STATIONARY MEASURE OF THE OPEN KPZ EQUATION THROUGH THE ENAUD-DERRIDA REPRESENTATION

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Abstract. Recent works of Barraquand and Le Doussal [BLD22] and Bryc, Kuznetsov, Wang, and Wesolowski [BKWW23] gave a description of the open KPZ stationary measure as the sum of a Brownian motion and a Brownian motion reweighted by a Radon-Nikodym derivative. Subsequent work of Barraquand and Le Doussal [BLD23] used the Enaud-Derrida [ED04] representation of the DEHP algebra to formulate the open ASEP stationary measure in terms of the sum of a random walk and a random walk reweighted by a Radon-Nikodym derivative. They show that this Radon-Nikodym derivative converges pointwise to the Radon-Nikodym derivative that characterizes the open KPZ stationary measure. This article proves that the corresponding sequence of measures converges weakly to the open KPZ stationary measure. This provides an alternative proof of the probabilistic formulation of the open KPZ stationary measure, which avoids dealing explicitly with finite dimensional distributions. We also provide the first construction of the measure on intervals of a general length L and for the full range of parameters in the fan region ($u+v>0$).

1. Introduction

The KPZ equation is a non-linear stochastic partial differential equation (SPDE) originally introduced by Kardar, Parisi, and Zhang [KPZ86] to describe the behavior of random interfaces under relaxation and lateral growth. The open KPZ equation is the term for the same SPDE with spatial coordinate restricted to an interval of finite length, and with fixed boundary conditions. For $t \geq 0, x \in [0,L]$, the equation takes the form

$$\partial_t H_{u,v}(t,x) = \partial_{xx} H_{u,v}(t,x) + \left(\partial_x H_{u,v}(t,x)\right)^2 + \sqrt{2}\xi(t,x). \tag{1.1}$$

The boundary conditions are $\partial_x H_{u,v}(t,x)|_{x=0} = u$, and $\partial_x H_{u,v}(t,x)|_{x=L} = -v$. We sometimes refer to these as Neumann boundary conditions $u, v \in \mathbb{R}$. The equation above needs some additional explanation in order to be well-posed. In particular, we need to make sense of the nonlinear term $(\partial_x H_{u,v}(t,x))^2$. A work of Mueller [Mue91] demonstrates that any mild solution of the multiplicative stochastic heat equation

$$\partial_t Z(t,x) = \partial_x^2 Z(t,x) + \sqrt{2} Z(t,x)\xi(t,x),$$

which is almost surely positive at time zero ($Z(0,x) > 0$) will then remain positive for all x and $t > 0$. Consequently, the "Hopf-Cole" transformation $H(t,x) = \log(Z(t,x))$ gives a well-defined solution to the KPZ equation (the standard equation without open boundaries). Solutions to the open KPZ equation are defined via the same Hopf-Cole transform which we use in the standard KPZ setting, with the added condition that the solution to the multiplicative stochastic heat equation on $[0,L]$, now parameterized in terms of boundary conditions $u, v \in \mathbb{R}$, must also satisfy $\partial_x Z_{u,v}(t,x)|_{x=0} = \left(u - \frac{1}{2}\right) Z_{u,v}(t,0)$, and $\partial_x Z_{u,v}(t,x)|_{x=L} = -\left(v - \frac{1}{2}\right) Z_{u,v}(t,L)$. A well-defined notion of what it means to be a solution of the open KPZ equation appears in Corwin and Shen [CS18].

Definition 1.1. For a Hopf-Cole solution to the open KPZ equation with Neumann boundary parameters $u, v \in \mathbb{R}$, $H_{u,v}(t,x)$, we say that the law of the process $\{H_{u,v}(x) - H_{u,v}(0)\}_{x\in[0,L]}$ on $C[0,L]$ is stationary for the open KPZ increment process if, when the open KPZ equation is started with the initial data $H_{u,v}(x)$ at time $t = 0$, then for all $t \geq 0$, the law of the increment $\{H_{u,v}(t,x) - H_{u,v}(t,0)\}_{x\in[0,L]}$ is equal to the law of the initial data. We refer to the law of such a process as a stationary measure of the open KPZ increment process with initial data $H_{u,v}(x) - H_{u,v}(0)$.

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Corwin and Knizel [CK24] provided the first proof of the existence of a stationary measure for the open KPZ equation for all $u, v \in \mathbb{R}$, and gave a characterization of the measure in the setting where $u+v \geq 0$ in terms of a multipoint Laplace transform. Later works by Barraquand and Le Doussal [BLD22] and Bryc, Kuznetsov, Wang, and Wesolowski [BKWW23], gave a probabilistic description of this measure in terms of a stochastic process. They showed that the stationary measure of the open KPZ increment process could be described (for $L = 1$ and $u + v > 0$ and $u, v > -1$) in terms of a Radon-Nikodym transform on the joint measure of a pair of Brownian motions. The following definition gives a precise characterization of the appropriate measure, in the slightly more general setting (for any L, and $u + v > 0$) needed to state our main theorem.

We will use the notation $\mathbb{P}_L := \mathbb{L} \times \mathbb{W}_L$ for the product of the Lebesgue measure on $\mathbb{R}$, which we denote by $\mathbb{L}$, and the measure of a two-dimensional Brownian motion (with covariance matrix equal to the identity) parameterized by time $t \in [0, L]$, which we denote by $\mathbb{W}_L$. We will use the notation $C_0[0, L]$ to denote continuous functions on $[0, L]$, started at 0.

Definition 1.2. We introduce a function $H(x, g, h) : \mathbb{R} \times C_0[0, L] \times C_0[0, L] \to \mathbb{R}$ and a partition function $\mathcal{Z}_{u,v}$.

$$H(x, g, h) := \exp\left(-2(u+v)x - 2vg(L) - e^{-2x} \int_0^L e^{-2g(t)} dt\right), \qquad \mathcal{Z}_{u,v} := \mathbb{E}_{\mathbb{P}_L}\left[H(x, g, h)\right].$$

By $\mathbb{E}_{\mathbb{P}_L}[\cdot]$ in the definition above, we mean integrating with respect to the measure $\mathbb{P}_L$ (this is a slight abuse of notation, since $\mathbb{P}_L$ is an infinite measure). For all $u + v > 0$, and any A in the product Borel σ-algebra on $\mathbb{R} \times C_0[0, L] \times C_0[0, L]$, we define the measure

$$\mathbb{Q}_{L;u,v}(A) := \mathbb{E}_{\mathbb{P}_L}[\mathcal{Z}_{u,v}^{-1} H(x, g, h) \mathbf{1}_A].$$

Remark 1.3. To recover the expression in [BLD23, Equation (40)], define $U(t) := g(t)+x$. It is advantageous for the calculations in our paper to separate the starting value x from the function $g(t)$.

We can now state our main result.

Theorem 1.4. *The stationary measure of the open KPZ increment process on the interval $[0, L]$ with Neumann boundary conditions $u, v \in \mathbb{R}$ which satisfy $u + v > 0$ is given by $\{H(x) - G_{u,v}(x)\}_{x\in[0,L]}$ where $(X, G_{u,v}(x), H(x))$ are sampled from the distribution defined by $\mathbb{Q}_{L;u,v}$ (Definition 1.2). The process $H(x)$ is a Brownian motion with diffusion coefficient $1/\sqrt{2}$.*

That there is a unique such measure is due to the work of Knizel and Matetski [KM22] and Parekh [Par22]. That this is a stationary measure for the open KPZ increment process when $L = 1$ and $u + v > 0$, $u, v > -1$ is known through work of Barraquand and Le Doussal [BLD22] and Bryc, Kuznetsov, Wang, and Wesolowski [BKWW23], who predicted that the same would hold in the general case that $u + v > 0$. The contributions of this paper are to prove this theorem for general L and all $u + v > 0$, as well as provide a rigorous proof of a new path to obtaining this stationary measure, through the Enaud-Derrida representation of the DEHP algebra [ED04] (see Section 2.2.1 for background on the open ASEP, the DEHP algebra, and the Enaud-Derrida representation).

Barraquand and Le Doussal [BLD23] demonstrate that, through the Enaud-Derrida representation, the stationary measure of the open ASEP increment process (Definition 2.1) can be described as the sum of reweighted random walk processes, and that the functional which reweights the processes, under appropriate scaling, converges pointwise to $H(x, g, h)$ (Theorem 2.8). Our result builds on their calculation to show weak convergence of measures. This approach has the advantage of circumventing the several steps involved in Corwin and Knizel's construction of the open KPZ stationary measure. As noted above, it also provides a probabilistic construction of the open KPZ stationary measure in a wider range of parameters than in previous works. This article also provides an alternative proof that the partition function in Definition 1.2 is finite, a fact which previously appeared in work of Bryc, Kuznetsov, Wang, and Wesolowski [BKWW23] for $L = 1$. We provide an alternative argument (for general L), following a paper of Yor [Yor92].

1.1. **Future Directions.** One direction which the author hopes to explore is the application of the Enaud-Derrida representation and path decomposition of the open ASEP stationary measure to asymptotics of open multi-species ASEPs. The techniques in this paper open a possible route to understand the asymptotics of these systems.

We also note that when the ASEP stationary measure is framed in terms of Askey-Wilson polynomials, the formula has a restriction (see [BW17, Theorem 1] for the origin of this constraint) $t_0 \leq t_1 \leq \cdots \leq t_n$, which makes it hard to study large deviations of the height function in the limit. The Enaud-Derrida formula avoids this restriction, and may provide a more tractable path to study the large deviations.

1.2. **Outline.** In Section 2, we introduce the open ASEP, the matrix product ansatz, and the Enaud-Derrida representation. We discuss the work of Derrida, Enaud, and Lebowitz [DEL04] and Barraquand and Le Doussal [BLD23] which uses the Enaud-Derrida representation to describe the stationary measure of the open ASEP increment process as the sum of reweighted random walks. We establish the rescaling that is needed to obtain the stationary measure of the open KPZ increment process from their formula. In the process, we define a sequence of measures $\mathbb{Q}^{(N)}_{L;u,v}$ (Definition 2.9) which are related to the rescaled stationary measure of open ASEP in the same way as $\mathbb{Q}_{L;u,v}$ is related to the stationary measure of open KPZ. We state the main technical result (Theorem 2.10), that $\mathbb{Q}^{(N)}_{L;u,v}$ converge weakly to $\mathbb{Q}_{L;u,v}$. Finally, we prove Theorem 1.4 from Theorem 2.10.

Most of the paper is devoted to the proof of Theorem 2.10, which relies on a crucial bound established in Lemma 3.1. In Section 3, in addition to proving Lemma 3.1, we demonstrate that $\mathbb{Q}^{(N)}_{L;u,v}$ and $\mathbb{Q}_{L;u,v}$ are probability measures, which involves showing that the associated partition functions are finite. The main challenge of the paper is the proof of Lemma 3.1, other arguments are essentially standard applications of KMT embedding arguments and dominated convergence.

In Section 4, we establish a bound on the distance between $\mathbb{E}_{\mathbb{W}^{(N)}_L}[H^{(N)}(x,g,h)]$ and $\mathbb{E}_{\mathbb{W}_L}[H(x,g,h)]$ as a function of x. In Section 5 we use the bound from Section 4 to prove Theorem 2.10. Appendix A contains the proof of several lemmas which are used in Section 3, and Appendix B contains the proof of pointwise convergence of the Radon-Nikodym derivatives defined by Barraquand and Le Doussal to $H(x,g,h)$.

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2. Background and Main Technical Result

In this section, we build up the definitions and notation which we need to state the main technical result which goes into the proof of Theorem 1.4 (Theorem 2.10). We will finish the section by giving the proof of Theorem 1.4 from Theorem 2.10, and further sections of the paper will be devoted to the proof of Theorem 2.10. We begin by defining notation which will be used throughout the paper.

2.1. **Notation.** We will typically use the notation $t_i := iLN^{-1}$, as well as $[n]_q = \frac{1-q^n}{1-q}$. When discussing the Enaud-Derrida representation and the matrix product ansatz, we will use bra-ket notation, with $\langle \cdot |$ indicating a row vector and $| \cdot \rangle$ indicating a column vector. The subscript N will indicate that a process has N steps or that a function takes as input something with N steps, as appropriate, whereas the superscript (N) will denote that a process of the former kind has been rescaled by $N^{-\frac{1}{2}}$ spatial scaling and N^{-1} temporal scaling, or, when applied to a function, that the input the function takes has been rescaled in the same way. We will use $C_0[0,L]$ to denote continuous functions on the domain $[0,L]$ which start at 0, and $C[0,L]$ to denote continuous functions on the domain $[0,L]$ which can start at any real value.

We use $\mathbb{L}$ to denote the Lebesgue measure on $\mathbb{R}$ and $\mathbb{W}_L$ to denote the two-dimensional Wiener measure on two-dimensional continuous functions on the domain $[0,L]$, starting at $(0,0)$. We use the notation $\mathbb{P}_L := \mathbb{L} \times \mathbb{W}_L$, and $\mathbb{L}^{(N)} := \sum_{i \in N^{-\frac{1}{2}}\mathbb{Z}} N^{-\frac{1}{2}} \delta_i$, both of which denote infinite measures. Throughout the paper, we use the notation $\mathbb{E}_P[\cdot]$ to denote integrating with respect to a measure P. The measure P will not always be a probability measure (for example, $\mathbb{P}_L$), so this is a slight abuse of notation.

2.2. **The open ASEP.** The underlying lattice of the open ASEP has finite size N. We use α, γ to denote the rate of particle movement into and out of the leftmost lattice point from, respectively to, the left reservoir. Similarly, we use β, δ to denote the rate of particle movement out of, respectively into, the rightmost lattice point of the lattice to, respectively from, the right reservoir. Within the lattice, particles move right at rate 1 and left at rate $q \in (0,1)$. We will often re-parameterize the open ASEP in terms of (A, B, C, D, q).

$$A := \kappa^+(q, \beta, \delta), \qquad B := \kappa^-(q, \beta, \delta), \qquad C := \kappa^+(q, \alpha, \gamma), \qquad D := \kappa^-(q, \alpha, \gamma),$$

$$\kappa^{\pm}(q, x, y) = \frac{1}{2x}\left(1 - q - x + y \pm \sqrt{(1 - q - x + y)^2 + 4xy}\right).$$

The state space of the open ASEP is $\tau = (\tau_1(t), ..., \tau_N(t)) \in \{0,1\}^N$ where $\tau_i(t) = 1$ when there is a particle in position i at time t and $\tau_i(t) = 0$ when position i on the lattice is empty at time t. The open ASEP process $\tau(t)$ is a Markov process defined by the state space $\tau(t) \in \{0,1\}^N$ and an infinitesimal generator $\mathcal{L}(\cdot)$ which acts on functions of the open ASEP state space $f : \{0,1\}^N \to \mathbb{R}$. The time evolution of ASEP induces a Markov process, the height function process, $h_{N;\alpha,\beta,\gamma,\delta}(t, r) = h_{N;\alpha,\beta,\gamma,\delta}(t, 0) + \sum_{i=1}^{r}(2\tau_i(t) - 1)$, for $r \in [[1, N]]$ where $h_{N;\alpha,\beta,\gamma,\delta}(t, 0)$ is the net number of particles which have entered through the left boundary at time t.

Definition 2.1. We say that the law of the process $\{h_{N;\alpha,\beta,\gamma,\delta}(r)\}_{r\in[[1,N]]}$ with $h_{N;\alpha,\beta,\gamma,\delta}(0) = 0$ is stationary for the open ASEP increment process if, when the open ASEP is started with the initial data $h_{N;\alpha,\beta,\gamma\delta}(x)$ at time $t = 0$, then for all $t \geq 0$, the law of the increment $\{h_{N;\alpha,\beta,\gamma,\delta}(t, r) - h_{N;\alpha,\beta,\gamma,\delta}(t, 0)\}_{r\in[[1,N]]}$ is equal to the law of the initial data. We refer to the law of such a process as a stationary measure of the open ASEP increment process with initial data $h_{N;\alpha,\beta,\gamma,\delta}(r)$.

As the size of the lattice grows, the value of the limiting stationary current depends on two parameters, ρ_ℓ and ρ_r, which can be interpreted physically as effective densities at the left and right boundaries. See the survey by Corwin [Cor22] for more information about the boundary densities. For the purposes of this article, it is only important that they can be expressed in terms of the open ASEP boundary parameters (A, B, C, D) as

$$\rho_\ell = \frac{1}{1 + C}, \qquad \rho_r = \frac{A}{1 + A}. \tag{2.1}$$

Previous work of Corwin and Knizel [CK24] constructed the stationary distribution of the open KPZ equation as the limit of the stationary distribution of the open ASEP under a weak asymmetry scaling. In this article, we will use the same set of scaling assumptions as Corwin and Knizel.

Assumptions 2.2. *The following assumptions define our scaling limit*

1. *Weak asymmetry scaling* $q = e^{-2/\sqrt{N}}$
2. *Boundary parameter scaling*
$$A = q^v \qquad B = -q \qquad C = q^u \qquad D = -q$$
3. *Height function scaling*
$$h^{(N)}_{L;u,v}(x) - h^{(N)}_{L;u,v}(0) = N^{-\frac{1}{2}}\left(h_{N;\alpha,\beta,\gamma,\delta}(\lfloor NL^{-1}x\rfloor) - h_{N;\alpha,\beta,\gamma,\delta}(0)\right) \tag{2.2}$$

We note that conditions (1) *and* (2) *imply triple point scaling*

$$\rho_\ell = \frac{1}{2} + \frac{u}{2}N^{-\frac{1}{2}} + o(N^{-1}) \qquad \rho_r = \frac{1}{2} - \frac{v}{2}N^{-\frac{1}{2}} + o(N^{-1})$$

The Enaud-Derrida [ED04] representation of the stationary measure of open ASEP, defined in the next section, provides a way of characterizing the stationary measure of open ASEP as the sum of two random walk processes. We will use this representation to prove Theorem 1.4.

2.2.1. *Matrix Product Ansatz Techniques and the Enaud-Derrida Representation.* A work of Derrida, Evans, Hakim and Pasquier [DEHP93] introduced a method, the Matrix Product Ansatz, for obtaining the stationary solutions to open ASEP. The idea behind this technique is to put forth the ansatz that there exists an algebra (the "DEHP" algebra, after the authors of the original paper) with representation in terms of matrices $\mathbf{D}$ and $\mathbf{E}$ and vectors $\langle W|, |V\rangle$ (possibly infinite dimensional) such that the probability of a given state τ on the open interval is given by

$$\mathbb{P}(\tau) = \frac{\langle W| \prod_{i=1}^{N} \mathbf{D}\tau_i + \mathbf{E}(1-\tau_i)|V\rangle}{\langle W|(\mathbf{D}+\mathbf{E})^N|V\rangle}.$$

Imposing the condition that this distribution is stationary results in the relations

$$\begin{aligned}\mathbf{DE} - q\mathbf{ED} &= \mathbf{D} + \mathbf{E},\\ (\beta\mathbf{D} - \delta\mathbf{E})|V\rangle &= |V\rangle,\\ \langle W|(\alpha\mathbf{E} - \gamma\mathbf{D}) &= \langle W|.\end{aligned}$$

Mathematicians and physicists have discovered several representations of the DEHP algebra. The original representation [DEHP93] was used to solve the open TASEP. Later, foundational papers by Sasamoto [Sas99] and Uchiyama, Sasamoto, and Wadati [USW04] gave representations of the DEHP algebra in terms of Askey-Wilson polynomials. Enaud and Derrida [ED04] studied a different representation of the DEHP algebra, now called the Enaud-Derrida representation (there is a closely related representation which was discovered independently by Corteel and Nunge [CN20]). The Enaud-Derrida representation is given by

$$\mathbf{D} = \begin{bmatrix} [1]_q & [1]_q & 0 & 0 & 0 & \dots \\ 0 & [2]_q & [2]_q & 0 & 0 & \dots \\ 0 & 0 & [3]_q & [3]_q & 0 & \dots \\ \dots & \dots & .. & \dots & \dots & \dots \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} [1]_q & 0 & 0 & 0 & \dots \\ [2]_q & [2]_q & 0 & 0 & \dots \\ 0 & [3]_q & [3]_q & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix},$$

$$\langle W| = \sum_{n\geq 1} \left(\frac{1-\rho_\ell}{\rho_\ell}\right)^n \langle n|, \qquad |V\rangle = \sum_{n\geq 1} \left(\frac{\rho_r}{1-\rho_r}\right)^n [n]_q |n\rangle,$$

where $[\cdot]_q$ is defined in Section 2.1, and $|n\rangle$ and $\langle n|$ denote the basis vectors. Derrida, Enaud, and Lebowitz [DEL04] noticed that it is possible to write the increment of the stationary state of the open ASEP as the sum of two random walks,

$$\{h_{N;\alpha,\beta,\gamma,\delta}(t,r) - h_{N;\alpha,\beta,\gamma,\delta}(t,0)\}_{r\in[[1,N]]} = \left\{\sum_{j=1}^{r}(2\tau_i - 1)\right\}_{r\in[[1,N]]} = \{n_r - n_0 + m_r\}_{r\in[[1,N]]}.$$

Barraquand and Le Doussal used the Enaud-Derrida representation to define a measure on $(n_r, m_r)_{r=1}^N$ by a reweighting of a two-dimensional random walk started at $(n_0, 0)$ for any $n_0 \in \mathbb{Z}_{>0}$.

Definition 2.3 (Equation (25) [BLD23])**.** We denote the space of two-dimensional N-step simple random walks started at $(0,0)$ by $W_{N,0}$ and define a measure on $\mathbb{Z}_{>0} \times W_{N,0}$

$$\mathbb{P}_N(r, \vec{n}, \vec{m}) := \frac{\mathbf{1}_{r>0} 4^N}{\widetilde{Z}_N(q)} \left(\frac{1-\rho_\ell}{\rho_\ell}\right)^r \left(\frac{\rho_r}{1-\rho_r}\right)^{n_N + r} \prod_{i=0}^{N} [n_i + r]_q \mathbb{P}^{\mathrm{SSRW}}(\vec{n}, \vec{m}), \tag{2.3}$$

where $\widetilde{Z}_N(q)$ is the partition function which normalizes this measure on $r \in \mathbb{Z}_{>0}$ and two-dimensional random walks of N steps started at $(0,0)$ given by $(\vec{n}, \vec{m})$. The notation ρ_r and ρ_ℓ is defined in (2.1), and $\mathbb{P}^{\mathrm{SSRW}}(\cdot,\cdot)$ is the law of a two-dimensional simple symmetric random walk started at $(0,0)$.

Remark 2.4. The terms $[n_i + r]_q$ guarantee that $n_i + r > 0$ for all $i \in [[0, N]]$.

Theorem 2.5 (Section 3 [BLD23])**.** *The stationary measure of the open ASEP increment process on the finite lattice of size N is given by $\{n_i + m_i\}_{i\in[[1,N]]}$, where $(r, \vec{n}, \vec{m})$ are sampled from the measure in Definition 2.3.*

The paper by Barraquand and Le Doussal [BLD23] which first proposed that this representation of the stationary measure of the open ASEP increment process could be used to obtain the stationary measure of the open KPZ increment process made several additional contributions. They found the rescaling of (2.3) that was appropriate for obtaining the stationary measure [BLD23, Equation (33)] and, after formulating both the ASEP stationary measure and the KPZ stationary measure in terms of Radon-Nikodym reweightings of standard measures $\mathbb{P}_L$ and $\mathbb{P}$, they demonstrated that the Radon-Nikodym derivatives corresponding to the appropriately rescaled open ASEP stationary measure converge pointwise to the function $H(x,g,h)$ defined in Definition 1.2 (Theorem 2.8). We will now describe their rescaling, as preparation for the proof of Theorem 1.4. We extend the two-dimensional random walk to a two-dimensional continuous function by rescaling in time and space and linearly interpolating between that lattice points it occupies. To be precise, for $i \in [[1, N-1]]$, we extend the function by allowing it to take the value $N^{-\frac{1}{2}}(n_i, m_i)$ on the time interval $[t_i, t_{i+1})$. The rest of the introduction will be devoted to setting up and defining the appropriately rescaled sequence of measures. We use continuous two-dimensional functions $(f,h) \in C[0,L] \times C_0[0,L]$ to denote the paths of these processes. Due to the fact that n_0 can take any value in $\mathbb{Z}_{>0}$, we see that $f(0) \in N^{-\frac{1}{2}}\mathbb{Z}_{>0}$. For ease of notation, we define

$$x := f(0), \qquad\qquad g(t) := f(t) - f(0).$$

We will denote the space of paths (g,h) by $W^{(N)}[0,L]$, with the notation motivated by the fact that, sans Radon-Nikodym derivative factor, the process would converge to a standard two-dimensional Brownian motion. All permissible values of x will lie in $N^{-\frac{1}{2}}\mathbb{Z}_{>0}$. We will use $\mathbb{W}_L^{(N)}$ to denote the sequence of measures which assign the weight of the two-dimensional simple symmetric random walk to the rescaled path $(g,h) \in W^{(N)}[0,L]$, meaning that if $\vec{g} = (g(t_0), ..., g(t_N))$, and similarly for $\vec{h}$, that $\mathbb{W}_L^{(N)}[(g,h)] = \mathbb{P}^{\text{SSRW}}(\sqrt{N}\vec{g}, \sqrt{N}\vec{h})$. As before, $\mathbb{P}^{\text{SSRW}}(\cdot,\cdot)$ is the law of the two-dimensional simple symmetric random walk. We denote the product measure at each $N \in \mathbb{N}$ by $\mathbb{P}_L^{(N)} := \mathbb{L}^{(N)} \times \mathbb{W}_L^{(N)}$, and note that this is an infinite measure, due to the first factor. The measure $\mathbb{L}^{(N)}$ is defined in Section 2.1. Using this notation, we can write the sequence of measures which are obtained by a weak asymmetry rescaling of (2.3) via a Radon-Nikodym transformation.

Definition 2.6. We define a sequence of functions $R^{(N)} : \mathbb{R} \times C_0[0,L] \times C_0[0,L] \to \mathbb{R}$, which give a reweighting of the density of $\mathbb{P}^{(N)}$ by

$$R^{(N)}(x,g,h) := \mathbf{1}_R \left(\frac{1-\rho_\ell}{\rho_\ell}\right)^{\sqrt{N}x} \left(\frac{\rho_r}{1-\rho_r}\right)^{\sqrt{N}(g(L)+x)} \prod_{i=0}^{N} [\sqrt{N}(g(t_i)+x)]_q,$$

The conditions in Assumptions 2.2 imply that $\left(\frac{(1-\rho_\ell)\rho_r}{\rho_\ell(1-\rho_r)}\right) = e^{-2(u+v)/\sqrt{N}}$, and $\frac{\rho_r}{1-\rho_r} = e^{-2v/\sqrt{N}}$, so the expression above becomes

$$R^{(N)}(x,g,h) = \mathbf{1}_R \exp\left(-2(u+v)x - 2vg(L)\right) \prod_{i=0}^{N} [\sqrt{N}(g(t_i)+x)]_q.$$

The set R is defined $R := \{x \in N^{-\frac{1}{2}}\mathbb{Z}_{>0}; (g,h) \in W^{(N)}[0,L]\}$, and ρ_ℓ and ρ_r are as in Assumptions 2.2. For the definition of the notation $[\cdot]_q$, see Section 2.1. We also rewrite the formula for the partition function rescaled from (2.3) in terms of $R^{(N)}(x,g,h)$,

$$\widetilde{Z}_{L;u,v}^{(N)} := \mathbb{E}_{\mathbb{P}_L^{(N)}}\left[R^{(N)}(x,g,h)\right] = N^{-\frac{1}{2}} \sum_{x \in N^{-\frac{1}{2}}\mathbb{Z}_{>0}} \sum_{(g,h) \in W^{(N)}[0,L]} R^{(N)}(x,g,h)\mathbb{W}^{(N)}(g,h).$$

In the current notation, which matches Barraquand and Le Doussal's [BLD23] notation for the open ASEP stationary measure, it is not true that $R^{(N)}(x,g,h) \to H(x,g,h)$ pointwise. It is first necessary to renormalize the function $R^{(N)}(x,g,h)$. This corresponds to "recentering" the Radon-Nikodym derivative to the appropriate height in x in order to obtain the correct pointwise limit. To that end, we define $H^{(N)}(x,g,h)$ and $Z_{L;u,v}^{(N)}$ such that

$$(Z_{L;u,v}^{(N)})^{-1} H^{(N)}(x,g,h) = (\widetilde{Z}_{L;u,v}^{(N)})^{-1} R^{(N)}(x + \log(\sqrt{N}), g, h). \tag{2.4}$$

Definition 2.7. We define a new sequence of functions $H^{(N)} : \mathbb{R} \times C_0[0,L] \times C_0[0,L] \to \mathbb{R}$.

$$
\begin{aligned}
H^{(N)}(x,g,h) &:= R^{(N)}(x + \log(\sqrt{N}), g, h) \left(\frac{(1-\rho_\ell)\rho_r}{\rho_\ell(1-\rho_r)} \right)^{-\sqrt{N}\log(\sqrt{N})} (1-q)^{N+1}, \\
&= \mathbf{1}_H \exp\left(-2(u+v)x - 2vg(L)\right) \prod_{i=0}^{N} \left(1 - \frac{e^{-2(g(t_i)+x)}}{N} \right),
\end{aligned}
$$

where the set H is defined $H := \{x \in \widetilde{\mathbb{Z}}(N); (g,h) \in W^N[0,L]\}$, and

$$
\widetilde{\mathbb{Z}}(N) := \{x | \sqrt{N}(x + \log(\sqrt{N})) \in \mathbb{Z}_{>0}\}. \tag{2.5}
$$

The indicator function guarantees that this function will always be positive. We also define a new partition function $Z^{(N)}_{L;u,v}$ such that $H^{(N)}(x,g,h)$ and $Z^{(N)}_{L;u,v}$ satisfy (2.4),

$$
Z^{(N)}_{L;u,v} := \mathbb{E}_{\mathbb{P}^{(N)}_L} \left[H^{(N)}(x,g,h) \right] = N^{-\frac{1}{2}} \sum_{x \in \widetilde{\mathbb{Z}}(N)} \sum_{(g,h) \in W^{(N)}[0,L]} H^{(N)}(x,g,h) \mathbb{W}^{(N)}(g,h).
$$

Barraquand and Le Doussal [BLD23] give a sketch of the proof of pointwise convergence $H^{(N)}(x,g,h) \to H(x,g,h)$ and conjecture weak convergence of the sequence of measures given by these Radon-Nikodym derivatives. We provide a proof of pointwise convergence in Appendix B.

Theorem 2.8. *For all $u+v>0$, $H^{(N)}(x,g,h)$ converges pointwise to $H(x,g,h)$ under Assumptions 2.2.*

In this article, we demonstrate weak convergence of the associated measures. For the rest of the article, we will deal with $H^{(N)}(x,g,h)$ instead of $R^{(N)}(x,g,h)$.

Definition 2.9. We define a sequence of measures $\mathbb{Q}^{(N)}_{L;u,v}$ which are given by the Radon-Nikodym derivative defined above applied to $\mathbb{P}^{(N)}_L$. Here, the arguments are $x \in \widetilde{\mathbb{Z}}(N)$, $(g,h) \in W^N[0,L]$.

$$
\mathbb{Q}^{(N)}_{L;u,v}(A) : = \mathbb{E}_{\mathbb{P}^{(N)}_L} \left[(Z^{(N)}_{L;u,v})^{-1} H^{(N)}(x,g,h) \mathbf{1}_A \right] = (Z^{(N)}_{L;u,v})^{-1} N^{-\frac{1}{2}} \sum_{(x,g,h) \in A} H^{(N)}(x,g,h) \mathbb{W}^{(N)}(g,h).
$$

In the equation above, A is any event in the product Borel σ-algebra on $R \times C_0[0,L] \times C_0[0,L]$. The rescaled stationary measure of the open ASEP increment process (2.2) can be written

$$
\left\{ h^{(N)}_{L;u,v}(x) \right\}_{x \in [0,L]} = \left\{ n^{(N)}(x) + m^{(N)}(x) \right\}_{x \in [0,L]}
$$

where $(r^{(N)}, n^{(N)}(x), m^{(N)}(x))$ are sampled from the measure $\mathbb{Q}^{(N)}_{L;u,v}$. This notation for functions $n^{(N)}(x)$ and $m^{(N)}(x)$ will not be used again, it is intended to suggest the notation in Definition 2.3, but these are not random walks: they are continuous functions on $[0,L]$.

We can now state the main technical result which allows us to prove Theorem 1.4.

Theorem 2.10. *Under Assumptions 2.2, the measures $\mathbb{Q}^{(N)}_{L;u,v}$ converge weakly in the uniform on compact topology to $\mathbb{Q}_{L;u,v}$ (Definition 1.2).*

The proof of Theorem 2.10 will be the subject of most of the rest of the paper. It follows from two different KMT-type embedding theorems. We demonstrate first that the partition functions $Z^{(N)}_{L;u,v}$ are uniformly bounded in N, using a KMT embedding result for random walk bridges and Brownian bridges established by Dimitrov and Wu [DW21]. We also provide an independent proof that $\mathcal{Z}_{u,v}$ is bounded, following a paper of Yor [Yor92]. Then we use the standard KMT embedding to establish a bound on the distance between $\mathbb{E}_{\mathbb{W}^{(N)}}[F(x,g,h)H^{(N)}(x,g,h)]$ and $\mathbb{E}_{\mathbb{W}}[F(x,g,h)H(x,g,h)]$ in terms of N, x, and $C_F := \sup|F(x,g,h)|$ for any bounded continuous function $F(x,g,h)$. We use this bound to show that $\mathbb{E}_{\mathbb{P}^{(N)}}[F(x,g,h)H^{(N)}(x,g,h)]$ converges to $\mathbb{E}_{\mathbb{P}}[F(x,g,h)H(x,g,h)]$. The case where $F(x,g,h) \equiv 1$ demonstrates that $Z^{(N)}_{L;u,v} \to \mathcal{Z}_{u,v}$. Combining these observations allows us to conclude that $\mathbb{Q}^{(N)}_{L;u,v}$ converges weakly to $\mathbb{Q}_{L;u,v}$.

The most technically challenging part of the paper is the proof of Lemma 3.1. This argument, and the applicability of the KMT embedding for random walk bridges, which is used to obtain the bound, crucially relies on the fact that the part of the function $H^{(N)}(x,g,h)$ which depends on the route of the path g is bounded between 0 and 1 for all permissible paths.

2.3. **Proof of Theorem 1.4.** In this section, we will use Theorem 2.10 to prove Theorem 1.4. The version of the open KPZ equation stated in (1.1) takes a different form than that in the papers by Corwin and Knizel [CK24], Corwin and Shen [CS18], and Parekh [Par19]. We can recover that form of the open KPZ equation (with coefficients of $1/2$ in front of both derivative terms on the right-hand side of the equation and a coefficient of 1 in front of the noise) by applying the transformation $t \mapsto 2t$ to (1.1). Since this is only a time transformation, these two versions of the open KPZ equation have the same stationary measure. This justifies applying the results of those papers. The result that we quote is proved in Corwin and Shen [CS18] and Parekh [Par19] for $L = 1$. We quote the statement as it appears in Corwin and Knizel [CK24, Proposition 3.2] for the sake of clarity. The proof by Parekh [Par19, Theorem 5.7], for instance, is unchanged when the interval length is altered to $[0, L]$.

Proposition 2.11. *Consider any N-indexed sequence of open ASEPs, which scale under Assumptions 2.2. Assume also that these ASEPs satisfy*

(1) $4:2:1$ *height function scaling: For $t \geq 0$ and $x \in [0, L]$, define*

$$h_{u,v}^{(N)}(t,x) := N^{-\frac{1}{2}} h_N\left(\frac{e^{\frac{1}{\sqrt{N}}} N^2 t}{2}, NL^{-1}x\right) + \frac{Nt}{2} + \frac{t}{24}$$
$$z_{u,v}^{(N)}(t,x) := \exp\left(h_{u,v}^{(N)}(t,x)\right)$$

where h_N is the height function process for ASEP of size N.

(2) Hölder bounds on initial data: the sequence of initial data $h_N(0,\cdot)$ satisfies that for all $\theta \in (0, 1/2)$ and $n \in \mathbb{Z}_{>0}$, there exist positive $C(n), C(\theta, n)$ such that for every $X, X' \in [0, 1]$ and $N \in \mathbb{Z}_{>0}$,

$$\|z_{u,v}^{(N)}(0,x)\|_n \leq C(n), \qquad \|z_{u,v}^{(N)}(0,x) - z_{u,v}^{(N)}(0,x')\|_n \leq C(\theta,n)|x-x'|^{\theta}.$$

Where $\|\cdot\|_n := (\mathbb{E}[|\cdot|^n])^{\frac{1}{n}}$ for expectation taken over $h_N(0,\cdot)$.

Then the sequence of laws of $z_{u,v}^{(N)}(\cdot,\cdot) \in D([0,T_0], C[0,L])$ is tight as $T \to \infty$ for any fixed $T_0 > 0$ and all limit points are in $C([0,T_0], C[0,L])$. If there exists a non-negative function $z(0,\cdot)$ such that as $N \to \infty$, $z^{(N)}(0,x)$ converges weakly to $z(0,x)$ in the space of continuous processes on $x \in [0,L]$, then $z_{u,v}^{(N)}(t,x)$ converges weakly to $z_{u,v}(t,x)$ in $D([0,T_0], C[0,L])$ for any $T_0 > 0$ as $N \to \infty$, where $z_{u,v}(t,x)$ is the unique mild solution to the stochastic heat equation with boundary parameters compatible with Neumann boundary conditions $u, v \in \mathbb{R}$, and initial data $z(0,x)$.

The Hölder bounds (2) follow from [CK24, Proposition 4.2], in which the coupling argument is unchanged if the interval $[0,1]$ is everywhere replaced by $[0,L]$. Therefore, the proposition above allows us to conclude that $z_{u,v}^{(N)}(t,x)$ converges weakly to $z_{u,v}(t,x) := \exp(h_{u,v}(t,x))$, the unique mild solution to the stochastic heat equation with appropriate boundary parameters and initial data. Since we began with stationary initial data, we see that the function $x \mapsto z_{u,v}^{(N)}(t,x)/z_{u,v}^{(N)}(t,0)$ is independent of t, and consequently, by the convergence to the stochastic heat equation, $x \to z_{u,v}(t,x)/z_{u,v}(t,0)$ is also stationary. Taking the logarithm implies that $x \mapsto h_{u,v}(t,x) - h_{u,v}(t,0)$ is independent of t. This justifies stating the following theorem.

Theorem 2.12. *For any $u, v \in \mathbb{R}$, all sub-sequential limits of the sequence of open ASEP stationary increment measures under Assumptions 2.2 are stationary measures of the open KPZ increment process.*

Proof of Theorem 1.4. By Theorem 2.10, we conclude that $h_{L;u,v}^{(N)}(x) - h_{L;u,v}^{(N)}(0)$ converges weakly to the measure of $(H(x) - G_{u,v}(x))$ defined in Theorem 1.4, where $(X, G_{u,v}(x), H(x))$ are sampled from $\mathbb{Q}_{L;u,v}$. What remains is to conclude that the resulting measure is the stationary measure of the open KPZ increment process on $[0,L]$. This follows from Theorem 2.12. ☐

3. Bounds on the Partition Functions

In this section, we prove estimates on the partition functions Z_N and $\mathcal{Z}_{u,v}$. To understand the contributions of the expectation over g and h, we first demonstrate that the expectation of $H^{(N)}(x,g,h)$ with respect to the simple random walk measure $\mathbb{W}_L^{(N)}$ is uniformly bounded in N for each x. We likewise show that

$\mathbb{E}_{\mathbb{W}_L}[H(x,g,h)]$ satisfies a similar bounded. In the next section, we will use these estimates to show that for any bounded function $F(x,g,h)$, $\mathbb{E}_{\mathbb{P}_L^{(N)}}[F(x,g,h)H^{(N)}(x,g,h)] \to \mathbb{E}_{\mathbb{P}_L}[F(x,g,h)H(x,g,h)]$.

We separate the computation of the expectation of $\mathbb{E}_{\mathbb{P}_L^{(N)}}[H^{(N)}(x,g,h)]$ into a sum over x of terms which depend on the paths under the measure $\mathbb{W}_L^{(N)}$. We will approach the bound on this partition function by first giving a bound on the component of the partition function which depends on the paths. One motivation for this approach is that we already know that the random walk measure is closely related to a Brownian measure, and in fact, converges weakly. Therefore, it seems likely that we could make sense of terms depending only on the path without regard to x. This turns out to be true. We recall Definition 2.7 and the formula for the partition function.

$$Z_{L;u,v}^{(N)} = N^{-\frac{1}{2}} \sum_{x\in\widetilde{\mathbb{Z}}(N)} e^{-2(u+v)x}\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[e^{-2vg(L)}\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)\right].$$

Considering the expression inside the expectation over the random walk measure, we might want to try to approximate it by limit expression using a KMT-type embedding. However, there is an obstacle to applying this technique directly, since the function inside the expectation is not necessarily bounded on every random walk path. To deal with this challenge, we separate the expectation into a sum over random walk endpoints, noting the probability that $g(L)$ lands at a value $N^{-\frac{1}{2}}k$ is given by $\mathbb{P}(g(L)=N^{-\frac{1}{2}}k) = 4^{-N}\binom{2N}{N+k}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{W}_L^{(N)}}&\left[e^{-2vg(L)}\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)\right]\\ &= 4^{-N}\sum_{k=-N}^{N}\exp\left(-\frac{2vk}{\sqrt{N}}\right)\binom{2N}{N+k}\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)\middle|g(L)=\frac{k}{\sqrt{N}}\right].\end{aligned}$$

The function inside the conditional expectation is now uniformly bounded by 1 for all random walk paths under the measure $\mathbb{W}^{(N)}$, so it may be easier to obtain a bound via the Brownian approximation of the random walks. Before we state the relevant random walk bridge KMT result and get into specifics, we will state the bound on $\mathbb{E}_{\mathbb{W}^{(N)}}[H^{(N)}(x,g,h)]$.

Lemma 3.1. *We can bound the component of the expectation which depends on (g,h) for all values of x as follows: There exist constants $C_{v,1}, C_{v,2}, K_v > 0$ depending only on v, and $\beta > 0$, such that for all $x\in\mathbb{R}$ and $N\in\mathbb{N}$,*

$$\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[H^{(N)}(x,g,h)\right] \le e^{-2(u+v)x}\left(C_{v,1}\exp\left(-\beta W\left(e^{-x}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2\right)+C_{v,2}e^{-y}\right),$$

where $W(\cdot)$ is Lambert's product-log function ($x=W(a)$ solves the equation $xe^x=a$), and we define $y := -\sqrt{N}\log(\sqrt{N})\log\left(\frac{\rho_r(1-\rho_\ell)}{(1-\rho_r)\rho_\ell}\right) = 2(u+v)\log(\sqrt{N})$.

We will use two intermediate results in the proof of Lemma 3.1. The first is the following approximation of the random walk bridge expectation in terms of an expectation over Brownian bridge paths.

Lemma 3.2. *For all $k\in[[-N,N]]$, there exists $K>0$ such that for any function of N, $y=y(N)$ satisfying $y\in O(\log(N))$,*

$$\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)\middle|g(L)=\frac{k}{\sqrt{N}}\right] \le 4\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-\frac{e^{-2x}}{2}\int_0^L e^{-2g(t)}dt\right)\middle|g(L)=\frac{k}{\sqrt{N}}\right]+Ke^{-y}.$$

Remark 3.1. In the bound above, we will ultimately choose $y := 2(u+v)\log(\sqrt{N})$. This choice will be necessary in order to prove Lemma 3.1.

The other intermediate result that we apply is a bound which arises due to change of measure calculations.

Lemma 3.3. *For all any bounded continuous functional $F : C[0, L] \to \mathbb{R}_{>0}$ such that $F(g(t))$ for $g(t)$ a path of a Brownian motion with diffusion coefficient $1/\sqrt{2}$ on $[0, L]$, has a continuous probability distribution on $\mathbb{R}_{>0}$, and any $m \in \mathbb{R}$, then there exists a constant $C_m > 0$ depending only on m such that*

$$\mathbb{E}_{\mathbb{W}_L}\left[F(g(t))\middle|g(L) = \frac{k}{\sqrt{N}}\right] = C_m \exp\left(-\frac{mk}{\sqrt{N}}\right)\mathbb{E}_{\mathbb{W}_L}\left[F(g(t) - mt)\middle|g(L) = \frac{k}{\sqrt{N}} + m\right]$$

The proof of this lemma appears in Appendix A. The following theorem of Dimitrov and Wu [DW21], adapted to the language of our problem, provides a KMT embedding result adapted for random walk bridges and Brownian bridges, and will allow us to prove Lemma 3.2.

Theorem 3.2 (Theorem 1.2, [DW21])**.** *Suppose $\{X_l\}_{l=0}^{\infty}$ are a collection of independent, identically distributed integer valued random variables, with $\mathbb{P}(X_l = -1) = \mathbb{P}(X_l = 1) = 1/4$ and $\mathbb{P}(X_l = 0) = 1/2$, and, for $t_i = iLN^{-1}$, $g(t_i) := N^{-\frac{1}{2}}\left(\sum_{l=0}^{t_iNL^{-1}} X_l\right)$, and $g_k(t_i)$ denotes $g(t_i)$ conditioned to end at $g(L) = N^{-\frac{1}{2}}k$. Then for every positive integer N there exists a probability space (we refer to the measure on this space as $\mathbb{P}_{\mathrm{BKMT}}$ for "bridge KMT") and constants $M, K, \lambda > 0$ such that*

$$\mathbb{P}_{\mathrm{BKMT}}\left(\sup_{0\le t_i\le L}\left|\widetilde{g}_k(t_i) - g_k(t_i)\right| \ge MN^{-\frac{1}{2}}\left(\log(N) + y\right)\right) \le Ke^{-\lambda y},$$

where $\widetilde{g}_k$ is a path sampled from the measure of a Brownian motion with diffusion coefficient $1/\sqrt{2}$ conditioned to end at $N^{-\frac{1}{2}}k$.

With this result in hand, we will give the proof of Lemma 3.2

Proof of Lemma 3.2. Throughout this proof, we will use $\widetilde{g}$ to denote the Brownian bridge path under the KMT measure, and g to denote the random walk path under the KMT measure. When the measure is specified as either $\mathbb{W}_L^{(N)}$ or $\mathbb{W}_L$, we will simply use g to denote a path under the appropriate measure. By Theorem 3.2,

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1 - \frac{e^{-2(g(t_i)+x)}}{N}\right)\middle|g(L) = \frac{k}{\sqrt{N}}\right] - \mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)\middle|g(L) = \frac{k}{\sqrt{N}}\right]\right| \\
&= \left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\prod_{i=0}^{N}\left(1 - \frac{e^{-2(g(t_i)+x)}}{N}\right) - \exp\left(-e^{-2x}\int_0^L e^{-2g'(t)}dt\right)\middle|g(L) = \widetilde{g}(L) = \frac{k}{\sqrt{N}}\right]\right|.
\end{aligned} \tag{3.1}$$

We separate this into two regions:

$$A_{k,N} := \left\{\max_{0\le t\le L}\|\widetilde{g} - g\| < MN^{-\frac{1}{2}}(\log(N) + y)\right\} \cap \left\{g(L) = \widetilde{g}(L) = \frac{k}{\sqrt{N}}\right\},$$
$$B_{k,N} := \left\{\max_{0\le t\le L}\|\widetilde{g} - g\| \ge MN^{-\frac{1}{2}}(\log(N) + y)\right\} \cap \left\{g(L) = \widetilde{g}(L) = \frac{k}{\sqrt{N}}\right\}.$$

By Theorem 3.2, $\mathbb{P}(B_{k,N}) \le Ke^{-\lambda y}$. Since both expectations in (3.1) are positive and bounded by 1, we conclude that for $K, \lambda > 0$ as in Theorem 3.2,

$$\left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\prod_{i=0}^{N}\left(1 - \frac{e^{-2(g(t_i)+x)}}{N}\right) - \exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)\middle|B_{k,N}\right]\right|\mathbb{P}(B_{k,N}) \le 2Ke^{-\lambda y}.$$

In the remaining region defined by $A_{k,N}$, we can expand the terms as follows

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\prod_{i=0}^{N}\left(1 - \frac{e^{-2(g(t_i)+x)}}{N}\right) - \exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)\middle|A_{k,N}\right]\right| \\
&\le \left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\prod_{i=0}^{N}\left(1 - \frac{e^{-2(g(t_i)+x)}}{N}\right) - \exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)\middle|A_{k,N}\right]\right| \\
&+ \left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right) - \exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)\middle|A_{k,N}\right]\right|.
\end{aligned} \tag{3.2}$$

We deal with the latter term first,

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)-\exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)\middle|A_{k,N}\right]\right|\\
&=\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}t)}dt\right)\middle|\exp\left(-e^{-2x}\left(\int_0^L e^{-2\widetilde{g}(t)}\left(e^{-2(g(t)-\widetilde{g}(t))}-1\right)dt\right)\right)-1\middle|\middle|A_{k,N}\right]\\
&\leq\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)\sup\left\{\begin{matrix}\left|\exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)^{\left(e^{2\delta}-1\right)}-1\right|,\\ \left|\exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)^{\left(e^{-2\delta}-1\right)}-1\right|\end{matrix}\right\}\middle|A_{k,N}\right].
\end{aligned}$$

We note that $X_k = e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt|_{\widetilde{g}(L)=N^{-\frac{1}{2}}k}$ can be considered as a random variable taking values $r\in[0,\infty)$, with some probability distribution function $\rho_{X,k}(r)$. The bound above can therefore be written as

$$\leq\sup\left\{\left|\int_0^\infty e^{-r}\left(e^{-r(e^{2\delta}-1)}-1\right)\rho_{X,k}(r)dr\right|,\left|\int_0^\infty e^{-r}\left(e^{-r(e^{-2\delta}-1)}-1\right)\rho_{X,k}(r)dr\right|\right\}.$$

There exists N_δ such that for all $N>N_\delta$, $0<e^{2\delta}-1<\frac{1}{4}$ and $0>e^{-2\delta}-1>\frac{1}{4}$, which implies that the expression above is bounded by

$$\leq\sup\left\{\int_0^\infty e^{-r}\left(1-e^{-r/4}\right)\rho_{X,k}(r)dr,\int_0^\infty e^{-r}\left(e^{r/4}-1\right)\rho_{X,k}(r)dr\right\}.$$

In particular, $e^{-r}(1-e^{-r/4})$ and $e^{-r}(e^{r/4}-1)$ are both bounded by $e^{-r/2}$ for all $r\geq 0$, and therefore, both terms in the supremum environment are themselves bounded by $\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}[\exp(-\frac{1}{2}e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt)]$. We similarly bound the first term on the right-hand side of (3.2) as follows:

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\prod_{i=0}^N\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)-\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)\middle|A_{k,N}\right]\right|\\
&\leq\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)\middle|\exp\left(e^{-2x}\int_0^L e^{-2g(t)}dt+\sum_{i=0}^N\log\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)\right)-1\middle|\middle|A_{k,N}\right]
\end{aligned}$$

The definition of the random walk paths implies $\int_0^L e^{-2g(t)}dt=N^{-1}\sum_{i=0}^N e^{-2g(t_i)}$. Therefore,

$$\sum_{i=0}^N\log\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)+\int_0^L e^{-2(g(t)+x)}dt=\sum_{i=0}^N\left(\log\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)+\frac{e^{-2(g(t_i)+x)}}{N}\right). \tag{3.3}$$

Since we are restricted to subset of paths $g(t_i)$ for each x where the Taylor expansion of the logarithm converges, we see that each term in this expression is equal to $-\sum_{l=2}^\infty\frac{e^{-2l(g(t_i)+x)}}{lN^l}$, which is always a negative value. This implies that

$$\left|\exp\left(e^{-2x}\int_0^L e^{-2g(t)}dt+\sum_{i=0}^N\log\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)\right)-1\right|\leq 1.$$

Applying this bound, and the calculation for the other term in (3.2), we see that

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\prod_{i=0}^N\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)-\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)\middle|A_{k,N}\right]\right|\\
&\qquad\leq\mathbb{E}_{\mathbb{P}_{\mathrm{KMT}}}\left[\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)\middle|A_{k,N}\right]\leq 2\mathbb{E}_{\mathbb{P}_{\mathrm{BKMT}}}\left[\exp\left(-e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\right)\middle|A_{k,N}\right].
\end{aligned}$$

Therefore, we can bound the full expression in (3.1) by

$$\begin{aligned}
&\leq 2\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-e^{-2x}\int_0^L e^{-2g(t)}dt\right)\Big| g(L)=\frac{k}{\sqrt{N}}\right]\\
&+\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-\frac{1}{2}e^{-2x}\int_0^L e^{-2g(t)}dt\right)\Big| g(L)=\frac{k}{\sqrt{N}}\right]+Ke^{-\lambda y}\\
&\leq 3\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-\frac{1}{2}e^{-2x}\int_0^L e^{-2g(t)}dt\right)\Big| g(L)=\frac{k}{\sqrt{N}}\right]+Ke^{-\lambda y}.
\end{aligned}$$

Finally, relabeling λy as y finishes the proof. □

We note that, applying essentially the same bounds as those which appear in the proof above, we can also obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-\frac{1}{2}e^{-2x}\int_0^L e^{-2g(t)}dt\right)\Big| g(L)=\frac{k}{\sqrt{N}}\right]\\
&\qquad\qquad\leq 4\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{4N}\right)\Big| g(L)=\frac{k}{\sqrt{N}}\right]+Ke^{-y}. \quad (3.4)
\end{aligned}$$

We note that we can choose y to be any value which still allows $\lim_{N\to\infty} MN^{-\frac{1}{2}}(\log(N)+y)=0$. The specific value of y will be engineered for arguments later in this section. The reason that it is crucial to do this random walk bridge KMT embeddding is because using the bound in Lemma 3.2 allows us to leverage an important property of the conditional expectation in the continuous setting which is not available to us in the discrete setting: that is, a nice expression for a change of measure, Lemma 3.3. We can now begin the proof of Lemma 3.1

Proof of Lemma 3.1. Conditioning on the endpoint of $g(t)$, as above, we find that

$$\begin{aligned}
\mathbb{E}_{\mathbb{W}_L^{(N)}}&\left[H^{(N)}(x,g,h)\right]=e^{-2(u+v)x}4^{-N}\\
&\cdot\sum_{k=-N}^{N}\exp\left(-\frac{2vk}{\sqrt{N}}\right)\binom{2N}{N+k}\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)\Big| g(L)=\frac{k}{\sqrt{N}}\right]. \quad (3.5)
\end{aligned}$$

Applying the equality above, as well as Lemma 3.2, and then Lemma 3.3, to (3.5), we can bound the entire expression by

$$\begin{aligned}
&\leq 4C_m e^{-2(u+v)x}4^{-N}\\
&\cdot\sum_{k=-N}^{N}\exp\left(-\frac{(2v+m)k}{\sqrt{N}}\right)\binom{2N}{N+k}\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-\frac{e^{-2x}}{2}\int_0^L e^{-2(g(t)-mt)}dt\right)\Big| g(L)=\frac{k}{\sqrt{N}}+m\right]\\
&+Ke^{-y}e^{-2(u+v)x}4^{-N}\sum_{k=-N}^{N}\exp\left(-\frac{2vk}{\sqrt{N}}\right)\binom{2N}{N+k}.
\end{aligned}$$

Now, applying the inequality from (3.4), this is bounded by

$$\begin{aligned}
&\leq 16C_m e^{-2(u+v)x}4^{-N}\\
&\cdot\sum_{k=-N}^{N}\exp\left(-\frac{(2v+m)k}{\sqrt{N}}\right)\binom{2N}{N+k}\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)-mt_i+x+\log(2))}}{N}\right)\Big| g(L)=\frac{k}{\sqrt{N}}+m\right] \quad (3.6)\\
&+Ke^{-y}e^{-2(u+v)x}4^{-N}\sum_{k=-N}^{N}\binom{2N}{N+k}\left(4C_m\exp\left(-\frac{(2v+m)k}{\sqrt{N}}\right)+\exp\left(-\frac{2vk}{\sqrt{N}}\right)\right).
\end{aligned}$$

To deal with the first term, we want to turn the sum back into an expression which we can neatly sum over k. We can think of the distribution $\mathbb{P}(g(L)=N^{-\frac{1}{2}}k)=4^{-N}\binom{2N}{N+k}$ as an approximation of a Gaussian on the domain $|k| \le N$. In this setting, we can understand the multiplicative factor of $\exp\left(-(2v+m)k/\sqrt{N}\right)$ as "shifting the center" of this Gaussian by a displacement of approximately $\left(v+\frac{m}{2}\right)\sqrt{N}$. We make this notion precise in the following proposition, which is proved in Appendix A.

Proposition 3.3. *For all $a \in \mathbb{R}$, and any k such that $|k|, |k+a\sqrt{N}| < N^{\frac{5}{6}}$, then there exists $D_a > 0$, depending only on a such that for all $N \in \mathbb{N}$,*

$$\exp\left(\frac{-2ak}{\sqrt{N}}\right)\binom{2N}{N+k} \le D_a \binom{2N}{N+k+a\sqrt{N}}.$$

Remark 3.4. The choice of N^c with $c = \frac{5}{6}$ as the threshold in Proposition 3.3 is somewhat arbitrary: all that is necessary for the lemma to hold is that $c \le \frac{5}{6}$, and all that is necessary for this lemma to be used in the ongoing proof is that $c > \frac{3}{4}$.

Setting $m = 2v$, this lemma implies that for all $|k|, |k+(2v+m)\sqrt{N}| < N^{\frac{5}{6}}$,

$$\exp\left(-\frac{(2v+m)k}{\sqrt{N}}\right)\binom{2N}{N+k} \le D_{v+\frac{m}{2}}\binom{2N}{N+k+\left(v+\frac{m}{2}\right)\sqrt{N}} = D_{2v}\binom{2N}{N+k+2v\sqrt{N}}.$$

Likewise, for all $|k|, |k+v\sqrt{N}| < N^{\frac{5}{6}}$,

$$\exp\left(-\frac{2vk}{\sqrt{N}}\right)\binom{2N}{N+k} \le D_v\binom{2N}{N+k+v\sqrt{N}}.$$

Therefore, we can bound (3.6) by

$$\begin{aligned}
&\le 16C_{2v}D_{2v}e^{-2(u+v)x}4^{-N} \\
&\cdot \sum_{|k|,|k+2v\sqrt{N}|<N^{\frac{5}{6}}} \binom{2N}{N+k+2v\sqrt{N}} \mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right)\middle| g(L)=\frac{k}{\sqrt{N}}+2v\right] \\
&+ Ke^{-y}e^{-2(u+v)x}4^{-N}\left(4D_{2v}C_{2v}\sum_{|k|,|k+2v\sqrt{N}|<N^{\frac{5}{6}}}\binom{2N}{N+k+2v\sqrt{N}} + D_4D_v\sum_{|k|,|k+v\sqrt{N}|<N^{\frac{5}{6}}}\binom{2N}{N+k+v\sqrt{N}}\right) \\
&+ e^{-2(u+v)x}\left(16C_{2v}\mathrm{ERR}_1 + 4C_{2v}Ke^{-y}\mathrm{ERR}_2 + Ke^{-y}\mathrm{ERR}_3\right).
\end{aligned}$$

The error terms are defined by

$$\begin{aligned}
\mathrm{ERR}_1 &:= 4^{-N}\sum_{|k|\ge N^{\frac{5}{6}} \vee |k+2v\sqrt{N}|\ge N^{\frac{5}{6}}} \exp\left(-\frac{2vk}{\sqrt{N}}\right)\binom{2N}{N+k} \\
&\quad \cdot \mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right)\middle| g(L)=\frac{k}{\sqrt{N}}+2v\right], \\
\mathrm{ERR}_2 &:= 4^{-N}\sum_{|k|\ge N^{\frac{5}{6}} \vee |k+2v\sqrt{N}|\ge N^{\frac{5}{6}}} \exp\left(-\frac{2vk}{\sqrt{N}}\right)\binom{2N}{N+k}, \\
\mathrm{ERR}_3 &:= 4^{-N}\sum_{|k|\ge N^{\frac{5}{6}} \vee |k+v\sqrt{N}|\ge N^{\frac{5}{6}}} \exp\left(-\frac{2vk}{\sqrt{N}}\right)\binom{2N}{N+k}.
\end{aligned}$$

In particular, $\mathrm{ERR}_1 \leq \mathrm{ERR}_2$. Since this sum only considers terms with $|k| \leq N$, and because the binomial coefficients are symmetric, we have the following bound on the error terms.

$$\mathrm{ERR}_{1,2} \leq e^{4|v|\sqrt{N}} 4^{-N} \sum_{k=N^{\frac{5}{6}}-4|v|\sqrt{N}}^{N} \binom{2N}{N+k} \leq (N - N^{\frac{5}{6}} + 2|v|\sqrt{N}) e^{4|v|\sqrt{N}} 4^{-N} \binom{2N}{N + N^{\frac{5}{6}} - 2|v|\sqrt{N}},$$

$$\mathrm{ERR}_3 \leq e^{2|v|\sqrt{N}} 4^{-N} \sum_{k=N^{\frac{5}{6}}-|v|\sqrt{N}}^{N} \binom{2N}{N+k} \leq (N - N^{\frac{5}{6}} + |v|\sqrt{N}) e^{2|v|\sqrt{N}} 4^{-N} \binom{2N}{N + N^{\frac{5}{6}} - |v|\sqrt{N}}.$$

Noting that $4^{-N}\binom{2N}{N+x} \leq 4^{-N}\binom{2N}{N} e^{-\frac{x^2}{2N}} < N^{-\frac{1}{2}} e^{-\frac{x^2}{2N}}$ for all $|x| \leq N$, we see that,

$$\mathrm{ERR}_{1,2} < \left(N^{\frac{1}{2}} - N^{\frac{3}{8}} + 4|v|\right) \exp\left(-\frac{1}{2}N^{\frac{3}{4}} + 4|v|N^{\frac{1}{2}} + 4|v|N^{\frac{3}{8}} - 8|v|^2\right),$$

$$\mathrm{ERR}_3 < \left(N^{\frac{1}{2}} - N^{\frac{3}{4}} + |v|\right) \exp\left(-\frac{1}{2}N^{\frac{3}{4}} + 2|v|N^{\frac{1}{2}} + |v|N^{\frac{3}{8}} - \frac{1}{2}|v|^2\right).$$

Both of which tend to 0 as $N \to \infty$. Furthermore, for any polynomial $p(N)$ of constant order in N, we see that $p(N)\mathrm{ERR}_1 \to 0$ as $N \to \infty$. In particular, with $y = 2(u+v)\log(\sqrt{N})$, we have $e^y = N^{u+v}$. We define $\mathrm{ERR}_4 := e^y \mathrm{ERR}_1$. Our calculation shows that $\mathrm{ERR}_4 \to 0$. Thus, relabeling constants, there exist $D_{v,1}, D_{v,2}, D_{v,3} > 0$ such that we can bound (3.6) by

$$\begin{aligned}
&\leq D_{v,1} e^{-2(u+v)x} 4^{-N} \\
&\cdot \sum_{k=-N-2v\sqrt{N}}^{N-2v\sqrt{N}} \binom{2N}{N+k+2v\sqrt{N}} \mathbb{E}_{\mathbb{W}_L^{(N)}} \left[\prod_{i=0}^{N} \left(1 - \frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right) \middle| g(L) = \frac{k}{\sqrt{N}} + 2v \right] \\
&+ D_{v,2} e^{-y} e^{-2(u+v)x} 4^{-N} \left(\sum_{k=-N-2v\sqrt{N}}^{N-2v\sqrt{N}} \binom{2N}{N+k+2v\sqrt{N}} + \sum_{k=-N-v\sqrt{N}}^{N-v\sqrt{N}} \binom{2N}{N+k+v\sqrt{N}} \right) \\
&+ e^{-y} D_{v,3} e^{-2(u+v)x} \left(\mathrm{ERR}_4 + \mathrm{ERR}_2 + \mathrm{ERR}_3\right).
\end{aligned}$$

From the equation above, we produce a looser upper bound by adding more (strictly positive) terms to each of the sums so that all k where the binomial coefficients are supported appear in the sum. This allows us to re-sum each term. We conclude that there exist $D_{v,4}, D_{v,5} > 0$ depending only on v such that the expression above is abounded by

$$\leq D_{v,4} e^{-2(u+v)x} \mathbb{E}_{\mathbb{W}_L^{(N)}} \left[\prod_{i=0}^{N} \left(1 - \frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right) \right] + D_{v,5} e^{-y} e^{-2(u+v)x}.$$

Setting A_s^N to be the event $A_s^N := \{\max_{t_i; i \in \{1,\ldots,N\}} |g(t_i)| > s\}$, we see that

$$\begin{aligned}
\mathbb{E}_{\mathbb{W}_L^{(N)}} \left[\prod_{i=0}^{N} \left(1 - \frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right) \right] &= \mathbb{E}_{\mathbb{W}_L^{(N)}} \left[\prod_{i=0}^{N} \left(1 - \frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right) \middle| A_s^N \right] \mathbb{P}\left(A_s^N\right) \\
&+ \mathbb{E}_{\mathbb{W}_L^{(N)}} \left[\prod_{i=0}^{N} \left(1 - \frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right) \middle| A_s^{N,c} \right] \mathbb{P}\left(A_s^{N,c}\right).
\end{aligned}$$

Noting that for all g which represent a simple random walk path of N steps, and for any $m \in \mathbb{R}$, there exists $D_1 > 0$ such that

$$\prod_{i=0}^{N} \left(1 - \frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right) \leq D_1 \exp\left(-e^{-2x-2\log(2)} \int_0^L e^{-2(g(t)-2vt)} dt\right).$$

We apply [LL10, Proposition 2.1.2] to obtain the bound

$$\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)-2vt_i+x+\log(2))}}{N}\right)\right]\le Ce^{-\beta s^2}+D_1\exp\left(-e^{-2(x+s+\log(2))}\int_0^L e^{4vt}dt\right).$$

The terms in the expression above are balanced at $s=W\left(e^{-x}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)$, where $K_v:=\frac{1}{16}\int_0^L e^{4vt}dt$. Setting $C_{v,1}:=(D_1+C)D_{v,4}$ and $C_{v,2}:=D_{v,5}$ finishes the proof. $\square$

Corollary 3.5. *The partition function $Z_{L;u,v}^{(N)}$ is bounded by a global constant for all $N\in\mathbb{N}$.*

Proof. We begin by noting that the previous lemma tells us that there exist constants $C_{v,1},C_{v,2},K_v>0$ depending only on v, and $\beta>0$, such that for all $x\in\mathbb{R}$ and $N\in\mathbb{N}$,

$$\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[H^{(N)}(x,g,h)\right]\le e^{-2(u+v)x}\left(C_{v,1}\exp\left(-\beta W\left(e^{-x}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2\right)+C_{v,2}e^{-y}\right). \tag{3.7}$$

Since $u+v>0$, $\left(\frac{\rho_r(1-\rho_\ell)}{(1-\rho_r)\rho_\ell}\right)=e^{-2(u+v)/\sqrt{N}}<1$. We begin by dealing with the second term of the sum. By setting $y=2(u+v)\log(\sqrt{N})$, we find

$$\frac{C_{v,2}e^{-y}}{\sqrt{N}}\sum_{x\in\widetilde{\mathbb{Z}}(N)}e^{-2(u+v)x}=\frac{C_{v,2}}{\sqrt{N}}\sum_{x\in\widetilde{\mathbb{Z}}(N)}e^{-2(u+v)(x+\log(\sqrt{N}))}=\frac{C_{v,2}}{\sqrt{N}}\sum_{x=1}^{\infty}e^{-\frac{2(u+v)x}{\sqrt{N}}}=\frac{C_{v,2}}{\sqrt{N}}\left(\frac{e^{-\frac{2(u+v)}{\sqrt{N}}}}{1-e^{-\frac{2(u+v)}{\sqrt{N}}}}\right),$$

which converges to a constant as $N\to\infty$, and which is therefore bounded by a constant $D_1>0$ for all $N\ge 1$. We split the first term of the sum in (3.7) into two pieces, corresponding to when $x\ge 0$ and $x<0$. When $x\ge 0$, the multiplicative factor with the product log function is bounded by a constant

$$\exp\left(-\beta W\left(e^{-x-\log(2)}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2\right)\le\exp\left(-\beta W\left(2^{-1}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2\right)=:D_{v,1}.$$

Therefore, when $x\ge 0$,

$$\frac{C_v}{\sqrt{N}}\sum_{x\in\widetilde{\mathbb{Z}}(N)\cap[0,\infty)}e^{-2(u+v)x}\exp\left(-\beta W\left(e^{-x-\log(2)}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2\right)\le\frac{C_vD_{v,1}}{\sqrt{N}}\sum_{x\in\widetilde{\mathbb{Z}}(N)\cap[0,\infty)}e^{-2(u+v)x}.$$

Likewise, when $x<0$, $W\left(e^{-x-\log(2)}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2$ grows super-linearly in $|x|$, and therefore there exists a constant, which we will denote by $D_{v,2}$, such that for all such x,

$$e^{-2(u+v)x}\exp\left(-\beta W\left(e^{-x-\log(2)}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2\right)\le D_{v,2}e^{2(u+v)x}.$$

Therefore,

$$\frac{C_v}{\sqrt{N}}\sum_{x\in\widetilde{\mathbb{Z}}(N)\cap(-\infty,0)}e^{-2(u+v)x}\exp\left(-\beta W\left(e^{-x-\log(2)}\beta^{-\frac{1}{2}}K_v^{\frac{1}{2}}\right)^2\right)\le\frac{C_vD_{v,2}}{\sqrt{N}}\sum_{x\in\widetilde{\mathbb{Z}}(N)\cap(-\infty,0)}e^{2(u+v)x}.$$

which is bounded by a constant depending only on u and v for all N. $\square$

We demonstrate a similar result for $\mathcal{Z}_{u,v}$, which allows us to conclude that $\mathbb{Q}$ is a probability measure. First we prove a bound for $\mathbb{E}_{\mathbb{W}_L}[H(x,g,h)]$.

Lemma 3.4. *There exist $C_2,C_3,C_4,C_5>0$ such that for all $x\in\mathbb{R}$ and for all $v\in\mathbb{R}$, when $v\ge 0$,*

$$\mathbb{E}_{\mathbb{W}_L}\left[H(x,g,h)\right]\le e^{-2(u+v)x}C_2K_0\left(\sqrt{2}e^{-x}\right),$$

and when $v<0$,

$$\mathbb{E}_{\mathbb{W}_L}\left[H(x,g,h)\right]\le e^{-2(u+v)x}C_3K_0\left(\sqrt{2}e^{-x}\right)+C_4e^{-2(u+v)+xv}K_v\left(2C_5e^{-x}\right).$$

Proof. By the Corollary in Section 6.2 of [Yor92], we note that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{W}}\left[f\left(e^{g(1)}\right)g\left(\int_0^L e^{-2g(t)}dt\right)\right],\\
&= D_1\int_0^\infty\int_0^\infty f(y)g\left(\frac{1}{r}\right)\exp\left(-\frac{r}{2}(1+y^2)\right)\psi_{yr}(L)drdy,\\
&= D_1\int_0^\infty\int_0^\infty f(y)g\left(\frac{1}{r}\right)\exp\left(-\frac{r}{2}(1+y^2)\right)\left(\int_0^\infty \exp\left(-\frac{p^2}{2u}-yr\cosh(y)\right)\sinh(p)\sin\left(\frac{\pi p}{L}\right)dp\right)drdy,
\end{aligned}$$

for a constant $D_1>0$. We achieve $\mathbb{E}_{\mathbb{W}_L}[H(x,g,h)]$ by setting $f(t)=t^{-2v}$ and $g(t)=e^{-e^{-2x}t}$. We can bound this quantity in steps. We begin by looking at the innermost integral,

$$\int_0^\infty \exp\left(-\frac{p^2}{2}-yr\cosh(p)\right)\sinh(p)\sin\left(\frac{\pi p}{L}\right)dp \le \frac{1}{2}\int_0^\infty \exp\left(-\frac{p^2+e^p yr-2p}{2}\right)dp.$$

We note that there exists a constant $D_2>0$ such that $\exp\left(-\frac{p^2+e^p yr-2p}{2}\right)\le \exp(-D_2 e^p yr)$. A sufficient choice of constant is any D_2 such that, for all $x>0$, $D_2\le \frac{1}{2}+\frac{x^2-2x}{2e^x yr}$. For reasons which will be apparent later in the argument, we also want to select D_2 as a function of v so that $\frac{1}{2}-D_2^2|v|\ge 0$, so we pick

$$D_2 := \min\left\{\min_{x\ge 0}\left\{\frac{1}{2}+\frac{x^2-2x}{2e^x yr}\right\},\frac{1}{\sqrt{2|v|}}\right\}.$$

Therefore,

$$\int_0^\infty \exp\left(-\frac{p^2}{2}-yr\cosh(p)\right)\sinh(p)\sin\left(\frac{\pi p}{L}\right)dp \le \frac{1}{2}\int_0^\infty \exp(-D_2 e^p yr)dp = \frac{1}{2}\Gamma(0,D_2 ry).$$

We can bound $\Gamma(0,D_2ry)$ by $\frac{e^{-D_2 ry}}{D_2 ry}$ when $y\ge 1$, and on the domain $[0,1]$, $|y^{-2v}\exp\left(-\frac{r}{2}y^2\right)|\le 1$, therefore,

$$\begin{aligned}
\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-2vg(1)-\int_0^L e^{-2g(t)}dt\right)\right] &\le D_1\int_0^\infty g\left(\frac{1}{r}\right)\exp\left(-\frac{r}{2}\right)\left(\int_0^1 \Gamma(0,D_2yr)dy\right)dr\\
&+\frac{D_1}{2D_2}\int_0^\infty \frac{1}{r}g\left(\frac{1}{r}\right)\exp\left(-\frac{r}{2}\right)\\
&\cdot\left(\int_1^\infty y^{-2v-1}\exp\left(-\frac{r}{2}y^2-D_2yr\right)dy\right)dr.
\end{aligned}\tag{3.8}$$

We deal with the first term,

$$\begin{aligned}
D_1\int_0^\infty g\left(\frac{1}{r}\right)\exp\left(-\frac{r}{2}\right)\int_0^1\Gamma(0,D_2ry)dydr &= D_1\int_0^\infty g\left(\frac{1}{r}\right)\exp\left(-\frac{r}{2}\right)\frac{-e^{-D_2r}+D_2r\Gamma(0,D_2r)+1}{D_2r}dr\\
&= \frac{D_1}{D_2}\int_0^\infty -\frac{1}{r}\exp\left(-\frac{e^{-2x}}{r}-\frac{r}{2}-D_2r\right)dr\\
&+ D_1\int_0^\infty \exp\left(-\frac{e^{-2x}}{r}-\frac{r}{2}\right)\Gamma(0,D_2r)dr\\
&+ \frac{D_1}{D_2}\int_0^\infty \frac{1}{r}\exp\left(-\frac{e^{-2x}}{r}-\frac{r}{2}\right)dr.
\end{aligned}$$

Each of these pieces is bounded by $\frac{D_1}{D_2}K_0(\sqrt{2}e^{-x})$, so the entire expression is bounded by $\frac{3D_1}{D_2}K_0(\sqrt{2}e^{-x})$. To deal with the second line of (3.8), we note that the interior integral over y is a Mellin transform of $\exp\left(-\frac{r}{2}y^2-D_2ry\right)$. Identity 3.13 in [Obe74] tells us that for all z with $Re(z)>0$,

$$\int_0^\infty x^{z-1}\exp(-ax^2-bx)dx = \frac{\Gamma(z)}{(2a)^{\frac{z}{2}}}\exp\left(\frac{b^2}{8a}\right)\mathcal{D}_{-z}\left(\frac{b}{\sqrt{2a}}\right),$$

where $\mathcal{D}_z(\cdot)$ is the parabolic cylinder function (see [Obe74] for further description). Therefore, when $v < 0$, this evaluates to

$$r^v \Gamma(-2v) \exp\left(\frac{D_2^2}{4} r\right) D_{2v}(C_2 \sqrt{r}).$$

The second line of (3.8) becomes

$$= \frac{D_1}{2D_2} \Gamma(-2v) \int_0^\infty r^{v-1} \exp\left(-\frac{e^{-2x}}{r} - \frac{r}{2} + \frac{D_2^2}{4} r\right) \mathcal{D}_{2v}(D_2 \sqrt{r}) dr.$$

This expression is bounded by

$$\leq \frac{D_1}{2D_2} \Gamma(-2v) 2^v \int_0^\infty r^{v-1} \exp\left(-\frac{e^{-2x}}{r} - r\left(\frac{1}{2} + D_2^2 v\right)\right) dr,$$

which is also a Mellin transform. Identity 3.16 from [Obe74] tells us that

$$\int_0^\infty x^{z-1} \exp\left(-ax - \frac{b}{x}\right) dr = 2\left(\frac{b}{a}\right)^{\frac{z}{2}} K_z\left(2\sqrt{ab}\right).$$

Using this identity, the equation above becomes

$$\leq \frac{D_1 \Gamma(2|v|) 2^v}{D_2} \left(\frac{1}{2} + D_2^2 v\right)^{v/2} e^{xv} K_v \left(2e^{-x}\left(\frac{1}{2} + D_2^2 v\right)^{\frac{1}{2}}\right).$$

Finally, we consider the second line of (3.8) in the case where $v > 0$. Then

$$\int_1^\infty \frac{\exp\left(-\frac{r}{2} y^2 - D_2 y r\right)}{y^{2|v|+1}} dy \leq \frac{1}{2|v|},$$

and therefore, the second line is bounded by $\frac{D_1}{4|v|D_2} K_0\left(\sqrt{2} e^{-x}\right)$. In conclusion, when $v \geq 0$,

$$\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-2vg(1) - \int_0^L e^{-2(g(t)+x)} dt\right)\right] \leq \left(\frac{3D_1}{D_2} + \frac{D_1}{4|v|D_2}\right) K_0\left(\sqrt{2} e^{-x}\right),$$

and when $v < 0$,

$$\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-2vg(1) - \int_0^L e^{-2(g(t)+x)} dt\right)\right]$$
$$\leq \frac{3D_1}{D_2} K_0\left(\sqrt{2} e^{-x}\right) + \frac{D_1 \Gamma(2|v|) 2^v}{D_2} \left(\frac{1}{2} + D_2^2 v\right)^{v/2} e^{xv} K_v\left(2\left(\frac{1}{2} + D_2^2 v\right)^{\frac{1}{2}} e^{-x}\right).$$

Choosing

$$C_2 := \frac{3D_1}{D_2} + \frac{D_1}{4|v|D_2}, \qquad C_3 := \frac{3D_1}{D_2}, \qquad C_4 := \frac{D_1 \Gamma(2|v|) 2^v}{D_2} C_5^v, \qquad C_5 := \left(\frac{1}{2} + D_2^2 v\right)^{\frac{1}{2}},$$

finishes the proof. ☐

From these bounds, we can provide an alternate, and very short, proof for the fact that $\mathcal{Z}_{u,v}$ is finite, which is also proved in other references, such as [BKWW23].

Corollary 3.6. *The partition function $\mathcal{Z}_{u,v}$ is finite for all $u + v > 0$.*

Proof. When $v \geq 0$,

$$\mathcal{Z}_{u,v} \leq C_2 \int_{-\infty}^\infty e^{-2(u+v)x} K_0(\sqrt{2} e^{-x}) dx.$$

Identity 16 in [GR15], Section 6.563 states that for all μ, ν, a such that $Re(\mu + 1 \pm \nu) > 0$ and $Re(a) > 0$,

$$\int_0^\infty x^\mu K_\nu(ax) dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{3.9}$$

Making the change of coordinates $r = e^{-x}$, we can use this identity to evaluate the integral

$$\mathcal{Z}_{u,v} \leq C_2 \int_0^\infty r^{2(u+v)-1} K_0(\sqrt{2}r)dr = C_2 2^{(u+v)-2} \left(\Gamma\left(u+v\right)\right)^2.$$

and when $v < 0$,

$$\begin{aligned}\mathcal{Z}_{u,v} &\leq C_3 \int_{-\infty}^\infty e^{-2(u+v)x} K_0(\sqrt{2}e^{-x})dx + C_4 \int_{-\infty}^\infty e^{-2(u+v)x+xv} K_v\left(2C_5 e^{-x}\right)dx, \\ &\leq C_3 2^{(u+v)-2} \left(\Gamma(u+v)\right)^2 + C_4 C_5^{-v(2u-v)/2} \Gamma\left(u+v\right)\Gamma\left(u+2v\right).\end{aligned}$$

We note that $2(u+v) - v > 0$ because $v < 0$. We use the same integral identity as before (3.9) to evaluate the integral. □

Corollary 3.7. *The measure $\mathbb{Q}$ (Definition 2.9) is a probability measure.*

4. Bounds on the Path Measure

We want to show weak convergence of the sequence $\mathbb{Q}^{(N)}$. In service of this goal, we will use this section to prove some bounds on the speed of convergence of the part of $\mathbb{Q}^{(N)}$ which depends only on the path of the random walk. We consider an arbitrary bounded continuous function $F(x, g, h)$, and set $C_F := \sup_{x,g,h} |F(x, g, h)|$. In this section, we will see our second application of a KMT theorem, this time the standard KMT embedding, which will help us compare $H^{(N)}(x, g, h)$ and $H(x, g, h)$ on the same probability space. The difference between the expectations over the paths is given by

$$\begin{aligned}&\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[F(x,g,h)H^{(N)}(x,g,h)\right] - \mathbb{E}_{\mathbb{W}_L}\left[F(x,g,h)H(x,g,h)\right], \\ &\leq \mathbb{E}_{\mathbb{W}_L^{(N)}}\left[F(x,g,h)H^{(N)}(x,g,h) - F(x,g,h)H(x,g,h)\right], \\ &+ \mathbb{E}_{\mathbb{W}_L^{(N)}}\left[F(x,g,h)H(x,g,h)\right] - \mathbb{E}_{\mathbb{W}_L}\left[F(x,g,h)H(x,g,h)\right].\end{aligned} \tag{4.1}$$

This section will be concerned with producing a bound on the final line of the equation above. For convenience, suppressing dependence on $F(x, g, h)$, u, and v, we define

$$\Pi_N := \left|\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[F(x,g,h)H(x,g,h)\right] - \mathbb{E}_{\mathbb{W}_L}\left[F(x,g,h)H(x,g,h)\right]\right|.$$

To show a bound on Π_N, we will use the (standard) KMT embedding theorem to put everything onto the same probability space.

Theorem 4.1 ([KMT75])**.** *Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbb{P}_{\mathbb{W}^{(N)}})$ be the probability space of two-dimensional Brownian motion with respect to the two-dimensional Wiener measure and let $(\Omega, \mathcal{F}, \mathbb{P}_{\mathbb{W}})$ be the probability space of two-dimensional simple random walks with N steps, scaled in time by N^{-1}, and in space by $N^{-\frac{1}{2}}$, with respect to the appropriately scaled two-dimensional simple random walk measure. We will use (g, h) to denote elements of Ω and $(\widetilde{g}, \widetilde{h})$ to denote elements of $\widetilde{\Omega}$. Then there exists a probability space with measure $\mathbb{P}_{\mathrm{KMT}}$ and a constant $C > 0$ such that for all $N, y \geq 0$,*

$$\mathbb{P}_{\mathrm{KMT}}\left[\sup_{0\leq t\leq L} \|(g,h) - (\widetilde{g},\widetilde{h})\| \geq CN^{-\frac{1}{2}}(\log(N) + y)\right] \leq e^{-y}.$$

With this theorem in hand, we can prove the following bound:

Proposition 4.2. *There exists $N_v, C_v > 0$ depending only on v such that for all $N > N_v$ for all bounded continuous functions $F(x, g, h)$ with $C_F := \sup |F(x, g, h)|$, and for all $x, v \in \mathbb{R}$,*

$$\begin{aligned}\Pi_N \leq 2C_F C_v \log(N) N^{-\frac{1}{2}} e^{-2(u+v)x} \mathbb{E}_{\mathbb{W}_L}&\left[\exp\left(-\frac{e^{-2x}}{4}\int_0^L e^{-2g(t)}dt\right)\right] \\ &+ 2C_F N^{-\frac{1}{2}} \sup\left\{\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[H(x,g,h)\right], \mathbb{E}_{\mathbb{W}_L}\left[H(x,g,h)\right]\right\}.\end{aligned}$$

Proof. We note that

$$\begin{aligned}&\left|\mathbb{E}_{\mathbb{W}_L}[F(x,g,h)H(x,g,h)] - \mathbb{E}_{\mathbb{W}_L^{(N)}}[F(x,g,h)H(x,g,h)]\right|\\&= \left|\mathbb{E}_{\mathbb{P}_{\mathrm{KMT}}}[F(x,g,h)H(x,g,h) - F(x,\widetilde{g},\widetilde{h})H(x,\widetilde{g},\widetilde{h})]\right|.\end{aligned}$$

We define the set

$$K_{N,y} := \left\{((g,h),(\widetilde{g},\widetilde{h}))|\sup_{0\le t\le L}\|(g,h)-(\widetilde{g},\widetilde{h})\| \ge CN^{-\frac{1}{2}}(\log(N)+y)\right\},$$

and split the expectation in terms of $K_{N,y}$ and its complement.

$$\begin{aligned}&\left|\mathbb{E}_{\mathbb{P}_{\mathrm{KMT}}}[F(x,g,h)H(x,g,h) - F(x,\widetilde{g},\widetilde{h})H(x,\widetilde{g},\widetilde{h})]\right|\\&= \left|\mathbb{E}_{\mathbb{P}_{\mathrm{KMT}}}[F(x,g,h)H(x,g,h) - F(x,\widetilde{g},\widetilde{h})H(x,\widetilde{g},\widetilde{h})|K_{N,y}^c]\right|\mathbb{P}(K_{N,y}^c)\\&+ \left|\mathbb{E}_{\mathbb{P}_{\mathrm{KMT}}}[F(x,g,h)H(x,g,h) - F(x,\widetilde{g},\widetilde{h})H(x,\widetilde{g},\widetilde{h})|K_{N,y}]\right|\mathbb{P}(K_{N,y}).\end{aligned}\tag{4.2}$$

By Theorem 4.1, $\mathbb{P}(K_{N,y}) \le e^{-y}$ and therefore the second term above is bounded by

$$\le 2C_F e^{-y}\sup\{\mathbb{E}_{\mathbb{W}_L^{(N)}}[H(x,g,h)], \mathbb{E}_{\mathbb{W}_L}[H(x,g,h)\}.\tag{4.3}$$

The terms inside this supremum are bounded in Lemma 3.1 and Lemma 3.4. To bound the first term on the left-hand side of (4.2), we note that when $\|g-\widetilde{g}\| < \delta$,

$$\begin{aligned}&\left|H(x,g,h) - H(x,\widetilde{g},\widetilde{h})\right|\\&\qquad = H(x,\widetilde{g},\widetilde{h})\left|\exp\left(-2v(g(L)-\widetilde{g}(L)) - e^{-2x}\int_0^L e^{-2g'(t)}\left(e^{-2(g(t)-\widetilde{g}(t))}-1\right)dt\right)-1\right|,\end{aligned}$$

$$\left|H(x,g,h)-H(x,\widetilde{g},\widetilde{h})\right| \le H(x,\widetilde{g},\widetilde{h})\sup\left\{\begin{array}{c}\exp\left(2|v|\delta - e^{-2x}(e^{-2\delta}-1)\int_0^L e^{-2\widetilde{g}(t)}dt\right)-1,\\ 1-\exp\left(-2|v|\delta - e^{-2x}(e^{-2\delta}-1)\int_0^L e^{-2\widetilde{g}(t)}dt\right),\\ \exp\left(-2|v|\delta - e^{-2x}(e^{-2\delta}-1)\int_0^L e^{-2\widetilde{g}(t)}dt\right)-1\end{array}\right\}.\tag{4.4}$$

We define the random variable $X_k := e^{-2x}\int_0^L e^{-2\widetilde{g}(t)}dt\Big|_{\widetilde{g}(L)=k}$ which takes values in $[0,\infty)$ and has distribution $\rho_{X,L}(m,k)\ge 0$, which, by calculations in Section 3, must satisfy

$$\mathbb{E}_{\mathbb{W}_L}[H(x,g,h)] = e^{-2(u+v)x}\int_{-\infty}^{\infty}\int_0^{\infty} e^{-2vk-m}\rho_{X,L}(m,k)dmdk < \infty.$$

We set $\delta = CN^{-\frac{1}{2}}(\log(\sqrt{N})+y)$ to match the KMT constraints, and choose $y=\log(\sqrt{N})$. Therefore, for all $N > N_\delta$, (4.4) is bounded by

$$\le \sup\left\{\begin{array}{c}e^{-2(u+v)x}\int_{-\infty}^{\infty}e^{-2vk}\int_0^{\infty}e^{-m}\left(e^{2|v|\delta+m(e^{-2\delta}-1)}-1\right)\rho_{X,L}(m,k)dmdk,\\ e^{-2(u+v)x}\int_{-\infty}^{\infty}e^{-2vk}\int_0^{\infty}e^{-m}\left(1-e^{-2|v|\delta+m(e^{-2\delta}-1)}\right)\rho_{X,L}(m,k)dmdk,\\ e^{-2(u+v)x}\int_{-\infty}^{\infty}e^{-2vk}\int_0^{\infty}e^{-m}\left(e^{-2|v|\delta+m(e^{-2\delta}-1)}-1\right)\rho_{X,L}(m,k)dmdk\end{array}\right\}.$$

There exist $N_\delta, D_v > 0$, depending only on $|v|$ such that for all $N > N_v$, all of the functions inside the interior integrand are bounded by $D_v\delta e^{-\frac{m}{4}}$ (for instance, when $N=1000$, and $D_v = \sup\{|v|^{-1}, 2|v|\}$ for all $m\ge 0$, this inequality holds). Thus, the supremum of the terms above is bounded by

$$\begin{aligned}&\le D_v\delta e^{-2(u+v)x}\int_{-\infty}^{\infty}e^{-2vk}\int_0^{\infty}e^{-\frac{m}{4}}\rho_{X,L}(m,k)dmdk,\\&\le D\delta e^{-2(u+v)x}\mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-\frac{e^{-2x}}{4}\int_0^L e^{-2g(t)}dt\right)\right].\end{aligned}$$

We conclude that there exists $C_v > 0$ such that equation above is bounded by

$$\leq C_v \log(N) N^{-\frac{1}{2}} e^{-2(u+v)x} \mathbb{E}_{\mathbb{W}_L} \left[\exp\left(-\frac{e^{-2x}}{4} \int_0^L e^{-2g(t)} dt \right) \right].$$

Therefore, with $y = \log(\sqrt{N})$, and noting that $\mathbb{P}(K_{N,y}^c) \leq 1$, for all $v \in \mathbb{R}$, we see that

$$\Pi_N \leq 2C_F C_v \log(N) N^{-\frac{1}{2}} e^{-2(u+v)x} \mathbb{E}_{\mathbb{W}_L} \left[\exp\left(-\frac{e^{-2x}}{4} \int_0^L e^{-2g'(t)} dt \right) \right] + 2C_F N^{-\frac{1}{2}} \sup \left\{ \mathbb{E}_{\mathbb{W}_L^{(N)}} \left[H(x,g,h) \right], \mathbb{E}_{\mathbb{W}_L} \left[H(x,g,h) \right] \right\}.$$

□

5. Proof of Weak Convergence

In this section, we prove Theorem 2.10. The proof of this theorem relies on the following intermediate proposition, the proof of which will occupy most of this section.

Proposition 5.1. *For all bounded continuous functions $F(x,g,h)$*

$$\lim_{N\to\infty} \mathbb{E}_{\mathbb{P}_L^{(N)}}[F(x,g,h)H^{(N)}(x,g,h)] = \mathbb{E}_{\mathbb{P}_L}[F(x,g,h)H(x,g,h)].$$

Before proving Proposition 5.1, we give the proof of Theorem 2.10 using this result. We state the following corollary of Proposition 5.1, which we obtain by setting $F(x,g,h) = 1$.

Corollary 5.2. *The partition functions converge, $\lim_{N\to\infty} Z_{L;u,v}^{(N)} = \mathcal{Z}_{u,v}$.*

With these propositions, we can prove Theorem 2.10.

Proof of Theorem 2.10. We combine the results of Corollary 5.2 and Proposition 5.1. This proves that for all bounded continuous functions $F(x,g,h)$.

$$\lim_{N\to\infty} (Z_{L;u,v}^{(N)})^{-1} \mathbb{E}_{\mathbb{P}_L^{(N)}} \left[F(x,g,h) H^{(N)}(x,g,h) \right] = \mathcal{Z}_{u,v}^{-1} \mathbb{E}_{\mathbb{P}_L} \left[F(x,g,h) H(x,g,h) \right].$$

□

Finally, we finish the proof of Proposition 5.1.

Proof of Proposition 5.1. In this proof, we will use x to denote lattice points in $\widetilde{\mathbb{Z}}(N)$ and r to denote elements of $\mathbb{R}$. We define the minimal element in $\widetilde{Z}(N)$ as $\widetilde{x} := \frac{1}{\sqrt{N}} - \log(\sqrt{N})$. We must show that the difference of these terms goes to zero.

$$\begin{aligned} &\sum_{x\in\widetilde{\mathbb{Z}}(N)} N^{-\frac{1}{2}} \mathbb{E}_{\mathbb{W}_L^{(N)}} \left[F(x,g,h) H^{(N)}(x,g,h) \right] - \mathbb{E}_{\mathbb{L}\times\mathbb{W}} \left[F(r,g,h) H(r,g,h) \right] \\ &= \sum_{x\in\widetilde{\mathbb{Z}}(N)} \mathbb{E}_{\mathbb{L}[x,x+1/\sqrt{N}]} \left[\mathbb{E}_{\mathbb{W}_L^{(N)}} \left[F(x,g,h) H^{(N)}(x,g,h) \right] - \mathbb{E}_{\mathbb{W}_L} \left[F(r,g,h) H(r,g,h) \right] \right] \\ &+ \mathbb{E}_{\mathbb{L}_{(-\infty,\widetilde{x}]}\times\mathbb{W}_L} \left[F(r,g,h) H(r,g,h) \right]. \end{aligned} \tag{5.1}$$

As before, we write $C_F := \sup_{x,g,h} |F(x,g,h)|$. We can deal with the final term of the bound simply by integrating and applying the bound from Lemma 3.4. when $v \geq 0$,

$$\begin{aligned} \mathbb{E}_{\mathbb{L}_{(-\infty,\widetilde{x}]}\times\mathbb{W}_L} \left[F(r,g,h) H(r,g,h) \right] &\leq \int_{-\infty}^{\frac{1}{\sqrt{N}} - \log(\sqrt{N})} e^{-2(u+v)r} C_F C_2 K_0 \left(\sqrt{2} e^{-r} \right) dr \\ &\leq C_F C_2 \int_{\sqrt{N} e^{-1/\sqrt{N}}}^{\infty} t^{2(u+v)-1} K_0(\sqrt{2} t) dt \\ &\leq C_F C_2 D_1 \int_{\sqrt{N} e^{-1/\sqrt{N}}}^{\infty} t^{2(u+v)-1} e^{-\sqrt{2} t} dt \\ &\leq C_F C_2 D_1 2^{-2(u+v)} \Gamma \left(2(u+v), \sqrt{N} e^{-1/\sqrt{N}} \right). \end{aligned}$$

The first change of coordinates was $t = e^{-x}$, and then we used the fact that for $t > 1$, there exists $D_1 > 0$ such that $K_0(\sqrt{2}t) \le D_1 e^{-\sqrt{2}t}$. And when $v < 0$,

$$\mathbb{E}_{\mathbb{L}_{(-\infty,\tilde{x}]}\times\mathbb{W}_L}\left[F(r,g,h)H(r,g,h)\right] \le \int_{-\infty}^{\frac{1}{\sqrt{N}}-\log(\sqrt{N})} e^{-2(u+v)r} C_F C_3 K_0\left(\sqrt{2}e^{-x}\right) dr + \int_{-\infty}^{\frac{1}{\sqrt{N}}-\log(\sqrt{N})} e^{-2(u+v)r} C_F C_4 e^{xv} K_v\left(2C_5 e^{-x}\right) dr.$$

We already have a bound on the first term similar to the one we obtained for $v \ge 0$, by $C_3 C_F D_2 2^{-2(u+v)} \Gamma(2(u+v), \sqrt{N}e^{-1/\sqrt{N}})$, so we proceed to bound the second term,

$$\begin{aligned}\int_{-\infty}^{\frac{1}{\sqrt{N}}-\log(\sqrt{N})} e^{-2(u+v)r} C_F C_4 e^{xv} K_v\left(2C_5 e^{-x}\right) dr &= C_4 C_F \int_{\sqrt{N}e^{-1/\sqrt{N}}}^{\infty} t^{2(u+v)-v-1} K_v(2C_5 t) dt \\ &\le C_4 D_3 C_F \int_{\sqrt{N}e^{-1/\sqrt{N}}}^{\infty} t^{2(u+v)-v-1} e^{-2C_5 t} dt \\ &\le C_4 D_3 C_F (2C_5)^{-2u-v} \Gamma\left(2(u+v)-v, 2C_5\sqrt{N}e^{-1/\sqrt{N}}\right).\end{aligned}$$

Where $D_3 > 0$ is chosen so that $K_v(2C_5 t) \le D_3 e^{-2C_5 t}$ for all $t \ge 1$. Since the limits as N goes to infinity of $\Gamma(2(u+v), \sqrt{N}e^{-1/\sqrt{N}})$ and $\Gamma(2(u+v)-v, 2C_5\sqrt{N}e^{-1/\sqrt{N}})$ are both 0, we conclude that $\lim_{N\to\infty} \mathbb{E}_{\mathbb{L}_{(-\infty,\tilde{x}]}\times\mathbb{W}}[F(r,g,h)H(r,g,h)] = 0$. Now, we bound the second line of (5.1),

$$\begin{aligned}&\sum_{x\in\widetilde{\mathbb{Z}}(N)} \mathbb{E}_{\mathbb{L}[x,x+1/\sqrt{N}]}\left[\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[F(x,g,h)H^{(N)}(x,g,h)\right] - \mathbb{E}_{\mathbb{W}_L}\left[F(r,g,h)H(r,g,h)\right]\right] \\ &\le \sum_{x\in\widetilde{\mathbb{Z}}(N)} \mathbb{E}_{\mathbb{L}[x,x+1/\sqrt{N}]}\left[\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[F(x,g,h)H^{(N)}(x,g,h)\right] - \mathbb{E}_{\mathbb{W}_L}\left[F(x,g,h)H(x,g,h)\right]\right] \\ &+ \sum_{x\in\widetilde{\mathbb{Z}}(N)} \mathbb{E}_{\mathbb{L}[x,x+1/\sqrt{N}]}\left[\mathbb{E}_{\mathbb{W}_L}\left[F(x,g,h)H(x,g,h)\right] - \mathbb{E}_{\mathbb{W}_L}\left[F(r,g,h)H(r,g,h)\right]\right].\end{aligned} \tag{5.2}$$

To deal with the first line on the right-hand side of (5.2), we use the expression in (4.1) and then Proposition 4.2 to write

$$\begin{aligned}&\sum_{x\in\widetilde{\mathbb{Z}}(N)} \mathbb{E}_{\mathbb{L}[x,x+1/\sqrt{N}]}\left[\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[F(x,g,h)H^{(N)}(x,g,h)\right] - \mathbb{E}_{\mathbb{W}_L}\left[F(x,g,h)H(x,g,h)\right]\right] \\ &= 2C_F \mathbb{E}_{\mathbb{P}^{(N)}}\left[H^{(N)}(x,g,h) - H(x,g,h)\right] \\ &+ 2C_F C_{v,1} \log(N) N^{-\frac{1}{2}} \sum_{x\in\widetilde{\mathbb{Z}}(N)} N^{-\frac{1}{2}} e^{-2(u+v)} \mathbb{E}_{\mathbb{W}_L}\left[\exp\left(-\frac{e^{-2x}}{4}\int_0^L e^{-2g(t)} dt\right)\right] \\ &+ 2C_F N^{-\frac{1}{2}} \sum_{x\in\widetilde{\mathbb{Z}}(N)} N^{-\frac{1}{2}} \sup\left\{\mathbb{E}_{\mathbb{W}_L^{(N)}}\left[H(x,g,h)\right], \mathbb{E}_{\mathbb{W}_L}\left[H(x,g,h)\right]\right\}.\end{aligned}$$

Now, by the pointwise convergence of $H^{(N)}(x,g,h)$ to $H(x,g,h)$ (Theorem 2.8) and the fact that $H^{(N)}(x,g,h) - H(x,g,h)$ is bounded by an integrable function due to the results in Section 3, the first line of the expression above converges to 0 by dominated convergence. The terms inside each of the sums in the equation above are bounded by a constant, by a calculation identical to that in Corollary 3.5 and Corollary 3.6. Therefore, the extra factors of $\log(N)N^{-\frac{1}{2}}$ and $N^{-\frac{1}{2}}$ guarantee that those terms also converge to zero as $N \to \infty$.

We proceed to deal with the second line of (5.2). For $r \in [x, x + \frac{1}{\sqrt{N}}]$, there exists $D_1 > 0$ such that the above is bounded by

$$|H(x,g,h) - H(r,g,h)| \le N^{-\frac{1}{2}} D_1 H(r,g,h).$$

Therefore,

$$2C_F \sum_{x\in\widetilde{\mathbb{Z}}(N)} \mathbb{E}_{\mathbb{L}[x,x+1/\sqrt{N}]}\left[\mathbb{E}_{\mathbb{W}_L}\left[H(x,g,h)-H(r,g,h)\right]\right] \leq 2C_F D_1 N^{-\frac{1}{2}} \int_{-\infty}^{\infty} \mathbb{E}_{\mathbb{W}_L}\left[H(r,g,h)\right] dr$$

Then, by Lemma 3.4, there exists $D_2 > 0$ such that the equation above is bounded by $D_2 N^{-\frac{1}{2}}$, which goes to 0 as $N \to \infty$. Combining these bounds, we see that the right-hand side of (5.1) goes to zero. $\square$

Appendix A. Technical Lemmas

In this appendix, we will prove Proposition 3.3, which essentially relies on Stirling's approximation.

Proof of Proposition 3.3. By Stirling's approximation, we see that for all $|k| \leq N$, there exists a global constant $D_1 > 0$ such that for all $N \in \mathbb{N}$,

$$\binom{2N}{N+k} \leq \frac{D_1}{\sqrt{2\pi N}}\left(1-\frac{k^2}{N^2}\right)^{-N}\left(1-\frac{k^2}{N^2}\right)^{-\frac{1}{2}}\left(1-\frac{2k}{N+k}\right)^{k}. \tag{A.1}$$

Likewise, for $|k+a\sqrt{N}| \leq N$ there is a global constant $D_2 > 0$ such that for all $N \in \mathbb{N}$,

$$\binom{2N}{N+k+a\sqrt{N}} \geq \frac{D_2}{\sqrt{2\pi N}}\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)^{-N}\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)^{-\frac{1}{2}}\left(1-\frac{2(k+a\sqrt{N})}{N+k+a\sqrt{N}}\right)^{k+a\sqrt{N}}. \tag{A.2}$$

We define a set

$$K_a := \{k \mid |k| < N, |k+a\sqrt{N}| < N^{\frac{5}{6}}\}.$$

We will show that on this domain, the following ratio can be bounded by a constant, possibly depending on a, for all $N \in \mathbb{N}$;

$$\begin{aligned}\frac{\exp\left(-\frac{(k+a\sqrt{N})^2}{N}+\frac{k^2}{N}\right)\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)^N}{\left(1-\frac{k^2}{N^2}\right)^N} &= \exp\left(-\frac{(k+a\sqrt{N})^2}{N}+N\log\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)\right),\\ &\cdot \exp\left(\frac{k^2}{N}-N\log\left(1-\frac{k^2}{N^2}\right)\right).\end{aligned}$$

Noting that $|k|, |k+a\sqrt{N}| < N^{\frac{5}{6}}$, we see that the Taylor expansion of the logarithm converges, for all finite N, in both cases. Therefore, the expression above becomes

$$\begin{aligned}&= \exp\left(-\frac{(k+a\sqrt{N})^2}{N}+\frac{(k+a\sqrt{N})^2}{N}\left(\sum_{l=1}^{\infty}\frac{(k+a\sqrt{N})^{2l-2}}{lN^{2l-2}}\right)+\frac{k^2}{N}-\frac{k^2}{N}\left(\sum_{l=1}^{\infty}\frac{k^{2l-2}}{lN^{2l-2}}\right)\right)\\ &= \exp\left(\frac{k^2}{N}\left(\sum_{l=1}^{\infty}\frac{(k+a\sqrt{N})^{2l-2}-k^{2l-2}}{lN^{2l-2}}\right)+\frac{2ak}{\sqrt{N}}\left(\sum_{l=1}^{\infty}\frac{(k+a\sqrt{N})^{2l-2}}{lN^{2l-2}}\right)+a^2\left(\sum_{l=1}^{\infty}\frac{(k+a\sqrt{N})^{2l-2}}{lN^{2l-2}}\right)\right)\\ &\leq \exp\left(\left(\frac{k^2}{N}+\frac{2|ak|}{\sqrt{N}}+a^2\right)\sum_{l=1}^{\infty}\frac{1}{lN^{\frac{l}{3}}}\right) \leq \exp\left(\frac{N^{\frac{2}{3}}+2|a|N^{\frac{1}{3}}+a^2}{N^{\frac{1}{3}}\left(N^{\frac{1}{3}}-1\right)}\right).\end{aligned}$$

As $N \to \infty$, the expression above will converge to a constant. In particular, we can choose $D_{a,1} > 0$, possibly depending on a such that, for all $N \in \mathbb{N}$, the expression above is bounded by $D_{a,1}$ This allows us to conclude that for any $a \in \mathbb{R}$

$$\exp\left(-\frac{(k+a\sqrt{N})^2}{N}+\frac{k^2}{N}\right)\frac{\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)^N}{\left(1-\frac{k^2}{N^2}\right)^N} = \exp\left(-\frac{2ak}{\sqrt{N}}-a^2\right)\frac{\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)^N}{\left(1-\frac{k^2}{N^2}\right)^N} \leq D_{a,1}.$$

Likewise, for $k \in K_a$,

$$\lim_{N\to\infty}\left(1-\frac{k^2}{N^2}\right)^{-\frac{1}{2}}\left(1-\frac{2k}{N+k}\right)^{k}=1,\quad \lim_{N\to\infty}\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)^{-\frac{1}{2}}\left(1-\frac{2(k+a\sqrt{N})}{N+k+a\sqrt{N}}\right)^{k+a\sqrt{N}}=1,$$

and, therefore, there exists a constant $D_{a,2}>0$ such that for all $N\in\mathbb{N}$,

$$\left(1-\frac{k^2}{N^2}\right)^{-\frac{1}{2}}\left(1-\frac{2k}{N+k}\right)^{k}\left(1-\frac{(k+a\sqrt{N})^2}{N^2}\right)^{\frac{1}{2}}\left(1-\frac{2(k+a\sqrt{N})}{N+k+a\sqrt{N}}\right)^{-k-a\sqrt{N}}<D_{a,2}.$$

Putting this together, we see that

$$\exp\left(-\frac{2ak}{\sqrt{N}}\right)\binom{2N}{N+k}\binom{2N}{N+k+a\sqrt{N}}^{-1}\leq e^{a^2}D_{a,1}D_{a,2}.$$

□

Finally, we give the proof of Lemma 3.3.

Proof. We begin by noting the Radon-Nikodym derivative changing between Brownian motion and Brownian motion with drift $-m$,

$$Z_t(g):=\frac{d\mathbb{W}\big|_{\mathcal{F}_t}}{d\mathbb{W}_m\big|_{\mathcal{F}_t}}(g)=\exp\left(mg(t)-\frac{1}{2}m^2t\right).$$

We study the conditional expectation expression

$$\mathbb{E}_{\mathbb{W}}\left[F(g(t))\middle|g(L)=\frac{k}{\sqrt{N}}\right]=\int_0^\infty r\rho_{F(g(t))|g(L)=\frac{k}{\sqrt{N}}}(r)dr.$$

By assumption, the distribution of $F(g(t))$ is continuous, therefore

$$\rho_{F(g(t))|g(L)=\frac{k}{\sqrt{N}}}(r)=\frac{\rho_{F(g(t)),g(L)}\left(r,\frac{k}{\sqrt{N}}\right)}{\rho_{g(L)}\left(\frac{k}{\sqrt{N}}\right)}.$$

Therefore,

$$\mathbb{E}_{\mathbb{W}}\left[F(g(t))\middle|g(L)=\frac{k}{\sqrt{N}}\right]=\frac{1}{\rho_{g(L)}\left(\frac{k}{\sqrt{N}}\right)}\int_0^\infty r\rho_{F(g(t)),g(L)}\left(r,\frac{k}{\sqrt{N}}\right)dr,$$

which allows us to conclude that

$$\mathbb{E}_{\mathbb{W}}\left[F(g(t))\mathbf{1}_{g(L)=\frac{k}{\sqrt{N}}}\right]=\mathbb{E}_{\mathbb{W}}\left[F(g(t))\middle|g(L)=\frac{k}{\sqrt{N}}\right]\exp\left(-\frac{k^2}{N}\right).$$

We can then perform the following change of measure calculation,

$$\begin{aligned}
\mathbb{E}_{\mathbb{W}}\left[F(g(t))\mathbf{1}_{g(L)=\frac{k}{\sqrt{N}}}\right]&=\mathbb{E}_{\mathbb{W}_m}\left[F(g_m(t))\mathbf{1}_{g_m(L)=\frac{k}{\sqrt{N}}}\right],\\
&=\mathbb{E}_{\mathbb{W}_m}\left[F(g(t)-mt)\mathbf{1}_{g(L)=\frac{k}{\sqrt{N}}+m}\right],\\
&=\mathbb{E}_{\mathbb{W}}\left[F(g(t)-mt)Z_L(g)\mathbf{1}_{g(L)=\frac{k}{\sqrt{N}}+m}\right],\\
&=\mathbb{E}_{\mathbb{W}}\left[F(g(t)-mt)Z_L(g)\middle|g(L)=\frac{k}{\sqrt{N}}+m\right]\exp\left(-\frac{\left(k+m\sqrt{N}\right)^2}{N}\right),\\
&=\mathbb{E}_{\mathbb{W}}\left[F(g(t)-mt)\middle|g(L)=\frac{k}{\sqrt{N}}+m\right]\exp\left(-\frac{mk}{\sqrt{N}}-\frac{L+2}{2}m^2-\frac{k^2}{N}\right).
\end{aligned}$$

The conclusion follows directly. □

Appendix B. Pointwise Convergence

In this appendix, we recap the argument in [BLD23] that gives the proof of pointwise convergence of the Radon-Nikodym derivatives. We quote a proposition from Durrett [Dur19].

Proposition B.1 (Exercise 3.1.1 [Dur19])**.** *If the following three conditions are met*

$$\lim_{n\to\infty}\max_{0\le j\le n}|c_{j,n}|=0,\qquad \lim_{n\to\infty}\sum_{j=0}^{n}c_{j,n}=\lambda,\qquad \sup_n\sum_{j=0}^{n}|c_{j,n}|<\infty,$$

then

$$\lim_{n\to\infty}\prod_{j=0}^{n}(1+c_{j,n})=e^{\lambda}.$$

Proposition B.2. *$H^{(N)}(x,g,h)\to H(x,g,h)$ for all $(x,g,h)\in\mathbb{R}\times C_0[0,L]\times C_0[0,L]$.*

Proof. We recall the expressions for $H^{(N)}(x,g,h)$ and $H(x,g,h)$, removing the restriction of $H^{(N)}(x,g,h)$ to a subset of the domain $\mathbb{R}\times C_0[0,L]\times C_0[0,L]$ (compare to Definition 2.7).

$$H^{(N)}(x,g,h)=\exp\left(-2(u+v)x-2vg(L)\right)\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right),$$

$$H(x,g,h)=\exp\left(-2(u+v)x-2vg(L)-e^{-2x}\int_0^L e^{-2g(t)}dt\right).$$

We consider the sequence $c_{i,N}=-N^{-1}e^{-2(g(t_i)+x)}$. For any fixed function $g(t)$ on the interval $[0,L]$, the value $e^{-2(g(t)+x)}$ is bounded by a constant $C_g>0$. Therefore,

$$\lim_{N\to\infty}|c_{i,N}|\le\lim_{N\to\infty}N^{-1}C_g=0.$$

Likewise, because $g\in C_0[0,L]$, we also know that $e^{-2g(t_i)}\in C[0,L]$. Therefore, by the definition of the Riemann integral,

$$\lim_{N\to\infty}-\sum_{i=0}^{N}\frac{e^{-2(g(t_i)+x)}}{N}=-e^{-2x}\int_0^L e^{-2g(t)}dt.$$

Finally, by the same reasoning,

$$\lim_{N\to\infty}\sum_{i=0}^{N}|c_{i,N}|=\lim_{N\to\infty}\sum_{i=0}^{N}N^{-1}e^{-2(g(t_i)+x)}=e^{-2x}\int_0^L e^{-2g(t)}dt,$$

and therefore $\sup_N\sum_{i=0}^{N}|c_{i,N}|<\infty$. Putting all of these calculations together, Proposition B.1 allows us to conclude that

$$\lim_{N\to\infty}\exp\left(-2(u+v)x-2vg(L)\right)\prod_{i=0}^{N}\left(1-\frac{e^{-2(g(t_i)+x)}}{N}\right)$$
$$=\exp\left(-2(u+v)x-2vg(L)-e^{-2x}\int_0^L e^{-2g(t)}dt\right).$$

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