cup L^{\prime \prime}}(v)+d_{L^{\prime}}(v) \leq\left(\gamma_{\mathbb{H}}+\alpha_{\mathbb{H}}\right)+d_{L^{\prime}}(v)<C_{\mathbb{H}}+d_{L^{\prime}}(v) .
$$

Since $\rho^{*} x_{v}=\sum_{u \in N_{G^{*}}(v)} x_{u} \leq d_{G^{*}}(v) x_{u^{*}}$, we have

$$
\sum_{v \in L^{\prime}} \rho^{*} x_{v} \leq \sum_{v \in L^{\prime}}\left(C_{\mathbb{H}}+d_{L^{\prime}}(v)\right) x_{u^{*}}=\left(C_{\mathbb{H}}\left|L^{\prime}\right|+2 e\left(L^{\prime}\right)\right) x_{u^{*}},
$$

where $e\left(L^{\prime}\right)<C_{\mathbb{H}}\left|L^{\prime}\right|$ by Lemma 2.2. Combining (2) gives $\rho^{*} \sum_{v \in L^{\prime}} x_{v} \leq 3 C_{\mathbb{H}}\left|L^{\prime}\right| x_{u^{*}} \leq$ $\frac{3 \gamma_{\mathbb{H}} n}{100 C_{\mathbb{H}}} x_{u^{*}}$. Again by $d_{L \cup L^{\prime \prime}}(v) \leq \gamma_{\mathbb{H}}+\alpha_{\mathbb{H}}<C_{\mathbb{H}}$, we have

$$
\begin{equation*}
\rho^{*} x_{v}=\sum_{u \in N_{G^{*}}(v)} x_{u}=\sum_{u \in N_{L \cup L^{\prime}}(v)} x_{u}+\sum_{u \in L^{\prime}} x_{u} \leq C_{\mathbb{H}} x_{u^{*}}+\frac{3 \gamma_{\mathbb{H}} n}{100 C_{\mathbb{H}} \rho^{*}} x_{u^{*}} . \tag{3}
\end{equation*}
$$

Dividing both sides of (3) by $\rho^{*}$ and combining $\rho^{* 2} \geq \gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)$, we obtain that $x_{v} \leq$ $\frac{4 x_{u^{*}}}{100 C_{H}}$ for $n$ sufficiently large, as desired.

Now we are ready to show a key lemma, which states that $L^{\prime}$ is empty. We proceed the proof by contradiction and absorbing method. To make the proof readable, we divide it into some claims and present a sketch as follows.
i) Construct a graph $G^{\prime}$ such that $\rho\left(G^{\prime}\right)>\rho\left(G^{*}\right)$ and $G^{\prime}-L^{\prime}=G^{*}-L^{\prime}$. Then $G^{\prime}$ admits an $H^{\prime}$ minor for some $H^{\prime} \in \mathbb{H}$, and thus an $H^{\prime}$-partition $\mathscr{V}$.
ii) Using the $H^{\prime}$-partition $\mathscr{V}$, we find an absorbing set $P^{*}$ in $G^{\prime}\left[L^{\prime \prime}\right]$, which should be a maximal linear path of length sufficiently large.
iii) Based on $G^{\prime}$, we construct a graph $G^{\prime \prime}$ by using $P^{*}$ to absorb vertices in $L^{\prime}$. Now we obtain a new maximal linear path $P$ of order $\left|P^{*}\right|+\left|L^{\prime}\right|$ in $G^{\prime \prime}\left[L^{\prime} \cup L^{\prime \prime}\right]$.
iv) Prove $\rho\left(G^{\prime \prime}\right)>\rho\left(G^{*}\right)$. Then $G^{\prime \prime}$ contains an $H^{\prime \prime}$ minor for some $H^{\prime \prime} \in \mathbb{H}$, and thus an $H^{\prime \prime}$-partition $\mathscr{V}^{\prime \prime}$. Based on $\mathscr{V}^{\prime \prime}$, we shall construct a graph $G^{\prime \prime \prime}$, by contracting $P$ in $G^{\prime \prime}$ as a new path of order $r \leq 2\left|H^{\prime \prime}\right|+1$, such that $G^{\prime \prime \prime}$ also contains an $H^{\prime \prime}$ minor.
$v$ ) Contract $P^{*}$ in $G^{*}-L^{\prime}$ as a new path of order $r$. The resulting graph is isomorphic to $G^{\prime \prime \prime}$. Hence, $G^{*}$ contains a $G^{\prime \prime \prime}$ minor and thus an $H^{\prime \prime}$ minor, a contradiction.

Lemma 3.3. $L^{\prime}$ is an empty set.
Proof. Suppose to the contrary that $L^{\prime} \neq \varnothing$. Let $G^{\prime}$ be the graph obtained from $G^{*}$ by deleting all edges incident to vertices in $L^{\prime}$ and joining all edges from $L^{\prime}$ to $L$.

Claim 3.1. Let $\rho^{\prime}:=\rho\left(G^{\prime}\right)$. Then $\rho^{\prime}>\rho^{*}$.
Proof. Since $e\left(L^{\prime}\right) \leq C_{\mathbb{H}}\left|L^{\prime}\right|$, there exists $v_{1} \in L^{\prime}$ with $d_{L^{\prime}}\left(v_{1}\right) \leq \frac{2 e\left(L^{\prime}\right)}{\left|L^{\prime}\right|} \leq 2 C_{\mathbb{H}}$. Set $L_{1}^{\prime}:=L^{\prime}$ and $L_{2}^{\prime}:=L_{1}^{\prime} \backslash\left\{v_{1}\right\}$. Then we also have $e\left(L_{2}^{\prime}\right) \leq C_{\mathbb{H}}\left|L_{2}^{\prime}\right|$, and thus there exists $v_{2} \in L_{2}^{\prime}$ with $d_{L_{2}^{\prime}}\left(v_{2}\right) \leq 2 C_{\mathbb{H}}$. Repeating this step, we obtain a sequence $L_{1}^{\prime}, \ldots, L_{\left|L^{\prime}\right|}^{\prime}$ such that $L_{i+1}^{\prime}=L_{i}^{\prime} \backslash\left\{v_{i}\right\}$ and $d_{L_{i}^{\prime}}\left(v_{i}\right) \leq 2 C_{\mathbb{H}}$ for each $i$. Now we can decompose $E\left(G^{*}\left[L^{\prime}\right]\right)$ into $\left|L^{\prime}\right|$ subsets $\left\{v_{i} v: v \in N_{L_{i}^{\prime}}\left(v_{i}\right)\right\}$, where $i=1, \ldots,\left|L^{\prime}\right|$. Thus,

$$
\begin{equation*}
\rho^{\prime}-\rho^{*} \geq X^{* T}\left(A\left(G^{\prime}\right)-A\left(G^{*}\right)\right) X^{*} \geq \sum_{i=1}^{\left|L^{\prime}\right|} 2 x_{v_{i}}\left(\sum_{u \in L} x_{u}-\sum_{v \in N_{L u L^{\prime \prime} \cup L_{i}^{\prime}}\left(v_{v}\right)} x_{v}\right) . \tag{4}
\end{equation*}
$$

Recall that $\gamma_{\mathbb{H}}+\alpha_{\mathbb{H}}<C_{\mathbb{H}}$. Moreover, by Proposition 3.1, we have $|L|=\gamma_{\mathbb{H}}$ and

$$
\begin{equation*}
\sum_{u \in L} x_{u} \geq \gamma_{H}\left(1-\frac{1}{2\left(10 C_{\mathbb{H}}\right)^{2}}\right) x_{u^{*}} \geq\left(\gamma_{H H}-\frac{1}{10}\right) x_{u^{*}} \tag{5}
\end{equation*}
$$

On the other hand, for each $v_{i} \in L^{\prime}$, we have $d_{L}\left(v_{i}\right) \leq|L|-1=\gamma_{\mathbb{H}}-1$ by the definition of $L^{\prime}, d_{L^{\prime \prime}}\left(v_{i}\right) \leq \alpha_{\mathbb{H}}$ by Lemma 3.1, and $d_{L_{i}^{\prime}}\left(v_{i}\right) \leq 2 C_{\mathbb{H}}$ by the choice of $v_{i}$. Moreover, Lemma 3.2 gives $x_{v} \leq \frac{4 x_{u^{*}}}{100 C_{H}}$ for $v \in L^{\prime} \cup L^{\prime \prime}$. Thus,

$$
\sum_{v \in N_{L \cup L^{\prime \prime} \cup L_{i}^{\prime}}\left(v_{i}\right)} x_{v} \leq d_{L}\left(v_{i}\right) x_{u^{*}}+d_{L^{\prime \prime} \cup L_{i}^{\prime}}\left(v_{i}\right) \frac{4 x_{u^{*}}}{100 C_{\mathbb{H}}} \leq\left(\gamma_{\mathbb{H}}-1\right) x_{u^{*}}+\left(\alpha_{\mathbb{H}}+2 C_{\mathbb{H}}\right) \frac{4 x_{u^{*}}}{100 C_{\mathbb{H}}},
$$

which implies that $\sum_{v \in N_{L U L^{\prime \prime L L_{i}^{\prime}}}\left(v_{i}\right)} x_{v}<\left(\gamma_{\mathbb{H}}-\frac{1}{10}\right) x_{u^{*}}$ as $\alpha_{\mathbb{H}}+2 C_{\mathbb{H}}<3 C_{\mathbb{H}}$. Combining with (4) and (5), we obtain $\rho^{\prime} \geq \rho^{*}$, with equality if and only if $X^{*}$ is also an eigenvector corresponding to $\rho\left(G^{\prime}\right)$ and $x_{v_{i}}=0$ for each $v_{i} \in L^{\prime}$. Observe that $G^{\prime}$ is connected. If $X^{*}$ is an eigenvector corresponding to $\rho\left(G^{\prime}\right)$. then $X^{*}$ is positive by the Perron-Frobenius theorem. Hence, $\rho^{\prime}>\rho^{*}$, as desired.

In view of Claim 3.1 and the choice of $G^{*}, G^{\prime}$ must contain an $H^{\prime}$ minor for some $H^{\prime} \in \mathbb{H}$. Let $\mathscr{V}=\left(V_{1}, \ldots, V_{\left|H^{\prime}\right|}\right)$ be a minimal $H^{\prime}$-partition of $G^{\prime}$. A set $V_{i}$, where $i \in$ $\left\{1, \ldots,\left|H^{\prime}\right|\right\}$, is called a good set if both $V_{i} \cap L$ and $V_{i} \backslash L$ are non-empty. Since $|L|=\gamma_{\mathbb{H}}$ and $V_{1}, \ldots, V_{\left|H^{\prime}\right|}$ are vertex-disjoint, there are at most $\gamma_{H-H}$ good sets in $\mathscr{V}$. We now give a precise characterization for good sets.

Claim 3.2. Every good set has exactly two vertices.
Proof. By the definition, we have $\left|V_{i}\right| \geq 2$ for each good set $V_{i}$. Now, suppose that there exists a good set $V_{i}$ with $\left|V_{i}\right| \geq 3$. Choose $u \in V_{i} \cap L$ and $v \in V_{i} \backslash L$. Note that $G^{\prime}\left[L, L^{\prime} \cup L^{\prime \prime}\right] \cong$ $K_{\left|L^{\prime}\right|,\left|L^{\prime} \cup L^{\prime \prime}\right|}$. Thus $L^{\prime} \cup L^{\prime \prime} \subseteq N_{G^{\prime}}(u)$ and $L \subseteq N_{G^{\prime}}(v)$. Thus, if we contract the edge $u v$ as a new vertex $w$ in $G^{\prime}$, then $w$ is a dominating vertex in the resulting graph. Let $V_{i}^{\prime}=\{u, v\}$ and $\mathscr{V}^{\prime}=\left(V_{1}, \ldots, V_{i}^{\prime}, \ldots, V_{\left|H^{\prime}\right|}\right)$. Then, $\mathscr{V}^{\prime}$ is also an $H^{\prime}$-partition of $G^{\prime}$, contradicting the fact that $\mathscr{V}$ is minimal. Therefore, the claim holds.

Claim 3.3. For $i \in\left\{1, \ldots,\left|H^{\prime}\right|\right\}$, every induced subgraph $G^{\prime}\left[V_{i} \cap\left(L^{\prime} \cup L^{\prime \prime}\right)\right]$ is connected provided that $V_{i} \cap\left(L^{\prime} \cup L^{\prime \prime}\right) \neq \varnothing$.

Proof. Suppose to the contrary that $G^{\prime}\left[V_{i} \cap\left(L^{\prime} \cup L^{\prime \prime}\right)\right]$ is not connected for some $i$. Then $\left|V_{i} \cap\left(L^{\prime} \cup L^{\prime \prime}\right)\right| \geq 2$. However, $G^{\prime}\left[V_{i}\right]$ is connected by the definition of model. Thus $V_{i} \cap L \neq$ $\varnothing$. This implies that $V_{i}$ is a good set and $\left|V_{i}\right| \geq 3$, contradicting Claim 3.2.

In the following, we further assume that $\mathscr{V}=\left(V_{1}, \ldots, V_{\left|H^{\prime}\right|}\right)$ is a minimal $H^{\prime}$-partition of $G^{\prime}$ such that $\left|L^{\prime} \cap\left(\cup_{i=1}^{\left|H^{\prime}\right|} V_{i}\right)\right|$ is also minimal. Moreover, assume that $G^{\prime}\left[L^{\prime \prime}\right]$ have $c$ connected components $G_{1}, \ldots, G_{c}$.
Claim 3.4. $L^{\prime \prime} \subseteq \cup_{i=1}^{\left|H^{\prime}\right|} V_{i}$ and $c \leq\left|H^{\prime}\right|$.

Proof. Note that $G^{*}\left[L \cup L^{\prime \prime}\right]=G^{\prime}\left[L \cup L^{\prime \prime}\right]$. If $\cup_{i=1}^{\left|H^{\prime}\right|} V_{i} \subseteq L \cup L^{\prime \prime}$, then $\mathscr{V}$ is an $H^{\prime}$-partition of $G^{*}$, contradicting the fact that $G^{*}$ is $H^{\prime}$-minor free. Hence, $L^{\prime} \cap\left(\cup_{i=1}^{\left|H^{\prime}\right|} V_{i}\right) \neq \varnothing$.

Now suppose to the contrary that there exists a vertex $u \in L^{\prime \prime} \backslash\left(\cup_{i=1}^{\left|H^{\prime}\right|} V_{i}\right)$. Choose a vertex $v \in L^{\prime} \cap\left(\cup_{i=1}^{\left|H^{\prime}\right|} V_{i}\right)$. We may assume without loss of generality that $v \in L^{\prime} \cap V_{1}$. Then, set $V_{1}^{\prime}:=\left(V_{1} \backslash\{v\}\right) \cup\{u\}$ and $\mathscr{V}^{\prime}:=\left(V_{1}^{\prime}, V_{2}, \ldots, V_{\left|H^{\prime}\right|}\right)$. Note that $N_{G^{\prime}}(v)=L \subseteq N_{G^{\prime}}(u)$. Then, $\mathscr{V}^{\prime}$ is also an $H^{\prime}$-partition of $G^{\prime}$, contradicting the assumption that $\left|L^{\prime} \cap\left(\cup_{i=1}^{\left|H^{\prime}\right|} V_{i}\right)\right|$ is minimal. Thus, $L^{\prime \prime} \subseteq \cup_{i=1}^{\left|H^{\prime}\right|} V_{i}$.

In the following, we show $c \leq\left|H^{\prime}\right|$. On the one hand, $\cup_{j=1}^{c} V\left(G_{j}\right)=L^{\prime \prime} \subseteq \cup_{i=1}^{\left|H^{\prime}\right|} V_{i}$. On the other hand, by Claim 3.3 we can see that for each $V_{i}$ (where $1 \leq i \leq\left|H^{\prime}\right|$ ), there exists at most one $G_{j}$ with $V\left(G_{j}\right) \cap V_{i} \neq \varnothing$. Therefore, $c \leq\left|H^{\prime}\right|$.

Claim 3.5. Let $\left|G_{1}\right|=\max _{1 \leq j \leq c}\left|G_{j}\right|$. Then $e\left(G_{1}\right) \leq\left|G_{1}\right|+\frac{1}{6} \sqrt{n}$.
Proof. Recall that $|L|=\gamma_{\text {Hi }}$. It follows from (2) that $\left|L^{\prime \prime}\right|=n-|L|-\left|L^{\prime}\right| \geq \frac{n}{2}$. Combining Claim 3.4, we have

$$
\begin{equation*}
\left|G_{1}\right| \geq \frac{1}{c}\left|L^{\prime \prime}\right| \geq \frac{1}{\left|H^{\prime}\right|}\left|L^{\prime \prime}\right| \geq \frac{n}{2\left|H^{\prime}\right|} \tag{6}
\end{equation*}
$$

which implies that $\left|G_{1}\right| \geq \gamma_{H^{*}}+\alpha_{H^{*}}+3$ (where $H^{*}$ is minimal with respect to $\mathbb{H}$ ). Now using a spanning tree of $G_{1}$, we can find a vertex subset $S \subseteq V\left(G_{1}\right)$ such that $|S|=\gamma_{H^{*}}$ and $G_{1}-S$ is also connected. Observe that $G^{*}\left[L^{\prime \prime}\right]=G^{\prime}\left[L^{\prime \prime}\right]$. Set $t:=\alpha_{H^{*}}+1$. Then by Lemma 3.1, $G_{1}$ is $\left(K_{1, t} \cup \gamma_{H^{*}} K_{1}\right)$-minor free and thus $G_{1}-S$ is $K_{1, t}$-minor free. Since $G_{1}-S$ is a connected graph of order at least $t+2$, it contains a $K_{1,2}$. This implies that $t \geq 3$. Now using Lemma 2.3, we obtain $e\left(G_{1}-S\right) \leq\binom{ t}{2}+\left|G_{1}-S\right|-t$. Moreover, Lemma 3.1 also gives $d_{L^{\prime \prime}}(v) \leq \alpha_{H^{*}}$ for each $v \in L^{\prime \prime}$. Consequently,

$$
e\left(G_{1}\right) \leq e\left(G_{1}-S\right)+\sum_{v \in S} d_{L^{\prime \prime}}(v) \leq \frac{1}{2} t(t-1)+\left|G_{1}\right|-|S|-t+|S| \alpha_{H^{*}}
$$

It follows that $e\left(G_{1}\right) \leq\left|G_{1}\right|+\frac{1}{6} \sqrt{n}$ for $n$ large enough.
Claim 3.6. Let $U_{1}$ be the set of vertices of degree one in $G_{1}$. Then $\left|U_{1}\right| \leq\left|H^{\prime}\right|$.
Proof. We first show $\left|V_{i} \cap U_{1}\right| \leq 1$ for each $i \in\left\{1, \ldots,\left|H^{\prime}\right|\right\}$. Suppose to the contrary that $\left|V_{i} \cap U_{1}\right| \geq 2$ for some $i$. Note that $V_{i} \cap U_{1} \subseteq V_{i} \backslash L$. Then $\left|V_{i} \backslash L\right| \geq\left|V_{i} \cap U_{1}\right| \geq 2$. Now if $V_{i} \cap L \neq \varnothing$, then $V_{i}$ is a good set, and thus $\left|V_{i}\right|=2$ by Claim 3.2, a contradiction. Hence, $V_{i} \cap L=\varnothing$, that is, $V_{i} \subseteq L^{\prime} \cup L^{\prime \prime}$.

Since $G^{\prime}\left[V_{i}\right]$ is connected and $\left|V_{i}\right| \geq\left|V_{i} \cap U_{1}\right| \geq 2$, there exist $u_{0} \in V_{i} \cap U_{1}$ and $w_{0} \in$ $N_{V_{i}}\left(u_{0}\right)$. Recall that $L \subseteq N_{G^{\prime}}(v)$ for each $v \in L^{\prime} \cup L^{\prime \prime}$. Consequently, $L \subseteq N_{G^{\prime}}\left(u_{0}\right) \cap N_{G^{\prime}}\left(w_{0}\right)$, as $u_{0}, w_{0} \in V_{i} \subseteq L^{\prime} \cup L^{\prime \prime}$. Since $G_{1}$ is a component of $G^{\prime}\left[L^{\prime} \cup L^{\prime \prime}\right]$ and $u_{0}$ is a vertex of degree one in $G_{1}$, we can see that $w_{0}$ is the unique neighbor of $u_{0}$ in $L^{\prime} \cup L^{\prime \prime}$, and thus $N_{G^{\prime}}\left(u_{0}\right) \backslash$ $\left\{w_{0}\right\} \subseteq N_{G^{\prime}}\left(w_{0}\right) \backslash\left\{u_{0}\right\}$. Hence, $\left(V_{1}, \ldots, V_{i} \backslash\left\{u_{0}\right\}, \ldots, V_{\left|H^{\prime}\right|}\right)$ is also an $H^{\prime}$-partition of $G^{\prime}$, contradicting the minimality of $\mathscr{V}$. Hence, $\left|V_{i} \cap U_{1}\right| \leq 1$ for each $i$, as desired.

Note that $U_{1} \subseteq V\left(G_{1}\right) \subseteq L^{\prime \prime}$. Moreover, $L^{\prime \prime} \subseteq \cup_{i=1}^{\left|H^{\prime}\right|} V_{i}$ by Claim 3.4. Thus, $U_{1} \subseteq \cup_{i=1}^{\left|H^{\prime}\right|} V_{i}$. Since $\left|V_{i} \cap U_{1}\right| \leq 1$ for each $i \in\left\{1, \ldots,\left|H^{\prime}\right|\right\}$, we obtain $\left|U_{1}\right| \leq\left|H^{\prime}\right|$.

Claim 3.7. $G_{1}$ admits a maximal linear path $P^{*}$ of order at least $\frac{\sqrt{n}}{2\left|H^{\top}\right|}$.
Proof. Let $U_{2}=\left\{v \in V\left(G_{1}\right): d_{G_{1}}(v)=2\right\}$ and $U_{3}=V\left(G_{1}\right) \backslash\left(U_{1} \cup U_{2}\right)$. Then, $3\left|G_{1}\right|-$ $2\left|U_{1}\right|-\left|U_{2}\right|=\left|U_{1}\right|+2\left|U_{2}\right|+3\left|U_{3}\right| \leq 2 e\left(G_{1}\right)$, which yields that $e\left(G_{1}\right)-\left|U_{2}\right| \leq 3\left(e\left(G_{1}\right)-\right.$ $\left.\left|G_{1}\right|\right)+2\left|U_{1}\right|$. Combining Claims 3.5 and 3.6 gives that

$$
\begin{equation*}
e\left(G_{1}\right)-\left|U_{2}\right| \leq \frac{1}{2} \sqrt{n}+2\left|H^{\prime}\right| \leq \sqrt{n} . \tag{7}
\end{equation*}
$$

Assume now that there are $\phi\left(G_{1}\right)$ maximal linear paths in $G_{1}$. If $G_{1}$ itself is a cycle, then $\phi\left(G_{1}\right)=1$ by Definition 3.1. If $G_{1}$ is not a cycle, then by Proposition 3.2, two ends of every maximal linear path occupies exactly two degrees of vertices in $V\left(G_{1}\right) \backslash U_{2}$. Thus, $2 \phi\left(G_{1}\right)=\sum_{v \in V\left(G_{1}\right) \backslash U_{2}} d_{G_{1}}(v)$. Combining (7) gives

$$
\phi\left(G_{1}\right)=\frac{1}{2} \sum_{v \in V\left(G_{1}\right) \backslash U_{2}} d_{G_{1}}(v)=e\left(G_{1}\right)-\left|U_{2}\right| \leq \sqrt{n} .
$$

But in view of (6), we have $\left|G_{1}\right| \geq \frac{n}{2\left|H^{\top}\right|}$. Moreover, by Proposition 3.3, $G_{1}$ admits an edge-decomposition of its maximal linear paths. Therefore, there exists a maximal linear path $P^{*}$ of length at least $\frac{e\left(G_{1}\right)}{\phi\left(G_{1}\right)}$, and thus $\left|P^{*}\right| \geq \frac{e\left(G_{1}\right)}{\phi\left(G_{1}\right)}+1 \geq \frac{\left|G_{1}\right|}{\phi\left(G_{1}\right)} \geq \frac{\sqrt{n}}{2\left|H^{\top}\right|}$.

Note that $V\left(P^{*}\right) \subseteq L^{\prime \prime}$ and $L^{\prime}$ is an independent set in $G^{\prime}$. Now we use $P^{*}$ to absorb vertices in $L^{\prime}$. Assume that $P^{*}=w_{1} w_{2} \ldots w_{a}$ and $L^{\prime}=\left\{v_{1}, v_{2}, \ldots, v_{b}\right\}$. Let $G^{\prime \prime}$ be the graph obtained from $G^{\prime}$ by replacing the edge $w_{1} w_{2}$ with a path $w_{1} v_{1} v_{2} \ldots v_{b} w_{2}$.

Claim 3.8. Let $\rho^{\prime \prime}:=\rho\left(G^{\prime \prime}\right)$. Then $\rho^{\prime \prime}>\rho^{*}$.
Proof. Recall that $\rho\left(G^{\prime}\right)=\rho^{\prime}>\rho^{*}$ by Claim 3.1. It suffices to show $\rho^{\prime \prime} \geq \rho^{\prime}$. Note that $G^{\prime}$ is connected. By the Perron-Frobenius theorem, there exists a positive unit eigenvector $Y=\left(y_{1}, \ldots, y_{n}\right)^{T}$ corresponding to $\rho\left(G^{\prime}\right)$. Set $\sigma_{L}:=\sum_{u \in L} y_{u}$ and $y_{L^{\prime \prime}}^{*}:=\max _{w \in L^{\prime \prime}} y_{w}$. For every $v_{i} \in L^{\prime}$, since $N_{G^{\prime}}\left(v_{i}\right)=L$, we have $\rho^{\prime} y_{v_{i}}=\sigma_{L}$. Consequently, $y_{v_{1}}=y_{v_{b}}=\frac{\sigma_{L}}{\rho^{\prime}}$. Moreover, by Lemma 3.1, $d_{L^{\prime \prime}}(w) \leq \alpha_{\mathbb{H}}$ for each $w \in L^{\prime \prime}$. Thus, $\rho^{\prime} y_{L^{\prime \prime}}^{*} \leq \sigma_{L}+\alpha_{\mathbb{H}} y_{L^{\prime \prime}}^{*}$, which yields $y_{L^{\prime \prime}}^{*} \leq \frac{\sigma_{L}}{\rho^{\prime}-\alpha_{\mathbb{H}}}$. By Lemma 2.6, $\rho^{*} \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}$, and thus $\rho^{\prime}>\sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)} \geq$ $2 \alpha_{\mathbb{H}}$. Combining these inequalities, we obtain

$$
\begin{equation*}
\max \left\{y_{w_{1}}, y_{w_{2}}\right\} \leq y_{L^{\prime \prime}}^{*} \leq \frac{\sigma_{L}}{\rho^{\prime}-\alpha_{\mathbb{H}}} \leq \frac{2 \sigma_{L}}{\rho^{\prime}}=2 y_{v_{1}}=2 y_{v_{b}} \tag{8}
\end{equation*}
$$

On the other hand, one can see that

$$
\rho^{\prime \prime}-\rho^{\prime} \geq 2\left(y_{w_{1}} y_{v_{1}}+y_{w_{2}} y_{v_{b}}-y_{w_{1}} y_{w_{2}}\right)=y_{w_{1}}\left(2 y_{v_{1}}-y_{w_{2}}\right)+y_{w_{2}}\left(2 y_{v_{b}}-y_{w_{1}}\right) .
$$

Combining (8), we have $\rho^{\prime \prime} \geq \rho^{\prime}$, and so $\rho^{\prime \prime}>\rho^{*}$.
We are ready to complete the proof of Lemma 3.3. Note that $G^{\prime \prime}\left[L^{\prime} \cup L^{\prime \prime}\right]$ has a maximal linear path $P=w_{1} v_{1} v_{2} \ldots v_{b} w_{2} \ldots w_{a-1} w_{a}$. In view of Claim 3.8 and the definition of $G^{*}$, $G^{\prime \prime}$ contains an $H^{\prime \prime}$ minor for some $H^{\prime \prime} \in \mathbb{H}$. Let $\mathscr{V}^{\prime \prime}=\left(V_{1}, \ldots, V_{\left|H^{\prime \prime}\right|}\right)$ be a minimal $H^{\prime \prime}$ partition of $G^{\prime \prime}$. Applying Claim 3.3 on $G^{\prime \prime}$ and $\mathscr{V}^{\prime \prime}$, we know that either $V_{i} \cap\left(L^{\prime} \cup L^{\prime \prime}\right)=\varnothing$
or $G^{\prime \prime}\left[V_{i} \cap\left(L^{\prime} \cup L^{\prime \prime}\right)\right]$ is connected for every $i \in\left\{1, \ldots,\left|H^{\prime \prime}\right|\right\}$. This implies that $V_{i} \cap V(P)=$ $\varnothing, G^{\prime \prime}\left[V_{i} \cap V(P)\right]$ is a subpath of $P$, or $G^{\prime \prime}\left[V_{i} \cap V(P)\right]$ consists of two subpaths $P^{\prime}, P^{\prime \prime}$ such that $w_{1} \in V\left(P^{\prime}\right)$ and $w_{a} \in V\left(P^{\prime \prime}\right)$. Now let $P(i)=G^{\prime \prime}\left[V_{i} \cap V(P)\right], i=1, \ldots,\left|H^{\prime \prime}\right|$. Then $P(1), \ldots, P\left(\left|H^{\prime \prime}\right|\right)$ separate $P$ into $r$ subpaths, where $r \leq 2\left|H^{\prime \prime}\right|+1$. Let $G^{\prime \prime \prime}$ be the graph obtained from $G^{\prime \prime}$ by contracting each of these $r$ subpaths as a vertex. Then by the definition of model, $G^{\prime \prime \prime}$ also has an $H^{\prime \prime}$ minor.

On the other hand, recall that $G^{*}\left[L^{\prime \prime}\right]=G^{\prime}\left[L^{\prime \prime}\right]$ and $P^{*} \subseteq G^{\prime}\left[L^{\prime \prime}\right]$. Hence, $P^{*}$ is also a maximal linear path in $G^{*}\left[L^{\prime \prime}\right]$. By Claim 3.7, $\left|P^{*}\right| \geq \frac{\sqrt{n}}{2\left|H^{\prime}\right|} \geq 2\left|H^{\prime \prime}\right|+1$. One can observe that if we contract the path $P^{*}$ as a new path of order $r$ in $G^{*}-L^{\prime}$, then the resulting graph is isomorphic to $G^{\prime \prime \prime}$. Hence, $G^{*}$ contains a $G^{\prime \prime \prime}$ minor and thus an $H^{\prime \prime}$ minor, a contradiction. Therefore, $L^{\prime}=\varnothing$. This completes the proof of Lemma 3.3.

In the following, we complete the proof of Theorem 1.1.
Proof. Now, we know that $|L|=\gamma_{\text {HI }}$ by Proposition 3.1 and $V\left(G^{*}\right)=L \cup L^{\prime \prime}$ by Lemma 3.3. Thus, $G^{*}$ contains $K_{\gamma_{\Pi}, n-\gamma_{\Pi}}$ as a spanning subgraph. To prove Theorem 1.1, that is, $G^{*}$ contains $B_{\gamma_{\Pi}, n-\gamma_{\Pi}}$, it suffices to show $G^{*}[L] \cong K_{\gamma_{\Pi}}$.

Suppose to the contrary that $L$ is not a clique in $G^{*}$. Then we can find $u_{1}, u_{2} \in L$ with $u_{1} u_{2} \notin E\left(G^{*}\right)$. Since $\left|L^{\prime \prime}\right|=n-|L|>\max _{H \in \mathbb{H}}(|H|+1)$, we can choose a subset $L^{\prime \prime \prime} \subset L^{\prime \prime}$ with $\left|L^{\prime \prime \prime}\right|=\max _{H \in \mathbb{H}}(|H|+1)$. Now set $L^{\prime \prime \prime \prime}:=L^{\prime \prime} \backslash L^{\prime \prime \prime}$. By Lemma 3.1, $d_{L^{\prime \prime}}(w) \leq \alpha_{\mathbb{H}}$ for each $w \in L^{\prime \prime}$. It follows that

$$
e\left(L^{\prime \prime}\right)-e\left(L^{\prime \prime \prime \prime}\right) \leq \sum_{v \in L^{\prime \prime \prime}} d_{L^{\prime \prime}}(v) \leq \max _{H \in \mathbb{H}}(|H|+1) \alpha_{\mathbb{H}} \leq \sqrt{n}
$$

Moreover, let $x_{L^{\prime \prime}}^{*}:=\max _{w \in L^{\prime \prime}} x_{w}$. Then we have $\rho^{*} x_{L^{\prime \prime}}^{*} \leq \sum_{u \in L} x_{u}+\alpha_{\mathbb{H}} x_{L^{\prime \prime}}^{*}$, Recall that $\rho^{*} \geq \sqrt{\gamma_{H}\left(n-\gamma_{H}\right)}$. Hence,

$$
x_{L^{\prime \prime}}^{*} \leq \frac{\sum_{u \in L} x_{u}}{\rho^{*}-\alpha_{\mathbb{H}}} \leq \frac{\gamma_{\mathbb{H}} x_{u^{*}}}{\rho^{*}-\alpha_{\mathbb{H}}} \leq \sqrt{\frac{2 \gamma_{\mathbb{H}}}{n}} x_{u^{*}} .
$$

Now let $G$ be the graph obtained from $G^{*}$ by deleting all edges in $E\left(L^{\prime \prime}\right) \backslash E\left(L^{\prime \prime \prime \prime}\right)$ and adding an edge $u_{1} u_{2}$. By Proposition 3.1, $\min \left\{x_{u_{1}}, x_{u_{2}}\right\} \geq\left(1-\frac{1}{2\left(10 C_{H}\right)^{2}}\right) x_{u^{*}}$, and thus

$$
\begin{aligned}
\rho(G)-\rho\left(G^{*}\right) & \geq 2\left(x_{u_{1}} x_{u_{2}}-\sum_{w_{1} w_{2} \in E\left(L^{\prime \prime}\right) \backslash E\left(L^{\prime \prime \prime \prime}\right)} x_{w_{1}} x_{w_{2}}\right) \\
& \geq 2\left(\left(1-\frac{1}{2\left(10 C_{\mathbb{H}}\right)^{2}}\right)^{2}-\sqrt{n} \cdot \frac{2 \gamma_{H}}{n}\right) x_{u^{*}}^{2} \\
& >0,
\end{aligned}
$$

which implies that $G$ contains an $H_{0}$ minor for some $H_{0} \in \mathbb{H}$.
Now let $\mathscr{V}=\left(V_{1}, \ldots, V_{\left|H_{0}\right|}\right)$ be a minimal $H_{0}$-partition of $G$. Observe that $N_{G}(v)=L$ for each vertex $v \in L^{\prime \prime \prime}$. Applying Claim 3.3 on $G$ and $\mathscr{V}$, we can see that $\left|V_{i} \cap L^{\prime \prime \prime}\right| \leq 1$ for $i=1, \ldots,\left|H_{0}\right|$. Since $\left|L^{\prime \prime \prime}\right|=\max _{H \in \mathbb{H}}(|H|+1) \geq\left|H_{0}\right|+1$, there exists a vertex $v \in$ $L^{\prime \prime \prime} \backslash \cup_{i=1}^{\left|H_{0}\right|} V_{i}$. Consequently, $\mathscr{V}=\left(V_{1}, \ldots, V_{\left|H_{0}\right|}\right)$ is also an $H_{0}$-partition of $G-\{v\}$. In other
words, $G-\{v\}$ contains an $H_{0}$ minor. Now let $G_{u_{1} v}^{*}$ be the graph obtained from $G^{*}$ by contracting the edge $u_{1} v$ as a new vertex $\bar{u}_{1}$. Then $\bar{u}_{1} u_{2} \in E\left(G_{u_{1} v}^{*}\right)$, and thus $G-\{v\}$ is isomorphic to some subgraph of $G_{u_{1} v}^{*}$. This implies that $G_{u_{1} v}^{*}$ also contains an $H_{0}$ minor. Correspondingly, $G^{*}$ contains an $H_{0}$ minor, a contradiction. Therefore, $L$ is a clique in $G^{*}$. This completes the proof of Theorem 1.1.

Having Theorem 1.1, we shall focus on the characterization of $G^{*}-L$. We now recall some notations and terminologies. For a member $H \in \mathbb{H}, \Gamma_{s}^{*}(H)$ denotes the family of $s$-vertex irreducible induced subgraphs of $H$ and $\Gamma(\mathbb{H})=\bigcup_{H \in \mathbb{H}} \Gamma_{|H|-\gamma_{\mathbb{H}}}^{*}(H)$, where $\gamma_{\mathbb{H}}=$ $\min _{H \in \mathbb{H}} \gamma_{H}$ and $\gamma_{H}=|H|-\alpha_{H}-1$.

Lemma 3.4. Let $G$ be a graph with a set $L$ of $\gamma_{\mathbb{H}}$ dominating vertices. Then, $G$ is $\mathbb{H}$-minor free if and only if $G-L$ is $\Gamma(\mathbb{H})$-minor free.

Proof. Firstly, assume that $G-L$ contains an $H_{0}$ minor for some $H_{0} \in \Gamma(\mathbb{H})$. Then there exists an $H \in \mathbb{H}$ such that $H_{0}$ is an $\left(|H|-\gamma_{\mathbb{H}}\right)$-vertex induced subgraph of $H$. Combining this $H_{0}$ minor with $\gamma_{\mathbb{H}}$ dominating vertices in $L$, we obtain an $H$ minor in $G$.

Conversely, assume that $G$ contains an $H$ minor for some $H \in \mathbb{H}$. Then by Definition 3.2, $G$ has an $H$-partition $\mathscr{V}=\left(V_{1}, \ldots, V_{|H|}\right)$. We may assume that $\mathscr{V}$ is a minimal $H$ partition such that $\left|L \cap\left(\cup_{i=1}^{|H|} V_{i}\right)\right|$ is maximal. Then there exist exactly $\gamma_{\mathbb{H}}$ members of $\mathscr{V}$, say $V_{1}, \ldots, V_{\gamma_{\Pi}}$, such that $\left|V_{i}\right|=\left|L \cap V_{i}\right|=1$ for $i \in\left\{1, \ldots, \gamma_{\Pi}\right\}$. Consequently, $\cup_{i=\gamma_{\Pi}+1}^{|H|} V_{i} \subseteq$ $V(G) \backslash L$. This implies that $G-L$ contains an $H_{0}$ minor, where $H_{0} \in \Gamma_{|H|-\gamma_{H}}(H)$. It follows that $G-L$ has an $H_{1}$ minor with $H_{1} \in \Gamma_{|H|-\gamma_{H}}^{*}(H)$.

In the following, we give the proof of Theorem 1.2.
Proof. Theorem 1.1 gives that $G^{*}$ has $\gamma_{\mathbb{H}}$ dominating vertices, which implies that $G^{*}$ is connected. Hence, adding an arbitrary edge within its independent set increases the spectral radius. Furthermore, by Lemma 3.4, $G^{*}-L$ is $\Gamma(\mathbb{H})$-minor free. Thus $G^{*}-L$ is $\Gamma(\mathbb{H})$-minor saturated, that is, $G^{*}-L \in \operatorname{SAT}\left(n-\gamma_{\mathbb{H}}, \Gamma(\mathbb{H})_{\text {minor }}\right)$.

Particularly, if $\mathbb{H}=\{H\}$, then $|H|-\gamma_{\mathbb{H}}=\alpha_{H}+1$. Hence, $\Gamma(\mathbb{H})=\Gamma_{\alpha_{H}+1}^{*}(H)$, and thus $G^{*}-L \in \operatorname{SAT}\left(n-\gamma_{H}, \Gamma_{\alpha_{H}+1}^{*}(H)_{\text {minor }}\right)$. This completes the proof of Theorem 1.2.

A subset $R$ of $V\left(G^{*}\right) \backslash L$ is called a component subset, if $G^{*}[R]$ consists of some connected components. Furthermore, $R$ is said to be small, if $|R| \leq C$ for a constant $C$. To prove Theorem 1.3, we need some more lemmas. We always assume that $\Gamma(\mathbb{H})$ is a family of connected graphs in the following lemmas.

Lemma 3.5. If $R$ is a small component subset of $V\left(G^{*}\right) \backslash$, then $e(R)=e x\left(|R|, \Gamma(\mathbb{H})_{\text {minor }}\right)$.
Proof. Let $H$ be a minimal member in $\mathbb{H}$. Recall that $\gamma_{H}=\gamma_{\mathbb{H}}$ and $\alpha_{H}=\alpha_{\mathbb{H}}$. Then $|H|-$ $\gamma_{\mathbb{H}}=\alpha_{\mathbb{H}}+1$, and thus $\Gamma_{\alpha_{\mathbb{H}}+1}^{*}(H) \subseteq \Gamma(\mathbb{H})$. By Lemma 3.4, $G^{*}-L$ is $\Gamma_{\alpha_{\mathbb{H}}+1}^{*}(H)$-minor free. Now, choose an $\left(\alpha_{\mathbb{H}}+1\right)$-subset $S$ of $V(H)$ such that it contains $\alpha_{\mathbb{H}}$ independent vertices of $H$. Then $H[S] \subseteq K_{1, \alpha_{\mathbb{H}}}$, and hence $G^{*}-L$ is $K_{1, \alpha_{\mathbb{H}}}$-minor free.

Similarly to the proof of Theorem 1.1, we let $L^{\prime \prime}=V\left(G^{*}\right) \backslash L$ and $x_{L^{\prime \prime}}^{*}=\max _{w \in L^{\prime \prime}} x_{w}$. We also set $\sum_{u \in L} x_{u}:=\sigma_{L}$. Clearly, $x_{w} \geq \frac{\sigma_{L}}{\rho^{*}}$ for each $w \in L^{\prime \prime}$. Since $G^{*}\left[L^{\prime \prime}\right]$ is $K_{1, \alpha_{\mathbb{H}}}$-minor free, we have $\rho^{*} x_{L^{\prime \prime}}^{*} \leq \sigma_{L}+\left(\alpha_{\mathbb{H}}-1\right) x_{L^{\prime \prime}}^{*}$, which gives $x_{L^{\prime \prime}}^{*} \leq \frac{\sigma_{L}}{\rho^{*}-\alpha_{\mathbb{H}}+1}$.

Now suppose to the contrary that there exists a small component subset $R$ of $L^{\prime \prime}$ such that $e(R)<e x\left(|R|, \Gamma(\mathbb{H})_{\text {minor }}\right)$. Then we can find a $\Gamma(\mathbb{H})$-minor free graph $G_{i}^{\prime \prime}$ on the vertex set $R$ with at least $e(R)+1$ edges. Let $G$ be the graph obtained from $G^{*}$ by replacing $E\left(G^{*}[R]\right)$ with $E\left(G_{i}^{\prime \prime}\right)$. Since $\Gamma(\mathbb{H})$ is a connected family, $G\left[L^{\prime \prime}\right]$ is still $\Gamma(\mathbb{H})$-minor free. By Lemma 3.4, $G$ is $\mathbb{H}$-minor free. Let $\rho=\rho(G)$. Then

$$
\begin{equation*}
\frac{1}{2}\left(\rho-\rho^{*}\right) \geq \sum_{u v \in E\left(G_{i}^{\prime \prime}\right)} x_{u} x_{v}-\sum_{u v \in E\left(G^{*}[R]\right)} x_{u} x_{v} \geq \frac{e\left(G_{i}^{\prime \prime}\right) \sigma_{L}^{2}}{\rho^{* 2}}-\frac{e(R) \sigma_{L}^{2}}{\left(\rho^{*}-\alpha_{\mathbb{H}}+1\right)^{2}} \tag{9}
\end{equation*}
$$

Since $R$ is small, $e(R)$ is also bounded by a constant. Recall that $e\left(G_{i}^{\prime \prime}\right) \geq e(R)+1$ and $\rho^{*} \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}$. It is clear that $\rho(G)>\rho^{*}$ for $n$ sufficiently large, a contradiction. The proof is completed.

Let $\mathbb{G}$ be the family of connected $\Gamma(\mathbb{H})$-minor free graphs on at most $n-\gamma_{H}$ vertices. In fact, the proof of Lemma 3.5 implies that every member in $\mathbb{G}$ is $K_{1, \alpha_{\mathbb{H}}}$-minor free. Given a member $G_{i} \in \mathbb{G}$, we denote by $d_{i}$ its average degree, and we say that $G_{i}$ is small if $\left|G_{i}\right| \leq c$ for some constant $c$ (in other words, $\left|G_{i}\right|$ is independent of $n$ ). Now, let $G_{0}$ be a member with $d_{0}=\max _{G_{i} \in \mathbb{G}} d_{i}$, and let $G_{1}, \ldots, G_{s}$ be all the non-isomorphic components in $G^{*}-L$. We may assume that $m\left(G_{1}\right) \geq \cdots \geq m\left(G_{s}\right)$, where $m\left(G_{i}\right)$ is the number of copies of $G_{i}$ in $G^{*}-L$. By Lemma 3.4, $G_{i} \in \mathbb{G}$ for every $i \in\{1, \ldots, s\}$.

Lemma 3.6. If all the members are small in $\left\{G_{i}: i=0,1, \ldots, s\right\}$, then we have $d_{1}=d_{0}$ and $m\left(G_{i}\right)<\left|G_{1}\right|$ for every $G_{i}$ with $d_{i}<d_{0}$.

Proof. In view of the choice of $G_{0}$, we know that $d_{i} \leq d_{0}$ for every $i \in\{0,1 \ldots, s\}$. Since $\max _{1 \leq i \leq s}\left|G_{i}\right|=c$ for some constant integer $c$, we have $s \leq \sum_{k=1}^{c} 22^{\binom{k}{2}}$, that is, $s$ is constant. However, $\left|G^{*}-L\right|$ is sufficiently large, thus $m\left(G_{1}\right)$ is a function on $n$.

We first prove that $d_{1}=d_{0}$. Otherwise, $d_{1}<d_{0}$, then we define $G$ to be the graph obtain from $G^{*}$ by replacing $\left|G_{0}\right|$ copies of $G_{1}$ with $\left|G_{1}\right|$ copies of $G_{0}$ in $G^{*}-L$. The component subset $V\left(\left|G_{0}\right| G_{1}\right)$ is small and $G-L$ is $\Gamma(\mathbb{H})$-minor free. However, $\left|G_{0}\right| e\left(G_{1}\right)-$ $\left|G_{1}\right| e\left(G_{0}\right)=\frac{1}{2}\left|G_{0}\right|\left|G_{1}\right|\left(d_{1}-d_{0}\right)<0$, contradicting Lemma 3.5. Hence, $d_{1}=d_{0}$.

Now suppose that there exists a component $G_{i}$ with $d_{i}<d_{0}$ but $m\left(G_{i}\right) \geq\left|G_{1}\right|$. Then we define $G$ to be a new graph obtain from $G^{*}$ by replacing $\left|G_{1}\right|$ copies of $G_{i}$ with $\left|G_{i}\right|$ copies of $G_{1}$ in $G^{*}-L$. Clearly, $\left|G_{1}\right| e\left(G_{i}\right)-\left|G_{i}\right| e\left(G_{1}\right)=\frac{1}{2}\left|G_{1}\right|\left|G_{i}\right|\left(d_{i}-d_{0}\right)<0$, and we similarly get a contradiction. Hence, the lemma holds.

Now, let $G^{\star}$ be a graph of order $n$ with a set $L$ of dominating vertices such that $G^{\star}-L \in$ $E X\left(n-|L|, \Gamma(\mathbb{H})_{\text {minor }}\right)$. Moreover, assume that $G_{i_{1}}, \ldots, G_{i_{t}}$ are all the non-isomorphic components in $G^{\star}-L$. Clearly, $G_{i_{j}} \in \mathbb{G}$ for every $j \in\{1, \ldots, t\}$.

Lemma 3.7. If all the members are small in $\left\{G_{i}: i=0,1, \ldots, s\right\}$ and $\left\{G_{i_{j}}: j=1, \ldots, t\right\}$, then we have $G^{*}-L \in E X\left(n-|L|, \Gamma(\mathbb{H})_{\text {minor }}\right)$.

Proof. Suppose to the contrary, then $e\left(G^{*}-L\right)<e\left(G^{\star}-L\right)$. Since $\max _{1 \leq j \leq t}\left|G_{i_{j}}\right|$ is a constant, $t$ is too. The proof of Lemma 3.6 also implies that $m\left(G_{i_{j}}\right)<\left|G_{1}\right|$ for every $G_{i_{j}}$ with $d_{i_{j}}<d_{0}$. Recall that $m\left(G_{i}\right)$ denotes the number of copies of $G_{i}$ in $G^{*}-L$. We now define $m^{\prime}\left(G_{i_{j}}\right)$ to be the number of copies of $G_{i_{j}}$ in $G^{\star}-L$. Write $m_{i} \equiv m\left(G_{i}\right)\left(\bmod \left|G_{1}\right|\right)$ for $i \in\{1, \ldots, s\}$ and $m_{i_{j}}^{\prime} \equiv m^{\prime}\left(G_{i_{j}}\right)\left(\bmod \left|G_{1}\right|\right)$ for $j \in\{1, \ldots, t\}$. Then, we can see that $m_{i}=m\left(G_{i}\right)$ if $d_{i}<d_{0}$ and $m_{i_{j}}^{\prime}=m\left(G_{i_{j}}^{\prime}\right)$ if $d_{i_{j}}<d_{0}$.

Now we denote $G_{1}^{\prime}=G^{*}-L-\cup_{i=1}^{s} V\left(m_{i} G_{i}\right)$ and $G_{2}^{\prime}=G^{\star}-L-\cup_{j=1}^{t} V\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)$ for simplicity. Then, both $\left|G_{1}^{\prime}\right|$ and $\left|G_{2}^{\prime}\right|$ are divisible by $\left|G_{1}\right|$, which gives that

$$
\begin{equation*}
\left|\cup_{j=1}^{t} V\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)\right|-\left|\cup_{i=1}^{s} V\left(m_{i} G_{i}\right)\right|=\left|G_{1}^{\prime}\right|-\left|G_{2}^{\prime}\right|=r\left|G_{1}\right| \tag{10}
\end{equation*}
$$

for an integer $r$. Moreover, since both $\left|\cup_{j=1}^{t} V\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)\right|$ and $\left|\cup_{i=1}^{s} V\left(m_{i} G_{i}\right)\right|$ are finite, $r$ is too. Note that every component in $G_{1}^{\prime}$ and $G_{2}^{\prime}$ has average degree $d_{0}$. Combining (10) and $d_{0}=d_{1}$, we have $e\left(G_{1}^{\prime}\right)-e\left(G_{2}^{\prime}\right)=\frac{1}{2} d_{0}\left(\left|G_{1}^{\prime}\right|-\left|G_{2}^{\prime}\right|\right)=r e\left(G_{1}\right)$. Consequently,

$$
\begin{equation*}
\sum_{j=1}^{t} e\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)-\sum_{i=1}^{s} e\left(m_{i} G_{i}\right)=e\left(G^{\star}-L\right)-e\left(G^{*}-L\right)+r e\left(G_{1}\right)>r e\left(G_{1}\right) . \tag{11}
\end{equation*}
$$

Recall that $m\left(G_{1}\right)$ is a function on $n$. If $r \geq 0$, we define $R=\left(\cup_{i=1}^{s} V\left(m_{i} G_{i}\right)\right) \cup V\left(r G_{1}\right)$. Then $R$ is a small component subset of $V\left(G^{*}\right) \backslash L$, and $|R|=\left|\cup_{j=1}^{t} V\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)\right|$ by (10). However, (11) gives $e(R)<\sum_{j=1}^{t} e\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)$, contradicting Lemma 3.5.

If $r<0$, we define $R=\cup_{i=1}^{S} V\left(m_{i} G_{i}\right)$. Then $R$ is still a small component subset of $V\left(G^{*}\right) \backslash L$, and by (10) $|R|=\left|\cup_{j=1}^{t} V\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)\right|+\left|V\left(-r G_{1}\right)\right|$. However, by (11) we have $e(R)<\sum_{j=1}^{t} e\left(m_{i_{j}}^{\prime} G_{i_{j}}\right)+(-r) e\left(G_{1}\right)$, also a contradiction. Hence, the lemma holds.

Lemma 3.8. If there exists a member in $\mathbb{G}$ which contains a bicyclic subgraph, then all the members are small in $\left\{G_{i}: i=0,1, \ldots, s\right\}$ and $\left\{G_{i_{j}}: j=1, \ldots, t\right\}$.

Proof. Recall that every member in $\mathbb{G}$ is $K_{1, \alpha_{\mathbb{H}}}$-minor free. If some member in $\mathbb{G}$ contains a bicyclic subgraph, then it has an $H_{0}$ minor, where $H_{0} \in\left\{K_{1} \nabla 2 P_{2}, K_{1} \nabla P_{3}\right\}$, and thus a $K_{1,3}$ minor. This implies that $\alpha_{\mathbb{H}} \geq 4$.

Let $\mathbb{G}^{\prime}$ be the subset of $\mathbb{G}$ in which any member is not small. Suppose to the contrary that $\mathbb{G}^{\prime}$ is non-empty and $G_{i_{0}} \in \mathbb{G}^{\prime}$. Since $G_{i_{0}}$ is $K_{1, \alpha_{\mathbb{H}}}$-minor free, by Lemma 2.3 we have $e\left(G_{i_{0}}\right) \leq\binom{\alpha_{\mathbb{H}}}{2}+\left|G_{i_{0}}\right|-\alpha_{\mathbb{H}}$. Thus, $d_{i_{0}}=2 e\left(G_{i_{0}}\right) /\left|G_{i_{0}}\right| \rightarrow 2\left(\left|G_{i_{0}}\right| \rightarrow \infty\right)$, and hence $d_{i_{0}}<2.2$ as $\left|G_{i_{0}}\right|$ is sufficiently large. Now we know that $d_{i}<2.2$ for every $G_{i} \in \mathbb{G}^{\prime}$.

Since a member in $\mathbb{G}$ contains an $H_{0}$ minor, $H_{0}$ is $\Gamma(\mathbb{H})$-minor free. A simple calculation gives that $2 e\left(H_{0}\right) /\left|H_{0}\right| \geq 2.4$. By the definition of $G_{0}$, we have $d_{0} \geq 2 e\left(H_{0}\right) /\left|H_{0}\right| \geq$ 2.4. Since $d_{i}<2.2$ for every $G_{i} \in \mathbb{G}^{\prime}$, it is clear that $G_{0} \notin \mathbb{G}^{\prime}$.

Now suppose that $G_{i_{j}} \in \mathbb{G}^{\prime}$ for some $j \in\{1, \ldots, t\}$. that is, $\left|G_{i_{j}}\right|$ is sufficiently large. Then $d_{i_{j}}<2.2$ and $\left|G_{i_{j}}\right|=a\left|H_{0}\right|+b$, where $0 \leq b<\left|H_{0}\right|$ and $a$ depends on $n$. Hence, $e\left(G_{i_{j}}\right)<1.1\left(a\left|H_{0}\right|+b\right)$. Now define $G_{i_{j}}^{\prime \prime}=a H_{0} \cup b K_{1}$. Then $G_{i_{j}}^{\prime \prime}$ is $\Gamma(\mathbb{H})$-minor free, and $e\left(G_{i_{j}}^{\prime \prime}\right)=a e\left(H_{0}\right) \geq 1.2 a\left|H_{0}\right|$. Thus, $e\left(G_{i_{j}}^{\prime \prime}\right)>e\left(G_{i_{j}}\right)$, contradicting the fact that $G^{\star}-L \in$ $E X\left(n-|L|, \Gamma(\mathbb{H})_{\text {minor }}\right)$. Hence, $G_{i_{j}} \notin \mathbb{G}^{\prime}$ for any $j \in\{1, \ldots, t\}$.

Finally, suppose $G_{i} \in \mathbb{G}^{\prime}$ for some $i \in\{1, \ldots, s\}$. Then $d_{i}<2.2$ and $\left|G_{i}\right|=a^{\prime}\left|H_{0}\right|+b^{\prime}$, where $0 \leq b^{\prime}<\left|H_{0}\right|$ and $a^{\prime}$ depends on $n$. Thus, $e\left(G_{i}\right)<1.1\left(a^{\prime}\left|H_{0}\right|+b^{\prime}\right)$. Now define $G_{i}^{\prime \prime}=a^{\prime} H_{0} \cup b^{\prime} K_{1}$. Then $G_{i}^{\prime \prime}$ is $\Gamma(\mathbb{H})$-minor free, and $e\left(G_{i}^{\prime \prime}\right)=a^{\prime} e\left(H_{0}\right) \geq 1.2 a^{\prime}\left|H_{0}\right|$. It is easy to see that $e\left(G_{i}^{\prime \prime}\right)>1.05 e\left(G_{i}\right)$. Let $G$ be obtained from $G^{*}$ by replacing $E\left(G_{i}\right)$ with $E\left(G_{i}^{\prime \prime}\right)$. By Lemma 3.4, $G$ is $\mathbb{H}$-minor free. Choosing $R=V\left(G_{i}\right)$ in (9), we have

$$
\frac{1}{2}\left(\rho(G)-\rho^{*}\right) \geq \sum_{u v \in E\left(G_{i}^{\prime \prime}\right)} x_{u} x_{v}-\sum_{u v \in E\left(G_{i}\right)} x_{u} x_{v} \geq \frac{e\left(G_{i}^{\prime \prime}\right) \sigma_{L}^{2}}{\rho^{* 2}}-\frac{e\left(G_{i}\right) \sigma_{L}^{2}}{\left(\rho^{*}-\alpha_{\mathbb{H}}+1\right)^{2}}
$$

Combining $e\left(G_{i}^{\prime \prime}\right)>1.05 e\left(G_{i}\right)$ yields that $\rho(G)>\rho^{*}$. This contradicts the fact that $G^{*} \in$ $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$. Therefore, $G_{i} \notin \mathbb{G}^{\prime}$ for any $i \in\{1, \ldots, s\}$.

Given a rooted tree $T$. A branching vertex in $T$ is a vertex of degree at least three. An edge-switching on $T$ means that we construct a new tree $T^{\prime}=T-\left\{u_{1} v_{1}\right\}+\left\{u_{2} v_{2}\right\}$, where $u_{1} v_{1} \in E(T)$ and $u_{2} v_{2}$ is a non-edge in $T$. A pruning on $T$ is an edge-switching $T^{\prime}=T-\left\{u_{1} v_{1}\right\}+\left\{v_{1} v_{2}\right\}$, where $u_{1}$ is a branching vertex, $v_{1}$ is its son and $v_{2}$ is a leaf (which is not a descendant of $v_{1}$ ). We end this section by proving Theorem 1.3.

Proof. Combining Lemmas 3.7 and 3.8, we can see that if all the members in $\mathbb{G}$ are small or some member has a bicyclic subgraph, then $G^{*}-L \in E X\left(n-|L|, \Gamma(\mathbb{H})_{\text {minor }}\right)$ and we are done. Thus we may assume that every member in $\mathbb{G}$ is either a tree or a unicyclic graph, and there exists a member $G_{i_{0}} \in \mathbb{G}$ such that $\left|G_{i_{0}}\right|$ is sufficiently large.

Let $G_{i}$ be an arbitrary member in $\mathbb{G}$ and $V\left(G_{i}\right)=\cup_{k=1}^{3} U_{k}$, where $U_{k}=\left\{v \in V\left(G_{i}\right)\right.$ : $\left.d_{G_{i}}(v)=k\right\}$ for $k \in\{1,2\}$ and $U_{3}=V\left(G_{i}\right) \backslash\left(U_{1} \cup U_{2}\right)$. Recall that every member in $\mathbb{G}$ is $K_{1, \alpha_{\mathbb{H}}}$-minor free. Then $\max \left\{\Delta\left(G_{i}\right),\left|U_{1}\right|\right\}<\alpha_{\mathbb{H}}$. Since $G_{i}$ is a tree or a unicyclic graph, we have $e\left(G_{i}\right) \leq\left|G_{i}\right|=\sum_{i=1}^{3}\left|U_{k}\right|$. Moreover, $e\left(G_{i}\right) \geq \frac{1}{2}\left(\left|U_{1}\right|+2\left|U_{2}\right|+3\left|U_{3}\right|\right)$. Combining the above three inequalities, we can deduce that $\left|U_{3}\right| \leq\left|U_{1}\right|<\alpha_{\mathbb{H}}$.

On the one hand, the inequality $\max \left\{\Delta\left(G_{i}\right),\left|U_{3}\right|\right\}<\alpha_{\mathbb{H}}$ implies that every $G_{i}$ can be transformed to a path or a lollipop graph by at most $\alpha_{\mathbb{H}}{ }^{2}$ steps of pruning. On the other hand, since $\left|U_{1}\right|+\left|U_{3}\right|<2 \alpha_{\mathbb{H}}$ for $G_{i_{0}}$ but $\left|G_{i_{0}}\right|$ is sufficiently large, we can see that $G_{i_{0}}$ contains a path of length large enough. Note that $G_{i_{0}}$ is $\Gamma(\mathbb{H})$-minor free. Thus, $P_{k} \notin \Gamma(\mathbb{H})$ for any positive integer $k$.

Recall that $G^{*}-L$ contains $s$ non-isomorphic components $G_{1}, \ldots, G_{s}$, each of which is a tree or a unicyclic graph. Suppose that $G^{*}-L$ contains at least two tree components, say $G_{1}$ and $G_{2}$. Then for $i \in\{1,2\}, G_{i}$ can be transformed to a path $P_{\left|G_{i}\right|}$ by at most $\alpha_{\mathbb{H}}{ }^{2}$ steps of pruning. This implies that $P_{\left|G_{i}\right|}=G_{i}-E_{i}^{\prime}+E_{i}^{\prime \prime}$, where $E_{i}^{\prime} \subseteq E\left(G_{i}\right), E_{i}^{\prime \prime}$ is set of non-edges of $G_{i}$ and $\left|E_{i}^{\prime}\right|=\left|E_{i}^{\prime \prime}\right| \leq \alpha_{\mathbb{H}^{2}}$. Let $P_{\left|G_{1} \cup G_{2}\right|}$ be a path obtained from $P_{\left|G_{1}\right|} \cup P_{\left|G_{2}\right|}$ by adding an edge. Moreover, denote by $G$ the graph obtained from $G^{*}$ by replacing $E\left(G_{1} \cup G_{2}\right)$ with $E\left(P_{\left|G_{1} \cup G_{2}\right|}\right)$. Since $P_{k} \notin \Gamma(\mathbb{H})$ for any positive integer $k, G-L$ is still $\Gamma(\mathbb{H})$-minor free. However, similarly as (9), we have

$$
\begin{equation*}
\frac{1}{2}\left(\rho(G)-\rho^{*}\right) \geq \sum_{u v \in E(P)} x_{u} x_{v}-\sum_{u v \in E\left(G_{1} \cup G_{2}\right)} x_{u} x_{v} \geq \frac{\left(\left|E_{1}^{\prime \prime} \cup E_{2}^{\prime \prime}\right|+1\right) \sigma_{L}^{2}}{\rho^{* 2}}-\frac{\left|E_{1}^{\prime} \cup E_{2}^{\prime}\right| \sigma_{L}^{2}}{\left(\rho^{*}-\alpha_{\mathbb{H}}+1\right)^{2}} \tag{12}
\end{equation*}
$$

which yields that $\rho(G)>\rho^{*}$, a contradiction. Therefore, $G^{*}-L$ contains at most one tree component. If every component in $G^{*}-L$ is unicyclic, then $e\left(G^{*}-L\right)=\left|G^{*}-L\right|$ and thus $G^{*}-L \in E X\left(n-|L|, \Gamma(\mathbb{H})_{\text {minor }}\right)$, as desired.

Now, we may assume that $G^{*}-L$ has exactly one tree component $G_{1}$. We first consider the case that $\mathbb{G}$ admits a unicyclic member $G_{j_{0}}$. Then $G_{j_{0}}$ contains an $\ell_{0}$-cycle for some $\ell_{0} \leq n-\gamma_{\mathbb{H}}$, and thus $C_{\ell} \notin \Gamma(\mathbb{H})$ for any positive integer $\ell \geq \ell_{0}$. Now if $\left|G_{1}\right| \geq \ell_{0}$, then we define a new $G$ to be the graph obtained from $G^{*}$ by replacing $E\left(G_{1}\right)$ with $E\left(C_{\left|G_{1}\right|}\right)$, where $C_{\left|G_{1}\right|}$ denotes a $\left|G_{1}\right|$-cycle obtained from $P_{\left|G_{1}\right|}$ by adding an edge. Then $G-L$ is still $\Gamma(\mathbb{H})$-minor free. But similarly as (12), we have $\rho(G)>\rho^{*}$, a contradiction. Hence, $\left|G_{1}\right|<\ell_{0} \leq n-\gamma_{\mathbb{H}}$, which implies that $G^{*}-L$ contains another component $G_{2}$ and $G_{2}$ is a unicyclic graph. Recall that $G_{1}$ can be transformed to a path $P_{G_{1} \mid}$ by at most $\alpha_{\mathbb{H}}{ }^{2}$ steps of pruning, and $G_{2}$ can be transformed to a lollipop graph by at most $\alpha_{\mathbb{H}}{ }^{2}$ steps of pruning. Obviously, $G_{1} \cup G_{2}$ can be transformed to a cycle $C_{\left|G_{1} \cup G_{2}\right|}$ by adding an edge and switching at most $2 \alpha_{\mathbb{H}}{ }^{2}+1$ edges. Let $G$ be the graph obtained from $G^{*}$ by replacing $E\left(G_{1} \cup G_{2}\right)$ with $E\left(C_{\left|G_{1} \cup G_{2}\right|}\right)$. We can similarly see that $G-L$ is $\Gamma(\mathbb{H})$-minor free and $\rho(G)>\rho^{*}$, a contradiction. Therefore, every member in $\mathbb{G}$ is a tree. Now $G^{*}-L=G_{1}$ and thus $G^{*}-L \in E X\left(n-|L|, \Gamma(\mathbb{H})_{\text {minor }}\right)$, completing the proof of Theorem 1.3.

## 4 Complete multi-partite minors

In this section, we will use Theorems 1.1, 1.2 and 1.3 to characterize $\operatorname{SPEX}\left(n, \mathbb{H}_{\text {minor }}\right)$ for $\mathbb{H}=\left\{K_{s_{1}, \ldots, s_{r}}\right\}$. Above all, we shall recall some notations. Let $\bar{G}$ be the complement of a graph $G$ and Pet $^{\star}$ be the Petersen graph. Let $H_{s_{1}, s_{2}}=(\beta-1) K_{1, s_{2}} \cup K_{1, s_{2}+\beta_{0}}$, where $\beta\left(s_{2}+1\right)+\beta_{0}=s_{1}+1$ and $0 \leq \beta_{0} \leq s_{2}$. Obviously, $H_{s_{1}, s_{2}}$ is a star forest of order $s_{1}+$ 1. Moreover, $H_{s_{1}, 1} \cong \frac{s_{1}+1}{2} K_{1,1}$ for odd $s_{1}$ and $H_{s_{1}, 1} \cong \frac{s_{1}-2}{2} K_{1,1} \cup K_{1,2}$ for even $s_{1}$. Let $S\left(\overline{H_{s_{1}, s_{2}}}\right)$ denote the graph obtained from $\overline{H_{s_{1}, s_{2}}}$ by subdividing an edge $u v$ with minimum degree sum $d(u)+d(v)$. Particularly, one can observe that $S\left(\overline{H_{4,1}}\right) \cong S^{2}\left(K_{4}\right)$.

For $\left(s_{1}, s_{2}\right) \in\{(2,2),(3,2),(3,3)\}, \operatorname{SPEX}\left(n,\left\{K_{s_{1}, s_{2}}\right\}_{\text {minor }}\right)$ was determined in [41, 42, 52, 55]. In [54], the authors characterized $\operatorname{SPEX}\left(n,\left\{K_{s_{1}, s_{2}}\right\}_{\text {minor }}\right)$ for $s_{1} \geq 4$ and $s_{2} \geq 2$. These results solved a conjecture proposed by Tait [46]. In fact, the above results can be rewritten as a slightly stronger version (see Theorem 4.1).

Theorem 4.1. Assume that $s_{1} \geq s_{2} \geq 2, \beta=\left\lfloor\frac{s_{1}+1}{s_{2}+1}\right\rfloor, \gamma \geq 1$ and $n-\gamma=p s_{1}+q\left(1 \leq q \leq s_{1}\right)$. Let $G$ be the join of a copy of $K_{\gamma}$ and an $(n-\gamma)$-vertex $\Gamma_{s_{1}+1}^{*}\left(K_{s_{1}, s_{2}}\right)$-minor free graph. Then $\rho(G) \leq \rho\left(K_{\gamma} \nabla G^{\mathbf{\Delta}}\right)$, with equality if and only if $G \cong K_{\gamma} \nabla G^{\mathbf{\Delta}}$, where

$$
G^{\mathbf{\Delta}}= \begin{cases}(p-1) K_{s_{1}} \cup S\left(\overline{H_{s_{1}, s_{2}}}\right) & \text { if }(q, \beta)=(2,2) \\ (p-1) K_{s_{1}} \cup \overline{P e t^{\star}} & \text { if }\left(q, \beta, s_{1}\right)=(2,1,8) ; \\ (p-q) K_{s_{1}} \cup q \overline{H_{s_{1}, s_{2}}} & \text { if } q \leq 2(\beta-1) \text { and }(q, \beta) \neq(2,2) \\ p K_{s_{1}} \cup K_{q} & \text { if } q>2(\beta-1) \text { and }\left(q, \beta, s_{1}\right) \neq(2,1,8) .\end{cases}
$$

Now we characterize $\operatorname{SPEX}\left(n,\{H\}_{\text {minor }}\right)$ for $H=K_{s_{1}, \ldots, s_{r}}$, where $s_{1} \geq \cdots \geq s_{r}$ and $n$
is sufficiently large. Clearly, $\alpha_{H}=s_{1}$ and $\Gamma_{\alpha_{H}+1}^{*}(H)=\Gamma_{s_{1}+1}^{*}(H)$. We first consider the case $s_{2} \geq 2$. Now we have $\sum_{2}^{r} s_{i}-1 \geq 1$.

Theorem 4.2. Let $s_{2} \geq 2, \beta=\left\lfloor\frac{s_{1}+1}{s_{2}+1}\right\rfloor, \gamma=\sum_{2}^{r} s_{i}-1$ and $n-\gamma=p s_{1}+q\left(1 \leq q \leq s_{1}\right)$. Then $\operatorname{SPEX}\left(n,\left\{K_{s_{1}, \ldots, s_{r}}\right\}_{\text {minor }}\right)=\left\{K_{\gamma} \nabla G^{\mathbf{\Delta}}\right\}$, where $G^{\mathbf{\Delta}}$ is defined as in Theorem 4.1.

Proof. Let $\cup_{i=1}^{r} S_{i}$ be the $r$-partite partition of $V(H)$, where $\left|S_{i}\right|=s_{i}$ for $i \in\{1, \ldots, r\}$. Given an arbitrary $\left(\alpha_{H}+1\right)$-subset $S$ of $V(H)$. We first claim that there exists an $\left(\alpha_{H}+1\right)$ subset $S^{\prime}$ such that $S^{\prime} \subseteq S_{1} \cup S_{2}$ and $H\left[S^{\prime}\right]$ is isomorphic to a subgraph of $H[S]$.

Let $t_{i}=\left|S_{i} \cap S\right|$ for $i \in\{1, \ldots, r\}$. Up to isomorphism of $H[S]$, one can assume that $t_{1} \geq$ $\cdots \geq t_{r}$. If $t_{3}=0$, then we choose $S^{\prime}=S$, as required. Suppose now that $t_{3} \geq 1$. Then $t_{2} \geq$ 1. Note that $\sum_{i=1}^{r} t_{i}=|S|=s_{1}+1$. Thus, $\sum_{i=3}^{r} t_{i}=s_{1}+1-t_{1}-t_{2} \leq s_{1}-t_{1}=\left|S_{1} \backslash S\right|$. Now let $S^{\prime}$ be obtained from $S$ by replacing $\sum_{i=3}^{r} t_{i}$ vertices in $S \backslash\left(S_{1} \cup S_{2}\right)$ with $\sum_{i=3}^{r} t_{i}$ vertices in $S_{1} \backslash S$. Then $S^{\prime} \subseteq S_{1} \cup S_{2}$, and it is easy to see that $H\left[S^{\prime}\right]$ is isomorphic to a subgraph of $H[S]$. Hence, the claim holds, which further implies $\Gamma_{\alpha_{H}+1}^{*}(H)=\Gamma_{s_{1}+1}\left(K_{s_{1}, s_{2}}\right)$.

Note that $\gamma_{H}=|H|-\alpha_{H}-1=\sum_{2}^{r} s_{i}-1$. By Theorem 1.1, the extremal graph $G^{*}$ has a set $L$ of dominating vertices, where $|L|=\sum_{2}^{r} s_{i}-1$. By Theorem 1.2, $G^{*}-L$ is $\Gamma_{\alpha_{H}+1}^{*}(H)$ minor free, that is, $\Gamma_{s_{1}+1}\left(K_{s_{1}, s_{2}}\right)$-minor free. Now setting $\gamma=\sum_{2}^{r} s_{i}-1$ in Theorem 4.1, we obtain that $\operatorname{SPEX}\left(n,\left\{K_{S_{1}, \ldots, s_{r}}\right\}_{\text {minor }}\right)=\left\{K_{\gamma} \nabla G^{\mathbf{\Delta}}\right\}$ immediately.

Having Theorem 4.2, it remains to characterize $\operatorname{SPEX}\left(n,\{H\}_{\text {minor }}\right)$ for a complete $r$-partite graph $H=K_{s_{1}, 1, \ldots, 1}$. If $s_{1}=1$, then $H \cong K_{r}$ and $\operatorname{SPEX}\left(n,\left\{K_{r}\right\}_{\text {minor }}\right)$ was determined by Tait (see Theorem 1.4). If $r=2$, then $H \cong K_{1, s_{1}}$ and $\operatorname{SPEX}\left(n,\left\{K_{1, s_{1}}\right\}_{\text {minor }}\right.$ ) was determined in [54]. In the following, we may assume that $\min \left\{r-1, s_{1}\right\} \geq 2$. In fact, we have $H \cong B_{r-1, s_{1}}$. Moreover, $\alpha_{H}=s_{1}, \gamma_{H}=|H|-\alpha_{H}-1=r-2$ and

$$
\begin{equation*}
\Gamma_{\alpha_{H}+1}^{*}(H)=\Gamma_{s_{1}+1}^{*}\left(B_{r-1, s_{1}}\right)=\left\{K_{1, s_{1}}\right\} . \tag{13}
\end{equation*}
$$

Assume that $G^{*} \in \operatorname{SPEX}\left(n,\left\{B_{r-1, s_{1}}\right\}_{\text {minor }}\right)$, and $X^{*}=\left(x_{1}, \ldots, x_{n}\right)^{T}$ is the positive unit eigenvector corresponding to $\rho^{*}:=\rho\left(G^{*}\right)$. Let $L$ be the set of dominating vertices in $G^{*}$. Then $|L|=\gamma_{H}=r-2$ by Theorem 1.1. In the following, we set $\gamma:=|L|=r-2$.

Lemma 4.1. Assume that $s_{1} \geq 2$ and $\gamma \geq 1$. Then we have

$$
\begin{equation*}
\rho^{* 2}-\left(s_{1}+\gamma-2\right) \rho^{*} \leq \gamma(n-\gamma)-(\gamma-1)\left(s_{1}-1\right) \tag{14}
\end{equation*}
$$

with equality if and only if $G^{*}-L$ is an $\left(s_{1}-1\right)$-regular $K_{1, s_{1}-\text { minor free graph. }}^{\text {g }}$.
Proof. By Theorem 1.2 and (13), we can see that $G^{*}-L$ is a $K_{1, s_{1}}$-minor saturated graph. Hence, $G^{*}-L$ is $K_{1, s_{1}}$-minor free.

By symmetry, $x_{u}$ is constant for $u \in L$. Choose $u^{*} \in L$ and $v^{*} \in V\left(G^{*}\right) \backslash L$ with $x_{v^{*}}=$ $\max _{v \in V\left(G^{*}\right) \backslash L} x_{v}$. Since $G^{*}-L$ is $K_{1, s_{1}}$-minor free, we have $\Delta\left(G^{*}-L\right) \leq s_{1}-1$. Note that $\rho^{*} x_{u}=\sum_{v \in N_{G^{*}}(u)} x_{v}$ for each $u \in V\left(G^{*}\right)$. Thus,

$$
\begin{equation*}
\rho^{*} x_{v^{*}} \leq \gamma x_{u^{*}}+\left(s_{1}-1\right) x_{v^{*}} \text { and } \rho^{*} x_{u^{*}} \leq(\gamma-1) x_{u^{*}}+(n-\gamma) x_{v^{*}} . \tag{15}
\end{equation*}
$$

Combining the two inequalities in (15), we obtain (14) immediately.
Next, we characterize equality case in (14). If the equality holds, then both inequalities in (15) become equalities. Hence, $d_{G^{*}-L}(v)=s_{1}-1$ for each $v \in V\left(G^{*}\right) \backslash L$, that is, $G^{*}-L$ is $\left(s_{1}-1\right)$-regular. Conversely, if $G^{*}-L$ is $\left(s_{1}-1\right)$-regular, then $G^{*}$ is the join of two regular graphs, which implies that both $X^{*} \|_{L}$ and $X^{*} \|_{V\left(G^{*}\right) \backslash L}$ are constant vectors. Hence, both inequalities of (15) hold in equality, and thus (14) too.

Given an arbitrary $v \in V\left(G^{*}\right) \backslash L$. Then $\rho^{*} x_{v}=\gamma x_{u^{*}}+\sum_{w \in N_{G^{*}}(v) \backslash L} x_{w}$. Clearly, $\rho^{*} x_{v} \geq$ $\gamma x_{u^{*}}$, and by (15) we have $\left(\rho^{*}-s_{1}+1\right) x_{v} \leq \gamma x_{u^{*}}$. Hence, $\frac{\gamma}{\rho^{*}} x_{u^{*}} \leq x_{v} \leq \frac{\gamma}{\rho^{*}-s_{1}+1} x_{u^{*}}$. Recall that $\Delta\left(G^{*}-L\right) \leq s_{1}-1$. We can further deduce that

$$
\begin{equation*}
x_{u}<x_{v} \tag{16}
\end{equation*}
$$

for any two vertices $u, v \in V\left(G^{*}\right) \backslash L$ with $d_{G^{*}}(u)<d_{G^{*}}(v)$.
Lemma 4.2. Assume that $s_{1} \geq 4$ and $G$ is a connected $K_{1, s_{1}}$-minor free graph. If $G$ is ( $s_{1}-1$ )-regular, then either $G \cong K_{s_{1}}$, or $G \cong \overline{H_{s_{1}, 1}}$ only for odd $s_{1}$.

Proof. We shall first note that both $K_{s_{1}}$ and $\overline{H_{s_{1}, 1}}$ are $K_{1, s_{1}}$-minor free. Indeed, recall that $H_{s_{1}, 1} \cong \frac{s_{1}+1}{2} K_{1,1}$ for odd $s_{1}$ and $H_{s_{1}, 1} \cong \frac{s_{1}-2}{2} K_{1,1} \cup K_{1,2}$ for even $s_{1}$. Thus, $\left|H_{s_{1}, 1}\right|=s_{1}+1$ and $\Delta\left(\overline{H_{s_{1}, 1}}\right)=s_{1}-1$. Therefore, $\overline{H_{s_{1}, 1}}$ is $K_{1, s_{1}}$-minor free, and $K_{s_{1}}$ is obviously too.

Since $G$ is $\left(s_{1}-1\right)$-regular, we have $|G| \geq s_{1}$. If $|G|=s_{1}$, then $G \cong K_{s_{1}}$. If $|G|=s_{1}+1$, then $G$ can only be obtained from $K_{S_{1}+1}$ by deleting a perfect matching, which implies that $s_{1}$ is odd and $\bar{G} \cong \frac{s_{1}+1}{2} K_{1,1} \cong H_{s_{1}, 1}$. Consequently, $G \cong \overline{H_{s_{1}, 1}}$.

Next assume that $|G| \geq s_{1}+2$. Since $G$ is connected and $K_{1, s_{1}}$-minor free, by Lemma 2.3 we have $e(G) \leq\binom{ s_{1}}{2}+|G|-s_{1}$, which implies that $e(G)<\frac{1}{2}\left(s_{1}-1\right)|G|$ for $s_{1} \geq 4$. Thus, $G$ is not ( $s_{1}-1$ )-regular, a contradiction. Hence, the lemma holds.

The following theorem follows immediately from Lemmas 4.1 and 4.2.
Theorem 4.3. Let $s_{1} \geq 3$ be odd and $\gamma \geq 1$. Then $\operatorname{SPEX}\left(n,\left\{B_{r-1, s_{1}}\right\}_{\text {minor }}\right)=\left\{K_{\gamma} \nabla G^{\mathbf{V}}\right\}$, where $G^{\mathbf{V}}$ takes over all the $(n-\gamma)$-vertex $\left(s_{1}-1\right)$-regular $K_{1, s_{1}}$-minor free graphs, more precisely, every component of $G^{\mathbf{V}}$ is a cycle for $s_{1}=3$, and is $K_{s_{1}}$ or $\overline{H_{s_{1}, 1}}$ for $s_{1} \geq 5$.

Remark 4.1. In view of Theorem 4.3, $\operatorname{SPEX}\left(n,\left\{B_{r-1, s_{1}}\right\}_{\text {minor }}\right)$ is an infinite family for odd $s_{1} \geq 3$. Indeed, assume that $s_{1} \geq 5$ and $n-\gamma=p s_{1}+q\left(1 \leq q \leq s_{1}\right)$, then $G^{\mathbf{V}}$ can be constructed as the disjoint union of $p-c-\left(c s_{1}+q\right)$ copies of $K_{s_{1}}$ and $c s_{1}+q$ copies of $\overline{H_{s_{1}, 1}}$ for an arbitrary non-negative integer $c$.

Next, we consider the case that $s_{1}$ is even. Let $\mathbb{G}_{i}$ denote the family of $i$-vertex components in $G^{*}-L$, and $\left|\mathbb{G}_{i}\right|$ be the number of components in $\mathbb{G}_{i}$.

Lemma 4.3. Assume that $s_{1} \geq 4$ is even and $\gamma \geq 1$. If $\mathbb{G}_{i} \neq \varnothing$, then we have $s_{1}-1 \leq i \leq$ $s_{1}+3$, where $i \in\left\{s_{1}+2, s_{1}+3\right\}$ only for $s_{1}=4$.

Proof. We first claim that $\mathbb{G}_{i}=\varnothing$ for each $i \geq 3 s_{1}$. Indeed, otherwise, $G^{*}-L$ contains a component $G_{0}$ with $\left|G_{0}\right|=a s_{1}+b$, where $a \geq 3$ and $0 \leq b<s_{1}$. Then $e\left(G_{0}\right) \leq\binom{ s_{1}}{2}+$ $\left|G_{0}\right|-s_{1}<\binom{s_{1}}{2}+a s_{1}$ by Lemma 2.3. Now let $G_{0}^{\prime}=a K_{s_{1}} \cup b K_{1}$. Clearly, $G_{0}^{\prime}$ is $K_{1, s_{1}}$-minor free and $e\left(G_{0}^{\prime}\right)=a\binom{s_{1}}{2}$. A straightforward calculation gives that $e\left(G_{0}\right)<e\left(G_{0}^{\prime}\right)$ for $a \geq 3$ and $s_{1} \geq 4$, which contradicts Theorem 1.3.

Secondly, we claim that $\left|\mathbb{G}_{i}\right| \leq s_{1}-1$ for $i \neq s_{1}$. Otherwise, there exists some $i_{1} \neq s_{1}$ with $\left|\mathbb{G}_{i_{1}}\right| \geq s_{1}$. Let $G_{1}$ be a union of $s_{1}$ components in $\mathbb{G}_{i_{1}}$. Then $\Delta\left(G_{1}\right) \leq s_{1}-1$ as $G_{1}$ is $K_{1, s_{1}}$-minor free. Moreover, $G_{1}$ is not $\left(s_{1}-1\right)$-regular (otherwise, $G_{1} \cong s_{1} K_{s_{1}}$ by Lemma 4.2). Thus, $e\left(G_{1}\right)<\frac{1}{2}\left(s_{1}-1\right)\left|G_{1}\right|=e\left(i_{1} K_{s_{1}}\right)$, also contradicting Theorem 1.3.

The above claims implies that $\sum_{i \neq s_{1}} i\left|\mathbb{G}_{i}\right|$ is constant. Thus $s_{1}\left|\mathbb{G}_{s_{1}}\right|=n-\gamma-\sum_{i \neq s_{1}} i\left|\mathbb{G}_{i}\right|$ $\geq \frac{n}{2 s_{1}}$. For an arbitrary $i_{2} \in\left\{1,2, \ldots, 3 s_{1}-1\right\} \backslash\left\{s_{1}, s_{1}+1\right\}$, we set $i_{2}=a s_{1}+b$, where $0 \leq a \leq 2$ and $0 \leq b<s_{1}$. If $\mathbb{G}_{i_{2}} \neq \varnothing$, we choose a subgraph $G_{2}$ in $G^{*}-L$, which consists of $b$ components in $\mathbb{G}_{s_{1}}$ and one in $\mathbb{G}_{i_{2}}$. Then $e\left(G_{2}\right) \leq b \cdot e\left(K_{s_{1}}\right)+e\left(K_{b}\right)$ for $a=0$, and by Lemma 2.3, e $\left(G_{2}\right) \leq b \cdot e\left(K_{s_{1}}\right)+\binom{s_{1}}{2}+\left(i_{2}-s_{1}\right)$ for $a \in\{1,2\}$. Now let $G_{2}^{\prime}=a K_{S_{1}} \cup b \overline{H_{s_{1}, 1}}$. Then $\left|G_{2}^{\prime}\right|=\left|G_{2}\right|$ and $e\left(G_{2}^{\prime}\right)=a \cdot e\left(K_{s_{1}}\right)+b \cdot \frac{1}{2}\left(s_{1}^{2}-2\right)$. Straightforward calculations give $e\left(G_{2}\right) \leq e\left(G_{2}^{\prime}\right)$, with equality if and only if $i_{2}=s_{1}-1$ or $i_{2} \in\left\{s_{1}+2, s_{1}+3: s_{1}=4\right\}$.

On the other hand, recall that both $K_{s_{1}}$ and $\overline{H_{s_{1}, 1}}$ are $K_{1, s_{1}}$-minor free, then $G_{2}^{\prime}$ is too. By Theorem 1.3, $e\left(G_{2}\right) \geq e\left(G_{2}^{\prime}\right)$. Hence, $e\left(G_{2}\right)=e\left(G_{2}^{\prime}\right)$. Consequently, $i_{2}=s_{1}-1$ or $i_{2} \in\left\{s_{1}+2, s_{1}+3: s_{1}=4\right\}$. In view of the choice of $i_{2}$, we completes the proof.

Lemma 4.4. Assume that $s_{1} \geq 4$ is even and $\gamma \geq 1$. Then $\left|\mathbb{G}_{s_{1}-1}\right| \leq 1$ and $\left|\mathbb{G}_{s_{1}+1}\right| \leq s_{1}-2$. Moreover, if $\left|\mathbb{G}_{s_{1}-1}\right|=1$, then $\mathbb{G}_{i}=\varnothing$ unless $i \in\left\{s_{1}-1, s\right\}$.

Proof. We first assume that there exists a $G_{0} \in \mathbb{G}_{s_{1}-1}$. By Theorem 1.2, $G^{*}-L$ is $K_{1, s_{1}}$ minor saturated. Hence, $G_{0} \cong K_{s_{1}-1}$. Choose another component $G_{1}$ arbitrarily in $G^{*}-L$. We now claim that $\left|G_{1}\right|=s_{1}$. Indeed, otherwise, $\left|G_{1}\right| \neq s_{1}$. Then by Lemma 4.2, $G_{1}$ is not $\left(s_{1}-1\right)$-regular, and thus there exists $v \in V\left(G_{1}\right)$ with $d_{G_{1}}(v) \leq s_{1}-2$. Let $G$ be the graph obtained from $G^{*}$ by replacing $G_{0} \cup G_{1}$ with $K_{S_{1}} \cup\left(G_{1}-\{v\}\right)$. Since $K_{S_{1}}$ and $G_{1}-\{v\}$ are $K_{1, s_{1}}$-minor free, $G$ is $B_{r-1, s_{1}}$-minor free by Lemma 3.4. Note that $e\left(K_{s_{1}}\right)-e\left(G_{0}\right)=$ $s_{1}-1$ but $e\left(G_{1}\right)-e\left(G_{1}-\{v\}\right) \leq s_{1}-2$. Then, $e(G)>e\left(G^{*}\right)$, contradicting Theorem 1.3. Therefore, $\left|G_{1}\right|=s_{1}$, as claimed.

Now we know that if there exists $G_{0} \in \mathbb{G}_{s_{1}-1}$, then every component in $G^{*}-L$ other than $G_{0}$ can only belong to $\mathbb{G}_{s_{1}}$. This implies that $\left|\mathbb{G}_{s_{1}-1}\right| \leq 1$, and if $\left|\mathbb{G}_{s_{1}-1}\right|=1$, then $\mathbb{G}_{i}=\varnothing$ unless $i \in\left\{s_{1}-1, s\right\}$.

It remains to show $\left|\mathbb{G}_{s_{1}+1}\right| \leq s_{1}-2$. Suppose to the contrary that $\left|\mathbb{G}_{s_{1}+1}\right| \geq s_{1}-1$, and let $G_{2}$ be a component in $\mathbb{G}_{s_{1}+1}$. By Theorem 1.3, $e\left(G_{2}\right)=e x\left(s_{1}+1,\left\{K_{1, s_{1}}\right\}_{\text {minor }}\right)$. Thus, $G_{2}$ can only be the complement of $\frac{s_{1}-2}{2} K_{1,1} \cup K_{1,2}$, that is, $G_{2} \cong \overline{H_{s_{1}, 1}}$. Now let $G$ be the graph obtained from $G^{*}$ by replacing $\left(s_{1}-1\right) G_{2}$ with $\left(s_{1}-1\right) K_{s_{1}} \cup K_{s_{1}-1}$. Then, $G$ is $B_{r-1, s_{1}}$-minor free by Lemma 3.4.

Since $\overline{G_{2}} \cong \frac{s_{1}-2}{2} K_{2} \cup K_{1,2}$, we may assume that $V\left(G_{2}\right)=\left\{v_{0}, v_{1}, \ldots, v_{s_{1}}\right\}$ such that $\left\{v_{0} v_{1}, v_{0} v_{2}\right\} \cup\left\{v_{3} v_{4}, v_{5} v_{6}, \ldots, v_{s_{1}-1} v_{s_{1}}\right\}$ is the set of non-edges in $G_{2}$. Then by symmetry, $x_{v_{i}}=x_{v_{3}}$ for each $i \in\left\{3, \ldots, s_{1}\right\}$. Furthermore, since $d_{G_{2}}\left(v_{0}\right)=s_{1}-2$ and $d_{G_{2}}\left(v_{3}\right)=s_{1}-1$, we have $x_{v_{0}}<x_{v_{3}}$ by (16).

Observe that $\left(s_{1}-1\right) K_{s_{1}} \cup K_{s_{1}-1}$ can be obtained from $\left(s_{1}-1\right) G_{2}$ by replacing the edge set $\left\{v_{0} v_{i}: i=3, \ldots, s_{1}\right\}$ with $\left\{v_{3} v_{4}, v_{5} v_{6}, \ldots, v_{s_{1}-1} v_{s_{1}}\right\}$ in every copy of $G_{2}$ and then forming $s_{1}-1$ copies of $v_{0}$ into a copy of $K_{s_{1}-1}$. Thus, $e(G)=e\left(G^{*}\right)$ and

$$
\begin{aligned}
\sum_{u v \in E(G)} 2 x_{u} x_{v}-\sum_{u v \in E\left(G^{*}\right)} 2 x_{u} x_{v} & =2 e\left(K_{s_{1}-1}\right) x_{v_{0}}^{2}+2(s-1)\left(\sum_{i=2}^{s_{1} / 2} x_{v_{2 i-1}} x_{v_{2 i}}-\sum_{i=3}^{s_{1}} x_{v_{0}} x_{v_{i}}\right) \\
& =\left(s_{1}-1\right)\left(s_{1}-2\right)\left(x_{v_{0}}^{2}+x_{v_{3}}^{2}-2 x_{v_{0}} x_{v_{3}}\right) .
\end{aligned}
$$

Since $x_{v_{0}}<x_{v_{3}}$, we have $\rho(G)>\rho^{*}$, a contradiction. Hence, $\left|\mathbb{G}_{s_{1}+1}\right| \leq s_{1}-2$.
Lemma 4.5. Assume that $s_{1}=4$ and $\gamma \geq 1$. Then $\mathbb{G}_{s_{1}+3}=\varnothing$ and $\sum_{i \in\{-1,1,2\}}\left|\mathbb{G}_{s_{1}+i}\right| \leq 1$.
Proof. Let $G_{0}$ be an arbitrary component in $G^{*}-L$. Then $\left|G_{0}\right| \leq s_{1}+3$ by Lemma 4.3. Furthermore, by Theorem 1.3, e( $\left.G_{0}\right)=e x\left(\left|G_{0}\right|,\left\{K_{1, s_{1}}\right\}_{\text {minor }}\right)$. Now $s_{1}=4$. By Lemma 2.4, every member in $\mathbb{G}_{s_{1}+i}$ is isomorphic to $S^{i}\left(K_{S_{1}}\right)$ for $i \in\{1,2,3\}$.

We first show $\mathbb{G}_{s_{1}+3}=\varnothing$. Suppose to contrary that there exists $G_{0} \in \mathbb{G}_{s_{1}+3}$. Then $G_{0} \cong$ $S^{3}\left(K_{4}\right)$, that is, $G_{0}$ is obtained from $K_{4}$ by replacing an edge $v_{1} v_{2}$ with a path $v_{1} w_{1} w_{2} w_{3} v_{2}$. Now let $G_{1}=G_{0}-\left\{v_{1} w_{1}, v_{2} w_{3}\right\}+\left\{v_{1} v_{2}, w_{1} w_{3}\right\}$. Then $G_{1} \cong K_{4} \cup K_{3}$, and obviously $G_{1}$ is $K_{1,4}$-minor free. Define $G$ to be the graph obtained from $G^{*}$ by replacing $G_{0}$ with $G_{1}$. Then $G$ is $B_{r-1, s_{1}}$-minor free by Lemma 3.4. Moreover,

$$
\rho(G)-\rho^{*} \geq 2\left(x_{v_{1}} x_{v_{2}}+x_{w_{1}} x_{w_{3}}\right)-2\left(x_{v_{1}} x_{w_{1}}+x_{v_{2}} x_{w_{3}}\right) .
$$

By symmetry, we have $x_{v_{1}}=x_{v_{2}}$ and $x_{w_{1}}=x_{w_{3}}$. Thus $\rho(G)-\rho^{*} \geq 2\left(x_{v_{1}}^{2}+x_{w_{1}}^{2}-2 x_{v_{1}} x_{w_{1}}\right)$. Note that $d_{G_{0}}\left(w_{1}\right)=2$ and $d_{G_{0}}\left(v_{1}\right)=3$. By (16) we obtain $x_{w_{1}}<x_{v_{1}}$, and hence $\rho(G)>$ $\rho^{*}$, a contradiction. Thus, $\mathbb{G}_{s_{1}+3}=\varnothing$.

Secondly, we claim that $\left|\mathbb{G}_{s_{1}+i}\right| \leq 1$ for $i \in\{1,2\}$. Indeed, if $\left|\mathbb{G}_{s_{1}+2}\right| \geq 2$, then we replace two copies of $S^{2}\left(K_{4}\right)$ in $\mathbb{G}_{s_{1}+2}$ with three copies of $K_{4}$. Now, $2 e\left(S^{2}\left(K_{4}\right)\right)=16<$ $3 e\left(K_{4}\right)$, contradicting Theorem 1.3. If $\left|\mathbb{G}_{s_{1}+1}\right| \geq 2$, we choose $G_{0}, G_{1} \in \mathbb{G}_{s_{1}+1}$. For $j \in$ $\{0,1\}, G_{j} \cong S^{1}\left(K_{4}\right)$ and thus $G_{j}$ is obtained from $K_{4}$ by replacing an edge $u_{j} w_{j}$ with a path $u_{j} v_{j} w_{j}$. Now let $G_{2}=\left(G_{0} \cup G_{1}\right)-\left\{u_{1} v_{1}, v_{1} w_{1}, u_{0} v_{0}\right\}+\left\{u_{1} w_{1}, u_{0} v_{1}, v_{1} v_{0}\right\}$. Then $G_{2} \cong K_{4} \cup S^{2}\left(K_{4}\right)$, and $G_{2}$ is clearly $K_{1,4}$-minor free. Define $G$ to be the graph obtained from $G^{*}$ by replacing $G_{0} \cup G_{1}$ with $G_{2}$. Then $G$ is $B_{r-1, s_{1}}$-minor free and

$$
\rho(G)-\rho^{*} \geq 2\left(x_{u_{1}} x_{w_{1}}+x_{u_{0}} x_{v_{1}}+x_{v_{1}} x_{v_{0}}\right)-2\left(x_{u_{1}} x_{v_{1}}+x_{v_{1}} x_{w_{1}}+x_{u_{0}} x_{v_{0}}\right) .
$$

By symmetry, $x_{v_{0}}=x_{v_{1}}, x_{u_{0}}=x_{u_{1}}=x_{w_{0}}=x_{w_{1}}$, and thus $\rho(G)-\rho^{*} \geq 2\left(x_{u_{0}}^{2}+x_{v_{0}}^{2}-\right.$ $\left.2 x_{u_{0}} x_{v_{0}}\right)$. Since $d_{G_{0}}\left(v_{0}\right)=2$ and $d_{G_{0}}\left(u_{0}\right)=3$, By (16) we similarly have $x_{v_{0}}<x_{u_{0}}$ and $\rho(G)>\rho^{*}$, a contradiction. Therefore, the claim holds.

Now we are ready to prove $\sum_{i \in\{-1,1,2\}}\left|\mathbb{G}_{s_{1}+i}\right| \leq 1$. If $\mathbb{G}_{s_{1}-1} \neq \varnothing$, then we are done by Lemma 4.4. Next, assume that $\mathbb{G}_{s_{1}-1}=\varnothing$. It suffices to show $\sum_{i=1}^{2}\left|\mathbb{G}_{s_{1}+i}\right| \leq 1$. Suppose to the contrary, then by the above claim, there simultaneously exist $G_{1} \in \mathbb{G}_{s_{1}+1}$ and $G_{2} \in$ $\mathbb{G}_{s_{1}+2}$, where $G_{i} \cong S^{i}\left(K_{4}\right)$ for $i \in\{1,2\}$. Let $G_{3}=\left(G_{1} \cup G_{2}\right)-\left\{u_{1} u_{2}, u_{2} u_{3}, v_{1} v_{2}, v_{3} v_{4}\right\}+$ $\left\{u_{1} u_{3}, v_{1} v_{4}, u_{2} v_{2}, u_{2} v_{3}\right\}$, where $u_{1} u_{2} u_{3}$ is the induced path of length two in $G_{1}$ and $v_{1} v_{2} v_{3} v_{4}$
is the induced path of length three in $G_{2}$. Then $G_{3} \cong 2 K_{4} \cup K_{3}$, and thus it is $K_{1,4}$-minor free. Define $G$ to be the graph obtained from $G^{*}$ by replacing $G_{1} \cup G_{2}$ with $G_{3}$. Then $G$ is $B_{r-1, s_{1}}$-minor free. By symmetry, $x_{u_{1}}=x_{u_{3}}, x_{v_{1}}=x_{v_{4}}$ and $x_{v_{2}}=x_{v_{3}}$. Thus,

$$
\begin{aligned}
\rho(G)-\rho^{*} & \geq 2\left(x_{u_{1}} x_{u_{3}}+x_{v_{1}} x_{v_{4}}+x_{u_{2}} x_{v_{2}}+x_{u_{2}} x_{v_{3}}\right)-2\left(x_{u_{1}} x_{u_{2}}+x_{u_{2}} x_{u_{3}}+x_{v_{1}} x_{v_{2}}+x_{v_{3}} x_{v_{4}}\right) \\
& =2\left(x_{u_{1}}^{2}+x_{v_{1}}^{2}+2 x_{u_{2}} x_{v_{2}}\right)-2\left(2 x_{u_{1}} x_{u_{2}}+2 x_{v_{1}} x_{v_{2}}\right) \\
& =2\left(x_{u_{1}}-x_{v_{1}}\right)^{2}+4\left(x_{u_{1}}-x_{v_{2}}\right)\left(x_{v_{1}}-x_{u_{2}}\right) .
\end{aligned}
$$

Note that $d_{G_{1}}\left(u_{2}\right)=d_{G_{2}}\left(v_{2}\right)=2$ and $d_{G_{1}}\left(u_{1}\right)=d_{G_{2}}\left(v_{1}\right)=3$. In view of (16), we get that $\max \left\{x_{u_{2}}, x_{v_{2}}\right\}<\min \left\{x_{u_{1}}, x_{v_{1}}\right\}$ and hence $\rho(G)>\rho^{*}$, a contradiction.

This completes the proof.
We now determine $\operatorname{SPEX}\left(n,\left\{B_{r-1, s_{1}}\right\}_{\text {minor }}\right)$ for even $s_{1}$.
Theorem 4.4. Let $s_{1} \geq 2$ be even, $\gamma \geq 1$, and $n-\gamma=p s_{1}+q$ (where $1 \leq q \leq s_{1}$ ). Then $\operatorname{SPEX}\left(n,\left\{B_{r-1, s_{1}}\right\}_{\text {minor }}\right)=\left\{K_{\gamma} \nabla G^{\Delta}\right\}$, where

$$
G^{\Delta}= \begin{cases}(p-1) K_{s_{1}} \cup S\left(\overline{H_{s_{1}, 1}}\right) & \text { if }\left(q, s_{1}\right)=(2,4) ; \\ (p-q) K_{s_{1}} \cup q \overline{H_{s_{1}, 1}} & \text { if } q \leq s_{1}-2 \text { and }\left(q, s_{1}\right) \neq(2,4) ; \\ p K_{s_{1}} \cup K_{q} & \text { if } q>s_{1}-2 .\end{cases}
$$

Proof. By Theorem 1.1, $G^{*}$ has a set $L$ of $\gamma$ dominating vertices, where $\gamma=r-2$. Furthermore, by Theorem 1.2 we know that $G^{*}-L$ is $K_{1, s_{1}}$-minor saturated.

Let $G_{0}$ be a component in $G^{*}-L$. By Theorem 1.3, $e\left(G_{0}\right)=e x\left(\left|G_{0}\right|,\left\{K_{1, s_{1}}\right\}_{\text {minor }}\right)$. Therefore, $G_{0} \cong K_{\left|G_{0}\right|}$ if $\left|G_{0}\right| \in\left\{s_{1}-1, s_{1}\right\}$, and $G_{0} \cong \overline{H_{s_{1}, 1}}$ if $\left|G_{0}\right|=s_{1}+1$. If $\left|G_{0}\right|=s_{1}+$ $2=6$, then $G_{0} \cong S^{2}\left(K_{4}\right)$ by Lemma 2.4, and one can further see that $S^{2}\left(K_{s_{1}}\right) \cong S\left(\overline{H_{s_{1}, 1}}\right)$ for $s_{1}=4$. In the following, we distinguish the proof into three cases.

If $s_{1}=2$, then $q \in\{1,2\}=\left\{s_{1}-1, s_{1}\right\}$ and $G^{*}-L$ is $K_{1,2}$-minor saturated. It is easy to see that $G^{*}-L \cong p K_{s_{1}} \cup K_{q}$, as desired.

If $s_{1}=4$, then $1 \leq q \leq 4$, and by Lemmas 4.3 and $4.5,\left|G_{0}\right| \in\left\{s_{1}+i:-1 \leq i \leq 2\right\}$. Lemma 4.5 also gives that $\sum_{i \in\{-1,1,2\}}\left|\mathbb{G}_{s_{1}+i}\right| \leq 1$. Consequently, if $q=1=s_{1}-3$, then $\left|G_{0}\right| \in\left\{s_{1}, s_{1}+1\right\}$, and $G_{0} \in\left\{K_{s_{1}}, \overline{H_{s_{1}, 1}}\right\}$ as stated above. Now $G^{*}-L \cong(p-1) K_{s_{1}} \cup$ $\overline{H_{s_{1}, 1}}$. If $q=2=s_{1}-2$, then $\left|G_{0}\right| \in\left\{s_{1}, s_{1}+2\right\}$, and $G_{0} \in\left\{K_{s_{1}}, S\left(\overline{H_{s_{1}, 1}}\right)\right\}$ as discussed above. Hence, $G^{*}-L \cong(p-1) K_{s_{1}} \cup S\left(\overline{H_{s_{1}, 1}}\right)$. If $q \in\{3,4\}=\left\{s_{1}-1, s_{1}\right\}$, then $\left|G_{0}\right| \in$ $\left\{s_{1}-1, s_{1}\right\}$, and thus $G_{0} \cong K_{\left|G_{0}\right|}$. It follows that $G^{*}-L \cong p K_{s_{1}} \cup K_{q}$.

If $s_{1} \geq 6$, then by Lemma 4.3, we have $\left|G_{0}\right| \in\left\{s_{1}+i:-1 \leq i \leq 1\right\}$. From Lemma 4.4, we know that $\left|\mathbb{G}_{s_{1}-1}\right| \leq 1,\left|\mathbb{G}_{s_{1}+1}\right| \leq s_{1}-2$, and $\mathbb{G}_{s_{1}+1}=\varnothing$ provided that $\left|\mathbb{G}_{s_{1}-1}\right|=1$. Thus, if $q \in\left\{s_{1}-1, s_{1}\right\}$, then $\left|G_{0}\right| \in\left\{s_{1}-1, s_{1}\right\}$ and so $G_{0} \cong K_{\left|G_{0}\right|}$, which implies that $G^{*}-L \cong p K_{s_{1}} \cup K_{q}$. If $q \leq s_{1}-2$, then $\left|G_{0}\right| \in\left\{s_{1}, s_{1}+1\right\}$ and thus $G_{0} \in\left\{K_{s_{1}}, \overline{H_{s_{1}}, 1}\right\}$, which implies that $G^{*}-L \cong(p-q) K_{s_{1}} \cup q \overline{H_{s_{1}, 1}}$.

This completes the proof.
We end this section with the proof of Theorem 1.8.

Proof. Let $H=K_{s_{1}, s_{2}, \ldots, s_{r}}$, where $s_{1} \geq s_{2} \geq \cdots \geq s_{r} \geq 1$, and $\gamma=\sum_{i=2}^{r} s_{i}-1 \geq 1$. Let $\beta=\left\lfloor\frac{s_{1}+1}{s_{2}+1}\right\rfloor$. If $s_{2} \geq 2$, then Theorem 1.8 holds by Theorem 4.2. If $s_{2}=1$ and $s_{1}$ is odd, then Theorem 1.8 holds by Theorem 4.3. If $s_{2}=1$ and $s_{1}$ is even, then $\beta=\left\lfloor\frac{s_{1}+1}{s_{2}+1}\right\rfloor=\frac{s_{1}}{2}$, and so $2(\beta-1)=s_{1}-2$. Moreover, the case $\left(q, \beta, s_{1}\right)=(2,1,8)$ never occurs. Hence, $G^{\Delta}=G^{\mathbf{\Delta}}$, and Theorem 1.8 follows from Theorem 4.4. This completes the proof.

## 5 Proof of Theorem 2.1

Let $G$ be a graph of order $n$ sufficiently large, and $X$ be a non-negative eigenvector corresponding to $\rho(G)$. Choose $u^{*} \in V(G)$ with $x_{u^{*}}=\max _{u \in V(G)} x_{u}$, and assume that $G$ is an $\mathbb{H}$-minor free graph with $\rho(G) \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}$. To prove Theorem 2.1, it suffices to find a set $L$ of exactly $\gamma_{H \mathbb{H}}$ vertices in $G$ such that $x_{u} \geq\left(1-\frac{1}{2\left(10 C_{H}\right)^{2}}\right) x_{u^{*}}$ and $d_{G}(u) \geq$ $\left(1-\frac{1}{\left(10 C_{H I}\right)^{2}}\right) n$ for every $u \in L$. To this end, we define a subset of $V(G)$ as follows:

$$
L^{\lambda}=\left\{u \in V(G): x_{u} \geq\left(10 C_{\mathbb{H}}\right)^{-\lambda} x_{u^{*}}\right\},
$$

where $\lambda$ is a positive constant. We now establish some lemmas on $L^{\lambda}$.
Lemma 5.1. $\left|L^{\lambda}\right|<\left(10 C_{\mathbb{H}}\right)^{\lambda-10} n$.
Proof. Given an arbitrary $u \in L^{\lambda}$. Then $\rho(G) x_{u} \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right)}\left(10 C_{\mathbb{H}}\right)^{-\lambda} x_{u^{*}}$, and thus $\rho(G) x_{u}>2 C_{\mathbb{H}}\left(10 C_{\mathbb{H}}\right)^{10-\lambda} x_{u^{*}}$ for $n$ large enough. On the other hand, $\rho(G) x_{u}=\sum_{v \in N_{1}(u)} x_{v}$ $\leq\left|N_{1}(u)\right| x_{u^{*}}$. Combining the above two inequalities gives that $2 C_{\mathbb{H}}\left(10 C_{\mathbb{H}}\right)^{10-\lambda}<\left|N_{1}(u)\right|$. Summing this inequality over all vertices $u \in L^{\lambda}$, we obtain

$$
2 C_{\mathbb{H}}\left(10 C_{\mathbb{H}}\right)^{10-\lambda}\left|L^{\lambda}\right|<\sum_{u \in L^{\lambda}}\left|N_{1}(u)\right| \leq \sum_{u \in V(G)}\left|N_{1}(u)\right|=2 e(G) .
$$

Note that $e(G)<C_{\mathbb{H}} n$ by Lemma 2.2. Thus $\left|L^{\lambda}\right|<\left(10 C_{\mathbb{H}}\right)^{\lambda-10} n$.
Now we introduce some notations. For a vertex $u \in V(G)$ and a positive integer $i$, let $N_{i}(u)$ be the set of vertices at distance $i$ from $u$ in $G$. We will frequently use $N_{1}(u)$ and $N_{2}(u)$. Furthermore, we use $L_{i}^{\lambda}$ and $\bar{L}_{i}^{\lambda}$ to denote $N_{i}(u) \cap L^{\lambda}$ and $N_{i}(u) \backslash L^{\lambda}$, respectively. We also denote $L_{i, j}^{\lambda}=L_{i}^{\lambda} \cup L_{j}^{\lambda}$ and $\bar{L}_{i, j}^{\lambda}=\bar{L}_{i}^{\lambda} \cup \bar{L}_{j}^{\lambda}$ for simplicity.

Lemma 5.2. For every $u \in V(G)$ and every positive constant $\lambda$, we have

$$
\begin{equation*}
\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right) x_{u} \leq\left|N_{1}(u)\right| x_{u}+\left(\frac{2 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{10-\lambda}}+\frac{2 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{\lambda}}\right) x_{u^{*}}+\sum_{v \in \bar{L}_{1}^{\lambda}, w \in N_{L_{1,2}^{\lambda}}(v)} x_{w} . \tag{17}
\end{equation*}
$$

Proof. Set $\rho:=\rho(G)$. Recall that $\rho \geq \sqrt{\gamma_{\mathbb{H}}\left(n-\gamma_{H}\right)}$. Then

$$
\begin{equation*}
\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right) x_{u} \leq \rho^{2} x_{u}=\sum_{v \in N_{1}(u)} \rho x_{v}=\left|N_{1}(u)\right| x_{u}+\sum_{v \in N_{1}(u), w \in N_{1}(v) \backslash\{u\}} x_{w} . \tag{18}
\end{equation*}
$$

For a given $u \in V(G)$ and each $v \in N_{1}(u)$, since $N_{1}(v) \backslash\{u\} \subseteq \cup_{i=1}^{2} N_{i}(u)$ and $N_{i}(u)=$ $L_{i}^{\lambda} \cup \bar{L}_{i}^{\lambda}$, we have $N_{1}(v) \backslash\{u\}=N_{L_{1,2}^{\lambda}}(v) \cup N_{\bar{L}_{1,2}^{\lambda}}(v)$. Moreover, by the definition of $L^{\lambda}$, we know that $x_{w}<\left(10 C_{\mathbb{H}}\right)^{-\lambda} x_{u^{*}}$ for each $w \in \bar{L}_{1,2}^{\lambda}$.

We now decompose $N_{1}(u)=L_{1}^{\lambda} \cup \bar{L}_{1}^{\lambda}$, and further decompose $\sum_{w \in N_{1}(v) \backslash\{u\}} x_{w}$ into two subitems according to $N_{1}(v) \backslash\{u\}=N_{L_{1,2}^{\lambda}}(v) \cup N_{\bar{L}_{1,2}^{\lambda}}(v)$. Consequently,

$$
\begin{align*}
\sum_{v \in L_{1}^{\lambda}, w \in N_{1}(v) \backslash\{u\}} x_{w} & \leq \sum_{v \in L_{1}^{\lambda}, w \in N_{L_{1,2}^{\lambda}}(v)} x_{u^{*}}+\sum_{v \in L_{1}^{\lambda}, w \in N_{L_{1,2}^{\lambda}}(v)}\left(10 C_{\mathbb{H}}\right)^{-\lambda} x_{u^{*}} \\
& \leq\left(2 e\left(L_{1}^{\lambda}\right)+e\left(L_{1}^{\lambda}, L_{2}^{\lambda}\right)\right) x_{u^{*}}+e\left(L_{1}^{\lambda}, \bar{L}_{1,2}^{\lambda}\right)\left(10 C_{\mathbb{H}}\right)^{-\lambda} x_{u^{*}} . \tag{19}
\end{align*}
$$

Similarly as above, we have

$$
\begin{equation*}
\sum_{v \in \bar{L}_{1}^{\lambda}, w \in N_{1}(v) \backslash\{u\}} x_{w} \leq \sum_{v \in \bar{L}_{1}^{\lambda}, w \in N_{L_{1,2}^{\lambda}}(v)} x_{w}+\left(2 e\left(\bar{L}_{1}^{\lambda}\right)+e\left(\bar{L}_{1}^{\lambda}, \bar{L}_{2}^{\lambda}\right)\right)\left(10 C_{\mathbb{H}}\right)^{-\lambda} x_{u^{*}} . \tag{20}
\end{equation*}
$$

By Lemma 2.2, we obtain $2 e\left(L_{1}^{\lambda}\right)+e\left(L_{1}^{\lambda}, L_{2}^{\lambda}\right) \leq 2 e\left(L^{\lambda}\right) \leq 2 C_{\mathbb{H}}\left|L^{\lambda}\right|$ and $e\left(L_{1}^{\lambda}, \bar{L}_{1,2}^{\lambda}\right)+$ $2 e\left(\bar{L}_{1}^{\lambda}\right)+e\left(\bar{L}_{1}^{\lambda}, \bar{L}_{2}^{\lambda}\right) \leq 2 e(G)<2 C_{\mathbb{H}} n$. Furthermore, $\left|L^{\lambda}\right|<\left(10 C_{\mathbb{H}}\right)^{\lambda-10} n$ by Lemma 5.1. Combining (18-20), we get the inequality (17) immediately.

We now choose $\lambda=4$ to get a better bound of $\left|L^{\lambda}\right|$, which only depends on $C_{\mathbb{H}}$.
Lemma 5.3. $\left|L^{4}\right|<\left(10 C_{\mathbb{H}}\right)^{6}$.
Proof. We first show $\left|N_{1}(u)\right| \geq\left(10 C_{\mathbb{H}}\right)^{-5} n$ for each $u \in L^{4}$. Suppose to the contrary that there exists $u_{0} \in L^{4}$ with $\left|N_{1}\left(u_{0}\right)\right|<\left(10 C_{\mathbb{H}}\right)^{-5} n$. Set $u=u_{0}$ and $\lambda=5$ in (17). Then

$$
\begin{equation*}
\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right) x_{u_{0}} \leq\left|N_{1}\left(u_{0}\right)\right| x_{u_{0}}+\frac{4 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{5}} x_{u^{*}}+\sum_{v \in \bar{L}_{1}^{5}, w \in N_{L_{1,2}^{5}}^{5}(v)} x_{w} \tag{21}
\end{equation*}
$$

Recall that $\bar{L}_{1}^{5} \subseteq N_{1}\left(u_{0}\right)$ and $L_{1,2}^{5} \subseteq L^{5}$. By Lemma 2.2, we have $e\left(\bar{L}_{1}^{5}, L_{1,2}^{5}\right) \leq e\left(N_{1}\left(u_{0}\right) \cup\right.$ $\left.L^{5}\right) \leq C_{\mathbb{H}}\left(\left|N_{1}\left(u_{0}\right)\right|+\left|L^{5}\right|\right)$, where $\left|N_{1}\left(u_{0}\right)\right|<\left(10 C_{\mathbb{H}}\right)^{-5} n$ by the assumption. By Lemma 5.1, we also have $\left|L^{5}\right| \leq\left(10 C_{\mathbb{H}}\right)^{-5} n$. Hence,

$$
\left|N_{1}\left(u_{0}\right)\right| x_{u_{0}}+\sum_{v \in \bar{L}_{1}^{5}, w \in N_{L_{1,2}^{5}}(v)} x_{w} \leq\left(\left|N_{1}\left(u_{0}\right)\right|+e\left(\bar{L}_{1}^{5}, L_{1,2}^{5}\right)\right) x_{u^{*}} \leq \frac{\left(1+2 C_{\mathbb{H}}\right) n}{\left(10 C_{\mathbb{H}}\right)^{5}} x_{u^{*}} .
$$

Combining this inequality and (21) gives that $\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right) x_{u_{0}} \leq\left(1+6 C_{\mathbb{H}}\right)\left(10 C_{\mathbb{H}}\right)^{-5} n x_{u^{*}}$.
On the other hand, recall that $\gamma_{\mathbb{H}} \geq 1$ and $x_{u_{0}} \geq\left(10 C_{\mathbb{H}}\right)^{-4} x_{u^{*}}$ as $u_{0} \in L^{4}$. Thus, $\gamma_{\mathbb{H}}(n-$ $\left.\gamma_{\mathbb{H}}\right) x_{u_{0}} \geq \frac{7}{10} n x_{u_{0}} \geq 7 C_{\mathbb{H}}\left(10 C_{\mathbb{H}}\right)^{-5} n x_{u^{*}}$, a contradiction. Therefore, $\left|N_{1}(u)\right| \geq\left(10 C_{\mathbb{H}}\right)^{-5} n$ for each $u \in L^{4}$.

Summing the above inequality over all vertices $u \in L^{4}$, we obtain

$$
\left|L^{4}\right|\left(10 C_{\mathbb{H}}\right)^{-5} n \leq \sum_{u \in L^{4}}\left|N_{1}(u)\right| \leq \sum_{u \in V(G)}\left|N_{1}(u)\right|=2 e(G)<2 C_{\mathbb{H}} n,
$$

which yields that $\left|L^{4}\right|<\left(10 C_{\mathbb{H}}\right)^{6}$.
Lemma 5.4. For each vertex $u \in L^{4}$, we have $\left|N_{1}(u)\right| \geq\left(\frac{x_{u}}{x_{u^{*}}}-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{3}}\right) n$.
Proof. Choose an arbitrary vertex $u \in L^{4}$ and a minimal graph $H^{*}$ with respect to $\mathbb{H}$. Then $\gamma_{\text {Hi }}=\gamma_{H^{*}}=\left|H^{*}\right|-\alpha_{H^{*}}-1$. Recall that $L_{i}^{4}=N_{i}(u) \cap L^{4}, \bar{L}_{i}^{4}=N_{i}(u) \backslash L^{4}$ and $L_{1,2}^{4}=L_{1}^{4} \cup L_{2}^{4}$. Let $L_{0}$ be the subset of $\bar{L}_{1}^{4}$ in which each vertex has at least $\gamma_{H^{*}}$ neighbors in $L_{1,2}^{4}$.

We first claim that $\left|L_{0}\right| \leq \varphi\left|H^{*}\right|$, where $\varphi:=\binom{\left|L_{1,2}^{4}\right|}{\gamma_{H^{*}}}$. Indeed, if $\left|L_{1,2}^{4}\right| \leq \gamma_{H^{*}}-1$, then $L_{0}=\varnothing$ and we are done. Now consider the case $\left|L_{1,2}^{4}\right| \geq \gamma_{H^{*}}$. Suppose to the contrary that $\left|L_{0}\right|>\varphi\left|H^{*}\right|$. Since there are only $\varphi$ options for vertices in $L_{0}$ to choose $\gamma_{H^{*}}$ neighbors from $L_{1,2}^{4}$, we can find $\gamma_{H^{*}}$ vertices in $L_{1,2}^{4}$ with at least $\left.\left|L_{0}\right| / \varphi\right\rangle\left|H^{*}\right|$ common neighbors in $L_{0}$. Furthermore, $u \notin L_{1,2}^{4}$ and $L_{0} \subseteq \bar{L}_{1}^{4} \subseteq N_{1}(u)$. Hence, $u_{0}$ and those $\gamma_{H^{*}}$ vertices have $\left|H^{*}\right|$ common neighbors, which implies that $G$ contains a bipartite subgraph $G[S, T]$ isomorphic to $K_{\gamma_{H^{*}}+1,\left|H^{*}\right|}$. As noted earlier, $\left|H^{*}\right|-\left(\gamma_{H^{*}}+1\right)=\alpha_{H^{*}}$. Then, contracting $\gamma_{H^{*}}+1$ independent edges in $G[S, T]$, we obtain a new graph isomorphic to $B_{\gamma_{H^{*}}+1, \alpha_{H^{*}}}$. By Lemma 2.5, $G$ has an $H^{*}$ minor, a contradiction. Thus, $\left|L_{0}\right| \leq \varphi\left|H^{*}\right|$.

By Lemma 5.3, we have $\left|L_{1,2}^{4}\right| \leq\left|L^{4}\right|<\left(10 C_{\mathbb{H}}\right)^{6}$, which implies that $\varphi$ is constant. Combining $\left|L_{0}\right| \leq \varphi\left|H^{*}\right|$, we obtain $e\left(L_{0}, L_{1,2}^{4}\right) \leq\left|L_{0}\right|\left|L_{1,2}^{4}\right| \leq\left(10 C_{\mathbb{H}}\right)^{-4} n$ for $n$ sufficiently large. On the other hand, by the choice of $L_{0}$ we know that $e\left(\bar{L}_{1}^{4} \backslash L_{0}, L_{1,2}^{4}\right) \leq$ $\left|\bar{L}_{1}^{4} \backslash L_{0}\right|\left(\gamma_{H^{*}}-1\right) \leq\left|N_{1}(u)\right|\left(\gamma_{\text {HI }}-1\right)$. Consequently,

$$
\begin{equation*}
e\left(\bar{L}_{1}^{4}, L_{1,2}^{4}\right) \leq\left|N_{1}(u)\right|\left(\gamma_{\mathbb{H}}-1\right)+\left(10 C_{\mathbb{H}}\right)^{-4} n . \tag{22}
\end{equation*}
$$

Notice that $\sum_{v \in \bar{L}_{1}^{4}} \sum_{w \in N_{L_{1,2}^{4}}(v)} x_{w} \leq e\left(\bar{L}_{1}^{4}, L_{1,2}^{4}\right) x_{u^{*}}$. Now, setting $\lambda=4$ in (17) and combining (22), we obtain that

$$
\begin{aligned}
\gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right) x_{u} & \leq\left(\left|N_{1}(u)\right|+\frac{2 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{6}}+\frac{2 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}+\left|N_{1}(u)\right|\left(\gamma_{\mathbb{H}}-1\right)+\frac{n}{\left(10 C_{\mathbb{H}}\right)^{4}}\right) x_{u^{*}} \\
& \leq \gamma_{\mathbb{H}}\left(\left|N_{1}(u)\right|+\frac{3 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}\right) x_{u^{*}} .
\end{aligned}
$$

Thus, $\left|N_{1}(u)\right| \geq\left(n-\gamma_{\mathbb{H}}\right) \frac{x_{u}}{x_{u^{*}}}-\frac{3 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}$, where $\gamma_{\mathbb{H}} \frac{x_{u}}{x_{u^{*}}} \leq \gamma_{\mathbb{H}} \leq \frac{7 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}$ for $n$ large enough. It follows that $\left|N_{1}(u)\right| \geq\left(\frac{x_{u}}{x_{u^{*}}}-\frac{1}{\left(10 C_{\text {Hit }}\right)^{3}}\right) n$, as desired.

Now, choose $\lambda=1$. By the definition of $L^{\lambda}$, it is clear that $u^{*} \in L^{1}$ and $L^{1} \subseteq L^{4}$.
Lemma 5.5. For every vertex $u \in L^{1}$, we have $x_{u} \geq\left(1-\frac{1}{2\left(10 C_{H}\right)^{2}}\right) x_{u^{*}}$ and $\left|N_{1}(u)\right| \geq$ $\left(1-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{2}}\right)$. Moreover, we have $\left|L^{1}\right|=\gamma_{\boldsymbol{H}}$.

Proof. We first show the lower bounds of $x_{u}$ and $\left|N_{1}(u)\right|$. Suppose to the contrary that there exists $u_{0} \in L^{1}$ with $x_{u_{0}}<\left(1-\frac{1}{2\left(10 C_{\mathrm{H}}\right)^{2}}\right) x_{u^{*}}$. According to the definition of $L^{1}$, we know that $x_{u_{0}} \geq \frac{x_{u^{*}}}{10 C_{\text {Hit }}}$. By Lemma 5.4, we get

$$
\left|N_{1}\left(u^{*}\right)\right| \geq\left(1-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{3}}\right) n \text { and }\left|N_{1}\left(u_{0}\right)\right| \geq\left(\frac{1}{10 C_{\mathbb{H}}}-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{3}}\right) n .
$$

Now let $L_{i}^{4}=N_{i}\left(u^{*}\right) \cap L^{4}$ and $\bar{L}_{i}^{4}=N_{i}\left(u^{*}\right) \backslash L^{4}$. By Lemma 5.3, we have $\left|L^{4}\right|<\left(10 C_{\mathbb{H}}\right)^{6}<$ $\frac{n}{\left(10 C_{H I}\right)^{3}}$ for $n$ large enough. Hence, $\left|\bar{L}_{1}^{4}\right| \geq\left|N_{1}\left(u^{*}\right)\right|-\left|L^{4}\right| \geq\left(1-\frac{2}{\left(10 C_{H I}\right)^{3}}\right) n$, and thus

$$
\begin{equation*}
\left|\bar{L}_{1}^{4} \cap N_{1}\left(u_{0}\right)\right| \geq\left|\bar{L}_{1}^{4}\right|+\left|N_{1}\left(u_{0}\right)\right|-n \geq\left(\frac{1}{10 C_{\mathbb{H}}}-\frac{3}{\left(10 C_{\mathbb{H}}\right)^{3}}\right) n \geq \frac{9 n}{100 C_{\mathbb{H}}} . \tag{23}
\end{equation*}
$$

In view of (23), $u_{0}$ has neighbors in $\bar{L}_{1}^{4}$. Since $\bar{L}_{1}^{4} \subseteq N_{1}\left(u^{*}\right)$, we can observe that $u_{0}$ is of distance at most two from $u^{*}$, that is, $u_{0} \in \cup_{i=1}^{2} N_{i}\left(u^{*}\right)$. Recall that $u_{0} \in L^{1} \subseteq L^{4}$. Then $u_{0} \in L_{1,2}^{4}$, where $L_{1,2}^{4}=L_{1}^{4} \cup L_{2}^{4}$. Now setting $u=u^{*}$ and $\lambda=4$ in (17), we can see that

$$
\begin{aligned}
& \gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right) x_{u^{*}} \\
\leq & \left(\left|N_{1}\left(u^{*}\right)\right|+\frac{2 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{6}}+\frac{2 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}+e\left(\bar{L}_{1}^{4}, L_{1,2}^{4} \backslash\left\{u_{0}\right\}\right)\right) x_{u^{*}}+e\left(\bar{L}_{1}^{4},\left\{u_{0}\right\}\right) x_{u_{0}} \\
\leq & \left(\left|N_{1}\left(u^{*}\right)\right|+\frac{2.5 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}+e\left(\bar{L}_{1}^{4}, L_{1,2}^{4}\right)\right) x_{u^{*}}+e\left(\bar{L}_{1}^{4},\left\{u_{0}\right\}\right)\left(x_{u_{0}}-x_{u^{*}}\right),
\end{aligned}
$$

where $x_{u_{0}}-x_{u^{*}}<-\frac{x_{u^{*}}}{2\left(10 C_{H}\right)^{2}}$ by the previous assumption.
From (22) we know that $e\left(\bar{L}_{1}^{4}, L_{1,2}^{4}\right) \leq\left(\gamma_{\mathbb{H}}-1\right)\left|N_{1}\left(u^{*}\right)\right|+\frac{C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}$. Moreover, it is easy to see $\gamma_{\mathbb{H}}^{2} \leq \frac{0.5 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}$ for $n$ large enough. Hence,

$$
\gamma_{\mathbb{H}} n \leq \gamma_{\mathbb{H}}\left|N_{1}\left(u^{*}\right)\right|+\frac{4 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}-\frac{e\left(\bar{L}_{1}^{4},\left\{u_{0}\right\}\right)}{2\left(10 C_{\mathbb{H}}\right)^{2}}<\gamma_{\mathbb{H}} n+\frac{4 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}-\frac{e\left(\bar{L}_{1}^{4},\left\{u_{0}\right\}\right)}{2\left(10 C_{\mathbb{H}}\right)^{2}},
$$

which yields that $e\left(\bar{L}_{1}^{4},\left\{u_{0}\right\}\right)<\frac{8 n}{100 C_{\text {Hi }}}$, contradicting (23). Hence, $x_{u} \geq\left(1-\frac{1}{2\left(10 C_{H I H}\right)^{2}}\right) x_{u^{*}}$ for each $u \in L^{1}$. Furthermore, it follows from Lemma 5.4 that for each $u \in L^{1}$,

$$
\left|N_{1}(u)\right| \geq\left(1-\frac{1}{2\left(10 C_{\mathbb{H}}\right)^{2}}-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{3}}\right) n \geq\left(1-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{2}}\right) n .
$$

It remains to show $\left|L^{1}\right|=\gamma_{\mathbb{H}}$. We first suppose that $\left|L^{1}\right| \geq \gamma_{\mathbb{H}}+1$. Then $\left|L^{1}\right| \geq \gamma_{H^{*}}+1$, where $H^{*}$ is minimal with respect to $\mathbb{H}$. Notice that every vertex $u \in L^{1}$ has at most $n /\left(10 C_{\mathbb{H}}\right)^{2}$ non-neighbors. Hence, every $\gamma_{H^{*}}+1$ vertices in $L^{1}$ have at least $n-\frac{\left(\gamma_{H^{*}}+1\right)}{\left(10 C_{\mathbb{H}}\right)^{2}} n \geq$ $\left|H^{*}\right|$ common neighbors, as $C_{\mathbb{H}}>\left|H^{*}\right|>\gamma_{H^{*}}$. Hence, $G$ contains a bipartite subgraph $G[S, T]$ isomorphic to $K_{\gamma_{H^{*}}+1,\left|H^{*}\right|}$. Note that $\left|H^{*}\right|-\left(\gamma_{H^{*}}+1\right)=\alpha_{H^{*}}$. Contracting $\gamma_{H^{*}}+1$ independent edges in $K_{\gamma_{H^{*}}+1,\left|H^{*}\right|}$, we get a copy of $B_{\gamma_{H^{*}}+1, \alpha_{H^{*}}}$. By Lemma 2.5, $G$ contains an $H^{*}$ minor, a contradiction. Therefore, $\left|L^{1}\right| \leq \gamma_{\Pi}$.

Next suppose $\left|L^{1}\right| \leq \gamma_{\mathbb{H}}-1$. Since $u^{*} \in L^{1} \backslash L_{1,2}^{4}$, we have $\left|L^{1} \cap L_{1,2}^{4}\right| \leq \gamma_{\mathbb{H}}-2$, and thus $e\left(\bar{L}_{1}^{4}, L_{1,2}^{4} \cap L^{1}\right) \leq\left(\gamma_{\text {Hil }}-2\right) n$. On the other hand, by Lemma 2.2 we have $e\left(\bar{L}_{1}^{4}, L_{1,2}^{4} \backslash L^{1}\right) \leq$ $e(G)<C_{\mathbb{H}} n$, and by the definition of $L^{1}$ we know that $x_{w}<\frac{x_{u^{*}}}{10 C_{\mathbb{H}}}$ for each $w \in L_{1,2}^{4} \backslash L^{1}$. Now, setting $u=u^{*}$ and $\lambda=4$ in (17), we can see that

$$
\begin{aligned}
& \gamma_{\mathbb{H}}\left(n-\gamma_{\mathbb{H}}\right) x_{u^{*}} \\
\leq & \left(\left|N_{1}\left(u^{*}\right)\right|+\frac{2.5 C_{\mathbb{H}} n}{\left(10 C_{\mathbb{H}}\right)^{4}}+e\left(\bar{L}_{1}^{4}, L_{1,2}^{4} \cap L^{1}\right)\right) x_{u^{*}}+e\left(\bar{L}_{1}^{4}, L_{1,2}^{4} \backslash L^{1}\right) \frac{x_{u^{*}}}{10 C_{\mathbb{H}}} \\
\leq & \left(n+\frac{n}{10}+\left(\gamma_{\mathbb{H}}-2\right) n+\frac{n}{10}\right) x_{u^{*}} \\
= & \left(\gamma_{\mathbb{H}}-\frac{4}{5}\right) n x_{u^{*}} .
\end{aligned}
$$

This gives $\gamma_{\mathbb{H}}^{2} \geq \frac{4}{5} n$, a contradiction. Therefore, $\left|L^{1}\right|=\gamma_{\mathbb{H}}$. The proof is completed.
Recall that, to prove Theorem 2.1, it suffices to find a set $L$ of exactly $\gamma_{\text {Hi }}$ vertices in $G$ such that $x_{u} \geq\left(1-\frac{1}{2\left(10 C_{\mathbb{H}}\right)^{2}}\right) x_{u^{*}}$ and $d_{G}(u) \geq\left(1-\frac{1}{\left(10 C_{\mathbb{H}}\right)^{2}}\right) n$ for every $u \in L$. By Lemma 5.5 , we immediately obtain the desired result by choosing $L=L^{1}$.

## References

[1] Y. Akama, B.B. Hua, Y.H. Su, L.L. Wang, A curvature notion for planar graphs stable under planar duality, Adv. Math. 385 (2021), Paper No. 107731, 44 pp.
[2] N. Alon, Eigenvalues and expanders, Combinatorica 6 (1986) 83-96.
[3] N. Alon, M. Krivelevich, B. Sudakov, Complete minors and average degree: a short proof, J. Graph Theory 103 (2023), no. 3, 599-602.
[4] L. Babai, Automorphism groups of planar graphs. II, Colloq. Math. Soc. János Bolyai 10 (1975) 29-84.
[5] B. Bollobás, J. Lee, S. Letzter, Eigenvalues of subgraphs of the cube, European J. Combin. 70 (2018), 125-148.
[6] B. Bollobás, V. Nikiforov, Cliques and the spectral radius, J. Comb. Theory, Ser. B 97 (2007) 859-865.
[7] B.N. Boots, G.F. Royle, A conjecture on the maximum value of the principal eigenvalue of a planar graph, Geogr. Anal. 23 (1991) 276-282.
[8] D.S. Cao, A. Vince, The spectral radius of a planar graph, Linear Algebra Appl. 187 (1993) 251-257.
[9] G. Chartrand, D. Geller, S. Hedetniemi, Graphs with forbidden subgraphs, J. Combin. Theory Ser. B 10 (1971) 12-41.
[10] M-Z. Chen, A-M. Liu, X-D. Zhang, The spectral radius of minor-free graphs, European J. Combin. 118 (2024) 103875.
[11] M. Chudnovsky, B. Reed, P. Seymour, The edge-density for $K_{2, t}$ minors, J. Combin. Theory Ser. B 101 (2011) 18-46.
[12] S. Cioabă, D.N. Desai, M. Tait, The spectral radius of graphs with no odd wheels, European J. Combin. 99 (2022) 103420.
[13] J.H. Conway, H. Burgiel, C. Goodman-Strauss, The symmetries of things, A K Peters, Ltd., Wellesley, MA, 2008.
[14] D. Cvetković, P. Rowlinson, The largest eigenvalue of a graph: a survey, Linear Multilinear Algebra 28 (1990) 3-33.
[15] G.L. Ding, T. Johnson, P. Seymour, Spanning trees with many leaves, J. Graph Theory 37 (2001) 189-197.
[16] G.L. Ding, B. Oporowski, D.P. Sanders, D. Vertigan, Surfaces, tree-width, cliqueminors, and partitions, J. Combin. Theory Ser. B 79 (2000), no. 2, 221-246.
[17] Z. Dvořák, B. Mohar, Spectral radius of finite and infinite planar graphs and of graphs of bounded genus, J. Combin. Theory Ser. B 100 (2010), no. 6, 729-739.
[18] M.N. Ellingham, X.Y. Zha, The spectral radius of graphs on surfaces, J. Combin. Theory Ser. B 78 (2000) 45-56.
[19] L. Ghidelli, On the largest planar graphs with everywhere positive combinatorial curvature, J. Combin. Theory Ser. B 158 (2023), part 2, 226-263.
[20] X.C. He, Y.T. Li, L.H. Feng, Spectral extremal graphs without intersecting triangles as a minor, arXiv:2301.06008 (2023).
[21] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, Vierteljschr. Naturforsch. Ges. Zürich 88 (1943) 133-142.
[22] Y. Higuchi, Combinatorial curvature for planar graphs, J. Graph Theory 38 (2001), no. 4, 220-229.
[23] Y. Hong, On the spectral radius and the genus of graphs, J. Combin. Theory Ser. B 65 (1995) 262-268.
[24] Y. Hong, Upper bounds of the spectral radius of graphs in terms of genus, J. Combin. Theory Ser. B 74 (1998) 153-159.
[25] Y. Hong, Tree-width, clique-minors, and eigenvalues, Discrete Math. 274 (2004) 281-287.
[26] S. Hoory, N. Linial, A. Widgerson, Expander graphs and their applications, Bull. Amer. Math. Soc. (N.S.), 43 (2006) 439-561.
[27] B.B. Hua, Y.H. Su, The set of vertices with positive curvature in a planar graph with nonnegative curvature, Adv. Math. 343 (2019), 789-820.
[28] B.B. Hua, Y.H. Su, The first gap for total curvatures of planar graphs with nonnegative curvature, J. Graph Theory 93 (2020), no. 3, 395-439.
[29] H. Huang, Induced subgraphs of hypercubes and a proof of the sensitivity conjecture, Ann. of Math. (2) 190 (2019), no. 3, 949-955.
[30] Z. Jiang, On spectral radii of unraveled balls, J. Combin. Theory Ser. B 136 (2019) 72-80.
[31] Z. Jiang, J. Tidor, Y. Yao, S. Zhang, Y. Zhao, Equiangular lines with a fixed angle, Ann. of Math. (2) 194 (2021), no. 3, 729-743.
[32] K.S. Kedlaya, Outerplanar partitions of planar graphs, J. Combin. Theory Ser. B 67 (1996) 238-248.
[33] A.V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, Metody Diskret. Anal. 38 (1982) 37-58.
[34] A.V. Kostochka, Lower bound for the Hadwiger number for graphs by their average degree, Combinatorica 4 (1984) 307-316.
[35] H.Q. Lin, B. Ning, A complete solution to the Cvetković-Rowlinson conjecture, J. Graph Theory 97 (2021), no. 3, 441-450.
[36] E. Lubetzky, B. Sudakov, V. Vu, Spectra of lifted Ramanujan graphs, Adv. Math. 227 (2011), no. 4, 1612-1645.
[37] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Ann. 174 (1967) 265-268.
[38] W. Mader, Homomorphiesätze für Graphen, Math. Ann. 178 (1968) 154-168.
[39] P. Mani, Automorphismen von polyedrischen Graphen, Math. Ann. 192 (1971) 279303.
[40] J.S. Myers, The extremal function for unbalanced bipartite minors, Discrete Math. 271 (2003) 209-222.
[41] V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra Appl. 427 (2007) 183-189.
[42] V. Nikiforov, The spectral radius of graphs with no $K_{2, t}$-minor, Linear Algebra Appl. 531 (2017) 510-515.
[43] S. Norin, L. Postle, Z. Song, Breaking the degeneracy barrier for coloring graphs with no $K_{t}$ minor, Adv. Math. 422 (2023), Paper No. 109020, 23 pp.
[44] N. Robertson, P.D. Seymour, Graph minors-a survey, Surveys in combinatorics 1985 (Glasgow, 1985), 153-171, London Math. Soc. Lecture Note Ser., 103, Cambridge Univ. Press, Cambridge, 1985.
[45] B. Servatius, H. Servatius, Symmetry, automorphisms, and self-duality of infinite planar graphs and tilings, in: International Scientific Conference on Mathematics. Proceedings, (Žilina, 1998), 83-116, Univ. Žilina, Žilina, 1998.
[46] M. Tait, The Colin de Verdière parameter, excluded minors, and the spectral radius, J. Combin. Theory Ser. A 166 (2019) 42-58.
[47] M. Tait, J. Tobin, Three conjectures in extremal spectral graph theory, J. Combin. Theory Ser. B 126 (2017) 137-161.
[48] A. Thomason, An extremal function for contractions of graphs, Math. Proc. Cambridge Philos. Soc. 95 (1984) 261-265.
[49] A. Thomason, The extremal function for complete minors, J. Combin. Theory Ser. B 81 (2001) 318-338.
[50] A. Thomason, Disjoint complete minors and bipartite minors, European J. Combin. 28 (2007), no. 6, 1779-1783.
[51] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570-590.
[52] B. Wang, W.W. Chen, L.F. Fang, Extremal spectral radius of $K_{3,3} / K_{2,4}$-minor free graphs, Linear Algebra Appl. 628 (2021) 103-114.
[53] H. Wilf, Spectral bounds for the clique and independence numbers of graphs, $J$. Comb. Theory, Ser. B 40 (1986) 113-117.
[54] M.Q. Zhai, H.Q. Lin, Spectral extrema of $K_{s, t}$-minor free graphs-On a conjecture of M. Tait, J. Combin. Theory, Ser. B 157 (2022) 184-215.
[55] M.Q. Zhai, B. Wang, Proof of a conjecture on the spectral radius of $C_{4}$-free graphs, Linear Algebra Appl. 437 (2012) 1641-1647.


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