GLOBAL BIFURCATION OF NON-RADIAL SOLUTIONS FOR SYMMETRIC SUB-LINEAR ELLIPTIC SYSTEMS ON THE PLANAR UNIT DISC

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Abstract. In this paper, we prove a global bifurcation result for the existence of non-radial branches of solutions to the paramterized family of Γ-symmetric problems $-\Delta u = f(\alpha, z, u)$, $u|_{\partial D} = 0$ on the unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ with $u(z) \in \mathbb{R}^k$, where $\mathbb{R}^k$ is an orthogonal Γ-representation, $f : \mathbb{R} \times \overline{D} \times \mathbb{R}^k \to \mathbb{R}^k$ is a sub-linear Γ-equivariant continuous function, differentiable with respect to u at zero and satisfying the conditions $f(\alpha, e^{i\theta} z, u) = f(\alpha, z, u)$ for all $\theta \in \mathbb{R}$ and $f(z, -u) = -f(z, u)$.

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1. Introduction

We draw our motivation from natural phenomena that are modeled by parameterized systems of autonomous Partial Differential Equations (PDEs), where the possible states of each phenomenon are represented by solutions of the associated system of equations at corresponding parameter values. These solutions can be categorized as *trivial* or *non-trivial.* Trivial solutions are so-named because they persist for all parameter values and are straightforward to determine. In contrast, Non-trivial solutions may only exist for a particular range of parameter values and often exhibit exotic properties which enhance our understanding of the model. The *classical bifurcation problem* is concerned with the existence and global behaviour of branches of non-trivial solutions emerging from their trivial counterparts. Should the phenomena admit the symmetries of a certain group G (represented by the G-equivariance of the associated equations), one may study the *equivariant bifurcation problem*, which additionally considers the symmetric properties of these branches (cf. [1, 3, 9, 11, 19, 21, 22, 23, 24, 25]).

The application of topological methods to the study of differential equations, as with most tools in the arsenal of the nonlinear analyst, traces back to *Maître* Poincare [13]. More recently, the *Local/Equivariant Brouwer Degrees* and their infinite dimensional generalizations – the *Local/Equivariant Leray-Schauder Degrees* – have proven prodigiously successful in obtaining existence results for solutions to a wide class of nonlinear differential equations. Somewhat remarkably, these degree theories have also been shown to be useful in solving classical and equivariant bifurcation problems. The first use of the Leray-Schauder degree for the detection of (local) bifurcation to a parameterized system of nonlinear differential equations is attributed to the seminal work of M.A. Krasnosel'skii, in which sufficient conditions for the existence of a branch of non-trivial solutions are established (cf. [17]). Only a decade later, P. Rabinowitz (cf. [20]) proposed his famous *Rabinowitz alternative*, which provides sufficient conditions for a given branch of non-trivial solutions to be unbounded. In this paper, equivariant analogues of Krasnosel'skii and Rabinowitz type results are employed to solve an equivariant bifurcation problem.

Specifically, our objective is to study the equivariant bifurcation problem for a symmetric system of parameterized nonlinear Laplace equations subject to Dirichlet boundary conditions. The bifurcation and global behaviour of solutions for such systems has been extensively studied, due to their significance in applied mathematics and physics. In the case that the nonlinearity arises as the gradient of some potential function, the bifurcation problem can be studied as a parameterized variational problem and appropriate topological methods such as Morse theory, index theory, and the equivariant gradient degree (cf. [14]) can be administered (cf. [6, 7, 8, 10, 12, 11, 21, 22, 23, 24, 25, 26]). However, these approaches are of no use for systems which do not admit variational structure. The degree theoretic methods we employ in this paper impose no variational requirements on the equations and, as such, are broadly applicable to a wider class of problems.

Consider the following parameterized family of symmetric Laplace equations:

$$\begin{cases} -\Delta u = f(\alpha, z, u), \quad \alpha \in \mathbb{R},\ z \in D,\ u(z) \in V \\ u|_{\partial D} = 0, \end{cases} \tag{1}$$

where $D := \{z \in \mathbb{C} : |z| < 1\}$ is the unit planar disc, $V := \mathbb{R}^k$ is an orthogonal representation of a finite group Γ and $f : \mathbb{R} \times \overline{D} \times V \to V$ is a continuous, odd, radially symmetric and Γ-equivariant family of functions of sub-linear growth, which are differentiable with respect to u at the origin in V. In particular, we assume that f satisfies the following conditions:

- (A_1) $f(\alpha, e^{i\theta} z, u) = f(\alpha, z, u)$ for all $\alpha \in \mathbb{R},\ z \in D,\ u \in V$ and $\theta \in \mathbb{R}$;
- (A_2) $f(\alpha, z, \gamma u) = \gamma f(\alpha, z, u)$ for all $\alpha \in \mathbb{R},\ z \in D,\ u \in V$ and $\gamma \in \Gamma$;
- (A_3) $f(\alpha, z, -u) = -f(\alpha, z, u)$ for all $\alpha \in \mathbb{R},\ z \in D$ and $u \in V$;
- (A_4) there exist continuous functions $A : \mathbb{R} \to L(k, \mathbb{R})$, $c : \mathbb{R} \to (0, \infty)$ and a number $\beta > 1$ such that for each $\alpha \in \mathbb{R}$ one has

$$|f(\alpha, z, u) - A(\alpha)u| \le c(\alpha)|u|^\beta \quad \text{for all } z \in \overline{D},\ u \in V;$$

(A_5) there exist continuous functions $a, b : \mathbb{R} \to (0, \infty)$ and a number $\nu \in (0, 1)$ such that for each $\alpha \in \mathbb{R}$ the following inequality holds

$$|f(\alpha, z, u)| < a(\alpha)|u|^\nu + b(\alpha) \quad \text{for all } z \in \overline{D}, u \in V.$$

Conditions (A_1) (A_2) and (A_3) imply the $O(2)\times\Gamma\times\mathbb{Z}_2$-symmetry of system (1) in the sense that $f : \mathbb{R} \times \overline{D} \times V \to V$ is $O(2)$-invariant with respect to the $O(2)$-action $O(2) \times D \to D$ given by $(\theta, z) \to e^{i\theta}z$, $(\kappa, z) \to \overline{z}$ and $\Gamma \times \mathbb{Z}_2$-equivariant with respect to the $\Gamma \times \mathbb{Z}_2$ action $(\Gamma \times \mathbb{Z}_2) \times V \to V$ given by $(\gamma, \pm 1, u) \to \pm\gamma u$. On the other hand, conditions (A_4) and (A_5) guarantee differentiability at the origin and sublinearity, respectively.

Remark 1.1. Since the zero function is a solution to (1) for all values $\alpha \in \mathbb{R}$, all trivial solutions to the system are of the form $(\alpha, 0)$. Non-trivial solutions to (1), on the other hand, fall into two categories, namely:

(i) *radial solutions*, which depend only on $|z|$ for any $z \in D$ (although less obvious than the trivial solution, radial solutions can often be identified using classical methods, eg. by reducing (1) to a second order ODE);

(ii) and *non-radial solutions*, which exhibit dependency on the angular variable.

Our objective is to identify branches of non-radial solutions to (1), describe their possible symmetric properties, and characterize their global behavior.

The methods used in this paper are inspired by [2], where an existence result was obtained for a similar sublinear elliptic system using equivariant degree theory. For a more thorough exposition of these topics, we direct readers to the recent monograph [4].

The equivariant degree is intimately connected with the classical Brouwer degree. Although its application is relatively simple, technical difficulties arise when dealing with algebraic computations related to unfamiliar group structures. Many of these issues can be resolved with usage of the G.A.P. system, and the G.A.P. package `EquiDeg` (created by Haopin Wu), which is available online at `https://github.com/psistwu/equideg` (cf. [28]).

In the remainder of this paper, we employ tools from the equivariant degree theory to determine (i) under what conditions branches of non-trivial solutions may bifurcate from a trivial solution, (ii) under what conditions these branches consist only of non-radial solutions and (iii) under what conditions these branches are unbounded. Subsequent sections are organized as follows: In Section 2 the problem (1) is reformulated in an appropriate functional setting. In Section 3 we recall the abstract equivariant bifurcation results, including the equivariant analogues of the classical Krasnosel'skii and Rabinowitz theorems. In Section 4, we apply the equivariant degree theory methods to establish local and global bifurcation results for (1). Finally, in Section 5 we present a motivating example of vibrating membranes where our main results, Theorem 4.1 and 4.3, are applied to demonstrate the existence of unbounded branches of non-radial solutions admitting all possible maximal orbit types. For convenience, the Appendices include an explanation of notations used, a summary of the spectral properties of the Laplace operator on the unit disc, and a brief introduction to the Brouwer equivariant degree theory.

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2. Functional Space Reformulation and *a priori* Bounds

Consider the Sobolev space $\mathscr{H} := H^2(D,V)\cap H^1_0(D,V)$ equipped with the usual norm

$$\|u\|_{\mathscr{H}} := \max\{\|D^s u\|_2 : |s|\leq 2\},$$

where $s=(s_1,s_2)$, $|s|:=s_1+s_2\leq 2$, and $D^s\varphi = \frac{\partial^{|s|}\varphi}{\partial^{s_1}x\partial^{s_2}y}$. It is well known that the *Laplacian operator* $\mathscr{L}:\mathscr{H}\to L^2(D,V)$ given by

$$\mathscr{L}u := -\Delta u, \tag{2}$$

is a linear isomorphism.

Let $\nu\in(0,1)$ be the scalar from condition (A_5). If one chooses $q>\max\{1,2\nu\}$ (for example, it is enough to take $q:=2\beta$ (cf. assumption (A_4)), then there is the standard Sobolev embedding $j:\mathscr{H}\to L^q(D,V)$

$$j(u)(z)=u(z),\quad u\in\mathscr{H}, \tag{3}$$

and also its associated *Nemytski operator* $N_\alpha : L^q(D,V)\to L^2(D,V)$

$$N_\alpha(v)(z) := f(\alpha,z,v(z)),\quad z\in D,\ \alpha\in\mathbb{R}. \tag{4}$$

Lemma 2.1. Assume that the conditions (A_1)–(A_5) are satisfied. The Nemystiki operator (4) is well-defined.

Proof. It suffices to demonstrate that for any $v \in L^q(D,V)$ and $\alpha \in \mathbb{R}$ one has $N_\alpha(v) \in L^2(D,V)$. Indeed, combining (A_5) with the Hölder inequality, one has:

$$\begin{aligned}
\|N_\alpha(v)\|_{L^2} &= \|f(\alpha,z,v)\|_{L^2}\\
&\le \|\,a(\alpha)|v|^\nu\|_{L^2} + \|\,b(\alpha)\|_{L^2}\\
&= a(\alpha)\|\,|v|^\nu\|_{L^2} + b(\alpha)\sqrt{\pi}\\
&= a(\alpha)\left(\int_D |v|^{2\nu}\right)^{\frac{1}{2}} + b(\alpha)\sqrt{\pi}\\
&\le a(\alpha)\pi^{\frac{1-2\nu/q}{2}}\left(\int_D |v|^{\frac{2\nu q}{2\nu}}\right)^{\frac{2\nu}{2}q} + b(\alpha)\sqrt{\pi}\\
&= a(\alpha)\pi^{1/2-\nu/q}\|v\|_{L^q}^\nu + b(\alpha)\sqrt{\pi},
\end{aligned} \tag{5}$$

where the result follows from the assumptions that $b(\alpha), a(\alpha), \|v\|_q < \infty$. □

Notice that the operator equation

$$\mathscr{L}u = N_\alpha(ju), \quad \alpha \in \mathbb{R}, u \in \mathscr{H}, \tag{6}$$

is equivalent to system (1) in the sense that $u \in \mathscr{H}$ is a solution to (1) for some $\alpha \in \mathbb{R}$ if and only if $(\alpha,u) \in \mathbb{R}\times\mathscr{H}$ satisfies (6). In turn, since $\mathscr{L}$ is an isomorphism, the map $\mathscr{F} : \mathbb{R}\times\mathscr{H} \to \mathscr{H}$ given by

$$\mathscr{F}(\alpha,u) := u - \mathscr{L}^{-1}N_\alpha(ju), \tag{7}$$

is well-defined so that system (1) is also equivalent to the equation

$$\mathscr{F}(\alpha,u) = 0, \quad \alpha \in \mathbb{R},\ u \in \mathscr{H}. \tag{8}$$

In what follows, we will call (8) the *operator equation* associated with (1).

Lemma 2.2. Let $f : \mathbb{R}\times\overline{D}\times V \to V$ be a continuous function satisfying the assumption (A_5). For every $\alpha \in \mathbb{R}$, there exists a constant $R(\alpha) > 0$ such that, if $(\alpha,u) \in \mathbb{R}\times\mathscr{H}$ is a solution to system (1), then $\|u\|_{\mathscr{H}} < R(\alpha)$.

Proof. Assume that $(\alpha,u) \in \mathbb{R}\times\mathscr{H}$ is a solution to system (1). Combining (5) with $u = \mathscr{L}^{-1}N_\alpha(ju)$, one has

$$\|u\|_{\mathscr{H}} \le a(\alpha)\pi^{1/2-\nu/q}\|\mathscr{L}^{-1}\|\,\|u\|_{L_2}^\nu + b(\alpha)\sqrt{\pi}\|\mathscr{L}^{-1}\|, \tag{9}$$

which, together with $\|u\|_{\mathscr{H}} \ge \|u\|_{L^2}$, implies

$$\|u\|_{L^2} \le c\|u\|_{L^2}^\nu + d, \tag{10}$$

where $c := a(\alpha)\pi^{1/2-\nu/q}\|\mathscr{L}^{-1}\|$, $d := b(\alpha)\sqrt{\pi}\|\mathscr{L}^{-1}\|$. Finally, since $0 < \nu < 1$, there exists $R_0(\alpha) > 0$ such that $\psi(t) := t - ct^\nu - d > 0$ for $t \ge R_0(\alpha)$. Consequently, $\|u\|_{L^2} < R_0(\alpha)$, and by (9),

$$\|u\|_{\mathscr{H}} \leq c\|u\|_{L^2}^{\nu} + d < cR_0(\alpha)^{\nu} + d =: R(\alpha)$$

is the required constant. □

3. Abstract Local and Global Equivariant Bifurcation

In this section we present a concise exposition, following [4], of an equivariant Brouwer degree method to study symmetric bifurcation problems. Given a compact Lie group $\mathcal{G}$, an isometric Banach $\mathcal{G}$-representation $\mathcal{H}$ and a completely continuous $\mathcal{G}$-equivariant field $\mathcal{F} : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$, we use the Leray-Schauder Equivariant $\mathcal{G}$-Degree to describe local and global properties of the solution set to the equation,

$$\mathcal{F}(\alpha, u) = 0, \quad \alpha \in \mathbb{R},\ u \in \mathcal{H}. \tag{11}$$

To simplify our exposition and for compatibility with the system of interest (8), we make the following two assumptions.

(B_1) The set of *trivial solutions* to (11) is given by

$$M := \{(\alpha, 0) \in \mathbb{R} \times \mathcal{H}\}.$$

(B_2) There exists a continuous family of linear operators $\mathcal{A}(\alpha) : \mathcal{H} \to \mathcal{H}$ such that, for every $\alpha \in \mathbb{R}$, the derivative $D_u\mathcal{F}(\alpha, 0)$ exists and

$$\mathcal{A}(\alpha) := D\mathcal{F}(\alpha, 0).$$

Solutions to (11) which do not belong to M are called *nontrivial*. Let $\mathcal{S} \subset \mathcal{F}^{-1}(0)$ denote the set of all non-trivial solutions, i.e.

$$\mathcal{S} := \{(\alpha, u) \in \mathbb{R} \times \mathcal{H} \ : \ \mathcal{F}(\alpha, u) = 0 \text{ and } u \neq 0\}.$$

Clearly, the set of non-trivial solutions $\mathcal{S} \subset \mathbb{R} \times \mathcal{H}$ is $\mathcal{G}$-invariant.

3.1. The Local Bifurcation Invariant and Krasnosel'skii's Theorem.

Formulation of a Krasnosel'skii type local bifurcation result for equation (11) necessitates the introduction of additional notations and terminology (for more details, the reader is referred to [1, 4]). Our first definition clarifies what is meant by a bifurcation of the equation (11).

Definition 3.1. A trivial solution $(\alpha_0, 0) \in M$ is said to be a *bifurcation point* for the equation (11) if every open neighborhood of the point $(\alpha_0, 0)$ has non-trivial intersection with $\mathcal{S}$.

It is well-known that a necessary condition for any trivial solution $(\alpha_0, 0) \in M$ to be a bifurcation point for the equation (11) is that the linear operator $\mathcal{A}(\alpha_0) : \mathcal{H} \to \mathcal{H}$ is not an isomorphism. This leads to the following definition.

Definition 3.2. A trivial solution $(\alpha_0, 0) \in M$ is said to be a *regular point* for the equation (11) if $\mathcal{A}(\alpha_0)$ is an isomorphism and a *critical point* otherwise. Moreover, a critical point $(\alpha_0, 0) \in M$ is said to be *isolated* if there exists a deleted ϵ-neighborhood $0 < |\alpha - \alpha_0| < \epsilon$ such that for all $\alpha \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon) \setminus \{\alpha_0\}$, the point $(\alpha, 0) \in M$ is regular.

The set of all critical points for equation (11), denoted Λ, is called the *critical set*, i.e.

$$\Lambda := \{(\alpha, 0) : \ \mathcal{A}(\alpha) : \mathcal{H} \to \mathcal{H} \text{ is not an isomorphism}\}. \tag{12}$$

The next definition concerns our interest in the continuation of non-trivial solution emerging from a bifurcation point $(\alpha_0, 0) \in M$.

Definition 3.3. A trivial solution $(\alpha_0, 0) \in M$ is said to be a *branching point* for the equation (11) if there exists a non-trivial continuum $K \subset \overline{S}$ with $K \cap M = \{(\alpha_0, 0)\}$ and the maximal connected set $\mathcal{C} \subset \overline{S}$ containing the branching point $(\alpha_0, 0)$ we call a *branch* of nontrivial solutions bifurcating from the point $(\alpha_0, 0)$.

Whereas the classical Krasnosiel'skii bifurcation result is only concerned with the existence of a branch of nontrivial solutions for the equation (11) bifurcating from a given critical point $(\alpha_0, 0) \in \Lambda$, the equivariant Krasnosiel'skii bifurcation result, which we employ in this paper, is also concerned with the symmetric properties of such a branch.

Definition 3.4. Let $H \leq \mathcal{G}$ be a given subgroup and denote by $\mathcal{S}^H$ the corresponding H-fixed point space of non-trivial solutions. A branch of solutions $\mathcal{C}$ is said to have *symmetries at least* (H) if $\mathcal{C} \cap \mathcal{S}^H \neq \emptyset$.

Take an isolated critical point $(\alpha_0, 0) \in \Lambda$ and choose $\alpha_0^{\pm} \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$ with $\alpha_0^- \leq \alpha_0 \leq \alpha_0^+$, where $\varepsilon > 0$ was chosen such that for all $0 < |\alpha - \alpha_0| < \varepsilon$ the solution $(\alpha, 0) \in M$ is a regular point. For convenience, we adopt the notations $\mathcal{F}_{\pm}(u) := \mathcal{F}(\alpha_0^{\pm}, u)$ and $B_\delta := \{u \in \mathscr{H} : \|u\| < \delta\}$. By construction, the linear operators $\mathcal{A}(\alpha^{\pm}) : \mathscr{H} \to \mathscr{H}$ are non-singular; so, from assumption (B_2), there exists $\delta > 0$ sufficiently small such that $\mathcal{F}_{\pm}^{-1}(0) \cap \partial B_\delta = \emptyset$ and $\mathcal{F}_{\pm}$ are B_δ-admissibly $\mathcal{G}$-homotopic to $\mathcal{A}(\alpha^{\pm})$. Moreover, since $\mathcal{G}$ acts isometrically on $\mathscr{H}$, the ball B_δ is clearly $\mathcal{G}$-invariant. It follows that $(\mathcal{F}_{\pm}, B_\delta)$ are admissible $\mathcal{G}$-pairs in $\mathscr{H}$ and also that $\mathcal{G}\text{-deg}(\mathcal{F}_{\pm}, B_\delta) = \mathcal{G}\text{-deg}(\mathcal{A}(\alpha_0^{\pm}), B(\mathscr{H}))$, where $B(\mathscr{H})$ is the open unit ball in $\mathscr{H}$. We define the the *local G-equivariant bifurcation invariant* $\omega_{\mathcal{G}}(\alpha_0) \in A(\mathcal{G})$ at the isolated critical solution $(\alpha_0, 0) \in \Lambda$ as follows,

$$\omega_{\mathcal{G}}(\alpha_0) = \mathcal{G}\text{-deg}(\mathcal{A}(\alpha_0^-), B(\mathscr{H})) - \mathcal{G}\text{-deg}(\mathcal{A}(\alpha_0^+), B(\mathscr{H})).$$

The invariant $\omega_{\mathcal{G}}(\alpha_0)$ does not depend on the choice of $\alpha_0^{\pm} \in \mathbb{R}$ or the radius $\delta > 0$. We refer to [4] for the proof of the following local bifurcation result, which is a consequence of the equivariant version of a classical result of K. Kuratowski (cf. [18], Thm. 3, p. 170).

Theorem 3.1. (M.A. Krasnosel'skii-Type Local Bifurcation) Let $\mathcal{F} : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ be a completely continuous $\mathcal{G}$-equivariant field satisfying the assumptions (B_1) and (B_2) and with an isolated critical point $(\alpha_0, 0)$. If $\omega_{\mathcal{G}}(\alpha_0) \neq 0$, then

(i) there exists a branch of nontrivial solutions $\mathcal{C}$ to system (11) with branching point $(\alpha_0, 0)$;

(ii) moreover, if $(H) \in \Phi_0(G)$ is an orbit type with

$$\operatorname{coeff}^H(\omega_{\mathcal{G}}(\alpha_0)) \neq 0,$$

then there exists a branch of non-trivial solutions bifurcating from $(\alpha_0, 0)$ with symmetries at least (H).

3.2. **Global Bifurcation and the Rabinowitz Alternative.** In order to employ the Leray-Schauder $\mathcal{G}$-equivariant degree to describe the global properties of a branch of nontrivial solutions bifurcating from an isolated critical point of the equation (11), we need to make an additional assumption.

(B_3) The critical set $\Lambda \subset M$ (given by (12)) is discrete.

Notice that the local bifurcation invariant $\omega_{\mathcal{G}}(\alpha_0)$ at any critical point $(\alpha_0, 0) \in \Lambda$ is well-defined under assumption (B_3). Moreover, if $\mathcal{U} \subset \mathbb{R} \times \mathscr{H}$ is an open bounded $\mathcal{G}$-invariant set, then its intersection with the critical set is finite. These observations will be important for the statement of the following global bifurcation result, the proof of which can be found in [4].

Theorem 3.2. (The Rabinowitz Alternative) Suppose that $\mathcal{F} : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ is a completely continuous $\mathcal{G}$-equivariant field satisfying conditions (B_1),(B_2) and (B_3). Let $\mathcal{U} \subset \mathbb{R} \times \mathscr{H}$ be an open bounded $\mathcal{G}$-invariant set with $\partial\mathcal{U} \cap \Lambda = \emptyset$ and $(\alpha_0, 0) \in \mathcal{U} \cap \Lambda$. If $\mathcal{C}$ is a branch of nontrivial solutions to (11) bifurcating from the critical point $(\alpha_0, 0)$, then one has the following alternative:

(a) either $\mathcal{C} \cap \partial\mathcal{U} \neq \emptyset$;

(b) or there exists a finite set

$$\mathcal{C} \cap \Lambda = \{(\alpha_0, 0), (\alpha_1, 0), \ldots, (\alpha_n, 0)\},$$

satisfying the following relation

$$\sum_{k=1}^{n} \omega_{\mathcal{G}}(\alpha_k) = 0.$$

Remark 3.1. Suppose that Theorem 3.1 is used to demonstrate the existence of a branch $\mathcal{C}$ of nontrivial solutions to (11) bifurcating from a critical point $(\alpha_0, 0)$. If, in addition, certain conditions are met such that for any open bounded $\mathcal{G}$-invariant neighborhood

$\mathscr{U} \ni (\alpha_0, 0)$ with $\partial\mathscr{U} \cap \Lambda = \emptyset$ the alternative (b) is impossible, then, according to Theorem 3.2, the branch $\mathcal{C}$ must be unbounded.

4. Local and Global Bifurcation of Non-radial Solutions in (1)

Returning to the functional reformulation of our original problem described in Section 2, put $G := O(2) \times \Gamma \times \mathbb{Z}_2$ and notice that the Sobolev space $\mathscr{H}$ introduced in Section 2 is a natural Banach G-representation with respect to the isometric G-action $G \times \mathscr{H} \to \mathscr{H}$ given by

$$(\theta, \gamma, \pm 1)u(z) := \pm\gamma u(e^{i\theta} \cdot z), \quad z \in D, \;\; u \in \mathscr{H};$$
$$(\kappa, \gamma, \pm 1)u(z) := \pm\gamma u(\overline{z}), \quad \gamma \in \Gamma, \; \kappa \in O(2), \; \theta \in SO(2),$$

where $\overline{z}$ is the complex conjugation of $z \in D$ and $e^{i\theta} \cdot z$ is the standard complex multiplication.

Remark 4.1. Under assumptions (A_1)-(A_5), the nonlinear operator $\mathscr{F} : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ given by (7) is a completely continuous G-equivariant field, differentiable at $0 \in \mathscr{H}$ with

$$D\mathscr{F}(\alpha, 0) = \mathrm{Id} - \mathscr{L}^{-1}A(\alpha) : \mathscr{H} \to \mathscr{H},$$

where $A(\alpha) : V \to V$ is the linearization of the map $f(\alpha, z, u)$ from the equation (1) at the origin (see for example see [16, 2]). For convenience, we introduce the notation

$$\mathscr{A}(\alpha)(u) := u - \mathscr{L}^{-1}A(\alpha)u, \quad u \in \mathscr{H}. \tag{13}$$

Choose a value $\alpha \in \mathbb{R}$ such that $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is an isomorphism and denote by $B(\mathscr{H}) := \{u \in \mathscr{H} : \|u\|_{\mathscr{H}} < 1\}$ the open unit ball in $\mathscr{H}$. Before approaching the question of bifurcation in the equation (1), we must derive a workable formula for the computation of the degree $G\text{-deg}(\mathscr{A}(\alpha), B(\mathscr{H}))$.

Assuming that a complete list of the irreducible Γ-representations $\{\mathcal{V}_j\}_{j=1}^r$ is made available we denote by $\{\mathcal{V}_j^-\}_{j=1}^r$ the corresponding list of irreducible $\Gamma \times \mathbb{Z}_2$-representations, where the superscript is meant to indicate that each irreducible Γ-representation has been equipped with the antipodal $\mathbb{Z}_2$-action in the standard way (cf. [4]). Now, as a $\Gamma \times \mathbb{Z}_2$-representation, $V = \mathbb{R}^k$ has the $\Gamma \times \mathbb{Z}_2$-isotypic decomposition

$$V = V_1 \oplus V_1 \oplus \cdots \oplus V_r,$$

where each $\Gamma \times \mathbb{Z}_2$-isotypic component component V_j is modeled on the irreducible $\Gamma \times \mathbb{Z}_2$-representation $\mathcal{V}_j^-$ in the sense that V_j is equivalent to the direct sum of some number of copies of $\mathcal{V}_j^-$, i.e.

$$V_j \simeq \mathcal{V}_j^- \oplus \cdots \oplus \mathcal{V}_j^-.$$

The exact number of irreducible Γ-representations $\mathcal{V}_j^-$ 'contained' in the $\Gamma \times \mathbb{Z}_2$-isotypic component V_j is called the $\mathcal{V}_j^-$-*isotypic multiplicity* of V and is calculated according to the ratio,

$$m_j := \dim V_j / \dim \mathcal{V}_j^-, \quad j \in \{1, 2, \dots, r\}.$$

To simplify computations, we introduce an additional condition on the linearization of (1):

(A_0) For each $j \in \{1, 2, \dots, r\}$ there exists a continuous map $\mu_j : \mathbb{R} \to \mathbb{R}$ with

$$A_j(\alpha) := A(\alpha)|_{V_j} = \mu_j(\alpha) \operatorname{Id}|_{V_j}.$$

In pursuit of a G-isotypic decomposition of $\mathscr{H}$, let us consider the spectrum of the Laplacian operator (2), understood in this context as an unbounded operator in $L^2(D; V)$. Namely, one has

$$\sigma(\mathscr{L}) = \{s_{nm} : n \in \mathbb{N},\ m = 0, 1, 2, \dots\},$$

where $\sqrt{s_{nm}}$ denotes the n-th positive zero of the m-th Bessel function of the first kind J_m. Corresponding to each eigenvalue $s_{nm} \in \sigma(\mathscr{L})$, there is an associated eigenspace $\mathscr{E}_{nm} \subset \mathscr{H}$ which can be expressed, using standard polar coordinates (r, θ), as follows

$$\mathscr{E}_{nm} := \left\{ J_m(\sqrt{s_{nm}} r)\Big(\cos(m\theta)\vec{a} + \sin(m\theta)\vec{b}\Big) : \vec{a},\, \vec{b} \in V \right\}.$$

Then, $\mathscr{H}$ admits the $O(2)$-isotypic decomposition

$$\mathscr{H} := \overline{\bigoplus_{m=0}^{\infty} \mathscr{H}_m}, \quad \mathscr{H}_m := \overline{\bigoplus_{n=1}^{\infty} \mathscr{E}_{nm}},$$

where the closure is taken in $\mathscr{H}$. In particular, adopting the notations

$$\mathscr{E}_{nm}^j := \left\{ J_m(\sqrt{s_{nm}} r)\Big(\cos(m\theta)\vec{a} + \sin(m\theta)\vec{b}\Big) : \vec{a},\, \vec{b} \in V_j \right\}, \tag{14}$$

and

$$\mathscr{A}_{n,m}^j(\alpha) := \mathscr{A}(\alpha)|_{\mathscr{E}_{nm}^j},$$

the G-isotypic of $\mathscr{H}$ is given by

$$\mathscr{H} = \bigoplus_{j=1}^{r} \overline{\bigoplus_{m=0}^{\infty} \mathscr{H}_{m,j}}, \quad \mathscr{H}_{m,j} := \overline{\bigoplus_{n=1}^{\infty} \mathscr{E}_{nm}^j}.$$

To be clear, each G-isotypic component $\mathscr{H}_{m,j}$ is modeled on the irreducible G-representation $\mathcal{V}_{m,j} := \mathscr{H}_m \otimes \mathcal{V}_j^-$, where $\mathscr{H}_m$ denotes the m-th irreducible $O(2)$-representation and $\mathcal{V}_j^-$ is the j-th irreducible Γ-representation $\mathcal{V}_j$ with antipodal $\mathbb{Z}_2$-action.

Lemma 4.1. *Under assumption (A_0), each eigenvalue of the linear operator $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is of the form*

$$\xi_{n,m,j}(\alpha) := 1 - \frac{\mu_j(\alpha)}{s_{nm}},$$

where $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $j \in \{1, 2, \dots, r\}$ *and* $\mu_j(\alpha) \in \sigma(A_j(\alpha))$.

Proof. Indeed, since $\mathscr{A}(\alpha)$ is G-equivariant, one has

$$\mathscr{A}^j_{n,m}(\alpha) : \mathscr{E}^j_{nm} \to \mathscr{E}^j_{nm}$$

such that

$$\sigma(\mathscr{A}(\alpha)) = \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{r} \sigma(\mathscr{A}^j_{n,m}(\alpha)),$$

where one easily obtains (see for example [2, 4])

$$\sigma(\mathscr{A}^j_{n,m}(\alpha)) = \left\{ 1 - \frac{\mu_j(\alpha)}{s_{nm}} \ : \ \mu_j(\alpha) \in \sigma(A_j(\alpha)) \right\}.$$

$\square$

Assume that, for a given $\alpha \in \mathbb{R}$, the operator $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is an isomorphism. The product property of the Leray-Schauder G-equivariant degree (cf. Appendix C) permits us to express $G\text{-deg}(\mathscr{A}(\alpha), B(\mathscr{H}))$ in terms of a Burnside ring product of the Leray-Schauder G-equivariant degrees of the various restrictions $\mathscr{A}^j_{n,m}(\alpha) : \mathscr{E}^j_{nm} \to \mathscr{E}^j_{nm}$ of the G-equivariant linear isomorphism $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ to the G-subrepresentations $\mathscr{E}^j_{nm}$ on their respective open unit balls $B(\mathscr{E}^j_{nm}) := \{u \in \mathscr{E}^j_{nm} \ : \ \|u\|_{\mathscr{H}} < 1\}$ as follows

$$G\text{-deg}(\mathscr{A}(\alpha), B(\mathscr{H})) = \prod_{j=1}^{r} \prod_{m=0}^{\infty} \prod_{n=1}^{\infty} G\text{-deg}(\mathscr{A}^j_{n,m}(\alpha), B(\mathscr{E}^j_{nm})). \tag{15}$$

Notice that, under assumption (A_0), one has $G\text{-deg}(\mathscr{A}^j_{n,m}(\alpha), B(\mathscr{E}^j_{nm})) = (G)$ for almost all indices m, n and j, so that the product (15) is well-defined. Indeed, each Leray-Schauder G-equivariant degree $G\text{-deg}(\mathscr{A}^j_{n,m}(\alpha), B(\mathscr{E}^j_{nm}))$ is fully specified by the $\mathcal{V}^-_j$-isotypic multiplicities $\{m_j\}_{j=1}^r$ together with the real spectra of $\mathscr{A}(\alpha)$ according to formula,

$$G\text{-deg}(\mathscr{A}^j_{n,m}(\alpha), B(\mathscr{E}^j_{nm})) = \begin{cases} (\deg_{\mathcal{V}_{m,j}})^{m_j} & \text{if } s_{nm} < \mu_j(\alpha); \\ (G) & \text{otherwise,} \end{cases} \tag{16}$$

where $\deg_{\mathcal{V}_{m,j}} \in A(G)$ is the basic degree (cf. Appendix C) associated with the irreducible G-representation $\mathcal{V}_{m,j}$ and $(G) \in A(G)$ is the unit element of the Burnside Ring. In addition, since each basic degree $\deg_{\mathcal{V}_{m,j}}$ is involutive in the Burnside ring (cf. Appendix C), one has

$$(\deg_{\mathcal{V}_{m,j}})^{m_j} = \begin{cases} \deg_{\mathcal{V}_{m,j}} & \text{if } 2 \nmid m_j; \\ (G) & \text{otherwise.} \end{cases} \tag{17}$$

Putting together (15), (16) and (17), we introduce some notations to keep track of the indices

$$\Sigma := \{(n,m,j) \;:\; n \in \mathbb{N},\; m \in \mathbb{N}\cup\{0\},\; j \in \{1,2,\dots,r\}\},$$

which contribute non-trivially to the Burnside Ring product (15). To begin, the *negative* spectrum of $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is accounted for with the index set

$$\Sigma_-(\alpha) := \left\{(n,m,j) \in \Sigma \;:\; 1 - \frac{\mu_j(\alpha)}{s_{nm}} < 0\right\}. \tag{18}$$

As is well known (see, for example, [27], p. 486), $s_{1m} > m(m+2)$, from which it follows that the set (18) is finite.

Combining (18) with formulas (15) and (16), one obtains

$$G\text{-deg}(\mathscr{A}(\alpha), B(\mathscr{H})) \;=\; \prod_{(n,m,j)\in\Sigma_-(\alpha)} (\deg_{\mathcal{V}_{m,j}})^{m_j}.$$

Computation of (15) can be further reduced by accounting for the even $\mathcal{V}_j$-isotypic multiplicities m_j, whose corresponding basic degrees contribute trivially to the Burnside Product (15). We put,

$$\Sigma(\alpha) := \{(n,m,j) \in \Sigma_-(\alpha) \;:\; 2 \nmid m_j\}, \tag{19}$$

which, together with (17), yields

$$G\text{-deg}(\mathscr{A}(\alpha), B(\mathscr{H})) \;=\; \prod_{(n,m,j)\in\Sigma(\alpha)} \deg_{\mathcal{V}_{m,j}}. \tag{20}$$

4.1. **Computation of the Local Bifurcation Invariant.** Under the assumptions (A_1)–(A_5), conditions (B_1) and (B_2) are satisfied for the operator equation (8) (cf. Remarks 1.1, 4.1). Therefore, the existence of a branch of non-trivial solutions to (1) bifurcating from an isolated critical point $(\alpha_0, 0) \in \Lambda$ is reduced, by Theorem 3.1, to computation of the local bifurcation invariant $\omega_G(\alpha_0)$. In addition, we assume (A_0), so that the local bifurcation invariant is well-defined at each critical point in Λ.

Adopting notations from Section 3, choose $\alpha_0^\pm \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$ with $\alpha_0^- \le \alpha_0 \le \alpha_0^+$, where $\epsilon > 0$ is chosen such that, for all $0 < |\alpha - \alpha_0| < \epsilon$, the solution $(\alpha, 0) \in M$ is a regular point and put $\mathscr{A}(\alpha_0^\pm) := D_u\mathscr{F}(\alpha_0^\pm, 0)$. Then, the local bifurcation invariant $\omega_G(\alpha_0)$ at the isolated critical point $(\alpha_0, 0) \in \mathbb{R}\times\mathscr{H}$ is given by

$$\omega_G(\alpha_0) = G\text{-deg}(\mathscr{A}(\alpha_0^-), B(\mathscr{H})) - G\text{-deg}(\mathscr{A}(\alpha_0^+), B(\mathscr{H})), \tag{21}$$

where $B(\mathscr{H}) := \{u \in \mathscr{H} \;:\; \|u\|_{\mathscr{H}} < 1\}$ is the open unit ball in $\mathscr{H}$. Notice that, since $(\alpha_0^\pm, 0) \in \mathbb{R}\times\mathscr{H}$ are regular points of (8), computation of the local bifurcation invariant $\omega_G(\alpha_0)$ amounts to computation of the Leray-Schuader G-equivariant degree of the G-equivariant linear isomorphism,

$$\mathscr{A}(\alpha_0^\pm) := D_u\mathscr{F}(\alpha_0^\pm, 0) = \mathrm{Id} - \mathscr{L}^{-1}(A(\alpha_0^\pm)) : \mathscr{H} \to \mathscr{H}.$$

Lemma 4.2. *Under the assumptions (A_1)-(A_5) and (A_0) and using the notation* (19)*, the local bifurcation invariant at an isolated critical point $(\alpha_0, 0) \in \Lambda$ with deleted regular neighborhood $\alpha_0^- \leq \alpha_0 \leq \alpha_0^+$ on which $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is an isomorphism is given by*

$$\omega_G(\alpha_0) = \prod_{(n,m,j)\in\Sigma(\alpha_0^-)} \deg_{\mathcal{V}_{m,j}} - \prod_{(n,m,j)\in\Sigma(\alpha_0^+)} \deg_{\mathcal{V}_{m,j}}. \tag{22}$$

In order to formulate the main local equivariant bifurcation result, we must first consider some additional properties of the basic degree. For a more thorough exposition of these topics we refer the reader to Appendix C.

Take $s \in \mathbb{N}$ and define the *s-folding map* as the Lie group homomorphism,

$$\psi_s(e^{i\theta}, \gamma, \pm 1) = (e^{sl\theta}, \gamma, \pm 1), \quad \psi_s(\kappa e^{i\theta}, \gamma, \pm 1) = (\kappa e^{is\theta}, \gamma, \pm 1).$$

Each $\psi_s : O(2) \times \Gamma \times \mathbb{Z}_2 \to O(2) \times \Gamma \times \mathbb{Z}_2$ induces a corresponding Burnside ring homomorphism $\Psi_s : A(G) \to A(G)$ defined the generators $(H) \in \Phi_0(G)$ by,

$$\Psi_s(H) = ({}^sH), \quad {}^sH := \psi_s^{-1}(H). \tag{23}$$

Notice that, for $j \in \{1, \dots, r\}$ and $m \geq 0$, there is the following relation between basic degrees

$$\Psi_s(\deg_{\mathcal{V}_{m,j}}) = \deg_{\mathcal{V}_{sm,j}}. \tag{24}$$

Remark 4.2. An orbit type which is maximal in $\Phi_0(G; \mathscr{H} \setminus \{0\})$ is also maximal in $\Phi_0(G; \mathscr{H}_0 \setminus \{0\})$. Therefore, any $u \in \mathscr{H} \setminus \{0\}$ with an isotropy $G_u \leq G$ such that (G_u) is maximal in $\Phi_0(G; \mathscr{H} \setminus \{0\})$ must be *radially symmetric*. It follows that in order to detect branches of solutions to (7) corresponding to maps which are both non-trivial *and* non-radial, we must restrict our focus to orbit types which are maximal in $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$ for some positive $m \in \mathbb{N}$. Indeed, to demonstrate the existence of a branch of non-radial solutions bifurcating from some isolated critical point $(\alpha_0, 0) \in \Lambda$, it is sufficient to show that for some $m > 0$ there is an orbit type $(H) \in \Phi_0(G; \mathscr{H}_m \setminus \{0\})$ with $\mathrm{coeff}^H(\omega_G(\alpha_0)) \neq 0$. Moreover, in such a case one can conclude that for some $s \geq 1$ there exists a branch non-radial solutions to (7) with symmetries at least $({}^sH)$ bifurcating from $(\alpha_0, 0)$.

For $m > 0$, denote by $\mathfrak{M}_m$ the set of all maximal orbit types in $\Phi_0(G; \mathscr{H}_m \setminus \{0\})$ and by $\mathfrak{M}_{m,j}$ the set of orbit types $\Phi_0(G; \mathscr{H}_{m,j} \setminus \{0\}) \cap \mathfrak{M}_m$. Since each $(H) \in \mathfrak{M}_m$ is also an orbit type in $\Phi_0(G;\mathscr{H}_{m,j}\setminus\{0\})$ for at least one $j\in\{1,2,\dots,r\}$, one has

$$\mathfrak{M}_m=\bigcup_{j=1}^{r}\mathfrak{M}_{m,j}.$$

Notice that $\Psi_s(\mathfrak{M}_{m,j})=\mathfrak{M}_{sm,j}$ for any $m>0,\ s\in\mathbb{N}$ and $j\in\{1,2,\dots,r\}$ (in particular, one has $\Psi_s(\mathfrak{M}_{1,j})=\mathfrak{M}_{s,j}$). Hence, any orbit type $(H_0)\in\mathfrak{M}_{m,j}$ can be recovered from an orbit type in $\mathfrak{M}_{1,j}$ by the relation $(H):=\Psi_s^{-1}(H_0)$.

Remark 4.3. Take $m>0$ and $j\in\{1,2,\dots,r\}$. For any basic degree $\deg_{\mathcal{V}_{m,j}}\in A(G)$ and orbit type $(H)\in\mathfrak{M}_{1,j}$, the recurrence formula for the Leray-Schauder G-equivariant degree (cf. Appendix C) implies

$$\deg_{\mathcal{V}_{m,j}}=(G)-y_j({}^mH)+a_j,$$

where $a_j\in A(G)$ is such that $\text{coeff}^{{}^sH}(a_j)=0$ for all $s\in\mathbb{N}$ and where the coefficient $y_j\in\mathbb{Z}$ is determined by the rule

$$y_j=\begin{cases}0 & \text{if } \dim\mathcal{V}_{1,j}^H \text{ is even;}\\ 1 & \text{if } \dim\mathcal{V}_{1,j}^H \text{ is odd and } |W(H)|=2;\\ 2 & \text{if } \dim\mathcal{V}_{1,j}^H \text{ is odd and } |W(H)|=1.\end{cases}$$

equivalently

$$y_j=\frac{x_0}{2}(1-(-1)^{\dim\mathcal{V}_{1,j}^H}),$$

where

$$x_0=\begin{cases}1 & \text{if } |W(H)|=2;\\ 2 & \text{if } |W(H)|=1.\end{cases}$$

It follows that non-triviality of the coefficient associated with an orbit type $({}^mH)\in\mathfrak{M}_{m,j}$ in the basic degree $\deg_{\mathcal{V}_{m,j}}$ is characterized by the parity of $\dim\mathcal{V}_{1,j}^H$ in the following way

$$\text{coeff}^{{}^mH}(\deg_{\mathcal{V}_{m,j}})\neq 0 \iff 2\nmid\dim\mathcal{V}_{1,j}^H.$$

We need one more result, this concerning the fate of orbit types belonging to $\mathfrak{M}_m$ in the Burnside Ring product of basic degrees such as (20). In particular, we find that for $m>0$ and $i,l\in\{1,2,\dots,r\}$, the coefficient of $(H)\in\mathfrak{M}_{m,i}\cap\mathfrak{M}_{m,l}$ is 2-*nilpotent* with respect to the Burnside Ring product $\deg_{\mathcal{V}_{m,i}}\cdot\deg_{\mathcal{V}_{m,l}}\in A(G)$ in the case that both $\dim\mathcal{V}_{m,i}^H$ and $\dim\mathcal{V}_{m,l}^H$ are odd.

Lemma 4.3. *Take $m>0$ and $i,l\in\{1,2,\dots,r\}$. For $(H)\in\mathfrak{M}_{1,i}\cap\mathfrak{M}_{1,l}$, one has*

$$\text{coeff}^{{}^mH}(\deg_{\mathcal{V}_{m,i}}\cdot\deg_{\mathcal{V}_{m,l}})=\begin{cases}0 & \textit{if } \dim\mathcal{V}_{1,i}^H \textit{ and } \dim\mathcal{V}_{1,l}^H \textit{ are of the same parity;}\\ -x_0 & \textit{else.}\end{cases}$$

equivalently

$$\mathrm{coeff}^{{}^mH}(\deg_{\mathcal{V}_{m,i}} \cdot \deg_{\mathcal{V}_{m,l}}) = \frac{-x_0}{2}(1-(-1)^{\dim \mathcal{V}^H_{1,i}+\dim \mathcal{V}^H_{1,l}}).$$

Proof. Consider the Burnside Ring product of the relevant basic degrees

$$\begin{aligned}\deg_{\mathcal{V}_{m,i}} \cdot \deg_{\mathcal{V}_{m,l}} &= ((G)-y_i({}^mH)+a_i)\cdot((G)-y_l({}^mH)+a_l)\\ &= (G)-(y_i+y_l-y_iy_l|W(H)|)(H)+a,\end{aligned}$$

where $a \in A(G)$ is such that $\mathrm{coeff}^{{}^sH}(a) = 0$ for all $s \in \{1,2,\dots\}$, and put $y_0 := y_i + y_l - y_iy_l|W(H)|$. Now, if $\dim \mathcal{V}^H_{1,i}$ and $\dim \mathcal{V}^H_{1,l}$ are both even, then one has $y_i = y_l = 0$ and the result follows. On the other hand, if $\dim \mathcal{V}^H_{1,i}$ and $\dim \mathcal{V}^H_{1,l}$ are both odd, then one has $y_i = y_l = x_0$ such that

$$y_0 = x_0(2 - x_0|W(H)|),$$

where in either of the cases, $x_0 = 2$ and $|W(H)| = 1$ or $x_0 = 1$ and $|W(H)| = 2$, one has $2 - x_0|W(H)| = 0$. If instead one supposes that $\dim \mathcal{V}^H_{1,i}$ and $\dim \mathcal{V}^H_{1,l}$ are of different parities, then one has the two corresponding cases, $y_i = x_0$ and $y_l = 0$ *or* $y_i = 0$ and $y_l = x_0$, both of which imply $y_0 = x_0$. □

Naturally, the Lemma (4.3) can be generalized to an orbit type $(H) \in \bigcap_{k=1}^N \mathfrak{M}_{1,j_k}$ and the Burnside Ring product of the basic degrees $\{\deg_{\mathcal{V}_{m,j_k}}\}_{k=1}^N$ with $\dim \mathcal{V}^H_{1,j_k}$ odd for an even number of $j_k \in \{1,2,\dots,r\}$.

Corollary 4.1. *Take $m > 0$, $N \in \mathbb{N}$ and $j_1,\dots,j_N \in \{1,2,\dots,r\}$. For $(H) \in \bigcap_{k=1}^N \mathfrak{M}_{1,j_k}$, one has*

$$\mathrm{coeff}^{{}^mH}\left(\prod_{k=1}^N \deg_{\mathcal{V}_{m,j_k}}\right) = \begin{cases} 0 & \text{if } |\{j_k : 2 \nmid \dim \mathcal{V}^H_{1,j_k}\}| \text{ is even;} \\ -x_0 & \text{otherwise}\end{cases}$$

equivalently

$$\mathrm{coeff}^{{}^mH}\left(\prod_{k=1}^N \deg_{\mathcal{V}_{m,j_k}}\right) = \frac{-x_0}{2}\left(1-(-1)^{\sum_{k=1}^N \dim \mathcal{V}^H_{1,j_k}}\right).$$

Remarks 4.2 and 4.3 permit us to further refine our index set (19) as follows.
Let $(\alpha_0, 0) \in \Lambda$ be an isolated critical point with a deleted regular neighborhood $\alpha_0^- < \alpha_0 < \alpha_0^+$ on which $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is an isomorphism and for a given $s \in \mathbb{N}$, $(H) \in \mathfrak{M}_1$ put

$$\Sigma^s(\alpha_0^\pm, H) := \{(n,m,j) \in \Sigma(\alpha_0^\pm) : \mathrm{coeff}^{{}^sH}(\deg_{\mathcal{V}_{m,j}}) \neq 0\}, \tag{25}$$

$$\mathfrak{n}^s(\alpha_0^\pm, H) := |\Sigma^s(\alpha_0^\pm, H)|, \tag{26}$$

and

$$\mathfrak{m}^s(\alpha_0^\pm, H) := |\{(n,m,j) \in \Sigma(\alpha_0^\pm) : ({}^sH) < ({}^mH) \text{ and } 2 \nmid \mathfrak{n}^m(\alpha_0^\pm, H)\}|. \tag{27}$$

Remark 4.4. Take $(H) \in \mathfrak{M}_1$, $(n,m,j) \in \Sigma$ and $s \in \mathbb{N}$. If $\text{coeff}^{{}^sH}(\deg_{\mathcal{V}_{m,j}}) \neq 0$, then $m = s$. On the other hand, if $({}^sH) \leq ({}^mH)$, then $s \mid m$.

In particular, we are interested in keeping track of the numbers $s \in \mathbb{N}$ for which the parities of $\mathfrak{n}^s(\alpha_0^\pm, H)$ disagree. With this in mind, put

$$\mathfrak{i}^s(\alpha_0, H) := \begin{cases} -1 & \text{if } \mathfrak{n}^s(\alpha_0^-, H) \text{ is odd and } \mathfrak{n}^s(\alpha_0^+, H) \text{ is even} \\ 1 & \text{if } \mathfrak{n}^s(\alpha_0^-, H) \text{ is even and } \mathfrak{n}^s(\alpha_0^+, H) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

and also

$$\mathfrak{s}(\alpha_0, H) := \max\{s \in \mathbb{N} : \mathfrak{i}^s(\alpha_0, H) \neq 0\}. \tag{29}$$

Remark 4.5. Notice that $\mathfrak{s}(\alpha_0, H)$ is well defined and that the numbers $\mathfrak{m}^{\mathfrak{s}(\alpha_0,H)}(\alpha_0^\pm, H)$ are of the same parity for any isolated critical point $(\alpha_0, 0) \in \Lambda$ and orbit type $(H) \in \mathfrak{M}_1$.

We are finally in a position to formulate our main local bifurcation result.

Theorem 4.1. *If $(\alpha_0, 0) \in \Lambda$ is an isolated critical point with a deleted regular neighborhood $\alpha_0^- \leq \alpha_0 \leq \alpha_0^+$ on which the linear operator $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is an isomorphism and if there is an orbit type $(H) \in \mathfrak{M}_1$ and a number $\mathfrak{s} := \mathfrak{s}(\alpha_0, H) \in \mathbb{N}$ such that $\mathfrak{i}^{\mathfrak{s}}(\alpha_0, H) \neq 0$ and $\mathfrak{i}^m(\alpha_0, H) = 0$ for all $m > \mathfrak{s}$, then*

$$\text{coeff}^{{}^sH}(\omega_G(\alpha_0)) = \begin{cases} (-1)^{\mathfrak{m}^{\mathfrak{s}}(\alpha_0^-, H)} \mathfrak{i}^{\mathfrak{s}}(\alpha_0, H) x_0 & \textit{if } s = s_0; \\ 0 & \textit{if } s > s_0. \end{cases}$$

Proof. For convenience, take $\alpha_0^* \in \{\alpha_0^\pm\}$ and introduce the notation

$$\rho_G(\alpha_0^*) := \prod_{(n,m,j) \in \Sigma(\alpha_0^*)} \deg_{\mathcal{V}_{m,j}}, \tag{30}$$

such that the local bifurcation invariant (cf. 3.1) becomes

$$\omega_G(\alpha_0) = \rho_G(\alpha_0^-) - \rho_G(\alpha_0^+).$$

Also, put

$$\Sigma^0(\alpha_0^*, H) := \{(n,m,j) \in \Sigma(\alpha_0^*) : \forall_{s' \in \mathbb{N}} \ \text{coeff}^{{}^{s'}H}(\deg_{\mathcal{V}_{m,j}}) = 0\},$$

and, for each $s = 0, 1, \ldots$ define

$$\rho_G^s(\alpha_0^*, H) := \prod_{(n,m,j) \in \Sigma^s(\alpha_0^*, H)} \deg_{\mathcal{V}_{m,j}}. \tag{31}$$

Clearly the index set (19) has the partition

$$\Sigma(\alpha_0^*) = \bigcup_{s\in\mathbb{N}\cup\{0\}} \Sigma^s(\alpha_0^*, H),$$

such that the product of basic degrees (30) becomes

$$\rho_G(\alpha_0^*) = \prod_{s\in\mathbb{N}\cup\{0\}} \rho_G^s(\alpha_0^*, H).$$

It follows, from Lemma (4.3) and its Corollary (4.1), that each of (31) is of the form

$$\rho_G^s(\alpha_0^*, H) = \begin{cases} (G) + b_0 & \text{if } s = 0; \\ (G) - y_s(\alpha_0^*)({}^sH) + b_s & \text{if } s \geq 1, \end{cases}$$

where $b_0, b_s \in A(G)$ are such that $\operatorname{coeff}^{s'H}(b_0) = \operatorname{coeff}^{s'H}(b_s) = 0$ for all $s' \in \mathbb{N}$ and where the coefficients $y_s(\alpha_0^*) \in \mathbb{Z}$ are determined by the rule

$$y_s(\alpha_0^*) = \begin{cases} 0 & \text{if } \mathfrak{n}^s(\alpha_0^*, H) \text{ is even}; \\ x_0 & \text{if } \mathfrak{n}^s(\alpha_0^*, H) \text{ is odd}, \end{cases}$$

equivalently

$$y_s(\alpha_0^*) = \frac{x_0}{2}(1 - (-1)^{\mathfrak{n}^s(\alpha_0^*, H)}).$$

Now, since $\mathfrak{n}^s(\alpha_0^{\pm}, H)$ are of the same parity for all $s > \mathfrak{s}$, one has

$$\rho_G^s(\alpha_0^-) \cdot \rho_G^s(\alpha_0^+) = (G) + c_s,$$

where $c_s \in A(G)$ is such that $\operatorname{coeff}^{s'H}(c_s) = 0$ for all $s' \in \mathbb{N}$. Therefore, the local bifurcation invariant can be expressed in terms of the quantities (31) as follows

$$\omega_G(\alpha_0) = \prod_{s>\mathfrak{s}} \rho_G^s(\alpha_0^-, H) \cdot \big(\rho_G^{\mathfrak{s}}(\alpha_0^-, H) - \rho_G^{\mathfrak{s}}(\alpha_0^+, H) + \boldsymbol{\beta}\big)$$

where $\boldsymbol{\beta} \in A(G)$ is such that $\operatorname{coeff}^{s'H}(\boldsymbol{\beta}) = 0$ for all $s' \geq \mathfrak{s}$.

To complete the proof of Theorem 4.1, we will need the following Lemma.

Lemma 4.4. *Take $s, s_0 \in \mathbb{N}$. Using the notation* (31)*, one has*

$$\operatorname{coeff}^{s_0H}(\rho_G^s(\alpha_0^*, H) \cdot \pm x_0({}^{s_0}H)) = \begin{cases} \mp x_0 & \textit{if } ({}^{s_0}H) \leq ({}^sH) \textit{ and } \mathfrak{n}^s(\alpha_0^*, H) \textit{ is odd}; \\ \pm x_0 & \textit{otherwise.} \end{cases}$$

Proof. Indeed, consider the relevant Burnside Ring product

$$\begin{aligned} \rho_G^s(\alpha_0^*, H) \cdot \pm x_0({}^{s_0}H) &= ((G) - y_s(\alpha_0^*)({}^sH) + b_s) \cdot (\pm x_0({}^{s_0}H)) \\ &= \pm x_0({}^{s_0}H) \mp y_s(\alpha^*)x_0({}^sH) \cdot ({}^{s_0}H) + \boldsymbol{\alpha} \end{aligned} \tag{32}$$

where $\boldsymbol{\alpha} \in A(G)$ is such that $\text{coeff}^{s' H}(\boldsymbol{\alpha}) = 0$ for all $s' \in \mathbb{N}$. Now, if $\mathfrak{n}^s(\alpha_0^*, H)$ is even, then $y_s(\alpha_0^*) = 0$ and the result follows. Suppose instead that $2 \nmid \mathfrak{n}^s(\alpha_0^*, H)$. Then (32) becomes

$$\rho_G^s(\alpha_0^*, H) \cdot \pm x_0(^{s_0}H) = \pm x_0(1 - x_0 d_0)(^{s_0}H) + \boldsymbol{\gamma}$$

where $\boldsymbol{\gamma} \in A(G)$ is such that $\text{coeff}^{s' H}(\boldsymbol{\gamma}) = 0$ for all $s' \in \mathbb{N}$ and $d_0 := \text{coeff}^{s_0 H}((^{s}H) \cdot (^{s_0}H))$ is given by the recursive formula (cf. Appendix C) as follows

$$d_0 := \frac{n(^{s_0}H, ^{s}H) n(^{s_0}H, ^{s_0}H) |W(H)|^2}{|W(H)|}.$$

Now, if $(^{s_0}H) \nleq (^{s}H)$, then $n(^{s_0}H, ^{s}H) = 0$ and the result follows. In the case that $(^{s_0}H) \leq (^{s}H)$, one has $n(^{s_0}H, ^{s}H) = n(^{s_0}H, ^{s_0}H) = 1$ such that $d_0 = |W(H)|$ and the result follows from the fact that $x_0|W(H)| = 2$. ☐

Completion of the proof of Theorem 4.1 The result follows from Lemma 4.4 together with the observation that

$$\text{coeff}^{s H}\left(\rho_G^{\mathfrak{s}}(\alpha_0^-, H) - \rho_G^{\mathfrak{s}}(\alpha_0^+, H) + \boldsymbol{\beta}\right) = \begin{cases} \mathfrak{i}^{\mathfrak{s}}(\alpha_0, H) x_0 & \text{if } s = \mathfrak{s}; \\ 0 & \text{if } s > \mathfrak{s}. \end{cases}$$

☐

Corollary 4.2. *Under the assumptions (A_1)-(A_5), if $(\alpha_0, 0) \in \Lambda$ is an isolated critical point with a deleted neighborhood $[\alpha^-, \alpha^+] \setminus \{\alpha_0\}$ on which the linear operator $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ is an isomorphism and if there is an orbit type $(H) \in \mathfrak{M}_1$ and a number $\mathfrak{s} := \mathfrak{s}(\alpha_0, H) \in \mathbb{N}$ such that $\mathfrak{i}^{\mathfrak{s}}(\alpha_0, H) \neq 0$ and $\mathfrak{i}^m(\alpha_0, H) = 0$ for all $m > \mathfrak{s}$, then system* (8) *admits a branch of non-radial solutions $\mathscr{C}$ with branching point* $(\alpha_0, 0)$ *and with $(G_u) \geq (^{\mathfrak{s}}H)$ for all $u \in \mathscr{C}$.*

4.2. **Resolution of the Rabinowitz Alternative.** Without an appropriate *fixed point reduction* of the bifurcation problem (8), one is unable to guarantee that a branch of non-trivial solutions $\mathcal{C}$ to (1) bifurcating from a given isolated critical point $(\alpha_0, 0) \in \mathbb{R} \times \mathscr{H}$ whose existence has been established using a Krasnosel'skii type result (e.g. by Theorem 4.1) is not comprised of *radial solutions*. With this in mind, consider the subgroup $\boldsymbol{K} := \{(-1, e, -1), (1, e, 1)\} \leq O(2) \times \Gamma \times \mathbb{Z}_2$ and denote by $\mathscr{F}^{\boldsymbol{K}} : \mathbb{R} \times \mathscr{H}^{\boldsymbol{K}} \to \mathscr{H}^{\boldsymbol{K}}$ the restriction

$$\mathscr{F}^{\boldsymbol{K}} := \mathscr{F}|_{\mathbb{R} \times \mathscr{H}^{\boldsymbol{K}}}, \tag{33}$$

of the operator (7) to the $\boldsymbol{K}$-fixed point space $\mathscr{H}^{\boldsymbol{K}}$. Clearly, any solution $(\alpha, u) \in \mathbb{R} \times \mathscr{H}^{\boldsymbol{K}}$ to the equation

$$\mathscr{F}^{\boldsymbol{K}}(\alpha, u) = 0, \tag{34}$$

is also solution to (8). Notice also that any radial solution to (34) belongs to the set of trivial solutions

$$M^{\boldsymbol{K}} := \{(\alpha, 0) : \alpha \in \mathbb{R} \times \mathscr{H}^{\boldsymbol{K}}\}.$$

Therefore, any branch of of non-trivial solutions to the bifurcation problem (34) consists solely of non-radial solutions. In this $\boldsymbol{K}$-fixed point setting, we adapt each of the notions introduced in Section 3.2 used to describe the Rabinowitz alternative for the equation (8) to the bifurcation map (33). To begin, notice that the $\boldsymbol{K}$ fixed point space $\mathscr{H}^{\boldsymbol{K}}$ is an isometric Hilbert representation of the group $\mathcal{G} := N(\boldsymbol{K})/\boldsymbol{K} = O(2) \times \Gamma$ with the $\mathcal{G}$-isotypic decomposition

$$\mathscr{H}^{\boldsymbol{K}} = \bigoplus_{j=1}^{r} \overline{\bigoplus_{m=1}^{\infty} \mathscr{H}_{2m-1,j}}, \quad \mathscr{H}_{2m-1,j} := \overline{\bigoplus_{n=1}^{\infty} \mathscr{E}_{n,2m-1}^{j}}.$$

We denote by $\mathscr{S}^{\boldsymbol{K}}$ the set of non-trivial solutions to (34), i.e.

$$\mathscr{S}^{\boldsymbol{K}} := \{(\alpha, u) \in \mathbb{R} \times \mathscr{H}^{\boldsymbol{K}} : \mathscr{F}^{\boldsymbol{K}}(\alpha, 0) = 0 \quad \text{and} \quad u \neq 0\},$$

and by $\mathscr{A}^{\boldsymbol{K}} : \mathbb{R} \times \mathscr{H}^{\boldsymbol{K}} \to \mathscr{H}^{\boldsymbol{K}}$ the restriction of the operator $\mathscr{A} : \mathbb{R} \times \mathscr{H} \to \mathscr{H}$ to $\mathscr{H}^{\boldsymbol{K}}$, i.e. for each $\alpha \in \mathbb{R}$

$$\mathscr{A}^{\boldsymbol{K}}(\alpha) := \mathscr{A}(\alpha)|_{\mathscr{H}^{\boldsymbol{K}}} : \mathscr{H}^{\boldsymbol{K}} \to \mathscr{H}^{\boldsymbol{K}}. \tag{35}$$

As before, the *critical set* of (34), now denoted $\Lambda^{\boldsymbol{K}}$, is the set of trivial solutions $(\alpha_0, 0) \in M^{\boldsymbol{K}}$ for which $\mathscr{A}^{\boldsymbol{K}}(\alpha_0)$ is not an isomorphism

$$\Lambda^{\boldsymbol{K}} := \{(\alpha, 0) \in \mathbb{R} \times \mathscr{H}^{\boldsymbol{K}} : \mathscr{A}^{\boldsymbol{K}}(\alpha) : \mathscr{H}^{\boldsymbol{K}} \to \mathscr{H}^{\boldsymbol{K}} \quad \text{is not an isomorphism}\}.$$

Next, we describe the spectrum of (35) in terms of the spectra $\sigma(\mathscr{A}_{n,m}^{j}(\alpha))$ (cf. Lemma 4.1) as follows

$$\sigma(\mathscr{A}^{\boldsymbol{K}}(\alpha)) = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{r} \sigma(\mathscr{A}_{n,2m-1}^{j}(\alpha)).$$

Assume that for a given $\alpha \in \mathbb{R}$, the operator $\mathscr{A}^{\boldsymbol{K}}(\alpha) : \mathscr{H}^{\boldsymbol{K}} \to \mathscr{H}^{\boldsymbol{K}}$ is an isomorphism. If we refine the index set (19) to include only those indices $(n, m, j) \in \Sigma(\alpha)$ relevant to the $\boldsymbol{K}$-fixed point setting with the notation

$$\Sigma^{\boldsymbol{K}}(\alpha) := \{(n, m, j) \in \Sigma(\alpha) : 2 \nmid m\},$$

then the $\mathcal{G}$-equivariant degree $\mathcal{G}\text{-deg}(\mathscr{A}^{\boldsymbol{K}}(\alpha), B(\mathscr{H}^{\boldsymbol{K}}))$ can be computed as follows

$$\mathcal{G}\text{-deg}(\mathscr{A}^{\boldsymbol{K}}(\alpha), B(\mathscr{H}^{\boldsymbol{K}})) = \prod_{(n,m,j) \in \Sigma^{\boldsymbol{K}}(\alpha)} \widetilde{\deg}_{\mathcal{V}_{j,m}},$$

where, to distinguish between G-basic degrees and $\mathcal{G}$-basic degrees, we have introduced the notation

$$\widetilde{\deg}_{\mathcal{V}_{j,m}} := \mathcal{G}\text{-deg}(-\operatorname{Id}, B(\mathcal{V}_{j,m})).$$

At this point, Lemma 4.2 can be reformulated for the map $\mathscr{F}^{\boldsymbol{K}}$ as follows:

Lemma 4.5. *Under the assumptions (A_0)–(A_5) and for any isolated critical point $(\alpha_0, 0) \in \Lambda^{\boldsymbol{K}}$ with deleted regular neighborhood $\alpha_0^- < \alpha_0 < \alpha_0^+$, the local bifurcation invariant $\omega_{\mathcal{G}}(\alpha_0) := \mathcal{G}\text{-}deg(\mathcal{A}^{\boldsymbol{K}}(\alpha_0^-), B(\mathscr{H}^{\boldsymbol{K}})) - \mathcal{G}\text{-}deg(\mathcal{A}^{\boldsymbol{K}}(\alpha_0^+), B(\mathscr{H}^{\boldsymbol{K}}))$ is given by*

$$\omega_{\mathcal{G}}(\alpha_0) = \prod_{(n,m,j) \in \Sigma^{\boldsymbol{K}}(\alpha^-)} \widetilde{\deg}_{\mathcal{V}_{m,j}} - \prod_{(n,m,j) \in \Sigma^{\boldsymbol{K}}(\alpha^+)} \widetilde{\deg}_{\mathcal{V}_{m,j}}. \tag{36}$$

Likewise, Theorem 3.2 becomes:

Theorem 4.2. (Rabinowitz' Alternative) *Under the assumptions (A_0)–(A_5), let $\mathcal{U} \subset \mathbb{R} \times \mathscr{H}^{\boldsymbol{K}}$ be an open bounded $\mathcal{G}$-invariant set with $\partial\mathcal{U} \cap \Lambda^{\boldsymbol{K}} = \emptyset$ and $(\alpha_0, 0) \in \mathcal{U} \cap \Lambda^{\boldsymbol{K}}$. If $\mathcal{C}$ is a branch of nontrivial solutions to* (34) *bifurcating from the critical point $(\alpha_0, 0)$, then one has the following alternative:*

- (a) *either $\mathcal{C} \cap \partial\mathcal{U} \neq \emptyset$;*
- (b) *or there exists a finite set*

$$\mathcal{C} \cap \Lambda^{\boldsymbol{K}} = \{(\alpha_0, 0), (\alpha_1, 0), \dots, (\alpha_n, 0)\},$$

satisfying the following relation

$$\sum_{k=1}^{n} \omega_{\mathcal{G}}(\alpha_k) = 0.$$

To further simplify our exposition, we replace assumption (A_0) with:

- $(\tilde{A}_0)$ For each $j \in \{1, 2, \dots, r\}$ there exists a continuous and bounded map $\mu_j : \mathbb{R} \to \mathbb{R}$ with

$$A_j(\alpha) := A(\alpha)|_{V_j} = \mu_j(\alpha) \operatorname{Id}|_{V_j}.$$

Take $(H) \in \mathfrak{M}_1$ and $(\alpha_0, 0) \in \Lambda^{\boldsymbol{K}}$. Under assumption $(\tilde{A}_0)$, there is some finite $m' = 0, 1, \dots$ for which $m > m'$ implies $(n, m, j) \notin \Sigma^{\boldsymbol{K}}(\alpha_0)$. Clearly, the quantity

$$\bar{\mathfrak{s}}(H) := \max_{(\alpha,0) \in \Lambda^{\boldsymbol{K}}} \{\mathfrak{s}(\alpha, H)\} \tag{37}$$

and the set

$$\mathfrak{J}(H) := \{(\alpha, 0) \in \Lambda^{\boldsymbol{K}} : \mathfrak{s}(\alpha, H) = \bar{\mathfrak{s}}(H)\}. \tag{38}$$

are well-defined. The elements of $\mathfrak{J}(H)$ can always be indexed $(\alpha_1, 0), (\alpha_2, 0), \dots \in \mathfrak{J}(H)$ in such a way that, if $i < j$, then $\alpha_i < \alpha_j$.

We are now in a position to formulate our main global bifurcation result.

Theorem 4.3. *If there is an orbit type $(H) \in \mathfrak{M}_1$ with $\bar{\mathfrak{s}} := \bar{\mathfrak{s}}(H)$ and for which $2 \nmid |\mathfrak{J}(H)|$, then the system* (1) *admits an unbounded branch $\mathscr{C}$ of non-radial solutions with symmetries at least $({}^{\bar{\mathfrak{s}}}H)$. Moreover, we have the following alternative: there exists $M > 0$ such that, either*

(a) for all $\alpha > M$ with $(\alpha, 0) \notin \Lambda^{\boldsymbol{K}}$ one has $\mathscr{C} \cap \{\alpha\} \times \mathscr{H} \neq \emptyset$, or
(b) for all $\alpha > -M$ with $(\alpha, 0) \notin \Lambda^{\boldsymbol{K}}$ one has $\mathscr{C} \cap \{\alpha\} \times \mathscr{H} \neq \emptyset$.

Remark 4.6. Conditions (a) and (b) in Theorem 4.3 guarantee that the branch $\mathscr{C}$ of non-radial solutions extends indefinitely either into the direction of increasing $\alpha \to \infty$ or into the decreasing $\alpha \to -\infty$.

Proof. Notice that, for any critical point $(\alpha_i, 0) \in \mathfrak{J}(H)$, the numbers $\mathfrak{m}^{\bar{\mathfrak{s}}}(\alpha_i^{\pm}, H)$ have the same parity (cf. Remark 4.5). Consider any other critical point $(\alpha_j, 0) \in \mathfrak{J}(H)$ with $\alpha_i < \alpha_j$. We will show that $\mathfrak{m}^{\bar{\mathfrak{s}}}(\alpha_i^{+}, H)$ and $\mathfrak{m}^{\bar{\mathfrak{s}}}(\alpha_j^{-}, H)$ are also of the same parity. Indeed, suppose for contradiction that $\mathfrak{m}^{\bar{\mathfrak{s}}}(\alpha_i^{+}, H)$ and $\mathfrak{m}^{\bar{\mathfrak{s}}}(\alpha_j^{-}, H)$ have different parities. Then there is a folding $m > \bar{\mathfrak{s}}$ such that the sets $\Sigma^m(\alpha_i^{+}, H)$, $\Sigma^m(\alpha_j^{-}, H)$ disagree for some odd numbers of indices (equivalently, the numbers $\mathfrak{n}^m(\alpha_i^{+}, H)$, $\mathfrak{n}^m(\alpha_j^{-}, H)$ have different parity). Consequently, there must be an intermediate critical point $(\alpha_k, 0) \in \mathfrak{J}(H)$ with $\alpha_i < \alpha_k < \alpha_j$ and $\mathfrak{i}^m(\alpha_k, H) \neq 0$. This is in contradiction with the assumption of maximality for $\bar{\mathfrak{s}}$. It follows that the quantity $(-1)^{\mathfrak{m}^{\bar{\mathfrak{s}}}(\alpha_*^{-}, H)}$ is constant for all critical points $(\alpha_*, 0) \in \mathfrak{J}(H)$.

On the other hand, let $(\alpha_k, 0), (\alpha_{k+1}, 0) \in \mathfrak{J}(H)$ be any two consecutive critical points. From the definition of (38), the numbers $\mathfrak{i}^{\bar{\mathfrak{s}}}(\alpha_k, H), \mathfrak{i}^{\bar{\mathfrak{s}}}(\alpha_{k+1}, H)$ must be non-zero. With an argument similar to that the one used above, it can be shown that the numbers $\mathfrak{n}^{\bar{\mathfrak{s}}}(\alpha_k^{+}, H)$ and $\mathfrak{n}^{\bar{\mathfrak{s}}}(\alpha_{k+1}^{-}, H)$ have the same parity, from which it follows that

$$\mathfrak{i}^{\bar{\mathfrak{s}}}(\alpha_k, H)\mathfrak{i}^{\bar{\mathfrak{s}}}(\alpha_{k+1}, H) = -1.$$

It follows then, from Theorem 4.1, that $\mathrm{coeff}^{\bar{\mathfrak{s}}\mathrm{H}}(\omega_{\mathrm{G}}(\alpha_{\mathrm{k}})) = -\,\mathrm{coeff}^{\bar{\mathfrak{s}}\mathrm{H}}(\omega_{\mathrm{G}}(\alpha_{\mathrm{k+1}}))$ and also, for any critical point $(\alpha_0, 0) \in \Lambda^{\boldsymbol{K}} \setminus \mathfrak{J}(H)$, that $\mathrm{coeff}^{\bar{\mathfrak{s}}\mathrm{H}}(\omega_{\mathrm{G}}(\alpha_0)) = 0$. Therefore, if $2 \nmid |\mathfrak{J}(H)|$, there exists an unbounded branch $\mathscr{C}$ of non-radial solutions bifurcating from each critical point in $\mathfrak{J}(H)$ with symmetries at least $({}^{\bar{\mathfrak{s}}}H)$. Moreover, by Lemma 2.2, the branch $\mathscr{C}$ cannot extend to infinity with respect to the magnitude of vectors u belonging to $\mathscr{C}$ (for any particular α), the only option is that the branch $\mathscr{C}$ extends to infinity with respect to α. $\square$

5. Motivating Example: Vibrating Membranes with Symmetric Coupling

Equations involving the Laplacian operator are sometimes used to describe time-invariant wave or diffusion processes, also called *steady-state phenomena*. As a preliminary model for steady-state phenomena, consider the Helmholtz equation

$$\begin{cases} -\Delta u = \lambda^2 u, \\ u|_{\partial D} = 0, \end{cases} \tag{39}$$

defined on the planar unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$. Solutions of (39) are called *normal modes* and take the form

$$u_{nm}(r,\theta) = J_m(\sqrt{s_{nm}}r)(A\cos(m\theta) + B\sin(m\theta)), \quad A, B \in \mathbb{R},\ n \in \mathbb{N},\ m = 0, 1, \dots$$

where each $\sqrt{s_{nm}}$ can be calculated as the n-th root of the m-th Bessel function of the first kind J_m. The classical Helmholtz equation, as is well-known, has limited applicability to real-world problems, which are often inherently nonlinear. For example, in order to study vibrations of a membrane, it is standard to perturb the system (39) with a nonlinearity $r : \overline{D} \times \mathbb{R} \to \mathbb{R}$ as follows

$$\begin{cases} -\Delta u = \lambda^2 u + r(z,u), \\ u|_{\partial D} = 0. \end{cases} \tag{40}$$

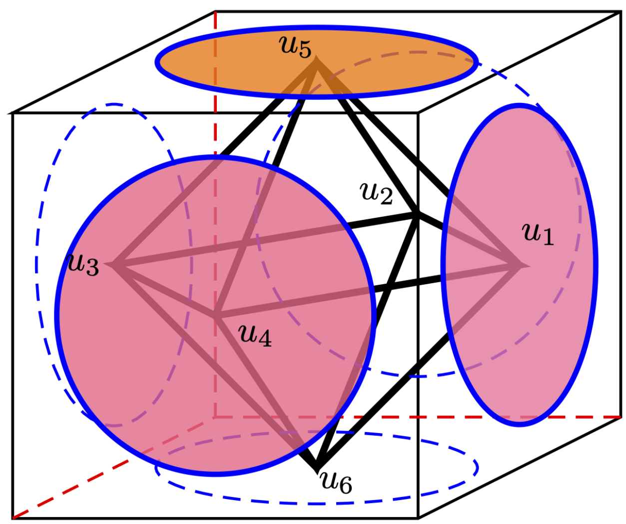

FIGURE 1. An octahedral configuration of six coupled membranes.

Of course, the single membrane model (40) is still too simplistic to model many physically integrated systems. A more realistic problem might instead involve a configuration of some number of coupled oscillating membranes. Consider, for example, a collection of six membranes arranged on corresponding faces of a cube, each coupled to its adjacent neighbors and modeled by a system of identical non-linear Helmholtz equations

$$\begin{cases} -\Delta u = f(\alpha, z, u), \quad u \in V \\ u|_{\partial D} = 0, \end{cases} \tag{41}$$

where $V=\mathbb{R}^6$ and $f:\mathbb{R}\times\overline{D}\times V\to V$ is the vector-valued function

$$f(\alpha,z,u)=\begin{bmatrix} f_1(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6)\\ f_2(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6)\\ f_3(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6)\\ f_4(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6)\\ f_5(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6)\\ f_6(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6)\end{bmatrix},\quad u=\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{bmatrix}. \tag{42}$$

satisfying the following assumptions:

(E_1) For all $\alpha\in\mathbb{R}$, $z\in D$, $u=(u_1,u_2,u_3,u_4,u_5,u_6)^T\in\mathbb{R}^6$ and $\tau\in\boldsymbol{O}$
$$\tau f(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6)=f(\alpha,z,u_{\tau(1)},u_{\tau(2)},u_{\tau(3)},u_{\tau(4)},u_{\tau(5)},u_{\tau(6)});$$

(E_2) For all $\alpha\in\mathbb{R}$, $z\in D$, $u=(u_1,u_2,u_3,u_4,u_5,u_6)^T\in\mathbb{R}^6$ and $e^{i\theta}\in O(2)$
$$f(\alpha,e^{i\theta}z,u_1,u_2,u_3,u_4,u_5,u_6)=f(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6).$$

(E_3) For all $\alpha\in\mathbb{R}$, $z\in D$, $u=(u_1,u_2,u_3,u_4,u_5,u_6)^T\in\mathbb{R}^6$,
$$f(\alpha,z,-u_1,-u_2,-u_3,-u_4,-u_5,-u_6)=-f(\alpha,z,u_1,u_2,u_3,u_4,u_5,u_6).$$

Here, $\boldsymbol{O}\le S_6$ is used to denote the chiral symmetry group of the cube, whose action on V is described by the set of orientation preserving permutations of the corresponding membranes. If we identify S_4 with $\boldsymbol{O}$ according to the rule

$$\begin{aligned}(1,2)\ &\leftrightarrow\ (1,4)(2,3)(5,6),\quad (1,2,3)\ \leftrightarrow\ (1,4,6)(3,5,2),\\ (1,2)(3,4)\ &\leftrightarrow\ (1,4)(2,3),\quad (1,2,3,4)\ \leftrightarrow\ (1,2,3,4),\end{aligned}$$

then the faces of the cube (see Figure **??**) are naturally permuted as follows

$$\forall_{\sigma\in S_4\le S_6}\ \sigma(u_1,u_2,u_3,u_4,u_5,u_6)^T=(u_{\sigma(1)},u_{\sigma(2)},u_{\sigma(3)},u_{\sigma(4)},u_{\sigma(5)},u_{\sigma(6)})^T,\quad u\in V. \tag{43}$$

In this way, V is an orthogonal S_4-representation with respect to the action (43). The table of characters for S_4

con. classes	(1)	(1,2)	(1,2)(3,4)	(1,2,3)	(1,2,3,4)
χ_0	1	1	1	1	1
χ_1	1	-1	1	1	-1
χ_2	2	0	2	-1	0
χ_3	3	-1	-1	0	1
χ_4	3	1	-1	0	-1
χ_V	6	0	2	0	2

TABLE 1. Character Table for S_4.

reveals the relation

$$\chi_V = \chi_0 + \chi_2 + \chi_3,$$

which can be used to obtain the following $S_4 \times \mathbb{Z}_2$-isotypic decomposition of V

$$V = \mathcal{V}_0^- \oplus \mathcal{V}_2^- \oplus \mathcal{V}_3^-.$$

Under the assumptions (E_1)-(E_3), the function f clearly satisfies conditions (A_1)-(A_3). For simplicity, let us also assume that f satisfies the conditions (A_4) and (A_5) with the matrix $A(\alpha) := D_u f(0) : V \to V$ given by

$$A(\alpha) = aI + \zeta(\alpha)C, \tag{44}$$

where $I : V \to V$ is the identity matrix, $C : V \to V$ is the weighted adjacency matrix

$$C = \begin{bmatrix} c & d & 0 & d & d & d \\ d & c & d & 0 & d & d \\ 0 & d & c & d & d & d \\ d & 0 & d & c & d & d \\ d & d & d & d & c & 0 \\ d & d & d & d & 0 & c \end{bmatrix}, \tag{45}$$

defined for some fixed $c, d \in \mathbb{R}$ satisfying $c > 0$, $d < 0$, $4d + c \geq 0$ and where $\zeta : \mathbb{R} \to \mathbb{R}$ is the *sigmoid function*

$$\zeta(\alpha) = \frac{1}{1 + e^{-\alpha}}. \tag{46}$$

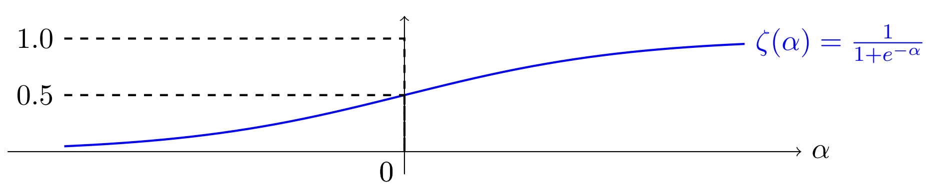

FIGURE 2. Graph of the sigmoid function $\zeta(\alpha)$.

Remark 5.1. The sigmoid function introduces a nonlinear dependence of the coupling in (41) on the bifurcation parameter $\alpha \in \mathbb{R}$. Moreover, the asymptotic behaviour of (46) imposes *coupling strength saturation limits* which might reflect physical constraints such as, for example, bounded ranges for certain material densities.

With the linearization (44), system (41) clearly satisfies the condition ($\tilde{A}_0$). Indeed, it can be shown that $A(\alpha): V \to V$ admits three eigenvalues with eigenspaces corresponding to the S_4-isotypic components as follows

$$\begin{aligned}
\mu_0(\alpha) &:= a + \zeta(\alpha)(c+4d), & E(\mu_0(\alpha)) &= \mathcal{V}_0^-, \\
\mu_2(\alpha) &:= a + \zeta(\alpha)(c-2d), & E(\mu_2(\alpha)) &= \mathcal{V}_2^-, \\
\mu_3(\alpha) &:= a + \zeta(\alpha)c, & E(\mu_3(\alpha)) &= \mathcal{V}_3^-.
\end{aligned} \tag{47}$$

If $c, d \in \mathbb{R}$ are such that $c+4d, c-2d, c \neq 0$, then (42) satisfies condition (B_3). Moreover, together with the strict monotonicity of the eigenvalues (47) and the distinctness of the numbers s_{nm}, it follows that each critical point of (41) can be specified by a unique triple $(n, m, j) \in \Sigma$ with the notation

$$\alpha_{n,m,j} := \mu_j^{-1}(s_{nm}). \tag{48}$$

Let us make the system (41) more specific by choosing the following values for the constants $a, c, d \in \mathbb{R}$

$$a = 32, \ c = 5, \ d = -1. \tag{49}$$

For convenience, we include graphs of the three eigenvalues (47) with the assignments (49) in Figure 3 and the numbers s_{nm} for $m = 0, \dots, 10$ and $n = 1, \dots, 9$ in Table 5. Notice

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
m=0	5.783	30.471	74.887	139.04	222.932	326.563	449.934	593.043	755.891
m=1	14.682	49.218	103.499	177.521	271.282	384.782	518.021	671	843.718
m=2	26.374	70.85	135.021	218.92	322.555	445.928	589.038	751.888	934.476
m=3	40.706	95.278	169.395	263.201	376.625	509.98	662.968	835.693	1028.15
m=4	57.583	122.428	206.57	310.322	433.761	576.913	739.79	922.398	1124.74
m=5	76.939	152.241	246.495	360.245	493.631	646.702	819.483	1011.99	1224.21
m=6	98.726	184.67	289.13	412.934	556.303	719.321	902.024	1104.44	1326.56
m=7	122.908	219.67	334.436	468.356	621.751	794.743	987.392	1199.73	1431.77
m=8	149.453	257.21	382.38	526.481	689.946	872.946	1075.56	1297.84	1539.81
m=9	178.337	297.26	432.933	587.281	760.863	953.907	1166.52	1398.77	1650.68
m=10	209.54	339.793	486.07	650.732	834.48	1037.6	1260.24	1502.48	1764.35

TABLE 2. Squared Zeros of Bessel functions s_{nm} for $0 \leq m \leq 10$, $1 \leq n \leq 9$.

that the critical set for (41) consists of exactly five critical points that can be expressed, using the notation (48), as follows

$$\Lambda = \{(\alpha_{1,3,0}, 0), (\alpha_{1,3,2}, 0), (\alpha_{1,3,3}, 0), (\alpha_{2,1,2}, 0), (\alpha_{2,1,3}, 0)\}. \tag{50}$$

In order to employ the results presented in previous sections to the bifurcation problem (41), we must first identify the maximal orbit types in $\Phi_0(G; \mathscr{H}_1 \setminus \{0\})$. The following G.A.P code can be used to generate a complete list of the sets $\mathfrak{M}_{1,j}$ and the corresponding Basic Degrees $\deg_{\mathcal{V}_{1,j}}$ for $j = 1, 2, 3$.

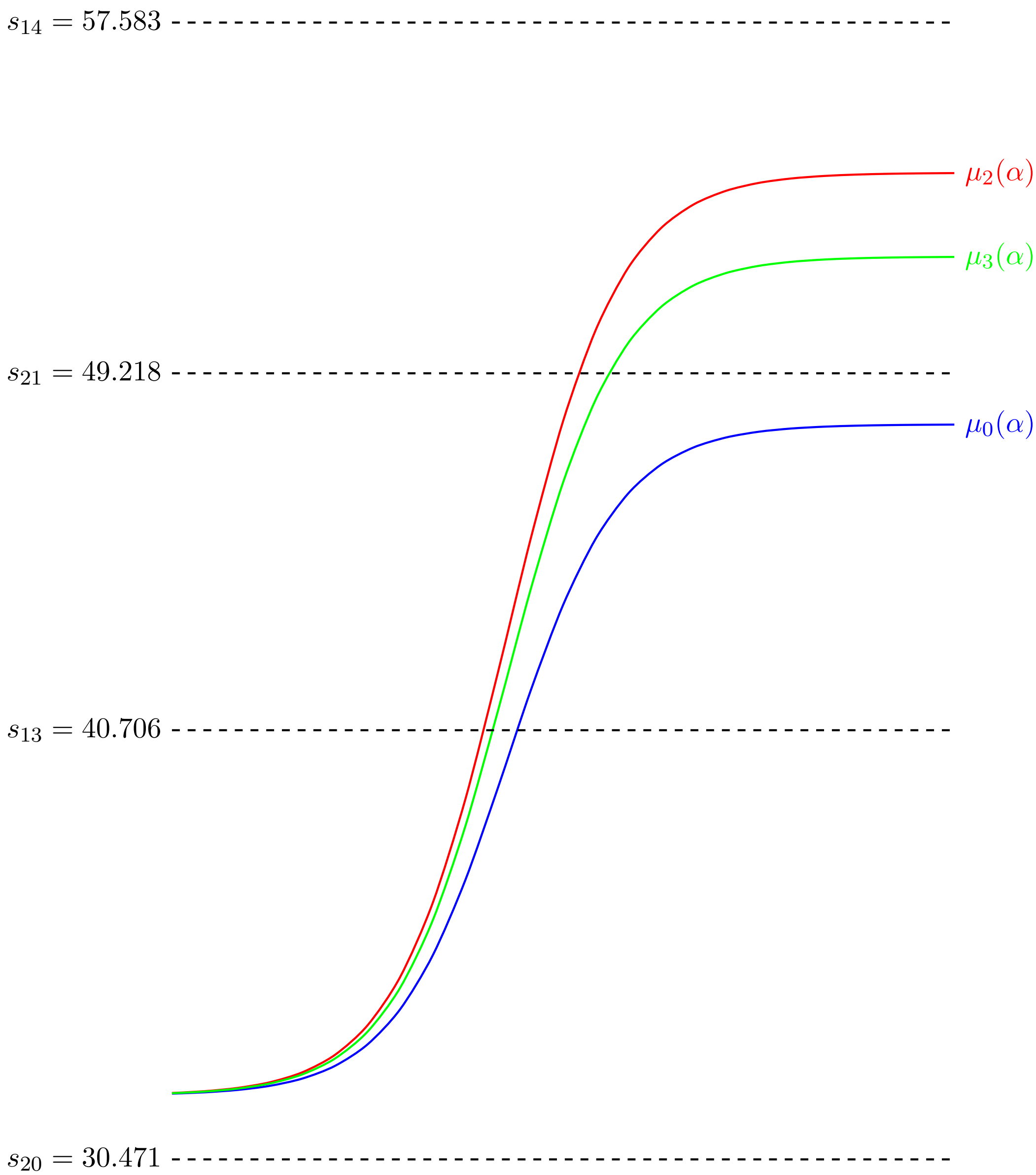

FIGURE 3. Graph of the eigenvalues $\mu_0(\alpha), \mu_2(\alpha)$ and $\mu_3(\alpha)$ and the consecutive numbers $s_{20} < s_{13} < s_{21} < s_{14}$.

```
# A G.A.P Program for the computation of Maximal Orbit Types and Basic Degrees associated with the G-isotypic decomposition  V = V_0 \times V_2 \times V_3

LoadPackage ("EquiDeg");
```

```
# create groups O(2)xS4xZ2
o2:=OrthogonalGroupOverReal(2);
s4:=SymmetricGroup(4);
z2:=pCyclicGroup(2);
# generate S4xZ2
g1:=DirectProduct(s4,z2);
# set names for subgroup conjugacy classes in S4xZ2
SetCCSsAbbrv(g1, [ "Z1", "Z2", "D1z","D1","Z2m", "Z1p",
"Z3", "V4", "D2z", "Z4", "D2", "D1p","D2d",
"V4m", "D2p", "Z4d", "D3", "D3z", "Z3p",
"V4p", "D4z", "D4d", "Z4p", "D4", "D4z",
 "D4hd", "D3p", "A4", "D4p", "A4p",  "S4", "S4m", "S4p"]);
# generate O(2)xS4xZ2
G:=DirectProduct(o2,g1);
ccs:=ConjugacyClassesSubgroups(G);
# find Maximal Orbit Types and Basic Degrees in first O(2)-isotypic component
irrs := Irr(G);
# The G-istoypic components 0,2,3 are indexed in G.A.P as 3,5,7
for i in [ 3,5,7 ] do
    max_orbtyps := MaximalOrbitTypes( irrs[1,i] );
    basic_deg := BasicDegree( irrs[1,i] );
    PrintFormatted( "\n Basic Degree associated with irrep V_{1,j} where j= \n", i );
    View(basic_deg);
    PrintFormatted( "\n Maximal Orbit Types in M_1,{} \n", i );
    View(max_orbtyps);
od;
Print( "Done!\n" );
```

The output of the above program can be expressed using amalgamated notation and an arbitrary $m > 0$ folding (cf. (23) and (24)) as follows:

$$
\begin{aligned}
\mathfrak{M}_{m,0} =& \Big\{ (D_{2m}^{D_m} \times^{S_4} S_4^p) \Big\}, \\
\mathfrak{M}_{m,2} =& \Big\{ (D_{6m}^{\mathbb{Z}_m} \times^{V_4} S_4^p), (D_{2m}^{D_m} \times^{D_4} D_4^p), (D_{2m}^{D_m} \times^{D_4^{\hat{d}}} D_4^p) \Big\}, \\
\mathfrak{M}_{m,3} =& \Big\{ (D_{2m}^{D_m} \times^{D_4^z} D_4^p), (D_{2m}^{D_m} \times^{D_3^z} D_3^p), (D_{2m}^{D_m} \times^{D_2^d} D_4^z), (D_{4m}^{\mathbb{Z}_m} \times^{\mathbb{Z}_2^-} D_4^p), (D_{6m}^{\mathbb{Z}_m} \times D_3^p) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\deg_{\mathcal{V}_{m,0}} &= (G) - (D_{2m}^{D_m} \times^{S_4} S_4^p), \\
\deg_{\mathcal{V}_{m,2}} &= (G) - 2(D_{6m}^{\mathbb{Z}_m} \times^{V_4} S_4^p) - (D_{2m}^{D_m} \times^{D_4} D_4^p) - (D_{2m}^{D_m} \times^{D_4^{\hat{d}}} D_4^p) \\
&\quad + (D_{2m}^{D_m} \times^{V_4} V_4^p) + 4(D_{2m}^{\mathbb{Z}_m} \times^{V_4} D_4^p), \\
\deg_{\mathcal{V}_{m,3}} &= (G) - (D_{2m}^{D_m} \times^{D_4^z} D_4^p) - (D_{2m}^{D_m} \times^{D_3^z} D_3^p) - (D_{2m}^{D_m} \times^{D_2^d} D_4^z) \\
&\quad - 2(D_{4m}^{\mathbb{Z}_m} \times^{\mathbb{Z}_2^-} D_4^p) - 2(D_{6m}^{\mathbb{Z}_m} \times D_3^p) + (D_{2m}^{\mathbb{Z}_m} \times^{\mathbb{Z}_2^-} D_2^p) \\
&\quad + 2(D_{2m}^{\mathbb{Z}_m} \times^{\mathbb{Z}_2^-} V_4^p) + 2(D_{2m}^{\mathbb{Z}_m} \times^{\mathbb{Z}_2^-} D_4^z) + 2(D_{2m}^{\mathbb{Z}_m} \times^{D_1^z} D_4^z) - (D_{2m}^{\mathbb{Z}_m} \times \mathbb{Z}_1^p) - 2(D_{2m}^{\mathbb{Z}_m} \times D_2^p).
\end{aligned}
$$

Take $j \in \{0, 2, 3\}$ and let (H) be any orbit type in $\mathfrak{M}_{1,j}$. From the coefficients of the Basic Degrees $\deg_{\mathcal{V}_{1,j}} \in A(G)$, it is clear that the fixed point space $\mathcal{V}_{1,j}^H$ has odd dimension. Since the critical set (50) is manageably small, we can compute the cardinalities (26) for each of the critical points $(\alpha_{n,m,j}, 0) \in \Lambda$ to obtain the quantities

$$
\begin{aligned}
&\mathfrak{s}(\alpha_{1,3,2}, H) = \mathfrak{s}(\alpha_{1,3,3}, H) = \mathfrak{s}(\alpha_{1,3,0}, H) = 3, \\
&\mathfrak{s}(\alpha_{2,1,2}, H) = \mathfrak{s}(\alpha_{2,1,3}, H) = 1.
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathfrak{i}^3(\alpha_{1,3,2}, H) = 1, \quad \mathfrak{i}^3(\alpha_{1,3,3}, H) = -1, \quad \mathfrak{i}^3(\alpha_{1,3,0}, H) = 1, \\
&\mathfrak{i}^1(\alpha_{2,1,2}, H) = 1, \quad \mathfrak{i}^1(\alpha_{2,1,3}, H) = -1.
\end{aligned}
$$

Therefore, from Theorem 4.1, it follows that every critical point $(\alpha_{n,m,j}, 0) \in \Lambda$ is a branching point for branches of non-trivial solutions to the problem (41) with symmetries at least $({}^3H)$ in the case of $(\alpha_{1,3,2}, 0), (\alpha_{1,3,3}, 0), (\alpha_{1,3,0}, 0) \in \Lambda$ and with symmetries at least (H) in the case of $(\alpha_{2,1,2}, 0), (\alpha_{2,1,3}, 0) \in \Lambda$. The following G.A.P code can be used to computationally verify the non-triviality of the local bifurcation invariant (specifically, the non-triviality of the relevant coefficients) at each of the critical points in terms of the basic degrees

$$
\deg_{\mathcal{V}_{3,2}}, \deg_{\mathcal{V}_{3,3}}, \deg_{\mathcal{V}_{3,0}}, \deg_{\mathcal{V}_{1,2}}, \deg_{\mathcal{V}_{1,3}} \in A(G),
$$

and the unit Burnside Ring element (G) according to the rule derived in Lemma (4.2)

$$\begin{aligned}
\omega_G(\alpha_{1,3,2}) &= (G) - \deg_{\mathcal{V}_{3,2}},\\
\omega_G(\alpha_{1,3,3}) &= \deg_{\mathcal{V}_{3,2}} - \deg_{\mathcal{V}_{3,2}} \cdot \deg_{\mathcal{V}_{3,3}},\\
\omega_G(\alpha_{1,3,0}) &= \deg_{\mathcal{V}_{3,2}} \cdot \deg_{\mathcal{V}_{3,3}} - \deg_{\mathcal{V}_{3,2}} \cdot \deg_{\mathcal{V}_{3,3}} \cdot \deg_{\mathcal{V}_{3,0}},\\
\omega_G(\alpha_{2,1,2}) &= \deg_{\mathcal{V}_{3,2}} \cdot \deg_{\mathcal{V}_{3,3}} \cdot \deg_{\mathcal{V}_{3,0}} - \deg_{\mathcal{V}_{3,2}} \cdot \deg_{\mathcal{V}_{3,3}} \cdot \deg_{\mathcal{V}_{3,0}} \cdot \deg_{\mathcal{V}_{1,2}},\\
\omega_G(\alpha_{2,1,3}) &= \deg_{\mathcal{V}_{3,2}} \cdot \deg_{\mathcal{V}_{3,3}} \cdot \deg_{\mathcal{V}_{3,0}} \cdot \deg_{\mathcal{V}_{1,2}} - \deg_{\mathcal{V}_{3,2}} \cdot \deg_{\mathcal{V}_{3,3}} \cdot \deg_{\mathcal{V}_{3,0}} \cdot \deg_{\mathcal{V}_{1,2}} \cdot \deg_{\mathcal{V}_{1,3}}.
\end{aligned}$$

```
# A G.A.P Program for the Burnside Ring product  of Basic
    Degrees to be used for the verification of non-triviality
    of local bifurcation invariants
# Initialize the Burnside Ring A(G)
AG:=BurnsideRing(G); H:=Basis(AG);
# Initialize the unit element (G) \in A(G)
U:=H[0,131];
# Initialize the relevant Basic Degrees
d_30 := BasicDegree(Irr(G)[3,3]);
d_32 := BasicDegree(Irr(G)[3,5]);
d_33 := BasicDegree(Irr(G)[3,7]);
d_12 := BasicDegree(Irr(G)[1,5]);
d_13 := BasicDegree(Irr(G)[1,7]);
# Compute Products of Basic Degrees Cumulatively
p_1 := d_32;
p_2 := p_1*d_33;
p_3 := p_2*d_30;
p_4 := p_3*d_12;
p_5 := p_4*d_13;
# Compute the local bifurcation invariants w_{n,m,j} at the
    corresponding critical points \alpha_{n,m,j}
w_132 := U - p_1;
w_133 := p_1 - p_2;
w_130 := p_2 - p_3;
w_212 := p_3 - p_4;
w_213 := p_4 - p_5;
# Sum the local bifurcation invariants to determine the
    Rabinowitz Alternative
sum := w_132 + w_133 + w_130 + w_212 + w_213
```

The output of the above program can be expressed using amalgamated notation as follows:

$$
\begin{aligned}
\omega_G(\alpha_{1,3,2}) &= -4(D_6^{\mathbb{Z}_3} \times^{V_4} D_4^p) - (D_6^{D_3} \times^{V_4} V_4^p) + (D_6^{D_3} \times^{D_4} D_4^p) + (D_6^{D_3} \times^{D_4^{\hat{d}}} D_4^p) + 2(D_{18}^{\mathbb{Z}_3} \times^{V_4} S_4^p),\\
\omega_G(\alpha_{1,3,3}) &= 4(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_1} D_1^p) + 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_1} D_2^p) + (D_6^{D_3} \times^{\mathbb{Z}_1} \mathbb{Z}_1^p) - 2(D_6^{\mathbb{Z}_3} \times^{D_1} D_4^z) - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2^-} D_4^z)\\
&\quad - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2^-} V_4^p) + 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2} \mathbb{Z}_4^p) + 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2} D_4^z) + 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2} V_4^p) - (D_6^{D_3} \times^{D_1} D_1^p)\\
&\quad - (D_6^{D_3} \times^{D_1^z} D_1^p) - (D_6^{D_3} \times^{\mathbb{Z}_2^-} D_2^p) + (D_6^{D_3} \times^{\mathbb{Z}_2} D_2^p) - 2(D_{18}^{\mathbb{Z}_3} \times^{\mathbb{Z}_1} D_3^p) + 2(D_{12}^{\mathbb{Z}_3} \times^{\mathbb{Z}_2^-} D_4^p)\\
&\quad - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_4} D_4^p) - 2(D_6^{\mathbb{Z}_3} \times^{D_2^z} D_4^p) + (D_6^{D_3} \times^{D_2^d} D_4^z) - (D_6^{D_3} \times^{\mathbb{Z}_4} \mathbb{Z}_4^p) - (D_6^{D_3} \times^{D_2^z} D_4^z)\\
&\quad + (D_6^{D_3} \times^{D_3^z} D_3^p) + (D_6^{D_3} \times^{D_4^z} D_4^p),\\
\omega_G(\alpha_{1,3,0}) &= 2(D_6^{\mathbb{Z}_3} \times^{D_1} D_4^z) - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2} \mathbb{Z}_4^p) - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2} D_4^z) - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2} V_4^p) + (D_6^{D_3} \times^{D_1} D_1^p)\\
&\quad - (D_6^{D_3} \times^{\mathbb{Z}_2} D_2^p) - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_3} D_3^p) - (D_6^{D_3} \times^{\mathbb{Z}_3} \mathbb{Z}_3^p) + 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_4} D_4^p) + 3(D_6^{\mathbb{Z}_3} \times^{V_4} D_4^p)\\
&\quad + (D_6^{D_3} \times^{\mathbb{Z}_4} \mathbb{Z}_4^p) + (D_6^{D_3} \times^{V_4} V_4^p) - 2(D_6^{D_3} \times^{D_4} D_4^p) + (D_6^{D_3} \times^{S_4} S_4^p)\\
\omega_G(\alpha_{2,1,2}) &= 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_2^p) + (D_2^{D_1} \times^{\mathbb{Z}_1} \mathbb{Z}_1^p) - 2(D_2^{\mathbb{Z}_1} \times^{D_1} D_4^z) - 2(D_2^{\mathbb{Z}_1} \times^{D_1^z} D_4^z) - (D_2^{D_1} \times^{D_1} D_1^p)\\
&\quad - (D_2^{D_1} \times^{D_1^z} D_1^p) + 4(D_6^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_3^p) - 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_4} D_4^p) + 2(D_2^{\mathbb{Z}_1} \times^{D_2^z} D_4^p) + 2(D_2^{\mathbb{Z}_1} \times^{V_4} D_4^p)\\
&\quad - (D_2^{D_1} \times^{\mathbb{Z}_4} \mathbb{Z}_4^p) + (D_2^{D_1} \times^{D_2^z} D_4^z) + (D_2^{D_1} \times^{D_4} D_4^p) - (D_2^{D_1} \times^{D_4^{\hat{d}}} D_4^p) - 2(D_6^{\mathbb{Z}_1} \times^{V_4} S_4^p)\\
\omega_G(\alpha_{2,1,3}) &= -4(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_1^p) - 4(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_2^p) - 2(D_2^{D_1} \times^{\mathbb{Z}_1} \mathbb{Z}_1^p) + 2(D_2^{\mathbb{Z}_1} \times^{D_1} D_4^z) + 2(D_2^{\mathbb{Z}_1} \times^{D_1^z} D_4^z)\\
&\quad + 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_2^-} D_4^z) + 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_2^-} V_4^p) + (D_2^{D_1} \times^{D_1} D_1^p) + 2(D_2^{D_1} \times^{D_1^z} D_1^p) + (D_2^{D_1} \times^{\mathbb{Z}_2^-} D_2^p)\\
&\quad - 2(D_6^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_3^p) + 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_3} D_3^p) + (D_2^{D_1} \times^{\mathbb{Z}_3} \mathbb{Z}_3^p) - 2(D_4^{\mathbb{Z}_1} \times^{\mathbb{Z}_2^-} D_4^p) + 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_4} D_4^p)\\
&\quad - (D_2^{D_1} \times^{D_2^d} D_4^z) + (D_2^{D_1} \times^{\mathbb{Z}_4} \mathbb{Z}_4^p) + (D_2^{D_1} \times^{D_3^z} D_3^p) - (D_2^{D_1} \times^{D_4^z} D_4^p).
\end{aligned}
$$

Next, we can determine the global behaviour of some of the branches predicted using Theorem (4.1) by establishing the parity of the set (38). Notice that $\max_{(\alpha,0)\in\Lambda}\{\mathfrak{s}(\alpha,H)\} \nmid 2$ is obtained for exactly three critical points $(\alpha_{1,3,0},0),(\alpha_{1,3,2},0),(\alpha_{1,3,3},0)$, i.e.

$$\bar{\mathfrak{s}}(H) = 3 \ \text{ and } \ \mathfrak{J}(H) = \{(\alpha_{1,3,0},0),(\alpha_{1,3,2},0),(\alpha_{1,3,3},0)\}.$$

Therefore, any branch of non-trivial solutions with symmetries at least (^3H) bifurcating from $(\alpha_{1,3,0},0),(\alpha_{1,3,2},0),(\alpha_{1,3,3},0)$ consists only of non-radial solutions and is unbounded. Of course, since the critical set (50) is finite, one needs only sum the local bifurcation invariants (**??**) to arrive at the same conclusion directly from the Rabinowitz alternative (cf. Theorems 4.2 and 4.3).

$$
\begin{aligned}
\sum_{(\alpha_0,0)\in\Lambda} \omega_G(\alpha_0) &= \omega_G(\alpha_{1,3,2}) + \omega_G(\alpha_{1,3,3}) + \omega_G(\alpha_{1,3,0}) + \omega_G(\alpha_{2,1,2}) + \omega_G(\alpha_{2,1,3}) \\
&= -4(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_1^p) - 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_2^p) - (D_2^{D_1} \times^{\mathbb{Z}_1} \mathbb{Z}_1^p) + 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_2^-} D_4^z) \\
&\quad + 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_2^-} V_4^p) + (D_2^{D_1} \times^{D_1^z} D_1^p) + (D_2^{D_1} \times^{\mathbb{Z}_2^-} D_2^p) + 2(D_6^{\mathbb{Z}_1} \times^{\mathbb{Z}_1} D_3^p) \\
&\quad + 2(D_2^{\mathbb{Z}_1} \times^{\mathbb{Z}_3} D_3^p) + (D_2^{D_1} \times^{\mathbb{Z}_3} \mathbb{Z}_3^p) - 2(D_4^{\mathbb{Z}_1} \times^{\mathbb{Z}_2^-} D_4^p) + 2(D_2^{\mathbb{Z}_1} \times^{D_2^z} D_4^p) \\
&\quad + 2(D_2^{\mathbb{Z}_1} \times^{V_4} D_4^p) - (D_2^{D_1} \times^{D_2^d} D_4^z) + (D_2^{D_1} \times^{D_2^z} D_4^z) - (D_2^{D_1} \times^{D_3^z} D_3^p) \\
&\quad - (D_2^{D_1} \times^{D_4^z} D_4^p) + (D_2^{D_1} \times^{D_4} D_4^p) - (D_2^{D_1} \times^{D_4^{\hat{d}}} D_4^p) - 2(D_6^{\mathbb{Z}_1} \times^{V_4} S_4^p) \\
&\quad + 4(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_1} D_1^p) + 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_1} D_2^p) + (D_6^{D_3} \times^{\mathbb{Z}_1} \mathbb{Z}_1^p) - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2^-} D_4^z) \\
&\quad - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_2^-} V_4^p) - (D_6^{D_3} \times^{D_1^z} D_1^p) - (D_6^{D_3} \times^{\mathbb{Z}_2^-} D_2^p) - 2(D_18^{\mathbb{Z}_3} \times^{\mathbb{Z}_1} D_3^p) \\
&\quad - 2(D_6^{\mathbb{Z}_3} \times^{\mathbb{Z}_3} D_3^p) - (D_6^{D_3} \times^{\mathbb{Z}_3} \mathbb{Z}_3^p) + 2(D_12^{\mathbb{Z}_3} \times^{\mathbb{Z}_2^-} D_4^p) - 2(D_6^{\mathbb{Z}_3} \times^{D_2^z} D_4^p) \\
&\quad - 2(D_6^{\mathbb{Z}_3} \times^{V_4} D_4^p) + (D_6^{D_3} \times^{D_2^d} D_4^z) - (D_6^{D_3} \times^{D_2^z} D_4^z) + (D_6^{D_3} \times^{D_3^z} D_3^p) \\
&\quad + (D_6^{D_3} \times^{D_4^z} D_4^p) - (D_6^{D_3} \times^{D_4} D_4^p) + (D_6^{D_3} \times^{D_4^{\hat{d}}} D_4^p) + 2(D_{18}^{\mathbb{Z}_3} \times^{V_4} S_4^p) \\
&\quad + (D_6^{D_3} \times^{S_4} S_4^p).
\end{aligned}
$$

In conclusion, let's consider an interpretation of the above results. Take the maximal orbit type $(H) = (D_2^{D_1} \times^{D_4} D_4^p) \in \mathscr{M}_{1,2}$ and let $\mathcal{C}$ be a branch of non-radial solutions with symmetries at least $({}^3H) = (D_6^{D_3} \times^{D_4} D_4^p) \in \mathscr{M}_{3,2}$ emerging from the first critical point $(\alpha_{1,3,2}, 0) \in \Lambda$. The kernel of $\mathscr{A}(\alpha) : \mathscr{H} \to \mathscr{H}$ at any critical point $(\alpha_{n,m,j}, 0) \in \Lambda$ is given by the corresponding irreducible G-representation $\mathscr{E}_{n,m}^j$ (cf. (14)), i.e.

$$
\operatorname{Ker} \mathscr{A}(\alpha_{1,3,2}) = \mathscr{E}_{1,3}^2.
$$

In particular, any vector $\hat{u} \in \mathscr{H}$ with $\mathscr{A}(\alpha_{1,3,2})\hat{u} = 0$ must be of the form

$$
\hat{u}(r,\theta) \in \left\{ J_3(\sqrt{s_{13}}r)\Big(\cos(3\theta)\vec{a} + \sin(3\theta)\vec{b}\Big) : \vec{a},\, \vec{b} \in \operatorname{span}\{\vec{w}_1, \vec{w}_2\} \right\},
$$

where $\vec{w}_1, \vec{w}_2 \in \mathcal{V}_2^-$ are the eigenvectors $\vec{w}_1 = (-1,1,-1,1,0,0)^T$ and $\vec{w}_2 = (1,0,1,0,-1,-1)^T$. Computing the generators of (H) and examining their action on $\mathscr{E}_{1,3}^2$, it can be shown that $\hat{u} \in \operatorname{Ker}\mathscr{A}(\alpha_0)^H$ if and only if $\vec{b} = 0$ and $\vec{a} = \vec{w}_1 + 2\vec{w}_2$. In particular, one has $\dim \operatorname{Ker} \mathscr{A}(\alpha_0)^H = 1$, which implies, from the celebrated theorem of Crandall-Rabinowitz (cf. [29]), that the branch $\mathcal{C}$ is tangent to the solution $\hat{u}(r,\theta) = J_3(\sqrt{s_{13}}r)\cos(3\theta)(\vec{w}_1 + 2\vec{w}_2)$ at the point $(\alpha_{1,3,2}, 0)$. Therefore, $\hat{u} \in \mathscr{H}$ can reasonably be used to estimate the behaviour of the branch $\mathcal{C}$ for parameter values near $\alpha_{1,3,2} \in \mathbb{R}$.

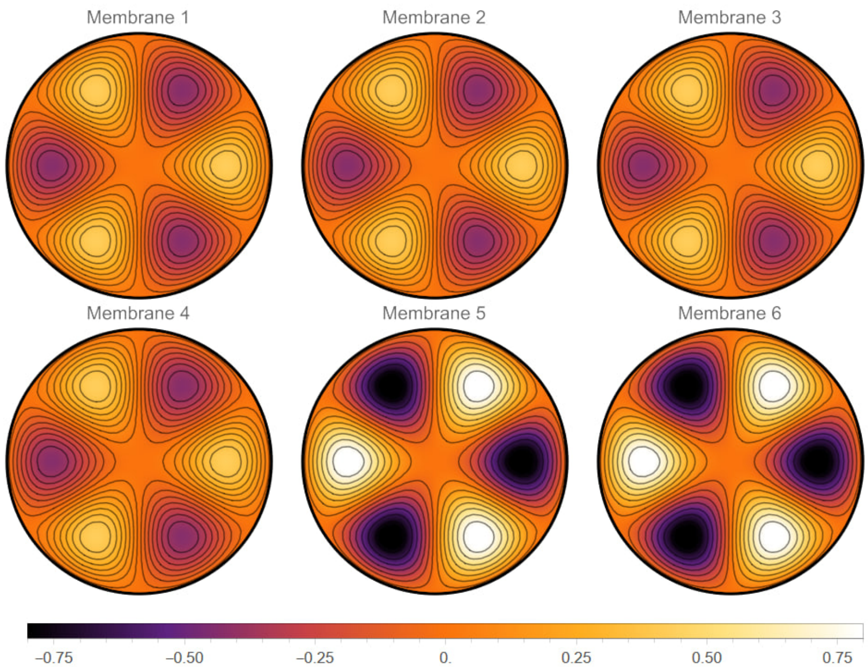

FIGURE 4. Graph of the map $\hat{u}(r,\theta) = J_3(\sqrt{s_{13}}r)\cos(3\theta)(\vec{w}_1 + 2\vec{w}_2)$.

APPENDIX A. AMALGAMATED NOTATION OF SUBGROUPS

The convention of *amalgamated notation* is a shorthand for the specification of subgroups and their conjugacy classes in a product group $G_1 \times G_2$. In order to introduce this convention, we recall a well-known consequence of Goursat's Lemma (cf. [15]). Namely that, for any closed subgroup in $H \leq G_1 \times G_2$, there are two subgroups $K_1 \leq G_1$, $K_2 \leq G_2$, a group L and a pair of epimorphisms $\varphi_2 : K_1 \to L$, $\varphi_2 : K_2 \to L$ such that

$$H = \{(x, y) \in K_1 \times K_2 \ : \ \varphi_1(x) = \varphi_2(y)\}. \tag{51}$$

Using amalgamated notation, the subgroup (51) is identified, up to its conjugacy class in $\Phi_0(G_1 \times G_2)$, by the quintuple (K_1, K_2, L, Z_1, Z_2), where $Z_1 := \operatorname{Ker}\varphi_1$ and $Z_2 := \operatorname{Ker}\varphi_2$, as follows

$$(H) = ({K_1}^{Z_1} \times_L^{Z_2} K_2).$$

In particular, one can always choose the group L such that $L \simeq K_1/Z_1$, in which case the conjugacy class of (51) is identified with the quadruple (K_1, K_1, Z_1, Z_2) as follows

$$(H) = ({K_1}^{Z_1} \times^{Z_2} K_2). \tag{52}$$

This *compact amalgamated decomposition* (52) is the form with which subgroup conjugacy classes are identified in this paper.

In addition to the notation defined above, the following list provides some ancillary shorthand used in this paper for the identification of subgroups in $S_4 \times \mathbb{Z}_2$

$$
\begin{aligned}
\mathbb{Z}_2^- &= \{((1),1),((12)(34),-1)\},\\
\mathbb{Z}_4^- &= \{((1),1),((1324),-1),((12)(34),1),((1423),-1)\},\\
D_1^z &= \{((1),1),((12),-1)\},\\
V_4^- &= \{((1),1),((12)(34),1),((13)(24),-1),((14)(23),-1)\},\\
D_2^d &= \{((1),1),((12)(34),-1),((12),1),((34),-1)\},\\
D_2^z &= \{((1),1),((12)(34),1),((12),-1),((34),-1)\},\\
D_3^z &= \{((1),1),((123),1),((132),1),((12),-1),((23),-1),((13),-1)\},\\
D_4^d &= \{((1),1),((1324),-1),((12)(34),1),((1423),-1),((34),1),\\
&\quad ((14)(23),-1),((12),1),((13)(24),-1)\},\\
D_4^{\hat{d}} &= \{((1),1),((1324),-1),((12)(34),1),((1423),-1),((34),-1),\\
&\quad ((14)(23),1),((12),-1),((13)(24),1)\},\\
D_4^z &= \{((1),1),((1324),1),((12)(34),1),((1423),1),((34),-1),\\
&\quad ((14)(23),-1),((12),-1),((13)(24),-1)\},\\
S_4^- &= \{((1),1),((12),-1),((12)(34),1),((123),1),((1234),-1),((13),-1),\\
&\quad ((13)(24),1),((132),1),((1342),-1),((14),-1),((14)(23),1),((142),1),\\
&\quad ((1324),-1),((23),-1),((124),1),((1243),-1),((24),-1),((134),1),\\
&\quad ((1423),-1),((34),-1),((143),1),((1432),-1),((243),1),((234),1)\}.
\end{aligned}
$$

Appendix B. Spectral Properties of Operator $\mathscr{A}$

In this section, we leverage the spectral properties of the Laplace operator $\mathscr{L} : \mathscr{H} \to \mathscr{H}$ to describe the spectrum of

$$\mathscr{A}(\alpha) := \mathrm{Id} - \mathscr{L}^{-1} A(\alpha) : \mathscr{H} \to \mathscr{H}.$$

To begin, consider the following eigenvalue problem defined on the planar unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda u, \quad u \in \mathscr{H}, \lambda \in \mathbb{R} \\ u|_{\partial D} = 0 \end{cases}$$

In polar coordinates, the Laplacian Δ takes the form

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

Assuming that $u \in \mathscr{H}$ can be separated $u(r,\theta) = R(r) \cdot T(\theta)$, the eigenvalue problem $-\Delta u = \lambda u$ becomes

$$\left(-R'' - \frac{1}{r}R'\right)T - \frac{1}{r^2}T''R = \lambda RT.$$

Rearranging terms, we obtain

$$\frac{r^2R'' + rR'}{R} + \lambda r^2 = -\frac{T''}{T}. \tag{53}$$

Since the left and right-hand sides of equation (53) depend on different variables, there must exist a constant c such that

$$r^2R'' + rR' + (\lambda r^2 - c)R = 0, \tag{54}$$

$$-T'' = cT. \tag{55}$$

To ensure that $u(r,\theta)$ is single-valued and continuous on D, we impose the periodicity condition $T(\theta+2\pi) = T(\theta)$. Then, the angular part (55) becomes the ordinary differential equation

$$\begin{cases} T''(\theta) = -cT(\theta) \\ T(\theta + 2\pi) = T(\theta). \end{cases}$$

which has eigenvalues $c = m^2$ for $m = 0, 1, 2, \ldots$ with corresponding eigenfunctions $T(\theta) = cos(m\theta)$ and $T(\theta) = \sin(m\theta)$.

Notice that, as a consequence of solving for $T(\theta)$, we have determined the values of constant c:

$$c = m^2, m = 0, 1, 2, \ldots$$

Then, equation (54) becomes

$$r^2R'' + rR' + (\lambda r^2 - m^2)R = 0. \tag{56}$$

Introducing the change of variables $\rho = \sqrt{\lambda}r$ and $\widetilde{R}(\rho) = R(\sqrt{\lambda}r)$ and substituting into (56), we obtain the classical Bessel equation

$$\rho^2\widetilde{R}''(\rho) + \rho\widetilde{R}'(\rho) + (\rho^2 - m^2)\widetilde{R}(\rho) = 0.$$

The bounded at zero solution is given by $\widetilde{R}(\rho) = J_m(\rho)$, where J_m stands for the m-th Bessel function of the first kind. It follows that solutions to (54) are constant multiples of $R(r) = J_m(\sqrt{\lambda}r)$ (notice that $R(r)$ is finite at zero).

Consequently, $u(r,\theta) = R(r)\Theta(\theta)$ satisfies the Dirichlet condition if $R(1) = 0$, i.e. $J_m(\sqrt{\lambda}) = 0$. In this way, we obtain that the spectrum of the Laplace operator is given by

$$\sigma(\mathscr{L}) = \{s_{nm} : \sqrt{s_{nm}} \text{ is the } n\text{-th positive zero of } J_m,\ n \in \mathbb{N}\ m = 0, 1, \ldots\}.$$

Moreover, corresponding to each eigenvalue $s_{nm} \in \sigma(\mathscr{L})$, there is the eigenfunction

$$u_{nm}(r,\theta) = J_m(\sqrt{s_{nm}}r)\left(A\cos(m\theta) + B\sin(m\theta)\right),$$

where $A, B \in \mathbb{R}$, and also the eigenspace

$$\mathscr{E}(s_{nm}) = \operatorname{span}\left\{J_m(\sqrt{s_{nm}}r)\Big(A\cos(m\theta) + B\sin(m\theta)\Big) : \vec{a},\, \vec{b} \in \mathbb{R}^k,\ 0 \le n \le k\right\}$$

In this way, we have determined the spectrum of our operator $\mathscr{A}$

$$\sigma(\mathscr{A}) = \left\{\xi_{n,m,j} := 1 - \frac{\mu_j(\alpha)}{s_{nm}} : j = 1, 2, \dots, k,\ n \in \mathbb{N},\ m = 0, 1, 2, \dots\right\}.$$

Appendix C. Equivariant Brouwer Degree Background

Equivariant notation. Let $\mathcal{G}$ be a compact Lie Group. For any subgroup $H \le \mathcal{G}$, denote by (H) its conjugacy class, by by $N(H)$ its normalizer by $W(H) := N(H)/H$ its Weyl group in $\mathcal{G}$. The set of all subgroup conjugacy classes in $\mathcal{G}$, denoted $\Phi(\mathcal{G}) := \{(H) : H \le \mathcal{G}\}$, has a natural partial order defined as follows

$$(H) \le (K) \iff \exists_{g\in\mathcal{G}}\ \ gHg^{-1} \le K.$$

In particular, we put $\Phi_0(\mathcal{G}) := \{(H) \in \Phi(\mathcal{G})\ :\ W(H) \text{ is finite}\}$.

For a $\mathcal{G}$-space X and $x \in X$, denote by $\mathcal{G}_x := \{g \in \mathcal{G} : gx = x\}$ the *isotropy group* of x and call $(\mathcal{G}_x) \in \Phi(\mathcal{G})$ the *orbit type* of $x \in X$. Put $\Phi(\mathcal{G}, X) := \{(H) \in \Phi_0(\mathcal{G}) : (H) = (\mathcal{G}_x) \text{ for some } x \in X\}$ and $\Phi_0(\mathcal{G}, X) := \Phi(\mathcal{G}, X) \cap \Phi_0(\mathcal{G})$. For a subgroup $H \le \mathcal{G}$, the subspace $X^H := \{x \in X : \mathcal{G}_x \ge H\}$ is called the *H-fixed-point subspace* of X. If Y is another $\mathcal{G}$-space, then a continuous map $f : X \to Y$ is said to be *$\mathcal{G}$-equivariant* if $f(gx) = gf(x)$ for each $x \in X$ and $g \in \mathcal{G}$.

The Burnside Ring and Axioms of Equivariant Brouwer Degree. Denote by $\mathcal{M}^G$ the set of all admissible G-pairs by $A(G)$ the Burnside ring of G. The following statement is the standard axiomatic definition of the *G-equivariant Brouwer degree* (cf. [5]).

Theorem C.1. *There exists a unique map G-deg : $\mathcal{M}^G \to A(G)$, that assigns to every admissible G-pair (f, Ω) the Burnside Ring element*

$$G\text{-deg}(f,\Omega) = \sum_{(H)\in\Phi_0(G)} n_H(H), \tag{57}$$

satisfying the following properties:

> ***(Existence)*** *If $n_H \neq 0$ for some $(H) \in \Phi_0(G)$ in (57), then there exists $x \in \Omega$ such that $f(x) = 0$ and $(G_x) \ge (H)$.*

(Additivity) *For any two disjoint open G-invariant subsets Ω_1 and Ω_2 with $f^{-1}(0)\cap\Omega\subset\Omega_1\cup\Omega_2$, one has*

$$G\text{-deg}(f,\Omega)=G\text{-deg}(f,\Omega_1)+G\text{-deg}(f,\Omega_2).$$

(Homotopy) *For any Ω-admissible G-homotopy, $h:[0,1]\times V\to V$, one has*

$$G\text{-deg}(h_t,\Omega)=\text{constant}.$$

(Normalization) *For any open bounded neighborhood of the origin in an orthogonal G-representation V with the identity operator $\mathrm{Id}:V\to V$, one has*

$$G\text{-deg}(\mathrm{Id},\Omega)=(G).$$

The following are additional properties of the map G-deg which can be derived from the four axiomatic properties defined above:

(Multiplicativity) *For any $(f_1,\Omega_1),(f_2,\Omega_2)\in\mathcal{M}^G$,*

$$G\text{-deg}(f_1\times f_2,\Omega_1\times\Omega_2)=G\text{-deg}(f_1,\Omega_1)\cdot G\text{-deg}(f_2,\Omega_2),$$

where the multiplication '$\cdot$' is taken in the Burnside ring $A(G)$.

(Recurrence Formula) *For an admissible G-pair (f,Ω), the G-degree (57) can be computed using the following Recurrence Formula:*

$$n_H=\frac{\deg(f^H,\Omega^H)-\sum_{(K)>(H)}n_K\,n(H,K)\;|W(K)|}{|W(H)|}, \tag{58}$$

where $|X|$ stands for the number of elements in the set X and $\deg(f^H,\Omega^H)$ is the Brouwer degree of the map $f^H:=f|_{V^H}$ on the set $\Omega^H\subset V^H$.

Computation of Brouwer equivariant degree. For an orthogonal G-representation V with the open unit ball $B(V):=\{x\in V:|x|<1\}$, denote by $\{\mathcal{V}_i\}_{i\in\mathbb{N}}$ the set of its irreducible G-subrepresentations. In particular, define the *basic degree associated with $\mathcal{V}_i$* as follows

$$\deg_{\mathcal{V}_i}:=G\text{-deg}(-\mathrm{Id},B(\mathcal{V}_i)),$$

Now, consider a G-equivariant linear isomorphism $T:V\to V$ and assume that V has a G-isotypic decomposition

$$V=\bigoplus_{i\in\mathbb{N}}V_i,$$

where each isotypic component V_i is equivalent to $m_i\in\mathbb{N}$ copies of the irreducible G-representation $\mathcal{V}_i$. From the Multiplicativity property of the G-equivariant Brouwer Degree, one has

$$G\text{-deg}(T,B(V))=\prod_{i\in\mathbb{N}}G\text{-deg}(T_i,B(V_i))=\prod_{i\in\mathbb{N}}\prod_{\mu\in\sigma_-(T)}\left(\deg_{\mathcal{V}_i}\right)^{m_i(\mu)}$$

where $T_i = T|_{V_i}$ and $\sigma_-(T)$ denotes the real negative spectrum of T.

Notice that the basic degrees can be effectively computed from (58):

$$\deg_{\mathcal{V}_i} = \sum_{(H)} n_H (H),$$

where

$$n_H = \frac{(-1)^{\dim \mathcal{V}_i^H} - \sum_{H<K} n_K \, n(H,K) \; |W(K)|}{|W(H)|}.$$

The following fact is well-known (see for example [3]).

Lemma C.1. *For any irreducible G-representation $\mathcal{V}$, the basic degree $\deg_{\mathcal{V}} \in A(G)$ is an involutive element of the Burnside Ring, i.e.*

$$(\deg_{\mathcal{V}_j})^2 = \deg_{\mathcal{V}_j} \cdot \deg_{\mathcal{V}_j} = (G).$$

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