On Modeling Multi-Criteria Decision Making with Uncertain Information using Probabilistic Rules

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Abstract. Decision-making processes often involve dealing with uncertainty, which is traditionally addressed through probabilistic models. However, in practical scenarios, assessing probabilities reliably can be challenging, compounded by diverse perceptions of probabilistic information among decision makers. To address this variability and accommodate diverse preferences regarding uncertainty, we introduce the Probabilistic Abstract Decision Framework (PADF). PADF offers a structured approach for reasoning across different decision criteria, encompassing the optimistic, pessimistic, and Laplace perspectives, each tailored to distinct perceptions of uncertainty. We illustrate how PADF facilitates the computation of optimal decisions aligned with these criteria by leveraging probabilistic rules. Furthermore, we present strategies for optimizing the computational efficiency of these rules, leveraging appropriate independence assumptions to navigate the extensive search space inherent in PADF. Through these contributions, our framework provides a robust and adaptable tool for effectively navigating the complexities of decision-making under uncertainty.

Keywords. Probabilistic Argumentation, Decision Making, Probabilistic Rules

1. Introduction

Probabilistic decision-making involves making informed choices amidst uncertainty [1]. It is pivotal for decision support systems not only to generate optimal decisions but also to consider various decision criteria that encapsulate different perspectives on uncertainty [2]. In this context, quantifying uncertainty and assessing the probabilities of diverse outcomes become essential tasks. This requires integrating multiple sources of information, including both objective data and subjective beliefs, and applying formal probabilistic models to represent and analyze uncertainty effectively. Additionally, accommodating differences in how different users perceive probabilistic information is crucial for ensuring the applicability and usability of decision support systems [3,4].

Probabilistic Rules (p-rules) [5] serve as the primary building blocks for probabilistic structured argumentation. They enable the construction of arguments that accommodate uncertainty and offer a conditional probability interpretation to rules commonly employed in structured argumentation.

The primary aim of this paper is to explore methods for accommodating different decision criteria within uncertain environments computed using p-rules. Initially, we introduce a probabilistic extension of the Abstract Decision Framework (ADF) [6], termed the Probabilistic Abstract Decision Framework (PADF), tailored specifically for modeling decision problems under uncertainty. PADF integrates the probability space to evaluate the likelihood of a decision being strongly or weakly dominant across various decision criteria, such as the optimistic, pessimistic, and Laplace criteria, reflecting diverse perceptions of probabilistic information among decision makers. Furthermore, we develop a procedure to map PADF onto p-rules, facilitating probabilistic inference to derive probability distributions under different decision criteria. To enhance computational efficiency within the extensive search space of PADF, we leverage on the concept of relative independence assumption. In summary, our contributions can be outlined as follows:

- We extend the existing notions of strongly dominant, dominant, and weakly dominant semantics within ADF to the probabilistic domain.
- Three distinct decision criteria are developed to capture diverse perceptions of uncertainty information among decision makers in probabilistic reasoning.
- We establish a mapping between PADF and p-rules, enabling the execution of probabilistic deductions to derive probability distributions under various decision criteria. This linkage is significant as it bridges argumentation formalism with decision theory, offering a formal connection that facilitates argumentative explanation within the decision-making process.

The rest of this paper is organised as the follows. Section 2 reviews concepts introduced in the literature that are used in this work. Section 3 introduces PADF formally. In Section 4, we propose how to map PADF to p-rules and how to optimise its computational complexity. Section 5 discusses our work in comparison with related work. We conclude in Section 6.

2. Background and Preliminaries

In this work, we need two notions, *Abstract Decision Framework (ADF)* and *probabilistic rule (p-rule)*.

Definition 2.1. [6] An Abstract Decision Framework (ADF) is a tuple $\langle D, G, \gamma \rangle$ such that D is set of decisions, G is a set of goals and γ is a mapping function for which goals met by a decision. For $d \in D$, it is

- strongly dominant iff $\gamma(d) = G$;
- dominant iff there is no $g' \in G \setminus \gamma(d)$ with $g' \in \gamma(d')$ for some $d' \in D \setminus \{d\}$;
- weakly dominant iff there is no $d' \in D \setminus \{d\}$ with $\gamma(d) \subset \gamma(d')$.

It has been shown in [6] that if a decision is strongly dominant, then it is dominant; if a decision is dominant, then it is weakly dominant.

Definition 2.2. [7] Given a language $\mathcal{L}$, a *probabilistic rule (p-rule)* is $\sigma_0 \leftarrow \sigma_1, ..., \sigma_k : [\theta]$, for $k \geq 0, \sigma_i \in \mathcal{L}, 0 \leq \theta \leq 1$.

Definition 2.3. [8] Given a language $\mathcal{L}$ with n sentences, the *Complete Conjunction Set (CC Set)* Ω of $\mathcal{L}$ is the set of 2^n conjunction of sentences such that each conjunction containsc n distinct sentences.

Definition 2.4. [7] Given a language $\mathcal{L}$ and p-rules $\mathcal{R}$, Ω is the CC set of $\mathcal{L}$. A function $\pi : \Omega \rightarrow [0,1]$ is a consistent probability distribution with respect to $\mathcal{R}$ on $\mathcal{L}$ for Ω iff:

1. For all $\omega_i \in \Omega$, $0 \le \pi(\omega_i) \le 1$, it holds that:

$$\sum_{\omega_i \in \Omega} \pi(\omega_i) = 1 \tag{1}$$

2. For each p-rule $\sigma_0 \longleftarrow: [\theta] \in \mathcal{R}$, it holds that:

$$\theta = \sum_{\omega_i \in \Omega, \omega_i \models \sigma_0} \pi(\omega_i) \tag{2}$$

3. For each p-rule $\sigma_0 \leftarrow \sigma_1, ..., \sigma_k : [\theta] \in \mathcal{R}, (k > 0)$, it holds that:

$$\sum_{\omega_i \in \Omega, \omega_i \models \sigma_1 \wedge, ..., \wedge \sigma_k} \pi(\omega_i) \times \theta = \sum_{\omega_i \in \Omega, \omega_i \models \sigma_0 \wedge, ..., \wedge \sigma_k} \pi(\omega_i) \tag{3}$$

Given a language $\mathcal{L}$ and a set of p-rule $\mathcal{R}$, let Π be the set of consistent probability distributions wrt $\mathcal{R}$ on $\mathcal{L}$. There are three kinds of reasoning asserts for $\sigma \in \mathcal{L}$:

- *σ-maixmal Solution*, a upper bounds distribution $\pi_0 \in \Pi$ of $\Pr(\sigma)$ as follows.

$$\pi_0 = \operatorname*{argmax}_{\pi \in \Pi} \sum_{\omega_i \in \Omega, \omega_i \models \sigma} \pi(\omega_i) \tag{4}$$

- *σ-minimal Solution*, a lower bounds distribution $\pi_0 \in \Pi$ of $Pr(\sigma_0)$ as follows.

$$\pi_0 = \operatorname*{argmin}_{\pi \in \Pi} \sum_{\omega_i \in \Omega, \omega_i \models \sigma} \pi(\omega_i) \tag{5}$$

- *Maximum Entropy Solution,* a Maximum Entropy Distribution $\pi_0 \in \Pi$ is:

$$\pi_0 = \operatorname*{argmax}_{\pi \in \Pi}(-\sum_{\omega_i \in \Omega} \pi(\omega_i) \log(\pi(\omega_i))) \tag{6}$$

As explained in [7], solving linear systems derived from p-rules to compute the joint distribution π may result in multiple solutions, as the linear system could be underdetermined. Thus, three reasoning paradigms are proposed to represent three potential selections of distributions. σ-maximal and σ-minimal solutions aim to maximize and minimize the probability of a chosen σ, respectively, while the maximum entropy solution aims to minimize selection bias in choosing solutions.

3. Probabilistic Abstract Decision Framework

In this section, we introduce an extension of ADFs called the Probabilistic Abstract Decision Frameworks (PADF). PADF describes the probabilistic relationship between decisions and goals in a decision problem while accommodating probabilistic information.

Definition 3.1. A *Probabilistic Abstract Decision Frameworks (PADF)* is a tuple $\langle R, H, \rho \rangle$ in which $\langle R, H \rangle$ are two graphs and $\rho \in [0, 1]$, with

- $R = (N, E)$ is a graph such that $N = D \cup A \cup G$ is a set of *nodes* such that $D \neq \emptyset$ is a set of *decisions*; A is a set of *attributes*; $G \neq \emptyset$ is a set of *goals*;

- $E = E_a \cup E_g$ is a set of *directed edges* such that E_a is a set of edges to represent the attributes that might *have for* the decision and E_g is a set of edges to represent the goal might *met by* the decision, it is the case that: (1) if $(n_i, n_j) \in E_a$, then $n_i \in D$ and $n_j \in A$; or (2) if $(n_i, n_j) \in E_g$, then $n_i \in D$ and $n_j \in G$;

- $H = (E, C)$ is a graph whose nodes E is the edges in R, and directed edges C such that $(e_i, e_j) \in C$ if and only if $e_i \in E_a$, $e_i \in E$ and $e_i \neq e_j$.
- ρ is a mapping from a set of nodes $S = \{e_0, \ldots, e_k\} \subseteq E$ in H to $[0, 1]$ such that if $k \geq 1$, then $(e_i, e_0) \in C$ for all $i \geq 1$.

In PADF, we use graph R to describe the attributes that the decisions may have and the goals that the decision may met, and then graph H to describe the probabilistic influence relationship between the edges in E. ρ maps the probability of one or more edges in E, where if the number of edges is greater than one, the conditional probability relationship of the edges is confirmed by the edges in C. The following example illustrates the notion of PADF.

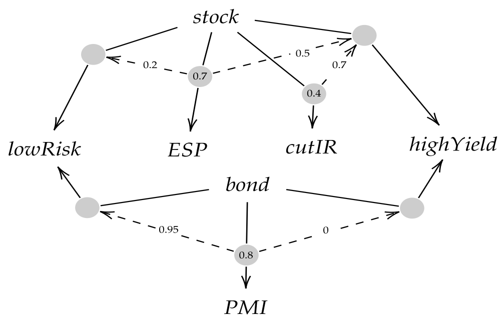

Figure 1. A PADF for Example 3.1. The solid line in the graph indicates the edge from E, the dashed line indicates the edge from C and the value indicates the ρ mapping.

Example 3.1. Figure 1 shows an example of a PADF. An agent is considering financial investment. The two candidate decisions are $stock$ and $bond$ representing two different investments. Agents consider two goals $highYield$ and $lowRisk$ as important. In agents' beliefs, there is 0.4 probability that the central bank will cut interest rates ($cutIR$), which would increase the probability of $highYield$ met by $stock$ to 0.7. In addition to this, Employment Situation Report (ESP) and Purchasing Manager's Index (PMI) are macro economic indicators for $stock$ and $bond$. The ESP indicates that there is a 0.5 probability that $stock$ has $highYield$, but only a 0.2 probability that it is $lowRisk$, while PMI show that $bond$ has a 0.95 probability of satisfying $lowRisk$, even though it does not satisfy the goal of $highYield$. However, the agent also assigns uncertainties to ESP and PMI as well, with $P(ESP) = 0.5$ and $P(PMI) = 0.8$, respectively, representing their reporting accuracy. Hence:

- $N = \{stock, bond\}_D \cup \{cutIR, ESP, PMI\}_A \cup \{highYield, lowRisk\}_G$
- E consists of:
  $e_0 = (stock, highYield)$, $e_1 = (stock, lowRisk)$, $e_2 = (stock, cutIR)$,
  $e_3 = (bond, highYield)$, $e_4 = (bond, lowRisk)$, $e_5 = (stock, ESP)$
  $e_6 = (bond, PMI)$
- C consists of:

(e_2,e_0), (e_5,e_0), (e_5,e_1), (e_6,e_3), (e_6,e_4)

- ρ mapping consists of:
  $\rho(\{e_2,e_0\})=0.7$, $\rho(\{e_5,e_0\})=0.5$, $\rho(\{e_5,e_1\})=0.2$, $\rho(\{e_6,e_3\})=0$,
  $\rho(\{e_6,e_4\})=0.95$, $\rho(\{e_2\})=0.4$, $\rho(\{e_5\})=0.7$, $\rho(\{e_6\}=0.8$

Definition 3.2. Given a PADF $\langle R,H,\rho\rangle$ where $D=\{d_1,...,d_i\}$ and $G=\{g_1,...,g_k\}$. $\mathcal{T}$ *target space* about all possible outcome of the decisions, such that:

- $\mathcal{T}=\{t_1,...,t_n\}$, its elements are *states* and there exist $n=2^{ik}$ *states* of $\mathcal{T}$ such that each state is a conjunction containing ik distinct goals met by distinct decisions $(d,g)\in E_g$ (or its neation $\neg(d,g)$ denotes that decision d does not meet goal g).

Example 3.2. (Example 3.1 continued) There exist 16 states of $\mathcal{T}=\{t_0,...,t_{15}\}$ for Example 3.1, such as:
$t_0=\neg(stock,highYield)\wedge\neg(stock,lowRisk)\wedge\neg(bond,Yield)\wedge\neg(bond,lowRisk)$,
$t_1=\neg(stock,highYield)\wedge\neg(stock,lowRisk)\wedge\neg(bond,Yield)\wedge(bond,lowRisk)$,
and $t_{15}=(stock,highYield)\wedge(stock,lowRisk)\wedge(bond,Yield)\wedge(bond,lowRisk)$.

Definition 3.3. Given a PADF $\langle R,H,\rho\rangle$. A probability function $\mathcal{F}:\mathcal{T}\rightarrow[0,1]$ is a joint probability distribution with PADF for the target set $\mathcal{T}$ such that $\sum_{t_i\in\mathcal{T}}\mathcal{F}(t_i)=1$. Let Π be the set of joint probability distributions wrt $\mathcal{T}$.

In this section, we focus on the basic concepts of PADF, so we will ignore the process of solving the joint probability distribution of $\mathcal{T}$ for now (we will present the its computation in the next section). Intuitively, it is natural to apply several decision criteria from Definition 2.1 of ADFs (*strongly dominant, dominant, weakly dominant*) to identify "good" decisions in each $t_i\in\mathcal{T}$. The probability of decision dominant in PADF can be inferred by summarizing the probability of states t_i, as follows:

Definition 3.4. Given a PADF $\langle R,H,\rho\rangle$. For each $t_i\in\mathcal{T}$, let $sDom\text{-}t_i, dom\text{-}t_i, wDom\text{-}t_i$ denote the sets of strongly dominant, dominant and weakly dominant decisions in t_i. Then, the *strongly dominant, dominant and weakly dominant probabilities* of decision $d\in D$ are obtained as follows, for $\chi=sDom,dom,wDom$, respectively:

$$\Pr_\chi(d)=\sum_{t_i\in\mathcal{T},d\in\chi\text{-}t_i}\mathcal{F}(t_i) \tag{7}$$

In Definition 3.4, $\Pr_{sDom}(d)$ is referred to as the strong dominant probability of d; $\Pr_{dom}(d)$ is referred to as the dominant probability of d; $\Pr_{wDom}(d)$ is referred to as the weakly dominant probability of d.

Example 3.3. (Example 3.1 continued.) Given a PADF $\langle R,H,\rho\rangle$ for Example 3.1. Table 1a shows one of the sets of consistent probability distributions for $\mathcal{T}$. For the decision $stock$ in each states $t_i\in\mathcal{T}$:

- $sDom: t_{12},t_{13},t_{14},t_{15}$;
- $dom: t_4,t_5,t_8,t_{10},t_{12},t_{13},t_{14},t_{15}$;
- $wDom: t_0,t_4,t_5,t_6,t_8,t_9,t_{10},t_{12},t_{13},t_{14},t_{15}$;

Consequently, the strong dominant probability of $stock$ is $\Pr_{sDom}(stock)=0.2904$, the dominant probability is $\Pr_{dom}(stock)=0.4256064$ and the weakly dominant probaility is $\Pr_{wDom}(stock)=0.7160064$. Note that these probabilities do not add up to one as $sDom, dom$, and $wDom$ are not mutually exclusive.

Table 1. This table shows the sixteen possible worlds in Example 3.3 and 4.2, there exist three sets of joint probability distributions for target set $\mathcal{T}$ under different decision certeria[1].

(a) Table for Optimistic Criterion

	$\mathcal{T}$	$\mathcal{F}(t)$
t_0	0000	0.0060928
t_1	0001	0.1462272
t_2	0010	0.0015232
t_3	0011	0.0365568
t_4	0100	0.0047872
t_5	0101	0.1148928
t_6	0110	0.0011968
t_7	0111	0.0287232
t_8	1000	0.0118272
t_9	1001	0.2838528
t_{10}	1010	0.0029568
t_{11}	1011	0.0709632
t_{12}	1100	0.0092928
t_{13}	1101	0.2230272
t_{14}	1110	0.0023232
t_{15}	1111	0.0557568

(b) Table for Pessimistic Criterion

	$\mathcal{T}$	$\mathcal{F}(t)$
t_0	0000	0.132096
t_1	0001	0.418304
t_2	0010	0.0
t_3	0011	0.0
t_4	0100	0.021504
t_5	0101	0.068096
t_6	0110	0.0
t_7	0111	0.0
t_8	1000	0.074304
t_9	1001	0.235296
t_{10}	1010	0.0
t_{11}	1011	0.0
t_{12}	1100	0.012096
t_{13}	1101	0.038304
t_{14}	1110	0.0
t_{15}	1111	0.0

(c) Table for Laplace Criterion

	$\mathcal{T}$	$\mathcal{F}(t)$
t_0	0000	0.0393624
t_1	0001	0.2417976
t_2	0010	0.0043736
t_3	0011	0.0268664
t_4	0100	0.0160776
t_5	0101	0.0987624
t_6	0110	0.0017864
t_7	0111	0.0109736
t_8	1000	0.0500976
t_9	1001	0.3077424
t_{10}	1010	0.0055664
t_{11}	1011	0.0341936
t_{12}	1100	0.0204624
t_{13}	1101	0.1256976
t_{14}	1110	0.0022736
t_{15}	1111	0.0139664

Proposition 3.1. Given a PADF $\langle R, H, \rho \rangle$, for any $d \in D$, $0 \le \Pr_{sDom}(d) \le \Pr_{dom}(d) \le \Pr_{wDom}(d) \le \sum_{t\in\mathcal{T}} \mathcal{F}(t) = 1$

It has been shown in [6] that if a decision is strongly dominant, then it is dominant; if a decision is dominant, then it is weakly dominant. This leads to a sequential relationship between the different dominant probabilities of a decision as Proposition 3.1. This sequential relationship can be read to mean that $\Pr_{wDom}(d)$ represents the most conservative reasoning, while $\Pr_{sDom}(d)$ represents the most radical reasoning.

With this, we define three decision criteria for the PADF, describing different perceptions of uncertainty information by decision makers in probabilistic reasoning:

Definition 3.5. (Optimistic Criterion) Given a PADF $\langle R, H, \rho \rangle$, $\delta(D)$ is the *best decision* in PADF with the *Optimistic Criterion* defines as follows:

1. Maximize the overall probability of all goals $\Pr(g_1 \vee ... \vee g_k)$:

$$\mathcal{F}_0 = \underset{\mathcal{F}\in\Pi}{\operatorname{argmax}} \Pr(g_1 \vee ... \vee g_k) \tag{8}$$

2. Select the decision with the highest probability of $\Pr_{sDom}(d_i)$:

$$\delta(D) = \underset{d\in D}{\operatorname{argmax}} \sum_{t_i\in\mathcal{T}, d\in sDom\text{-}t_i} \mathcal{F}_0(t_i) \tag{9}$$

The Optimistic Criterion describes that decision makers always have optimistic perspectives on probabilistic decision problems. Under Optimisyic Criterion, the decision

[1] To simplify the presentation, Boolean values are used as shorthand for conjunctions. E.g., 1111 denotes that $(stock, highYield) \wedge (stock, lowRisk) \wedge (bond, Yield) \wedge (bond, lowRisk)$.

maker are radical in their reasoning and tries to maximise the unknow outcome as Equation 8. Then, the decision with the highest strong dominant probability is selected as the best decision by Equation 9.

Definition 3.6. (Pessimistic Criterion) Given a PADF $\langle R, H, \rho \rangle$, $\delta(D)$ is the *best decision* in PADF with the *Pessimistic Criterion* defines as follows:

1. Minimize the overall probability of all goals $\Pr(g_1 \vee ... \vee g_k)$:

$$\mathcal{F}_0 = \operatorname*{argmin}_{\mathcal{F} \in \Pi} \Pr(g_1 \vee ... \vee g_k) \tag{10}$$

2. Select the decision with the highest probability of $\Pr_{wDom}(d_i)$:

$$\delta(D) = \operatorname*{argmax}_{d \in D} \sum_{t_i \in \mathcal{T}, d \in wDom\text{-}t_i} \mathcal{F}_0(t_i) \tag{11}$$

The opposite of the Optimistic Criterion is the Pessimistic Criterion. In this Criterion, the decision maker's reasoning is conservative and tries to minimise the unknow outcome as Equation 10 then choose the best decision of the worst preconceived notions by Equation 11.

Definition 3.7. (Laplace Criterion) Given a PADF $\langle R, H, \rho \rangle$, $\delta(D)$ is the *best decision* in PADF with the *Laplace Criterion* defines as follows:

1. Choosing the joint probability distribution with maximum entropy:

$$\mathcal{F}_0 = \operatorname*{argmax}_{\mathcal{F} \in \Pi} (-\sum_{t_i \in \mathcal{T}} \mathcal{F}(t_i) \log(\mathcal{F}(t_i))) \tag{12}$$

2. Select the decision with maximum expected utility:

$$\delta(D) = \operatorname*{argmax}_{d_i \in D} \sum_{t_k \in \mathcal{T}} \mathcal{F}_0(t_k) U(d_i, t_k) \tag{13}$$

   where the utility function $U(d_i, t_k)$ is defined in terms of $d_i \in D$ occupying several dominant decisions in t_k. Formally,

$$U(d_i, t_k) = \begin{cases} 3, & d_i \in sDom\text{-}t_k \\ 2, & d_i \in dom\text{-}t_k \;\&\; d_i \notin sDom\text{-}t_k \\ 1, & d_i \in wDom\text{-}t_k \;\&\; d_i \notin dom\text{-}t_k \\ 0, & d_i \notin wDom\text{-}t_k \end{cases} \tag{14}$$

Laplace Criterion assumes that the absence of information means that all outcomes have equal probability. When there are unknown probabilities, a unique set of consistent probability distributions can be determined using the principle of maximum entropy, and it characterises the known information well and distributes the probabilities fairly [9]. After confirming the maximum entropy distribution (Equation 12), Laplace Criterion selects the decision of maximum utility as the best decision (Equation 13).

4. Probabilistic Deduction on PADF

In this section, we explore how to use probabilistic rules to solve for the best decision in PADF. First, we map PADF to the language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$.

Definition 4.1. Given a PADF $\langle R, H, \rho \rangle$, its corresponding language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$ are:
- $\mathcal{L} = \{have(d,a) | d \in D, a \in A\} \cup \{met(d,g) | d \in D, g \in G\}$;
- $\mathcal{R} = \{n_0 \leftarrow n_1, ..., n_k : [\theta] | \theta = \rho(n_0, ..., n_k)\}$

Namely, the language $\mathcal{L}$ consists of two types of sentences $have(d,a)$ and $met(d,g)$, representing the attributes a that decision d have and the goal g that decision d may meet, respectively. The set of p-rules $\mathcal{R}$ are mapped from ρ. We illustrate how to map PADF to a language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$ with the following example.

Example 4.1. (Example 3.1 continued) A language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$ corresponding to the PADF $\langle R, H, \rho \rangle$ in Example 3.1 is:
- $\mathcal{L}$ consists of:
  $have(stock, cutIR)$, $have(stock, ESP)$, $have(bond, PMI)$,
  $met(stock, highYield)$, $met(stock, lowRisk)$,
  $met(bond, highYield)$, $met(bond, lowRisk)$;
- $\mathcal{R}$ consists of:
  $met(stock, highYield) \leftarrow have(stock, cutIR) : [0.7]$, $have(stock, cutIR) \leftarrow : [0.4]$,
  $met(stock, highYield) \leftarrow have(stock, ESP) : [0.5]$, $have(stock, ESP) \leftarrow : [0.7]$,
  $met(bond, highYield) \leftarrow have(bond, PMI) : [0]$, $have(bond, PMI) \leftarrow : [0.8]$,
  $met(stock, lowRisk) \leftarrow have(stock, ESP) : [0.2]$,
  $met(bond, lowRisk) \leftarrow have(bond, PMI) : [0.95]$;

The study from [5] points out that the time to solve Rule-PSAT increases exponentially as the size of $\mathcal{L}$ grows. This growth phenomenon is more severe for PADF, the size of the corresponding $\mathcal{L}$ possible worlds are 2^{ik+j} (where i is the number of decisions, k is the number of goals and j is the number of attributes). A simple PADF like Example 4.1 will lead to 2^7 possible worlds in CC set. The study [10] proposes that although there are various links between goals, they are often considered independently in decision. On this basis, we introduce the Relative Independence Assumption (RIA) for PADF. First, we define the notion of *reachable* node.

Definition 4.2. Given a PADF $\langle R, H, \rho \rangle$ with its corresponding language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$. For any $n_i, n_j \in \mathcal{L}$, we say that n_i is *reachable* from n_j if and only if the following two conditions holds:
C1. there exists a p-rule $n_i \leftarrow n_j, ..., n_k : [\theta]$ or
C2. there exists $n' \in \mathcal{L}$ such that n_i is *reachable* from n' and n' is *reachable* from n_j.

Definition 4.2 is given recursively with C1 being the base case. With this, RIA on $\mathcal{L}$ corresponding to PADF can be explained as follows.

Assumption 4.1. (Relative Independence Assumption) Given a PADF $\langle R, H, \rho \rangle$ with its corresponding language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$. For all $n_i, n_j \in \mathcal{L}$ are *independent* if and only if n_i is not *reachable* from n_j and n_j is not *reachable* from n_i.

Proposition 4.1. For all $met(d,g) \in \mathcal{L}$ are *independent* with each other.

The proof of Proposition 4.1 comes from Definition 3.1, where all $(d,g) \in E_g$ are not allowed to point to other nodes in the graph H, and thus are mutually unreachable when mapped to $met(d,g) \in L$. With this, we can construct the reachable set of $met(d,g)$.

Definition 4.3. Given a PADF $\langle R, H, \rho \rangle$ with its corresponding language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$. For each $met(d,g) \in \mathcal{L}$, have a *reachable set* $\mathcal{L}_{(d,g)}$ such that $n_i \in \mathcal{L}_{(d,g)}$ if and only if $n_i = met(d,g)$ or the $met(d,g)$ is reachable from n_i.

For $\mathcal{L}_{(d,g)}$, its p-rules denoted as $\mathcal{R}_{(d,g)}$ and CC Set denoted as $\Omega_{(d,g)}$.

Proposition 4.2. Given a *reachable set* $\mathcal{L}_{(d,g)} \subseteq \mathcal{L}$ such that the $\Pr(met(d,g))$ solved by $\mathcal{L}_{(d,g)}$ is consistent with that solved by $\mathcal{L}$.

To compute $\Pr(met(d,g))$, we need to consider the other sentences. Let $\mathcal{L}_{(d,g)} = \{met(d,g), n_1, ..., n_i\}$ and $\mathcal{L} = \{met(d,g), n_1, ..., n_k\}$ such that $i \leq k$,. The sentences of $n_{i+1}, ..., n_k \in \mathcal{L}$ are independent with $met(d,g)$. According to probability theory, if a, b are independent of each other, then $\Pr(a|b) = \Pr(a)$. That is, $\Pr(met(d,g)|n_1, ..., n_i) = \Pr(met(d,g)|n_1, ..., n_k)$ can be shown that Proposition 4.2.

With Proposition 4.2, we can compute the probability of each $met(d,g)$ locally as following process. Given a language $\mathcal{L}$ and p-rules $\mathcal{R}$ corresponding to PADF $\langle R, H, \rho \rangle$. For any $met(d,g)$ (or its negation $notMet(d,g)$) in $\mathcal{L}_{(d,g)}$, the $\Pr(met(d,g))$ is:

$$\Pr(met(d,g)) = \sum_{\omega_i \in \Omega_{(d,g)}, \omega_i \models met(d,g)} \pi(\omega_i) \tag{15}$$

Note that $\Pr(met(d,g))$ is equals to $\rho((d,g))$ in PADF. Given a PADF $\langle R, H, \rho \rangle$ with the target set $\mathcal{T}$, the joint probability distribution of each states $t \in \mathcal{T}$ can be calculated by solving each reachable set $\mathcal{L}_{(d,g)}$ locally and multiplying the probability of distinct $met(d,g)$ (or its negation $notMet(d,g)$) based on RIA as following equation:

$$\mathcal{F}(t) = \mathcal{F}(n_0 \wedge ... \wedge n_k) = \prod_{n_i \in t} \Pr(n_i) \tag{16}$$

For each $\mathcal{L}_{(d,g)} \subseteq \mathcal{L}$, it is easy to observe that σ-maximal Solution, σ-minimal Solution, and Maximum Entropy Solution is corresponding to the three decision criteria in PADF, respectively. We use an example to illustrate how to solve the best decision under different decision criteria by our process.

Example 4.2. (Example 4.1 continued) For each reachable set $\mathcal{L}_{(d,g)} \subseteq \mathcal{L}$, the corresponding set of p-rules $\mathcal{R}_{(d,g)}$ is:

- $\mathcal{R}_{(stock,highYield)}$ consists of:
  $met(stock, highYield) \leftarrow have(stock, cutIR) : [0.7]$, $\quad have(stock, cutIR) \leftarrow: [0.4]$
  $met(stock, highYield) \leftarrow have(stock, ESP) : [0.5]$, $\quad have(stock, ESP) \leftarrow: [0.7]$;
  and set up its equations as follows[2] by Definition 2.4:
  $(\pi(010) + \pi(011) + \pi(110) + \pi(111)) \times 0.7 = \pi(111) + \pi(110)$,

[2] Boolean values are used as shorthand to simplify presentation. E.g., $\pi(111)$ in $\mathcal{R}_{(stock,highYield)}$ denotes that $\pi(met(stock, highYield) \wedge have(stock, cutIR) \wedge have(stock, ESP))$.

$(\pi(001)+\pi(011)+\pi(101)+\pi(111))\times 0.5 = \pi(111)+\pi(101)$,
$0.4 = \pi(010)+\pi(011)+\pi(110)+\pi(111)$,
$0.7 = \pi(001)+\pi(011)+\pi(101)+\pi(111)$,
$1 = \pi(000)+\pi(001)+\pi(010)+\pi(011)+\pi(100)\pi(101)+\pi(110)+\pi(111)$.

- $\mathcal{R}_{(stock,lowRisk)}$ consists of:
  $met(stock,lowRisk) \leftarrow have(stock,ESP) : [0.2]$, $\quad have(stock,ESP) \leftarrow: [0.7]$;
  and set up its equations as follows:
  $(\pi(01)+\pi(11))\times 0.2 = \pi(11),\ 0.7 = \pi(01)+\pi(11),\ 1 = \pi(00)+\pi(01)+\pi(10)+\pi(11)$.
- $\mathcal{R}_{(bond,highYield)}$ consists of:
  $met(bond,highYield) \leftarrow have(bond,PMI) : [0]$ $\quad have(bond,PMI) \leftarrow: [0.8]$;
  and set up its equations as follows:
  $(\pi(01)+\pi(11))\times 0 = \pi(11),\ 0.8 = \pi(01)+\pi(11),\ 1 = \pi(00)+\pi(01)+\pi(10)+\pi(11)$.
- $\mathcal{R}_{(bond,lowRisk)}$ consists of:
  $met(bond,lowRisk) \leftarrow have(bond,PMI) : [0.95]$ $\quad have(bond,PMI) \leftarrow: [0.8]$;
  and set up its equations as follows:
  $(\pi(01)+\pi(11))\times 0.95 = \pi(11),\ 0.8 = \pi(01)+\pi(11),\ 1 = \pi(00)+\pi(01)+\pi(10)+\pi(11)$.

In probabilistic deduction, we have chosen the σ-maximal Solution to solve the best decision under *Optimistic Criterion* in PADF from Definition 3.5, it have:

1. Maximising $Pr(met(d,g))$ with σ-maximal Solution in each $\mathcal{L}_{(d,g)} \subseteq \mathcal{L}$ as follows:
   $\Pr(met(stock,highYield)) = 0.66$ $\quad \Pr(notMet(stock,highYield)) = 0.34$
   $\Pr(met(stock,lowRisk)) = 0.44$ $\quad \Pr(notMet(stock,lowRisk)) = 0.56$
   $\Pr(met(bond,highYield)) = 0.2$ $\quad \Pr(notMet(bond,lowRisk)) = 0.8$
   $\Pr(met(bond,lowRisk)) = 0.96$ $\quad \Pr(notMet(bond,lowRisk)) = 0.04$
2. The joint probability distributions for $\mathcal{T}$ under *Optimistic Criterion* calculated by Equation 16 can be seen in Table 1a.
3. Compute and compare the strongly dominant probabilities of decisions $stock$ and $bond$ by Equation 9:
   $Pr_{sDom}(stock) = 0.2904$ $\qquad Pr_{sDom}(bond) = 0.12672$
   Therefore, the best decision $\delta(D) = stock$ under Optimistic Criterion.

By similar steps with σ-minimal Solution and Maximum Entropy Solution respectively, it can be solved that the best decision under Pessimistic Criterion and Laplace Criterion are both $\delta(D) = bond$ (To save space, we'll leave out the intermediate processes) and their joint probability for $\mathcal{T}$ can be seen in Table 1b and 1c.

The Theorem 4.1 identifies a formal semantic link between the PADF and p-rule, and this link provides guidance for solving the consistent probability distribution for the target set $\mathcal{T}$ under different decision criteria on the PADF.

Theorem 4.1. Given a PADF $\langle R,H,\rho\rangle$ with corresponding language $\mathcal{L}$ and a set of p-rules $\mathcal{R}$. For each $\Pr(met(d,g))$ in its corresponding $\mathcal{L}_{(d,g)} \subseteq \mathcal{L}$, let π_{max} be its σ-maximal Solution, π_{min} be its σ-minimal Solution and π_e be its Maximimum Entropy Solution. $\mathcal{F}_0$ is a joint probability distribution for $\mathcal{T}$. For each $t_i \in \mathcal{T}$, the $\mathcal{F}_0(t)$ under different decision criteria can be calculated as follows, with $\phi = \{max, min, e\}$, respectively:

$$\mathcal{F}_0(t_i) = \prod_{\substack{t_i \models n_j,\\ n_j \in \mathcal{L}_{(d,g)}}} \sum_{\substack{\omega \in \Omega_{(d,g)},\\ \omega \models n_j}} \pi_\phi(n_j) \tag{17}$$

If $\phi = max$, then $\mathcal{F}_0$ is a set of joint probability distribution for $\mathcal{T}$ under the *Optimistic Criterion*; if $\phi = min$, then $\mathcal{F}_0$ is a set of joint probability distribution for $\mathcal{T}$ under the *Pessimistic Criterion*; if $\phi = e$, then $\mathcal{F}_0$ is a set of joint probability distribution for $\mathcal{T}$ under the *Laplace Criterion*.

Proof. (Sketch.) For $\phi = max$ or $\phi = min$, according to the Assumption 4.1 of independence between the goals, the overall probability of all goals can be written as follows:

$$\Pr(g_1 \vee ... \vee g_k) = 1 - \Pr(\neg g_1 \wedge ... \wedge \neg g_k) = 1 - \prod_{g_i \in G}(1 - \Pr(g_i)) \tag{18}$$

$$= 1 - \prod_{g_i \in G}(1 - \sum_{d_j \in D} \Pr(met(d_j, g_i))) \tag{19}$$

It is easy to see that $Pr(g_1 \vee ... \vee g_k)$ reaches an upper (or lower) bound as maximising (or minimising) each $\Pr(met(d,g))$ as Equation 8 and 10 in the Optimistic Criterion and Pessimistic Criterion. In addition to this, Proposition 4.2 shows that under RIA, the probability of locally computing each met(d,g) is the consistent with global. Therefore, $\mathcal{F}_0$ in Optimistic Criterion and Pessimistic Criterion can be solved by Theorem 4.1.

For $\phi = e$, let $H(\mathcal{F}_0)$ be the maximum entropy as Equation 12. We have $H(\mathcal{F}_0(t)) = H(n_0, ..., n_k)$ such that $t \in \mathcal{T}$. According to RIA that each $n_i \in t$ are independent with each other so that their mutual information is 0. That is, $H(n_0, ..., n_k) = \sum_{n_i \in t} H(n_i)$. When $\chi = e$, it will maximise each $H(n_i)$. Thus, such a distribution is the maximum entropy distribution consistent with $\mathcal{F}_0$. □

5. Related Works

A variety of research studies have explored probabilistic argumentation-based decision making, e.g. [11,12,13,14]. Keshavarzi-Zafarghandi et al. [15] propose numerical abstract dialectical frameworks (nADFs) for decision-making, representing actions, states, and outcomes with numerical acceptance conditions. However, our approach surpasses nADFs in handling unknown values and provides optimal decision choices across different probability perceptions, rather than only choosing the decision with the highest expected utility.

Toni et al. [16] present a decision-making method based on jury-based probabilistic argumentation, integrating quantitative reasoning with qualitative argumentation to support forecasting by aggregating subjective probability estimates. Their approach defines probability distributions directly on assumption-based argumentation, whereas our work provides insights into solving for such distributions.

The work by Hadoux [4] is particularly relevant to our study. This article introduces a framework for persuasion dialogues using decision trees and evaluates various decision rules for optimal selection in uncertain scenarios regarding the persuadee's beliefs. They employ Bayesian networks to bypass the computation of joint probability distributions, leading their decision rules to constrain only the connected nodes at each step. In contrast, we optimize our computational approach by adopting the relative independence assumption, solving locally before merging to derive the joint probability distribution. This ensures that the decision criterion can globally constrain the nodes. Thus, we believe the relative independence assumption is a superior choice over the conditional independence assumption in probabilistic reasoning.

6. Conclusion

In this study, we have studied the decision-making problem when faced with uncertainty, a common scenario in many real-world situations. We introduce the Probabilistic Abstract Decision Frameworks (PADF) as a novel approach. Through PADF, we've developed three distinct decision criteria to help select the best course of action when uncertainty is present. Our method involves linking PADF with Probabilistic Deduction (PD), revealing a meaningful connection between the two. This connection allows us to simplify computational approaches for resolving consistent probability distributions under different decision criteria, making the process less complex. By taking an argumentative perspective, our work not only addresses uncertainty in decision-making but also provides a structured framework for deductive reasoning. Looking ahead, we aim to explore specific applications and refine our methods to improve efficiency.

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