# A CONGRUENTIAL RECURRENCE CHARACTERIZES THE INVERSES OF SÓS PERMUTATIONS 

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#### Abstract

In a proof of the three gaps theorem, a class of permutations known as the Sós permutations was introduced. It is known that a Sós permutation, as a sequence, satisfies a certain recurrence (Sós's recurrence), however, whether the converse holds remains unknown. On the other hand, the inverses of Sós permutations have been studied also. It has been reported that such a permutation satisfies a congruential recurrence as a sequence. The converse problem of this fact, i.e., whether a permutation satisfying the congruential recurrence is the inverse of a Sós permutation, is also unsolved, except for a finite number of the degrees of the permutations. This paper relates the set of permutations satisfying the congruential recurrence to other sets of permutations and gives upper bounds for their cardinalities. The upper bounds are in fact tight. In particular, the set of permutations satisfying the congruential recurrence has the same cardinality as that of Sós permutations, giving the affirmative answer to the above unsolved problem as a corollary. As another corollary, it is shown that all permutation, which is regarded as a congruential quasi-progression of diameter 1 in that the set formed by the first order differences modulo the degree of the permutation is a singleton or a set of two successive integers, is the inverse of a Sós permutation with an elementary operation called as shift applied. As an application of these facts, we present a procedure that lifts the set of the inverses of the Sós permutations of a given degree to the set of the same kind with the degree increased by one, without referring to the underlying parameters defining the Sós permutations or the Farey sequence associated to them.


## 1. Introduction

For a positive integer $m$ and a real number $\alpha$, the permutation that sorts the fractional parts of $m$ real numbers $\alpha, 2 \alpha, \ldots, m \alpha$ in the increasing order is referred to as a Sós permutation of degree $m$. This class of permutations was introduced in [1] to prove so-called three gaps theorem and [6] describes Sós permutations in detail. A known property of the Sós permutations is that their terms satisfy a certain recurrence [1, Theorem I], to which we refer as Sós's recurrence in what follows. Another known fact is the existence of Surányi's bijection [2, Satz I], which maps the pair formed by the denominators of two successive terms in the $m$-th Farey sequence to the pair of the first and last terms of a Sós permutation of degree $m$. Thus, if a permutation satisfies Sós's recurrence as a sequence and if its first and last terms form the pair of the denominators of two successive terms in the $m$-th Farey sequence, then it is a Sós permutation. However, it is not known whether a permutation satisfying Sós's recurrence is always a Sós permutation.

The inverses of the Sós permutations have been also studied [7, 8, 6] and it is reported that they satisfy a certain congruential recurrence. The converse problem of this fact, i.e., whether a permutation satisfying the congruential recurrence is the inverse of a Sós permutation, is still unsolved, except that it has been verified using computers for a finite number of the degrees $m$.

This paper presents an upper bound of the cardinality of the set of the permutations satisfying the congruential recurrence, by relating the set to another set of permutations which has a different defining relation and an increased degree. In fact the upper bound is the same as the number of Sós permutations

[^0]which is already known, as a corollary, we obtain the affirmative answer to the converse problem that all permutations satisfying the congruential recurrence are the inverses of Sós ones.

We also obtain another corollary which states, roughly speaking, that all permutation which forms a congruential quasi-progression of diameter 1 essentially comes from the inverse of a Sós permutation.

Furthermore, as an application, we present a lifting procedure which lifts the set of permutations satisfying the congruential recurrence of degree $m-1$ to the set of the same kind whose degree is $m$. Though the resulting set of our procedure is the set of the inverses of Sós permutations of degree $m$, it is not a product made from the underlying parameters $\alpha$ of Sós permutations or the Farey sequence. Only the information of each sequence in the set of degree $m-1$ and integer operations are used by the procedure.

The rest of the paper is organized as follows. Section 2 is preliminary, where we introduce notations and known results on which we will depend. In Section 3, the main results and their proofs are presented. In Section 4, the aforementioned application is described.

## 2. Some known results on the inverses of Sós permutations

In this section, we recall some known properties about the inverses of Sós permutations.
Suppose that $m$ is an integer $\geq 2$. Let $[m]$ denote the set $\{1,2, \ldots, m\}$ of consecutive integers from 1 to $m$. In this paper, the term "permutation" means a bijective map from $[m]$ to $[m]$ and $m$ is referred to as the degree of the permutation. Let $\mathfrak{S}_{m}$ denote the set of all permutations of degree $m$. For a real number $\alpha$, its integral part $\lfloor\alpha\rfloor$ denotes the largest integer that is not larger than $\alpha$, whereas $\{\alpha\}$ denotes the fractional part $\alpha-\lfloor\alpha\rfloor$ of $\alpha$.

In this paper referring to the name of the author of [1] , a permutation $\sigma$ of degree $m$ is said to be a Sós permutation, if there exists a real number $\alpha$ such that

$$
\begin{equation*}
0<\{\sigma(1) \alpha\}<\{\sigma(2) \alpha\}<\cdots<\{\sigma(m) \alpha\}<1 . \tag{1}
\end{equation*}
$$

Hereafter the set of all Sós permutations of degree $m$ is denoted by $\mathcal{S}_{m}$. In addition, let $\mathcal{S}^{*}{ }_{m}$ be the collection of the inverses of them:

$$
\mathcal{S}^{*}{ }_{m}:=\left\{\sigma^{-1} \in \mathfrak{S}_{m}: \sigma \in \mathcal{S}_{m}\right\} .
$$

For the properties of Sós permutations, the readers are referred to [6] as well as [1, 2, 5]. In particular, Surányi's bijection [2, Satz I] describes the crucial correspondence between a Sós permutation $\sigma \in \mathfrak{S}_{m}$ satisfying (1) and the interval formed by consecutive two terms in the Farey sequence into which the real parameter $\alpha$ falls. Since the main interest of the current paper is in the inverses of the Sós permutations rather than themselves, below, the description of the facts/properties of the Sós permutations is kept to the minimum necessary.

For a Boolean predicate $P$, let

$$
\mathbb{1}(P):= \begin{cases}1 & \text { if } P \text { is true } \\ 0 & \text { if } P \text { is false }\end{cases}
$$

be its truth value converted to the integer (i.e., another notation of Iverson's bracket). Now we quote a few known facts on the Sós permutations and their inverses.

Theorem A. ([1, Theorem I]) Suppose that $m \geq 2$ and suppose that $\sigma$ is a Sós permutation of degree $m$. Then

$$
\sigma(i+1)=\sigma(i)+ \begin{cases}\sigma(1) & \text { if } \sigma(i) \leq m-\sigma(1)  \tag{2}\\ \sigma(1)-\sigma(m) & \text { if } m-\sigma(1)<\sigma(i)<\sigma(m) \\ -\sigma(m) & \text { if } \sigma(m) \leq \sigma(i)\end{cases}
$$

holds for $i \in[m-1]$.

It remains unknown whether the converse, i.e., a permutation satisfying the recurrence (2) is a Sós permutation, is true.

On the other hand, the inverses of Sós permutations satisfy a congruential recurrence.
Theorem B. (cf. [7, Corollary 1]) Suppose that $m \geq 2$ and suppose that $\theta$ is the inverse of a Sós permutation of degree $m$. Then

$$
\begin{equation*}
\theta(i+1)-\theta(i) \equiv \theta(1)-\mathbb{1}(\theta(m) \leq \theta(i)) \quad \bmod m \tag{3}
\end{equation*}
$$

holds for $i \in[m-1]$.
In other words, the inverse of a Sós permutation is a slightly generalized congruential arithmetic progression in that the differences $\theta(i+1)-\theta(i)$ take, in modulo $m$, at most two values $\theta(1)$ and $\theta(1)-1$. The first question in this paper is whether the converse of Theorem B is the case. Eq. (3), a requirement on the congruential difference set of a permutation $\theta$, is an interesting property of a permutation in its own right and we define the following set of permutations, not necessarily Sós ones, with this property. For $m \geq 2$, a subset $\mathcal{V}_{m} \subset \mathfrak{S}_{m}$ is defined as

$$
\mathcal{V}_{m}:=\left\{\theta \in \mathfrak{S}_{m}: \theta(i+1)-\theta(i) \equiv \theta(1)-\mathbb{1}(\theta(m) \leq \theta(i)) \quad \bmod m \text { for } i \in[m-1]\right\}
$$

Then, Theorem B is shortly described as

$$
\mathcal{S}_{m}^{*} \subset \mathcal{V}_{m} .
$$

We introduce an action $\lambda$ over permutations and an equivalence relation based on $\lambda$. Let $\operatorname{Mod}_{m}(j)$ be the standard residue of an integer $j$ modulo $m$ taking the value in the set $\{0,1, \ldots, m-1\}$. In addition, let $\overline{\operatorname{Mod}}_{m}(j):=\operatorname{Mod}_{m}(j-1)+1$ be a variant of it, in which 0 of the range is replaced with $m$ keeping the $\bmod m$ class. For $\theta \in \mathfrak{S}_{m}$, the map

$$
\begin{equation*}
\lambda(\theta):=\overline{\operatorname{Mod}}_{m}(\theta(\cdot)+1) \tag{4}
\end{equation*}
$$

is also an element of $\mathfrak{S}_{m}$ (We note that in [6, Lemma 1] the same concept as $\lambda$ is introduced with the notation $c^{-1}$ ). The action $\lambda: \mathfrak{S}_{m} \ni \theta \mapsto \lambda(\theta) \in \mathfrak{S}_{m}$ forms a cyclic group $\langle\lambda\rangle$ by iterative applications $\lambda \circ \cdots \circ \lambda(\theta)$. It is easy to check that $\lambda^{k}(\theta)=\overline{\operatorname{Mod}}_{m}(\theta(\cdot)+k)$ for $k \in \mathbb{Z}$ and that $\lambda^{m}(\theta)=\theta$ for all $\theta \in \mathfrak{S}_{m}$, while $\lambda^{k}(\theta) \neq \lambda^{l}(\theta)$ for $0 \leq k<l<m$ considering that $\lambda^{k}(\theta)(1)(k=0, \ldots, m-1)$ take the $m$ different values of $[m]$. In other words, for any $\theta \in \mathfrak{S}_{m}$ the orbit $\langle\lambda\rangle \cdot \theta$ always contains $m$ different permutations. Any permutation $\lambda^{k}(\theta)$ in the orbit is referred to as a shift of $\theta$. Then, for $\theta, \theta^{\prime} \in \mathfrak{S}_{m}$, a binary relation

$$
\theta \sim \theta^{\prime} \stackrel{\text { def }}{\Leftrightarrow} \theta^{\prime} \text { is a shift of } \theta
$$

is defined. Observe that it has equivalent definitions

$$
\begin{equation*}
\theta \sim \theta^{\prime} \Leftrightarrow \exists k \in[m] \text { s.t. } \theta^{\prime}(i) \equiv \theta(i)+k \quad \bmod m \text { for } i \in[m] \tag{5}
\end{equation*}
$$

or

$$
\theta \sim \theta^{\prime} \Leftrightarrow \theta(i)-\theta(j) \equiv \theta^{\prime}(i)-\theta^{\prime}(j) \quad \bmod m \text { for } i, j \in[m]
$$

and that $\sim$ is an equivalence relation over $\mathfrak{S}_{m}$. We say that $\theta$ and $\theta^{\prime}$ are shift-equivalent if $\theta \sim \theta^{\prime}$. Given an arbitrary subset $T$ of permutations, its closure with respect to the equivalence relation $\sim$ is defined. Formally, the shift-closure operator $\widetilde{\sim}: 2^{\mathfrak{S}_{m}} \rightarrow 2^{\mathfrak{S}_{m}}$ is defined through

$$
\widetilde{T}:=\left\{\theta^{\prime} \in \mathfrak{S}_{m}: \exists \theta \in T \text { s.t. } \theta \sim \theta^{\prime}\right\}, \text { for } T \subset \mathfrak{S}_{m}
$$

Of course, it holds that $\widetilde{\widetilde{T}}=\widetilde{T}$. We also note that $\{\theta \in \widetilde{T}: \theta(1)=1\}$ is a complete representative system of $\widetilde{T} / \sim$, because $\widetilde{\{\theta\}}=\langle\lambda\rangle \cdot \theta$ contains $m$ different permutations whose values $\left\{\lambda^{k}(\theta)(1): k=0,1, \ldots m-1\right\}$ coincide with the set $[m]$ for any $\theta \in \mathfrak{S}_{m}$.

Remark 1. It seems appropriate to mention here that the term Sós permutation in [6] refers to a bijection $\pi:\{0, \ldots, m-1\} \rightarrow\{0, \ldots, m-1\}$ satisfying $0<\{\pi(0) \alpha+\beta\}<\{\pi(1) \alpha+\beta\}<\cdots<\{\pi(m-1) \alpha+\beta\}<1$ for some $\alpha$ and $\beta$, whereas in [7], a permutation $\sigma \in \mathfrak{S}_{m}$ satisfying $0<\{\sigma(1) \alpha+\beta\}<\{\sigma(2) \alpha+\beta\}<$ $\cdots<\{\sigma(m) \alpha+\beta\}<1$ for some $\alpha$ and $\beta$ is referred to as a permutation of Sós-type. They define a class of permutations wider than that of [1] using (1], by the presence of $\beta$. In [7], the distinction of the original and the wider concepts is made by using the term Sós permutations and Sós-type permutations respectively. This paper also inherits this distinction of the terminology. Therefore, a Sós permutation means a permutation of Sós-type with $\beta=0$. By adding the $\beta$ term, a Sós permutation is rotated to form a permutation of Sós-type. Taking the inverse, it turns out that the inverses of the permutations of Sós-type coincide with $\widetilde{\mathcal{S}^{*}{ }_{m}}$. In the following section, it will be revealed that our approach essentially requires introducing both of $\mathcal{S}^{*}{ }_{m}$ and $\widetilde{\mathcal{S}^{*}{ }_{m}}$.

Remark 2. The above definition of Sós-type permutations may remind some readers of Beatty sequences $(\lfloor i \alpha+\beta\rfloor)_{i=1}^{\infty}$ and/or Sturmian words. Indeed, [4] relates Sturmian words and Sós permutations. In contrast, we restrict ourselves to the permutations of a finite degree, in this paper.

In the following, $\phi$ denotes Euler's totient function, i.e., $\phi(x)=|\{a \in[x]: \operatorname{gcd}(a, x)=1\}|$ for any positive integer $x$. Aforementioned Surányi's bijection [2] (see also [5]) relates the pair formed by the denominators of two successive terms in the $m$-th Farey sequence to the pair $(\sigma(1), \sigma(m))$ of the first and last terms of a Sós permutation $\sigma$. An important and direct consequence of this one-to-one correspondence is the cardinality formula $\left|\mathcal{S}_{m}\right|=\sum_{k=1}^{m} \phi(k)$. Taking the inverse, we have the following.

Theorem C. ([2, 5]) Suppose that $m \geq 2$. Then the cardinality of $\mathcal{S}^{*}{ }_{m}$ is $\left|\mathcal{S}^{*}{ }_{m}\right|=\sum_{k=1}^{m} \phi(k)$.
Furthermore, the cardinality of $\widetilde{\mathcal{S}^{*}}{ }_{m}$ has been obtained as follows:
Theorem D. (6, Theorem 4] see also [7, Theorem 5]) Suppose that $m \geq 2$. Then the cardinality of $\widetilde{\mathcal{S}^{*}{ }_{m}}$ is $\left|\widetilde{\mathcal{S}^{*}}{ }_{m}\right|=m \sum_{k=1}^{m-1} \phi(k)$.

We note that the assertion [6, Theorem 4] was stated for Sós-type permutations, i.e., the inverses of the permutations in $\widetilde{\mathcal{S}^{*}{ }_{m}}$. Also, note that $\left|\widetilde{\mathcal{S}^{*}{ }_{m}}\right|$ is not $m$ times $\left|\mathcal{S}^{*}{ }_{m}\right|$.

Lastly, we quote a result with respect to $\widetilde{\mathcal{V}_{m}}$ from [8]: Let

$$
\mathcal{X}_{m}:=\left\{\theta \in \mathfrak{S}_{m}: \exists k \in[m-1] \forall i \in[m-1]\left[\operatorname{Mod}_{m}(\theta(i+1)-\theta(i)) \in\{k, k+1\}\right]\right\},
$$

for which $\mathcal{X}_{m}=\widetilde{\mathcal{X}_{m}}$ obviously holds. An element of $\mathcal{X}_{m}$ may be regarded as a modulo- $m$ version of an $m$-term quasi-progression of diameter 1 . An $m$-term sequence $x_{1}<\cdots<x_{m}$ is said to be an m-term quasi-progression of diameter $d$ [3] if there exists $N$ such that $N \leq x_{i+1}-x_{i} \leq N+d$ for $i \in[m-1]$. We borrow this term removing the increasing condition of the sequence, restricting the range of the sequence to $[m]$ and applying $\operatorname{Mod}_{m}$ to the differences. We say that $x_{1}, \ldots, x_{m} \in[m]$ is a mod-m congruential m-term quasi-progression of diameter $d$ if there exists an integer $N$ such that $N \leq \operatorname{Mod}_{m}\left(x_{i+1}-x_{i}\right) \leq N+d$ for $i \in[m-1]$. Using this term, we can say that $\mathcal{X}_{m}$ consists of all permutations of degree $m$ which are mod- $m$ congruential $m$-term quasi-progressions of diameter 1 . As we noted just after introducing Theorem B, $\theta \in \mathcal{V}_{m}$ satisfies the stricter condition that the differences $\operatorname{Mod}_{m}(\theta(i+1)-\theta(i))$ are $\theta(1)-\mathbb{1}(\theta(m) \leq \theta(i))$. Therefore it holds that $\mathcal{V}_{m} \subset \mathcal{X}_{m}$, from which the inclusion $\widetilde{\mathcal{V}_{m}} \subset \mathcal{X}_{m}$ follows by $\mathcal{X}_{m}=\widetilde{\mathcal{X}_{m}}$. In fact, [8, Corollary 5] asserted the reverse inclusion:

Theorem E. ([8, Corollary 5]) Suppose that $m \geq 2$. Then the set $\mathcal{X}_{m}$ coincides with $\widetilde{\mathcal{V}_{m}}$.
In the next section, our main results will be shown without depending on the known results quoted above, then the corollaries will be shown using both the main results and the known results quoted above.

## 3. MAIN RESUlTS

Before stating our results, we need to introduce two sets, each of which consists of permutations that satisfy their own difference equations. For $m \geq 2$, let
$\mathcal{W}_{m}:=\left\{\theta \in \mathfrak{S}_{m}: \theta(i+1)-\theta(i)=\theta(1)-\mathbb{1}(\theta(m) \leq \theta(i))+m(\mathbb{1}(\theta(i) \leq \theta(i+1))-1)\right.$ for $\left.i \in[m-1]\right\}$, which satisfies the inclusion $\mathcal{W}_{m} \subset \mathcal{V}_{m}$ clearly. Next, for $m \geq 3$, let

$$
\mathcal{Y}_{m}:=\left\{\theta \in \mathfrak{S}_{m}: \Delta_{\theta}(i+1)-\Delta_{\theta}(i)=0 \text { for } i \in[m-2]\right\}
$$

introducing the notation
(6) $\Delta_{\theta}(i):=\theta(i+1)-\theta(i)+\mathbb{1}(\theta(m) \leq \theta(i))-\mathbb{1}(\theta(1) \leq \theta(i+1))-(m-1) \mathbb{1}(\theta(i) \leq \theta(i+1))$ for $\theta \in \mathfrak{S}_{m}$.

In addition, for the case of $m=2$, let $\mathcal{Y}_{2}:=\mathfrak{S}_{2}$. We remark that a set which is defined by a similar but different condition is considered in [9, Section 5.3]. Nevertheless, we need the introduction of $\mathcal{Y}_{m}$ above to approach Theorem 1 below. An important property satisfied by $\mathcal{Y}_{m}$ is the closure property $\widetilde{\mathcal{Y}_{m}}=\mathcal{Y}_{m}$ with respect to the shift, which will be presented as Proposition 2 in Section 3.2.2.

Here we state our main results, Theorems 1 and 2. Their proofs in this paper do not use the properties of $\mathcal{S}_{m}, \mathcal{S}^{*}{ }_{m}$ and $\mathcal{X}_{m}$ in the previous section. In other words, they are formally independent of Sós permutations and the inverses of Sós permutations.

Theorem 1. Suppose that $m \geq 2$. Then

$$
\mathcal{V}_{m}=\mathcal{W}_{m} \quad \text { and } \quad \mathcal{W}_{m} \subset \mathcal{Y}_{m}
$$

Theorem 2. Suppose that $m \geq 2$. Then

$$
\left|\mathcal{V}_{m}\right| \leq \sum_{k=1}^{m} \phi(k) \quad \text { and } \quad\left|\mathcal{Y}_{m}\right|=\left|\widetilde{\mathcal{Y}_{m}}\right| \leq m \sum_{k=1}^{m-1} \phi(k)
$$

By Theorems B, C and Din the previous section, Theorems 1 and 2 immediately produce the following corollary.

Corollary 1. Suppose that $m \geq 2$. Then

$$
\mathcal{S}^{*}{ }_{m}=\mathcal{V}_{m}=\mathcal{W}_{m} \quad \text { and } \quad \widetilde{\mathcal{S}^{*}{ }_{m}}=\mathcal{Y}_{m}=\widetilde{\mathcal{V}_{m}}=\widetilde{\mathcal{W}_{m}}
$$

In particular, the set $\mathcal{S}^{*}{ }_{m}$ of the inverses of Sós permutations of degree $m$ coincides with the set $\mathcal{V}_{m}$, and the set $\widetilde{\mathcal{S}^{*}{ }_{m}}$ of the inverses of permutations of Sós-type of degree $m$ is the set $\mathcal{Y}_{m}$.

Note that Corollary 1 in particular asserts the converse of Theorem B. Therefore now we have: For a permutation $\theta \in \mathfrak{S}_{m}$, satisfying the congruential recurrence Eq. (3) is equivalent to $\sigma:=\theta^{-1}$ being a Sós permutation which is characterized by Ineq. (1) for some real number $\alpha$.

Moreover, Theorem E and Corollary 1 allow one to deduce the following directly.
Corollary 2. Suppose that $m \geq 2$. Then the set $\mathcal{X}_{m}$ is the same as the set $\widetilde{\mathcal{S}^{*}}{ }_{m}$ of the inverses of permutations of Sós-type of degree m:

$$
\mathcal{X}_{m}=\widetilde{\mathcal{S}_{m}^{*}}
$$

In other words, all permutation which forms a mod- $m$ congruential $m$-term quasi-progression of diameter 1 is the inverse of a Sós-type permutation.

In the rest of this section, we prove Theorems 1 and 2 . Our proofs are self-contained.
3.1. Proof of Theorem 1. In this subsection, we show our proof of Theorem 1 by splitting it to the first part $\mathcal{V}_{m}=\mathcal{W}_{m}$ and the second part $\mathcal{W}_{m} \subset \mathcal{Y}_{m}$.
3.1.1. Proof of $\mathcal{V}_{m}=\mathcal{W}_{m}$.

Proof. It is clear that the inclusion relationship $\mathcal{W}_{m} \subset \mathcal{V}_{m}$ holds by the definitions. We will show that the inverted relationship $\mathcal{V}_{m} \subset \mathcal{W}_{m}$ holds.

Let $\theta$ be in $\mathcal{V}_{m}$. Then there exists an integer $k_{\theta, i}$ which satisfies that

$$
\begin{equation*}
\theta(i+1)-\theta(i)-\theta(1)+\mathbb{1}(\theta(m) \leq \theta(i))=k_{\theta, i} m \tag{7}
\end{equation*}
$$

for $i \in[m-1]$. We divide the LHS of Eq. (7) into two parts. Each part satisfies that

$$
\begin{equation*}
1-m \leq \theta(i+1)-\theta(i) \leq m-1 \text { and } 0 \leq \theta(1)-\mathbb{1}(\theta(m) \leq \theta(i)) \leq m \tag{8}
\end{equation*}
$$

because $\theta(1), \theta(i)$ and $\theta(i+1) \in[m]$.
Thus, the RHS of Eq. (7) satisfies that $1-m-m \leq k_{\theta, i} m \leq m-1-0$, that is,

$$
-2+\frac{1}{m} \leq k_{\theta, i} \leq 1-\frac{1}{m}
$$

which show bounds on $k_{\theta, i}$. It follows that the integer $k_{\theta, i}$ must be either -1 or 0 . We consider each case as follows.

When $k_{\theta, i}=-1$, Eq. (7) implies that

$$
\theta(i+1)-\theta(i)=\theta(1)-\mathbb{1}(\theta(m) \leq \theta(i))-m
$$

whose RHS is at most 0 , because it satisfies $\theta(1)-\mathbb{1}(\theta(m) \leq \theta(i))-m \leq m-m$ by the second inequalities of (8). By $\theta(i+1) \neq \theta(i)$, we obtain $\theta(i+1)-\theta(i) \leq-1$, that is, $\mathbb{1}(\theta(i) \leq \theta(i+1))=0$, which implies $k_{\theta, i}=\mathbb{1}(\theta(i) \leq \theta(i+1))-1$ in this case of $k_{\theta, i}=-1$.

The case of $k_{\theta, i}=0$ is similar: Eq. (7) means that $\theta(i+1)-\theta(i)=\theta(1)-\mathbb{1}(\theta(m) \leq \theta(i))$, whose RHS is at least 0 , because of the second inequalities of (8). By $\theta(i+1) \neq \theta(i)$, we have $\theta(i+1)-\theta(i) \geq 1$, that is, $\mathbb{1}(\theta(i) \leq \theta(i+1))=1$ and then $k_{\theta, i}=\mathbb{1}(\theta(i) \leq \theta(i+1))-1$ in this case of $k_{\theta, i}=0$.

In both cases,

$$
\theta(i+1)-\theta(i)=\theta(1)-\mathbb{1}(\theta(m) \leq \theta(i))+m(\mathbb{1}(\theta(i) \leq \theta(i+1))-1)
$$

holds. Therefore we conclude that $\theta \in \mathcal{W}_{m}$ which implies $\mathcal{V}_{m} \subset \mathcal{W}_{m}$.
3.1.2. Proof of $\mathcal{W}_{m} \subset \mathcal{Y}_{m}$.

Proof. Let $\theta$ be in $\mathcal{W}_{m}$. By the definition of $\mathcal{W}_{m}$,

$$
\begin{equation*}
\theta(i+1)-\theta(i)+\mathbb{1}(\theta(m) \leq \theta(i))-m(\mathbb{1}(\theta(i) \leq \theta(i+1))-1)=\theta(1) \tag{9}
\end{equation*}
$$

holds for $i \in[m-1]$. In particular, by the special case

$$
\theta(2)-\theta(1)+\mathbb{1}(\theta(m) \leq \theta(1))-m(\mathbb{1}(\theta(1) \leq \theta(2))-1)=\theta(1)
$$

for $i=1$, we have

$$
\begin{aligned}
\theta(i+1)-\theta(i)+\mathbb{1}(\theta(m) & \leq \theta(i))-\mathbb{1}(\theta(i) \leq \theta(i+1))-(m-1) \mathbb{1}(\theta(i) \leq \theta(i+1)) \\
& =\theta(2)-\theta(1)+\mathbb{1}(\theta(m) \leq \theta(1))-\mathbb{1}(\theta(1) \leq \theta(2))-(m-1) \mathbb{1}(\theta(1) \leq \theta(2)) .
\end{aligned}
$$

By using the notation $\Delta_{\theta}$ of Eq. (6), the last equality is equivalent to

$$
\Delta_{\theta}(i)-\mathbb{1}(\theta(i) \leq \theta(i+1))+\mathbb{1}(\theta(1) \leq \theta(i+1))=\Delta_{\theta}(1)
$$

for $i \in[m-1]$. Now we claim that

$$
\begin{equation*}
\mathbb{1}(\theta(i) \leq \theta(i+1))=\mathbb{1}(\theta(1) \leq \theta(i+1)) \tag{10}
\end{equation*}
$$

holds for $i \in[m-1]$. If our claim, Eq. 10 , is valid, then we have $\Delta_{\theta}(i)=\Delta_{\theta}(1)$ or equivalently $\Delta_{\theta}(i+1)=\Delta_{\theta}(i)$ for $i \in[m-2]$, that is, $\theta \in \mathcal{Y}_{m}$ which implies $\mathcal{W}_{m} \subset \mathcal{Y}_{m}$.

Let us show the validity of our claim, that is, Eq. 10 holds for $i \in[m-1]$ as follows. For $i \in[m-1]$, Eq. (9) is equivalent to

$$
\begin{equation*}
\theta(i+1)-\theta(1)=\theta(i)-\mathbb{1}(\theta(m) \leq \theta(i))+m(\mathbb{1}(\theta(i) \leq \theta(i+1))-1) . \tag{11}
\end{equation*}
$$

Because $\theta \in \mathfrak{S}_{m}$ and $\mathbb{1}(\theta(m) \leq \theta(i)) \in\{0,1\}$, we have $0 \leq \theta(i)-\mathbb{1}(\theta(m) \leq \theta(i)) \leq m$. If $\theta(i)-\mathbb{1}(\theta(m) \leq$ $\theta(i))=0$, then both $\theta(i)=1$ and $\mathbb{1}(\theta(m) \leq \theta(i))=1$ hold. That is, $\theta(i)=1$ and $\theta(m) \leq \theta(i)$. On the other hand, because $\theta(m) \neq \theta(i)$ for $i \in[m-1]$, it is impossible that $\theta(m) \leq \theta(i)$ for $\theta \in \mathfrak{S}_{m}$. Therefore $1 \leq \theta(i)-\mathbb{1}(\theta(m) \leq \theta(i))$. Similarly, if $\theta(i)-\mathbb{1}(\theta(m) \leq \theta(i))=m$, then both $\theta(i)=m$ and $\mathbb{1}(\theta(m) \leq \theta(i))=0$ hold, that is, $\theta(i)=m$ and $\theta(m)>\theta(i)$. But, by $\theta(m) \neq \theta(i)$ for $i \in[m-1]$, it is impossible that $\theta(m)>\theta(i)$ for $\theta \in \mathfrak{S}_{m}$. Then $\theta(i)-\mathbb{1}(\theta(m) \leq \theta(i)) \leq m-1$.

Consequently, the inequalities

$$
\begin{equation*}
1 \leq \theta(i)-\mathbb{1}(\theta(m) \leq \theta(i)) \leq m-1 \tag{12}
\end{equation*}
$$

hold for $i \in[m-1]$. Since $\mathbb{1}(\theta(i) \leq \theta(i+1))$ is 0 or 1 , we have to consider each case as follows. When $\mathbb{1}(\theta(i) \leq \theta(i+1))=1$, Eq. (11) means that $\theta(i+1)-\theta(1)=\theta(i)-\mathbb{1}(\theta(m) \leq \theta(i))$. Then from the inequalities (12), it follows that $\theta(i+1)-\theta(1) \geq 1$, that is, $\mathbb{1}(\theta(1) \leq \theta(i+1))=1$. When $\mathbb{1}(\theta(i) \leq \theta(i+1))=0$ is the case, Eq. (11) means that $\theta(i+1)-\theta(1)=\theta(i)-\mathbb{1}(\theta(m) \leq \theta(i))-m$ and from the inequalities 12 , it follows that $\theta(i+1)-\theta(1) \leq-1$, that is, $\mathbb{1}(\theta(1) \leq \theta(i+1))=0$.

In either case, Eq. 10] holds for $i \in[m-1]$. Therefore our claim is valid.
3.2. Some preliminaries and proof of Theorem 2. Our proof of Theorem 2 requires some preliminaries.
3.2.1. An embedding of $\mathcal{W}_{m-1}$ into $\mathcal{Y}_{m}$. We suppose that $m \geq 2$. It is common to embed the set of permutations of degree $m-1$ into the set of permutations of degree $m$ having one particular fixed point, say 1. Here we introduce an explicit notation for such an embedding and its inverse. Let $\mathfrak{S}_{m}^{1}$ denote

$$
\mathfrak{S}_{m}^{1}:=\left\{\theta \in \mathfrak{S}_{m}: \theta(1)=1\right\} .
$$

We introduce the following map $\Psi_{m}: \mathfrak{S}_{m}^{1} \rightarrow \mathfrak{S}_{m-1}$. For $\theta \in \mathfrak{S}_{m}^{1}, \Psi_{m}(\theta) \in \mathfrak{S}_{m-1}$ is defined by

$$
\Psi_{m}(\theta)(i):=\theta(i+1)-1 \text { for } i \in[m-1] .
$$

It is easy to check that $\Psi_{m}: \mathfrak{S}_{m}^{1} \rightarrow \mathfrak{S}_{m-1}$ is well-defined and bijective where the inverse $\Psi_{m}^{-1}: \mathfrak{S}_{m-1} \rightarrow$ $\mathfrak{S}_{m}^{1}$ satisfies, for given $\pi \in \mathfrak{S}_{m-1}$, that

$$
\Psi_{m}^{-1}(\pi)(i):=\left\{\begin{array}{ll}
1 & \text { if } i=1, \\
\pi(i-1)+1 & \text { if } i \geq 2
\end{array} \quad \text { for } i \in[m] .\right.
$$

The following Proposition 1 is similar to [9, Theorem 37]. We give its proof as below.
Proposition 1. Suppose that $m \geq 3$. Then $\mathcal{W}_{m-1}$ is the bijective image of $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ by $\Psi_{m}$ :

$$
\Psi_{m}\left(\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}\right)=\mathcal{W}_{m-1}
$$

Proof. The bijectivity as the map $\Psi_{m}: \mathfrak{S}_{m}^{1} \rightarrow \mathfrak{S}_{m-1}$ has been mentioned. Below, the equality of the image and $\mathcal{W}_{m-1}$ is shown.

First, we show that $\Psi_{m}\left(\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}\right) \supset \mathcal{W}_{m-1}$. It is enough to show that $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m} \supset \Psi_{m}^{-1}\left(\mathcal{W}_{m-1}\right)$ by using the inverse map $\Psi_{m}^{-1}: \mathfrak{S}_{m-1} \rightarrow \mathfrak{S}_{m}^{1}$ of $\Psi_{m}$. Let $\pi$ be in $\mathcal{W}_{m-1}$. Then it satisfies that

$$
\begin{equation*}
\pi(i+1)-\pi(i)=\pi(1)-\mathbb{1}(\pi(m-1) \leq \pi(i))+(m-1)(\mathbb{1}(\pi(i) \leq \pi(i+1))-1) \tag{13}
\end{equation*}
$$

for $i \in[m-2]$. That is,

$$
\begin{equation*}
\pi(i)-\pi(i-1)=\pi(1)-\mathbb{1}(\pi(m-1) \leq \pi(i-1))+(m-1)(\mathbb{1}(\pi(i-1) \leq \pi(i))-1) \tag{14}
\end{equation*}
$$

for $i \in[m-1]$ with $i \neq 1$. We put $\theta:=\Psi_{m}^{-1}(\pi)$. Since $\theta(i)-1=\pi(i-1)$ for $i \in[m-1]$ with $i \neq 1$, Eq. (14) means that

$$
\begin{equation*}
\theta(i+1)-\theta(i)=\theta(2)-1-\mathbb{1}(\theta(m) \leq \theta(i))+(m-1)(\mathbb{1}(\theta(i) \leq \theta(i+1))-1) . \tag{15}
\end{equation*}
$$

To ease the rest of our argument, let us rewrite it as

$$
\begin{equation*}
\theta(i+1)-\theta(i)+\mathbb{1}(\theta(m) \leq \theta(i))-1-(m-1) \mathbb{1}(\theta(i) \leq \theta(i+1))=\theta(2)-1+0-1-(m-1) . \tag{16}
\end{equation*}
$$

By $i+1 \geq 2$ and by $\theta(1)=1, \theta(i+1) \geq 2$ etc., we have $\mathbb{1}(\theta(1) \leq \theta(i+1))=1, \mathbb{1}(\theta(1) \leq \theta(2))=1$, $\mathbb{1}(\theta(m) \leq \theta(1))=0, \mathbb{1}(\theta(1) \leq \theta(2))=1$. Therefore Eq. (16) is the same as

$$
\begin{align*}
\theta(i+1)-\theta(i)+\mathbb{1}(\theta(m) \leq \theta(i))-\mathbb{1}(\theta(1) \leq \theta(i+1))-(m-1) \mathbb{1}(\theta(i) \leq \theta(i+1))  \tag{17}\\
=\theta(2)-\theta(1)+\mathbb{1}(\theta(m) \leq \theta(1))-\mathbb{1}(\theta(1) \leq \theta(2))-(m-1) \mathbb{1}(\theta(1) \leq \theta(2)),
\end{align*}
$$

which, with Eq. (6), shows that $\theta=\Psi_{m}^{-1}(\pi)$ satisfies $\Delta_{\theta}(i)=\Delta_{\theta}(1)$ for $i \in[m-1]$. Thus, it follows that $\Delta_{\theta}(i+1)=\Delta_{\theta}(i)$ for $i \in[m-2]$ and then $\theta \in \mathcal{Y}_{m}$, which, with $\theta(1)=1$, implies that $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m} \supset$ $\Psi_{m}^{-1}\left(\mathcal{W}_{m-1}\right)$.

We now show the reverse inclusion $\Psi_{m}\left(\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}\right) \subset \mathcal{W}_{m-1}$. The argument below simply follows the path of the above argument backward. Let $\theta \in \mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$, then $\theta$ satisfies $\Delta_{\theta}(i+1)=\Delta_{\theta}(i)$ for $i \in[m-2]$, that is, $\Delta_{\theta}(i)=\Delta_{\theta}(1)$ for $i \in[m-1]$ with $\theta(1)=1$. So $\theta$ satisfies Eq. (17) with $\theta(1)=1$. Since $\theta \in \mathfrak{S}_{m}$ and $\theta(1)=1$, we have $\theta(i+1) \geq 2$ for $i+1 \geq 2$. From the facts that $\mathbb{1}(\theta(1) \leq \theta(i+1))=1$, $\mathbb{1}(\theta(m) \leq \theta(1))=0$, etc., we have Eq. (16). Thus, $\theta \in \mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ satisfies Eq. (15) for $i \in[m-1]$ and $\theta(1)=1$. Put $\pi:=\Psi_{m}(\theta)$, then $\pi(i)=\theta(i+1)-1$ for $i \in[m-1]$. Therefore Eq. (15) implies Eq. (14), i.e., Eq. (13) for $i \in[m-2]$, which implies $\pi \in \mathcal{W}_{m-1}$.
3.2.2. The closure property of $\mathcal{Y}_{m}$ with respect to the shift. Our proof of Theorem 2 will use the property that $\mathcal{Y}_{m}$ is closed about the shift-equivalence. In [9, Proposition 35], similar properties are considered in various generalized settings. We give a proof of what we need as follows. Suppose that $m \geq 2$.

For a given permutation $\theta \in \mathfrak{S}_{m}$, we put the number of its ascents as

$$
A_{\theta}:=\sum_{j=1}^{m-1} \mathbb{1}(\theta(j) \leq \theta(j+1)),
$$

then use it to define a set

$$
\mathcal{Y}_{m}^{\prime}:=\left\{\theta \in \mathfrak{S}_{m}: \Delta_{\theta}(i)+A_{\theta}=0 \text { for } i \in[m-1]\right\}
$$

of permutations. The condition $\Delta_{\theta}(i)=-A_{\theta}$ is a special case of the first equality in [9, Proposition 21]. The following Lemma 1 states that this condition is in fact equivalent to (though sounds stricter than) the defining condition of $\mathcal{Y}_{m}$.

Lemma 1. Suppose that $m \geq 3$. Then $\mathcal{Y}_{m}^{\prime}=\mathcal{Y}_{m}$.
Proof. Let $\theta$ be in $\mathcal{Y}_{m}^{\prime}$. Then $\Delta_{\theta}(i+1)=-A_{\theta}=\Delta_{\theta}(i)$ for $i \in[m-2]$, that is, $\theta \in \mathcal{Y}_{m}$ which implies $\mathcal{Y}_{m}^{\prime} \subset \mathcal{Y}_{m}$. Let us show $\mathcal{Y}_{m}^{\prime} \supset \mathcal{Y}_{m}$. In the general case of $\theta \in \mathfrak{S}_{m}, \sum_{i=1}^{m-1} \mathbb{1}(\theta(m)>\theta(i))$ is the number of $\theta(1), \ldots, \theta(m-1)$ that is less than $\theta(m)$. It is equal to $\theta(m)-1$. Similarly, $\sum_{i=1}^{m-1} \mathbb{1}(\theta(1)>\theta(i+1))$ is equal to $\theta(1)-1$. Hence

$$
\begin{aligned}
& \sum_{i=1}^{m-1} \Delta_{\theta}(i)=\theta(m)-\theta(1)+\sum_{i=1}^{m-1} \mathbb{1}(\theta(m) \leq \theta(i))-\sum_{i=1}^{m-1} \mathbb{1}(\theta(1) \leq \theta(i+1))-(m-1) \sum_{i=1}^{m-1} \mathbb{1}(\theta(i) \leq \theta(i+1)) \\
&= \theta(m)-\theta(1)+\sum_{i=1}^{m-1}(1-\mathbb{1}(\theta(m)>\theta(i)))-\sum_{i=1}^{m-1}(1-\mathbb{1}(\theta(1)>\theta(i+1)))-(m-1) \sum_{i=1}^{m-1} \mathbb{1}(\theta(i) \leq \theta(i+1)) \\
&=-(m-1) \sum_{i=1}^{m-1} \mathbb{1}(\theta(i) \leq \theta(i+1))=-(m-1) A_{\theta} .
\end{aligned}
$$

Let $\theta$ be in $\mathcal{Y}_{m}$. Because $\Delta_{\theta}(i)=\Delta_{\theta}(1)$ for $i \in[m-1]$, we have $\sum_{i=1}^{m-1} \Delta_{\theta}(i)=(m-1) \Delta_{\theta}(1)$. It follows that $\Delta_{\theta}(1)=-A_{\theta}$, that is, $\Delta_{\theta}(i)=-A_{\theta}$ for $i \in[m-1]$, which implies $\theta \in \mathcal{Y}_{m}^{\prime}$.

The above alternative definition of $\mathcal{Y}_{m}$ helps proving its closure property with respect to the shift. We give its proof as below.

Proposition 2. Suppose that $m \geq 3$. If $\theta^{\prime} \in \mathfrak{S}_{m}$ satisfies that $\theta^{\prime} \sim \theta$ for some $\theta \in \mathcal{Y}_{m}$, then $\theta^{\prime} \in \mathcal{Y}_{m}$. That is, $\widetilde{\mathcal{Y}_{m}}=\mathcal{Y}_{m}$.
Proof. It is obvious that $\widetilde{\mathcal{Y}_{m}} \supset \mathcal{Y}_{m}$. We prove that $\widetilde{\mathcal{Y}_{m}} \subset \mathcal{Y}_{m}$.
For $\theta \in \mathfrak{S}_{m}$, let $\theta^{\prime}=\lambda(\theta)$ be the shift-by-one of $\theta$ defined in Eq. (4), then it satisfies that

$$
\theta^{\prime}(i)=\left\{\begin{array}{ll}
\theta(i)+1 & \text { if } \theta(i) \leq m-1, \\
1 & \text { if } \theta(i)=m
\end{array} \quad \text { for } i \in[m]\right.
$$

To prove $\widetilde{\mathcal{Y}_{m}} \subset \mathcal{Y}_{m}$, it is enough to show that $\widetilde{\{\theta\}} \subset \mathcal{Y}_{m}$ for each $\theta \in \mathcal{Y}_{m}$. Since $\widetilde{\{\theta\}}=\langle\lambda\rangle \cdot \theta$, showing that $\theta^{\prime}=\lambda(\theta) \in \mathcal{Y}_{m}$ suffices.

First of all, we show that

$$
\begin{equation*}
A_{\theta}+\mathbb{1}(\theta(m) \leq \theta(1))=A_{\theta^{\prime}}+\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(1)\right) \tag{18}
\end{equation*}
$$

for $\theta \in \mathfrak{S}_{m}$. We introduce the shorthand $(j)_{m}:=\overline{\operatorname{Mod}}_{m}(j)$ for integers $j$. Then

$$
(j)_{m}=\left\{\begin{array}{lll}
\operatorname{Mod}_{m}(j) & \text { if } j \not \equiv 0 & \bmod m \\
m & \text { if } j \equiv 0 & \bmod m
\end{array}\right.
$$

By using this notation, the LHS and the RHS of Eq. 18) are written as

$$
\begin{equation*}
\sum_{j \in[m]} \mathbb{1}\left(\theta\left((j)_{m}\right) \leq \theta\left((j+1)_{m}\right)\right), \quad \sum_{j \in[m]} \mathbb{1}\left(\theta^{\prime}\left((j)_{m}\right) \leq \theta^{\prime}\left((j+1)_{m}\right)\right) \tag{19}
\end{equation*}
$$

respectively. For the validity of Eq. (18), let us show that both values of (19) are equal as follows.
We put $k:=\theta^{-1}(m)$, i.e., $k$ satisfies $k \in[m]$ and $\theta(k)=m$. Using $k$, the first sum in (19) is split to 3 parts, $j=(k-1)_{m}, j=k$ and others, namely

$$
\mathbb{1}\left(\theta\left((k-1)_{m}\right) \leq \theta\left((k)_{m}\right)\right)+\mathbb{1}\left(\theta\left((k)_{m}\right) \leq \theta\left((k+1)_{m}\right)\right)+\sum_{j \in[m] \backslash\left\{k,(k-1)_{m}\right\}} \mathbb{1}\left(\theta\left((j)_{m}\right) \leq \theta\left((j+1)_{m}\right)\right) .
$$

The second sum for $\theta^{\prime}$ in (19) also allows the same decomposition. In evaluating the first and the second parts with respect to $\theta$, since both $\theta\left((k-1)_{m}\right)$ and $\theta\left((k+1)_{m}\right)$ are different values from $\theta(k)$, they are less than $m=\theta(k)$. Similarly, for the corresponding parts with respect to $\theta^{\prime}$, both $\theta^{\prime}\left((k-1)_{m}\right)$ and $\theta^{\prime}\left((k+1)_{m}\right)$ are greater than $1=\theta^{\prime}(k)$. Then, for $\theta$, the sum $\mathbb{1}\left(\theta\left((k-1)_{m}\right) \leq \theta\left((k)_{m}\right)\right)+\mathbb{1}\left(\theta\left((k)_{m}\right) \leq\right.$ $\left.\theta\left((k+1)_{m}\right)\right)$ of the first and the second parts is equal to $\mathbb{1}\left(\theta\left((k-1)_{m}\right) \leq m\right)+\mathbb{1}\left(m \leq \theta\left((k+1)_{m}\right)\right)$. For $\theta^{\prime}$, the sum $\mathbb{1}\left(\theta^{\prime}\left((k-1)_{m}\right) \leq \theta^{\prime}\left((k)_{m}\right)\right)+\mathbb{1}\left(\theta^{\prime}\left((k)_{m}\right) \leq \theta^{\prime}\left((k+1)_{m}\right)\right)$ of the corresponding two terms is equal to $\mathbb{1}\left(\theta^{\prime}\left((k-1)_{m}\right) \leq 1\right)+\mathbb{1}\left(1 \leq \theta^{\prime}\left((k+1)_{m}\right)\right)$. All of these are equal to 1 .

For the third part, neither $(j)_{m}$ nor $(j+1)_{m}$ is $k$. Thus, for this part with respect to $\theta^{\prime}$, each term $\mathbb{1}\left(\theta^{\prime}\left((j)_{m}\right) \leq \theta^{\prime}\left((j+1)_{m}\right)\right)$ is the same as $\mathbb{1}\left(\theta\left((j)_{m}\right)+1 \leq \theta\left((j+1)_{m}\right)+1\right)$ which is equal to $\mathbb{1}\left(\theta\left((j)_{m}\right) \leq \theta\left((j+1)_{m}\right)\right)$. Therefore we conclude that both values of (19) are equal, that is, Eq. 18) is valid.

By Lemma 1 and Eq. 18], for the validity of " $\theta \in \mathcal{Y}_{m}$ implies $\theta^{\prime} \in \mathcal{Y}_{m}$ ", it is enough to show that

$$
\Delta_{\theta}(i)-\mathbb{1}(\theta(m) \leq \theta(1))=\Delta_{\theta^{\prime}}(i)-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(1)\right)
$$

or an equivalent equality, in the expanded and rearranged form,

$$
\begin{align*}
& \left(\theta(i+1)-\theta^{\prime}(i+1)\right)-\left(\theta(i)-\theta^{\prime}(i)\right)-(m-1)\left(\mathbb{1}(\theta(i) \leq \theta(i+1))-\mathbb{1}\left(\theta^{\prime}(i) \leq \theta^{\prime}(i+1)\right)\right)  \tag{20}\\
& \quad=\left(\mathbb{1}(\theta(m) \leq \theta(1))-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(1)\right)\right) \\
& -\left(\mathbb{1}(\theta(m) \leq \theta(i))-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right)\right)+\left(\mathbb{1}(\theta(1) \leq \theta(i+1))-\mathbb{1}\left(\theta^{\prime}(1) \leq \theta^{\prime}(i+1)\right)\right)
\end{align*}
$$

holds for $i \in[m-1]$.
Let us consider it for all possible cases separately. First, we check what kind of cases can occur. For each $i \in[m-1]$, we put $A d j_{i}:=(\mathbb{1}(\theta(i)=m), \mathbb{1}(\theta(i+1)=m))$. Then there are three cases of $A d j_{i}$ :
$(0,0),(1,0)$ and $(0,1)$. We also put $H T:=(\mathbb{1}(\theta(1)=m), \mathbb{1}(\theta(m)=m))$. Then there are also three cases of HT: $(0,0),(1,0)$ and $(0,1)$. Therefore there are 9 possible cases, from Case A to Case I, in the following table

| $H T \backslash A d j_{i}$ | $(0,0)$ | $(1,0)$ | $(0,1)$ |
| :---: | :---: | :---: | :---: |
| $(0,0)$ | A | B | C |
| $(1,0)$ | D | E | F |
| $(0,1)$ | G | H | I |

but Case F and Case H are impossible. So, we consider the remaining 7 cases below.
Before considering these 7 cases, we will find the value of the LHS of Eq. 20) in each of the three possible cases of $A d j_{i}$.
The case of $A d j_{i}=(0,0)$ : In this case, it follows that $\theta^{\prime}(i)=\theta(i)+1$ and $\theta^{\prime}(i+1)=\theta(i+1)+1$. Then the LHS of 20 is 0 .
The case of $A d j_{i}=(1,0)$ : In this case, it follows that $\theta(i)=m, \theta^{\prime}(i)=1$ and $\theta^{\prime}(i+1)=\theta(i+1)+1$. Then the LHS of 20 is

$$
\begin{aligned}
& \left(\theta(i+1)-\theta^{\prime}(i+1)\right)-\left(\theta(i)-\theta^{\prime}(i)\right)-(m-1)\left(\mathbb{1}(\theta(i) \leq \theta(i+1))-\mathbb{1}\left(\theta^{\prime}(i) \leq \theta^{\prime}(i+1)\right)\right) \\
& \quad=-1-(m-1)-(m-1)\left(\mathbb{1}(m \leq \theta(i+1))-\mathbb{1}\left(1 \leq \theta^{\prime}(i+1)\right)\right)=-1-(m-1)-(m-1)(0-1)
\end{aligned}
$$

which is equal to -1 .
The case of $A d j_{i}=(0,1)$ : In this case, it follows that $\theta^{\prime}(i)=\theta(i)+1, \theta(i+1)=m$ and $\theta^{\prime}(i+1)=1$. Then the LHS of 20 is

$$
(m-1)-(-1)-(m-1)\left(\mathbb{1}(\theta(i) \leq m)-\mathbb{1}\left(\theta^{\prime}(i) \leq 1\right)\right)=(m-1)-(-1)-(m-1)(1-0)
$$

which is equal to 1 .
Using the value for each of the three cases above, let us show the validity of Eq. (20) for 7 cases separately as follows.
Case A: In this case, it follows that $\theta^{\prime}(i)=\theta(i)+1, \theta^{\prime}(i+1)=\theta(i+1)+1, \theta^{\prime}(1)=\theta(1)+1$ and $\theta^{\prime}(m)=\theta(m)+1$. Then the RHS of Eq. 20) is

$$
\begin{aligned}
& \left(\mathbb{1}(\theta(m) \leq \theta(1))-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(1)\right)\right) \\
& \quad-\left(\mathbb{1}(\theta(m) \leq \theta(i))-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right)\right)+\left(\mathbb{1}(\theta(1) \leq \theta(i+1))-\mathbb{1}\left(\theta^{\prime}(1) \leq \theta^{\prime}(i+1)\right)\right)
\end{aligned}
$$

$$
=0-0+0
$$

which is equal to the LHS of Eq. (20). Therefore Eq. 20 holds.
Case B: In this case, it follows that $\theta(i)=m, \theta^{\prime}(i)=1, \theta^{\prime}(i+1)=\theta(i+1)+1, \theta^{\prime}(1)=\theta(1)+1$, $\theta^{\prime}(m)=\theta(m)+1$ and that $\theta(m)$ is not $m$, i.e., $\theta^{\prime}(m)$ is not 1 . Then the RHS of Eq. 20) is

$$
0-\left(\mathbb{1}(\theta(m) \leq m)-\mathbb{1}\left(\theta^{\prime}(m) \leq 1\right)\right)+0=0-(1-0)+0
$$

which is equal to the LHS of Eq. (20).
Case C: In this case, it follows that $\theta^{\prime}(i)=\theta(i)+1, \theta(i+1)=m, \theta^{\prime}(i+1)=1, \theta^{\prime}(1)=\theta(1)+1$, $\theta^{\prime}(m)=\theta(m)+1$ and that $\theta(1)$ is not $m$, i.e., $\theta^{\prime}(1)$ is not 1 . Then the RHS of Eq. (20) is

$$
0-0+\left(\mathbb{1}(\theta(1) \leq m)-\mathbb{1}\left(\theta^{\prime}(1) \leq 1\right)\right)=(1-0)
$$

which is equal to the LHS of Eq. 20).
Case D: In this case, it follows that $\theta^{\prime}(i)=\theta(i)+1, \theta^{\prime}(i+1)=\theta(i+1)+1, \theta(1)=m, \theta^{\prime}(1)=1$, $\theta^{\prime}(m)=\theta(m)+1$ and that $\theta(i+1)$ is not $m$, i.e., $\theta^{\prime}(i+1)$ is not 1 . Similarly, $\theta(m)$ is not $m$, i.e., $\theta^{\prime}(m)$ is not 1. Therefore the RHS of Eq. (20) is

$$
\left(\mathbb{1}(\theta(m) \leq m)-\mathbb{1}\left(\theta^{\prime}(m) \leq 1\right)\right)-0+\left(\mathbb{1}(m \leq \theta(i+1))-\mathbb{1}\left(1 \leq \theta^{\prime}(i+1)\right)\right)=(1-0)-0+(0-1)
$$

which is equal to the LHS of Eq. (20).

Case E: In this case, it follows that $\theta(i)=m, \theta^{\prime}(i)=1, \theta^{\prime}(i+1)=\theta(i+1)+1, \theta(1)=m, \theta^{\prime}(1)=1$, $\theta^{\prime}(m)=\theta(m)+1$ and that $\theta(i+1)$ is not $m$, i.e., $\theta^{\prime}(i+1)$ is not 1 . Similarly, $\theta(m)$ is not $m$, i.e., $\theta^{\prime}(m)$ is not 1. Therefore the RHS of Eq. (20) is

$$
\begin{aligned}
\left(\mathbb{1}(\theta(m) \leq m)-\mathbb{1}\left(\theta^{\prime}(m) \leq 1\right)\right) & -\left(\mathbb{1}(\theta(m) \leq m)-\mathbb{1}\left(\theta^{\prime}(m) \leq 1\right)\right) \\
& +\left(\mathbb{1}(m \leq \theta(i+1))-\mathbb{1}\left(1 \leq \theta^{\prime}(i+1)\right)\right)=(1-0)-(1-0)+(0-1)
\end{aligned}
$$

which is equal to the LHS of Eq. (20).
Case G: In this case, it follows that $\theta^{\prime}(i)=\theta(i)+1, \theta^{\prime}(i+1)=\theta(i+1)+1, \theta^{\prime}(1)=\theta(1)+1, \theta(m)=m$, $\theta^{\prime}(m)=1$ and that $\theta(1)$ is not $m$, i.e., $\theta^{\prime}(1)$ is not 1 . Similarly, $\theta(i)$ is not $m$, i.e., $\theta^{\prime}(i)$ is not 1 . Therefore the RHS of Eq. (20) is

$$
\begin{aligned}
\left(\mathbb{1}(m \leq \theta(1))-\mathbb{1}\left(1 \leq \theta^{\prime}(1)\right)\right)- & \left(\mathbb{1}(m \leq \theta(i))-\mathbb{1}\left(1 \leq \theta^{\prime}(i)\right)\right) \\
& +\left(\mathbb{1}(\theta(1) \leq \theta(i+1))-\mathbb{1}\left(\theta^{\prime}(1) \leq \theta^{\prime}(i+1)\right)\right)=(0-1)-(0-1)+0
\end{aligned}
$$

which is equal to the LHS of Eq. 20).
Case I: In this case, it follows that $\theta^{\prime}(i)=\theta(i)+1, \theta(i+1)=m, \theta^{\prime}(i+1)=1, \theta^{\prime}(1)=\theta(1)+1, \theta(m)=m$, $\theta^{\prime}(m)=1$ and that $\theta(i)$ is not $m$, i.e., $\theta^{\prime}(i)$ is not 1 . Similarly, $\theta(1)$ is not $m$, i.e., $\theta^{\prime}(1)$ is not 1 . Therefore the RHS of Eq. (20) is

$$
\begin{aligned}
\left(\mathbb{1}(m \leq \theta(1))-\mathbb{1}\left(1 \leq \theta^{\prime}(1)\right)\right)-(\mathbb{1}(m & \left.\leq \theta(i))-\mathbb{1}\left(1 \leq \theta^{\prime}(i)\right)\right) \\
& +\left(\mathbb{1}(\theta(1) \leq m)-\mathbb{1}\left(\theta^{\prime}(1) \leq 1\right)\right)=(0-1)-(0-1)+(1-0)
\end{aligned}
$$

which equal to the LHS of Eq. (20).
From the above arguments, Eq. 200 holds for all possible cases and for each $i \in[m-1]$. The proof is completed.
3.2.3. A property of $\mathcal{V}_{m}$. In this subsection, some key facts on the structure of the set $\mathcal{V}_{m}$ are shown. For any elements $a, b$ in the ring $\mathbb{Z}$ of integers, let

$$
\begin{equation*}
\theta_{a, b}(i):=\overline{\operatorname{Mod}}_{m}(a i+b) \text { for } i \in[m], \tag{21}
\end{equation*}
$$

which defines a map $\theta_{a, b}:[m] \rightarrow[m]$.
Proposition 3. Suppose that $m \geq 2$. Let $\mathcal{V}_{m}^{\mathrm{L}, b}:=\left\{\theta_{a, b}: a \in[m], \operatorname{gcd}(a, m)=1\right\}$ for $b \in\{0,1\}$. The following holds:
(i) It holds that $\mathcal{V}_{m}^{\mathrm{L}, 0} \cap \mathcal{V}_{m}^{\mathrm{L}, 1}=\emptyset$, that $\mathcal{V}_{m}^{\mathrm{L}, b} \subset \mathcal{V}_{m}$ for $b=0,1$, and that $\left|\mathcal{V}_{m}^{\mathrm{L}, b}\right|=\phi(m)$ for $b=0,1$. The cardinality of the set $\mathcal{V}_{m}^{-}:=\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, 1}$ is $\left|\mathcal{V}_{m}^{-}\right|=\left|\mathcal{V}_{m}\right|-\phi(m)$.
(ii) Let $m \geq 3$. Suppose that $\theta, \theta^{\prime} \in \mathcal{V}_{m}$ satisfy $\theta \sim \theta^{\prime}$ and $\theta \neq \theta^{\prime}$. Then, exactly one of $\left\{\theta, \theta^{\prime}\right\}$ is in $\mathcal{V}_{m}^{\mathrm{L}, 0}$ and the other is in $\mathcal{V}_{m}^{\mathrm{L}, 1}$. Especially, $\theta \sim \theta^{\prime}$ for $\theta, \theta^{\prime} \in \mathcal{V}_{m}^{-}$implies $\theta=\theta^{\prime}$.
Proof. (i) Well-known facts are that $\{a+m \mathbb{Z}: a \in[m], \operatorname{gcd}(a, m)=1\}$ is a complete representative system of the invertible elements in the residue ring $\mathbb{Z} / m \mathbb{Z}$, that $\theta_{a, 0} \in \mathfrak{S}_{m}$ if and only if $\operatorname{gcd}(a, m)=1$ and that $\left|\mathcal{V}_{m}^{\mathrm{L}, 0}\right|=\phi(m)$. Since $\mathcal{V}_{m}^{\mathrm{L}, 1}$ is the bijective image of $\mathcal{V}_{m}^{\mathrm{L}, 0}$ by a shift, it has the same cardinality. Suppose that $a, a^{\prime} \in \mathbb{Z}$ satisfy $\operatorname{gcd}(a, m)=\operatorname{gcd}\left(a^{\prime}, m\right)=1$. For $b, b^{\prime} \in \mathbb{Z}$, the relation $\theta_{a, b}=\theta_{a^{\prime}, b^{\prime}}$ implies $\left(a-a^{\prime}\right) i+\left(b-b^{\prime}\right) \equiv 0 \bmod m$ for $i \in[m]$, then we have $b \equiv b^{\prime} \bmod m$ by substituting $i$ with $m$, then $a \equiv a^{\prime} \bmod m$ by substituting $i$ with 1 . That is, $\theta_{a, b}=\theta_{a^{\prime}, b^{\prime}}$ occurs for $a, a^{\prime}, \in[m], b, b^{\prime} \in\{0\} \cup[m-1]$ only if $(a, b)=\left(a^{\prime}, b^{\prime}\right)$. Thus, in particular, $\mathcal{V}_{m}^{\mathrm{L}, 0} \cap \mathcal{V}_{m}^{\mathrm{L}, 1}=\emptyset$.

For all $\theta_{a, 0} \in \mathcal{V}_{m}^{\mathrm{L}, 0}$, by $\overline{\operatorname{Mod}}_{m}\left(\theta_{a, 0}(m)\right)=\overline{\operatorname{Mod}}_{m}(a m)=\overline{\operatorname{Mod}}_{m}(0)=m$, it follows that $\mathbb{1}\left(\theta_{a, 0}(m) \leq\right.$ $\left.\theta_{a, 0}(i)\right)=0$ for $i \in[m-1]$. Thus, by

$$
\theta_{a, 0}(i+1)-\theta_{a, 0}(i) \equiv a(i+1)-a i \equiv a \equiv \theta_{a, 0}(1)-\mathbb{1}\left(\theta_{a, 0}(m) \leq \theta_{a, 0}(i)\right) \quad \bmod m \text { for } i \in[m-1],
$$

we have $\theta_{a, 0} \in \mathcal{V}_{m}$, which implies $\mathcal{V}_{m}^{\mathrm{L}, 0} \subset \mathcal{V}_{m}$.

And for $\theta_{a, 1} \in \mathcal{V}_{m}^{\mathrm{L}, 1}$, from $\theta_{a, 1}(m)=\overline{\operatorname{Mod}}_{m}(a m+1)=\overline{\operatorname{Mod}}_{m}(1)=1$, we have that $\mathbb{1}\left(\theta_{a, 1}(m) \leq\right.$ $\left.\theta_{a, 1}(i)\right)=1(i \in[m-1])$ and that

$$
\begin{aligned}
\theta_{a, 1}(i+1)-\theta_{a, 1}(i) \equiv a(i+1)+1-(a i+1) & \equiv a \\
& \equiv \theta_{a, 1}(1)-\mathbb{1}\left(\theta_{a, 1}(m) \leq \theta_{a, 1}(i)\right) \quad \bmod m \text { for } i \in[m-1],
\end{aligned}
$$

which implies $\theta_{a, 1} \in \mathcal{V}_{m}$ and $\mathcal{V}_{m}^{\mathrm{L}, 1} \subset \mathcal{V}_{m}$. Thus, $\left|\mathcal{V}_{m}^{-}\right|=\left|\mathcal{V}_{m}\right|-\left|\mathcal{V}_{m}^{\mathrm{L}, 1}\right|=\left|\mathcal{V}_{m}\right|-\phi(m)$.
(ii) Suppose that $\theta, \theta^{\prime} \in \mathcal{V}_{m}$ satisfy $\theta \sim \theta^{\prime}$. Then by Eq. (5), there exists $k \in \mathbb{Z}$ such that

$$
\begin{equation*}
\theta^{\prime}(i) \equiv \theta(i)+k \quad \bmod m \text { for } i \in[m] . \tag{22}
\end{equation*}
$$

By $\theta^{\prime} \in \mathcal{V}_{m}$, we have

$$
(\theta(i+1)+k)-(\theta(i)+k) \equiv \theta^{\prime}(1)-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right) \quad \bmod m \text { for } i \in[m-1] .
$$

Comparing it with the following congruence coming from $\theta \in \mathcal{V}_{m}$,

$$
\begin{equation*}
\theta(i+1)-\theta(i) \equiv \theta(1)-\mathbb{1}(\theta(m) \leq \theta(i)) \quad \bmod m \text { for } i \in[m-1], \tag{23}
\end{equation*}
$$

we find that their LHS are the same, and we are led to

$$
\theta(1)-\mathbb{1}(\theta(m) \leq \theta(i)) \equiv \theta^{\prime}(1)-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right) \quad \bmod m \text { for } i \in[m-1] .
$$

By rearranging terms, with recalling $\sqrt{22}$, it follows that

$$
\begin{equation*}
k \equiv \theta^{\prime}(1)-\theta(1) \equiv \mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right)-\mathbb{1}(\theta(m) \leq \theta(i)) \quad \bmod m \text { for } i \in[m-1] . \tag{24}
\end{equation*}
$$

The rightmost hand of the above takes only three values $0,1,-1$. Thus, only the three cases $k \equiv 0,1,-1$ $\bmod m$ are possible.

When $k \equiv 0 \bmod m$, Eq. 22 ) is

$$
\theta^{\prime}(i) \equiv \theta(i) \quad \bmod m \text { for } i \in[m],
$$

then $\theta=\theta^{\prime}$ by $\theta^{\prime}(i), \theta(i) \in[m]$.
When $k \equiv 1 \bmod m$, by (24),

$$
1 \equiv \mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right)-\mathbb{1}(\theta(m) \leq \theta(i)) \quad \bmod m \text { for } i \in[m-1] .
$$

By $m \geq 3$ and the fact that the image of $\mathbb{1}$ is contained in $\{0,1\}$, this is possible only if

$$
\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right)=1 \text { and } \mathbb{1}(\theta(m) \leq \theta(i))=0 \text { for } i \in[m-1] .
$$

Then, from the latter half of the logical conjunction, with Eq. 23$), \theta(i+1)-\theta(i) \equiv \theta(1) \bmod m$ for $i \in$ $[m-1]$ follows. Put $a:=\theta(1) \in[m]$. Then we have that $\theta(i) \equiv a i \bmod m$ for $i \in[m]$ and that $\operatorname{gcd}(a, m)=1$ by $\theta \in \mathfrak{S}_{m}$. Thus, $\theta=\theta_{a, 0} \in \mathcal{V}_{m}^{\mathrm{L}, 0}$. Then, $\theta^{\prime}(i) \equiv \theta(i)+1 \bmod m$ for $i \in[m]$ implies $\theta^{\prime}=\theta_{a, 1}$.

When $k \equiv-1 \bmod m$, swapping $\theta$ and $\theta^{\prime}$ reduces this case to the previous case $k \equiv 1 \bmod m$. Thus we have $\theta^{\prime}=\theta_{a, 0}, \theta=\theta_{a, 1}$ by the same argument.

At this point it has been shown that one of $\theta, \theta^{\prime}$ must belong to $\mathcal{V}_{m}^{\mathrm{L}, 1}$ when $\theta, \theta^{\prime} \in \mathcal{V}_{m}$ satisfy $\theta \neq \theta^{\prime}, \theta \sim$ $\theta^{\prime}$. Therefore, when $\theta, \theta^{\prime} \in \mathcal{V}_{m}^{-}$satisfy $\theta \sim \theta^{\prime}$, the equality $\theta=\theta^{\prime}$ must hold since none of $\theta, \theta^{\prime}$ belongs to $\mathcal{V}_{m}^{\mathrm{L}, 1}$.
3.2.4. Proof of Theorem 国. We are now ready to provide the proof of Theorem 2 below.

Proof. By Theorem 1 and Proposition 3, there exists a set $\mathcal{W}_{m}^{-} \subset \mathcal{W}_{m}$ which satisfies the two conditions that " $\left|\mathcal{W}_{m}^{-}\right|=\left|\mathcal{W}_{m}\right|-\phi(m)$ " and " $\theta \sim \theta^{\prime}$ for $\theta, \theta^{\prime} \in \mathcal{W}_{m}^{-}$implies $\theta=\theta^{\prime}$." From the latter condition, for each $\theta, \theta^{\prime} \in \mathcal{W}_{m}^{-}$, we have $\widetilde{\{\theta\}} \cap \widetilde{\left\{\theta^{\prime}\right\}}=\emptyset$ unless $\theta=\theta^{\prime}$, since the existence of $\theta^{\prime \prime}$ in this intersection implies $\theta \sim \theta^{\prime \prime} \sim \theta^{\prime}$. Considering that the orbit $\widetilde{\{\theta\}}=\langle\lambda\rangle \cdot \theta$ for any $\theta \in \mathfrak{S}_{m}$ always contains exactly $m$ distinct elements corresponding to their values at 1 , we have $\left|\widetilde{\mathcal{W}_{m}^{-}}\right|=m\left|\mathcal{W}_{m}^{-}\right|$from the disjointness. By a similar argument, with $\widetilde{\mathcal{Y}_{m}}=\mathcal{Y}_{m}$ of Proposition 2, we also have $\left|\mathcal{Y}_{m}\right|=\left|\widetilde{\mathcal{Y}_{m}}\right|=\left|\widetilde{\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}}\right|=m\left|\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}\right|$.

By Theorem 1. we have $\mathcal{W}_{m}^{-} \subset \mathcal{Y}_{m}$. Thus, taking the shift-closure and applying Proposition 2 again, it follows that $\mathcal{W}_{m}^{-} \subset \widetilde{\mathcal{Y}_{m}}=\mathcal{Y}_{m}$. In particular, we have the inequality $m\left|\mathcal{W}_{m}^{-}\right| \leq\left|\mathcal{Y}_{m}\right|$ whose LHS is $m\left(\left|\mathcal{W}_{m}\right|-\phi(m)\right)$ from the former condition satisfied by $\mathcal{W}_{m}^{-}$.

By Proposition 1 and by the injectivity of the map $\Psi_{m}$, we have $\left|\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}\right|=\left|\Psi_{m}\left(\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}\right)\right|=\left|\mathcal{W}_{m-1}\right|$. Consequently, $m\left(\left|\mathcal{W}_{m}\right|-\phi(m)\right)=m\left|\mathcal{W}_{m}^{-}\right| \leq m\left|\mathcal{W}_{m-1}\right|$, that is, $\left|\mathcal{W}_{m}\right|-\left|\mathcal{W}_{m-1}\right| \leq \phi(m)$.

Again by Theorem 1, we have $\left|\mathcal{V}_{m}\right|=\left|\mathcal{W}_{m}\right|$. It is easy to check that $\mathcal{W}_{2}=\mathfrak{S}_{2}$ and that $\left|\mathcal{W}_{2}\right|=$ $2=\phi(1)+\phi(2)$, by a direct argument. Therefore we conclude that $\left|\mathcal{V}_{m}\right| \leq \sum_{k=1}^{m} \phi(k)$ and that $\left|\mathcal{Y}_{m}\right|=$ $m\left|\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}\right|=m\left|\mathcal{W}_{m-1}\right| \leq m \sum_{k=1}^{m-1} \phi(k)$.

## 4. An application

A key ingredient in the proof of Theorem 2 is the use of the map $\Psi_{m}$ which bijectively connects the two sets $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ and $\mathcal{V}_{m-1}$ of permutations of different degrees.

In this section, we look at another building block which connects $\mathcal{V}_{m}$ and $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$, then combine it with $\Psi_{m}$ to form a procedure that lifts the elements of $\mathcal{V}_{m-1}$ to $\mathcal{V}_{m}$ in a canonical manner. In this section we assume that $m \geq 3$ unless otherwise stated.
4.1. Connecting $\mathcal{V}_{m}$ and $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ by shift. The building block we consider here is the map

$$
\begin{equation*}
\Gamma_{m}: \mathfrak{S}_{m} \ni \theta \mapsto \overline{\operatorname{Mod}}_{m}(\theta(\cdot)-\theta(1)+1) \in \mathfrak{S}_{m}^{1} \tag{25}
\end{equation*}
$$

which applies a shift so that a permutations of degree $m$ is sent to the permutation of the same degree having 1 as a fixed point. From the definition, it is obvious that $\Gamma_{m}$ preserves the shift-equivalence class, i.e., $\theta \sim \Gamma_{m}(\theta)$.

Let us consider the restriction $\Gamma_{m} \mid \mathcal{V}_{m}$. From the definition (25), it holds that $\Gamma_{m}\left(\mathcal{V}_{m}\right) \subset \widetilde{\mathcal{V}_{m}} \cap \mathfrak{S}_{m}^{1}$. From Theorem 1, $\widetilde{\mathcal{V}_{m}} \subset \widetilde{\mathcal{Y}}_{m}$ holds, then by Proposition 2 , it follows that $\widetilde{\mathcal{V}_{m}} \subset \mathcal{Y}_{m}$. Thus, $\Gamma_{m} \mid \mathcal{V}_{m}$ is a map

$$
\Gamma_{m} \mid \mathcal{V}_{m}: \mathcal{V}_{m} \rightarrow \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}
$$

We note that $\mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}$ is a complete system of the representatives for $\mathcal{Y}_{m} / \sim$, again by Proposition 2 (i.e., for any $\theta \in \mathcal{Y}_{m}$ there exists a unique $\theta^{\prime} \in \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}$ such that $\theta^{\prime} \sim \theta$ ).

Then, let us consider what restriction does make $\Gamma_{m}$ injective. As we have shown in Proposition 3 (ii), $\theta \sim \theta^{\prime}$ for $\theta, \theta^{\prime} \in \mathcal{V}_{m}^{-}=\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, 1}$ implies $\theta=\theta^{\prime}$. Since $\Gamma_{m}$ preserves the equivalence class by $\sim$, $\Gamma_{m}(\theta)=\Gamma_{m}\left(\theta^{\prime}\right)$ implies $\theta \sim \theta^{\prime}$. Combining these, the restriction $\left.\Gamma_{m}\right|_{\mathcal{V}_{m}^{-}}$is injective.

When the domain is restored to $\mathcal{V}_{m}, \Gamma_{m} \mid \mathcal{V}_{m}$ is no longer injective. However, in fact, the manner that a collision occurs is quite well-controlled as shown below. Suppose that $\theta, \theta^{\prime} \in \mathcal{V}_{m}, \theta \neq \theta^{\prime}$ satisfy $\Gamma_{m}(\theta)=\Gamma_{m}\left(\theta^{\prime}\right)$. Then by the shift-equivalence preservation of $\Gamma_{m}$, we have $\theta \sim \theta^{\prime}$, however, again by Proposition 3 (ii), it occurs if and only if one of $\theta, \theta^{\prime}$ is in $\mathcal{V}_{m}^{\mathrm{L}, 0}$ and the other is in $\mathcal{V}_{m}^{\mathrm{L}, 1}$. In this case, it is easy to see, by the shift-equivalence preservation and $\Gamma_{m}\left(\mathcal{V}_{m}\right) \subset \mathfrak{S}_{m}^{1}$, that $\Gamma_{m}(\theta)(i)=\theta_{a, m-a+1}(i)$ for $\theta=\theta_{a, 0} \in \mathcal{V}_{m}^{\mathrm{L}, 0}$ and $\theta=\theta_{a, 1} \in \mathcal{V}_{m}^{\mathrm{L}, 1}$, where $\theta_{a, *}$ were defined in Eq. (21). On the other hand, if $\pi \in \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}$ is $\pi=\theta_{a, m-a+1}$ for some $a \in[m]$ with $\operatorname{gcd}(a, m)=1$, then it is clear $\pi=\Gamma_{m}\left(\theta_{a, 0}\right)=\Gamma_{m}\left(\theta_{a, 1}\right)$. Thus, $\left(\left.\Gamma_{m}\right|_{\mathcal{V}_{m}^{\mathrm{L}, 0} \cup \mathcal{L}_{m}^{\mathrm{L}, 1}}\right)^{-1}(\{\pi\})=\left\{\theta_{a, 0}, \theta_{a, 1}\right\}$ for $\pi \in \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}$ occurs if and only if $\pi=\theta_{a, m-a+1}$ for some $a$ with $\operatorname{gcd}(a, m)=1$.

For $\theta \in \mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}$ and $\theta^{\prime} \in \mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right), \Gamma_{m}(\theta)=\Gamma_{m}\left(\theta^{\prime}\right)$ is impossible as it implies $\theta \sim \theta^{\prime}$ which is refuted by Proposition 3 (ii) again.

We summarize these properties of $\Gamma_{m}$ and its restrictions in the following lemma. For convenience, we introduce a notation for the congruential difference set for a sequence $\theta$ over $[\mathrm{m}]$ as

$$
\operatorname{CDS}_{m}(\theta):=\left\{\operatorname{Mod}_{m}(\theta(i+1)-\theta(i)): i \in[m-1]\right\} .
$$

Lemma 2. Let $m \geq 3$. Let $\Gamma_{m}: \mathfrak{S}_{m} \rightarrow \mathfrak{S}_{m}^{1}$ be the map defined as 25). Then, the following holds:
(i) $\theta \sim \Gamma_{m}(\theta)$ for $\theta \in \mathfrak{S}_{m}$.
(ii) The restriction $\Gamma_{m} \mid \mathcal{V}_{m}$ is a map $\mathcal{V}_{m} \rightarrow \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}$.
(iii) The restriction $\left.\Gamma_{m}\right|_{\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, b}}$ is an injection from $\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, b}$ to $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ for each of $b=0,1$.
(iv) The restriction $\left.\Gamma_{m}\right|_{\mathcal{V}_{m}^{\mathrm{L}, 0}} \mathcal{V}_{m}^{\mathrm{L}, 1}$ is a $2: 1$ surjection from $\mathcal{V}_{m}^{\mathrm{L}, 0} \mathcal{V}_{m}^{\mathrm{L}, 1}$ onto $\left\{\theta_{a, m-a+1}: a \in[m], \operatorname{gcd}(a, m)=\right.$ $1\}$ whose cardinality is $\phi(m)$. And $\left(\left.\Gamma_{m}\right|_{\mathcal{V}_{m}^{\mathrm{L}, 0} \cup \mathcal{V}_{m}^{\mathrm{L}, 1}}\right)^{-1}\left(\left\{\theta_{a, m-a+1}\right\}\right)=\left\{\theta_{a, 0}, \theta_{a, 1}\right\}$.
(v) $\Gamma_{m}\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right) \cap \Gamma_{m}\left(\mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right)\right)=\emptyset$.
(vi) The cardinality of the congruential difference set $\operatorname{CDS}_{m}\left(\Gamma_{m}(\theta)\right)$ for $\theta \in \mathcal{V}_{m}$ is 1 if and only if $\theta \in \mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}$.

Proof. We already have shown (i)-(v). For (vi), "if" part easily follows from (iv). For "only if" part, suppose that a $\theta \in \mathcal{V}_{m}$ has the congruential difference set $\operatorname{CDS}_{m}\left(\Gamma_{m}(\theta)\right)$ of cardinality 1 . If the set is $\{a\}$, then $\Gamma_{m}(\theta(i))=\overline{\operatorname{Mod}}_{m}(a i+b)(i \in[m])$ for some $b \in \mathbb{Z}$, however, then $\operatorname{gcd}(a, m)=1$ holds because otherwise $\Gamma_{m}(\theta)$ is not a permutation. Thus, we have $\theta \sim \Gamma_{m}(\theta) \sim \theta_{a, 0} \sim \Gamma_{m}\left(\theta_{a, 0}\right)$ from (i). This implies $\Gamma_{m}(\theta)=\Gamma_{m}\left(\theta_{a, 0}\right)$ because $\mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}$ is a complete representative system of $\mathcal{Y}_{m} / \sim$. From (v) and $\operatorname{gcd}(a, m)=1, \theta \in \mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}$ holds.

The above lemma is established in terms of $\mathcal{V}_{m}$ and $\mathcal{Y}_{m}$, independent of the knowledge on $\mathcal{S}_{m}^{*}$. To complete the description of properties of $\mathcal{V}_{m}$, we will quote a few results on $\mathcal{S}_{m}^{*}$ (TheoremCand Corollary 1).

## Lemma 3. Let $m \geq 3$. The following holds:

(i) The injection $\left.\Gamma_{m}\right|_{\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, b}}$ is a bijection from $\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, b}$ to $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ for each of $b=0$, . The common cardinality of the domain and the image is $\sum_{k=1}^{m-1} \phi(k)$.
(ii) For every $\theta \in \mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right)$, the congruential difference set $\mathrm{CDS}_{m}\left(\Gamma_{m}(\theta)\right)$ is of the form $\{a, a+1\}$ for some $a \in[m-1]$. Moreover, $\left\{\theta^{\prime} \in \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}: \exists a \in[m-1]\right.$ s.t. $\operatorname{CDS}_{m}\left(\theta^{\prime}\right)=$ $\{a, a+1\}\}$ is the bijective image of $\mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right)$ by $\Gamma_{m}$.

Proof. (i) By Theorem C and Corollary 1, $\left|\mathcal{V}_{m}\right|=\left|\mathcal{S}_{m}^{*}\right|=\sum_{k=1}^{m} \phi(k)$. Thus we have, for each of $b=0,1$, $\left|\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, b}\right|=\sum_{k=1}^{m} \phi(k)-\phi(m)=\sum_{k=1}^{m-1} \phi(k)$. Since the map $\left.\Gamma_{m}\right|_{\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, b}}$ is injective as seen in Lemma 22 (iii), we have $\left|\mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}\right| \geq \sum_{k=1}^{m-1} \phi(k)$. The reversed inequality (multiplied by $m$ ) has already been shown in the very end of the proof of Theorem 22, yielding the equality $\left|\mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}\right|=\sum_{k=1}^{m-1} \phi(k)$. Therefore, $\left.\Gamma_{m}\right|_{\mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, b}}$ is bijective for $b=0,1$.
(ii) Let $A^{\prime}:=\left\{\theta^{\prime} \in \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}: \exists a \in[m-1]\right.$ s.t. $\left.\operatorname{CDS}_{m}\left(\theta^{\prime}\right)=\{a\}\right\}$ and $A^{\prime \prime}:=\left\{\theta^{\prime} \in \mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}: \exists a \in\right.$ $[m-1]$ s.t. $\left.\operatorname{CDS}_{m}\left(\theta^{\prime}\right)=\{a, a+1\}\right\}$ respectively. By the definition of $\mathcal{V}_{m}$, namely the defining congruential recurrence Eq. (3), the congruential difference set $\operatorname{CDS}_{m}(\theta)$ is of the form either $\{a, a+1\}$ or $\{a\}$ for some $a \in[m-1]$, for all $\theta \in \mathcal{V}_{m}$. Because $\Gamma_{m}(\theta)$ is a shift of $\theta$, it follows that $\operatorname{CDS}_{m}\left(\Gamma_{m}(\theta)\right)=\operatorname{CDS}_{m}(\theta)$. On the other hand, the assertion (i) just proved and Lemma 2 (ii) tell us that $\Gamma_{m}\left(\mathcal{V}_{m}\right)=\mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}$ in particular. Hence, we have

$$
\begin{equation*}
\mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}=\Gamma_{m}\left(\mathcal{V}_{m}\right)=A^{\prime} \sqcup A^{\prime \prime} \tag{26}
\end{equation*}
$$

By Lemma 2 (vi), for $\theta \in \mathcal{V}_{m}, \operatorname{CDS}_{m}\left(\Gamma_{m}(\theta)\right)$ is a singleton $\{a\}$ if and only if $\theta \in \mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}$. In other words, it follows that

$$
\begin{equation*}
\Gamma_{m}\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right)=A^{\prime} \tag{27}
\end{equation*}
$$

From Eqs. (26), (27) and Lemma 2 (v), it follows that $\Gamma_{m}\left(\mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right)\right)=A^{\prime \prime}$, i.e., $\left.\left.\Gamma_{m}\right|_{\mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0}\right.} \mathcal{V}_{m}^{\mathrm{L}, 1}\right)$ : $\mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right) \rightarrow A^{\prime \prime}$ is a surjection. For the bijectivity, $\left.\Gamma_{m}\right|_{\mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right)}$ is injective, again by the assertion (i) and $\mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right) \subset \mathcal{V}_{m} \backslash \mathcal{V}_{m}^{\mathrm{L}, 1}$.
Remark 3. The validity of Lemmas 2 and 3 is able to be verified for the case $m=2$ where $\mathcal{V}_{m}=\mathcal{Y}_{m}=\mathfrak{S}_{2}$, by a direct argument.

Remark 4. The above proof of (i) depends on Theorem C and Corollary 1. However, what we really needed in the proof was the fact $\left|\mathcal{V}_{m}\right| \geq \sum_{k=1}^{m} \phi(k)$ only. We used no detailed property of $\mathcal{S}_{m}^{*}, \mathcal{S}_{m}$ nor Farey sequence associated to them.
4.2. The lifting procedure. Now we have the building block $\Gamma_{m} \mid \mathcal{V}_{m}$ which maps $\mathcal{V}_{m}$ of cardinality $\sum_{k=1}^{m} \phi(k)$ onto $\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ of cardinality $\sum_{k=1}^{m-1} \phi(k)$. It is not injective but the collisions are wellcontrolled, as mentioned in Lemma 2 .

On the other hand, we already have had another ingredient, the bijection $\left.\Psi_{m}\right|_{\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}}: \mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m} \rightarrow$ $\mathcal{V}_{m-1}$. Then, taking the composition, we obtain the surjection $\left.\Psi_{m}\right|_{\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}} \circ \Gamma_{m} \mid \mathcal{V}_{m}: \mathcal{V}_{m} \rightarrow \mathcal{V}_{m-1}$ in which the only source of collisions is the $2: 1$ property of $\left.\Gamma_{m}\right|_{\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}}$.

The composition projects $\mathcal{V}_{m}$ to $\mathcal{V}_{m-1}$ nicely. Indeed, up to the $2: 1$ case of Lemma 2 (iv), the projection is invertible. Thus we are led to the following procedure to lift the permutations $\pi$ in $\mathcal{V}_{m-1}$ to form $\mathcal{V}_{m}$.

Procedure 1. Given $m \geq 2$, do the following:

```
Let X:= \emptyset;
For each }\pi\in\mp@subsup{\mathcal{V}}{m-1}{}\mathrm{ do {
        Let }\mp@subsup{0}{\pi}{}:=\mp@subsup{\Psi}{m}{-1}(\pi)
        If CDS 
            {Let Y:= {\mp@subsup{0}{a,0}{,},\mp@subsup{0}{a,1}{}};}
        If CDS 
            Let \hat{0}:=\mp@subsup{\overline{\operatorname{Mod}}}{m}{}(\mp@subsup{0}{\pi}{}(\cdot)+a)\mathrm{ and let Y:={{̂}; }}
        Let X:= X\cupY.
    9 }
    10 Let \mathcal{V}}:=X
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    Here we put \(\mathcal{V}_{1}:=\mathfrak{S}_{1}, \mathcal{W}_{1}:=\mathfrak{S}_{1}\) and \(\mathcal{Y}_{2}:=\mathfrak{S}_{2}\), for which Proposition 1 holds, when \(m=2\).
    Theorem 3. Procedure 1 lifts $\mathcal{V}_{m-1}$ to $\mathcal{V}_{m}$ correctly.
Proof. In the line $3, \theta_{\pi} \in \mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}$ is uniquely determined since $\Psi_{m}: \mathfrak{S}_{m}^{1} \rightarrow \mathfrak{S}_{m-1}$ is bijective and so is $\left.\Psi_{m}\right|_{\mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m}}: \mathfrak{S}_{m}^{1} \cap \mathcal{Y}_{m} \rightarrow \mathcal{V}_{m-1}$ by Proposition 1 and Theorem 1. From Lemma 2 (vi) and Lemma 3 (ii), exactly one "If" statement of the lines 4 and 6 is the case. When the statement in the line 4 is the case, $\theta_{\pi}=\theta_{a, b}$ for some $b \in[m]$, however, $b=m-a+1$ from the condition $\theta_{\pi}(1)=1$. Thus, from Lemma 2 (iv) and (vi), $Y=\left(\Gamma_{m} \mid \mathcal{V}_{m}\right)^{-1}\left(\left\{\theta_{\pi}\right\}\right)$ holds in the line 5. On the other hand, when the statement in the line 6 holds, by Lemma 3 (ii), a unique $\theta^{\prime}:=\Gamma_{m}^{-1}\left(\theta_{\pi}\right) \in \mathcal{V}_{m} \backslash\left(\mathcal{V}_{m}^{\mathrm{L}, 0} \sqcup \mathcal{V}_{m}^{\mathrm{L}, 1}\right)$ exists. From $\theta^{\prime} \in \mathcal{V}_{m}$ and the shift-equivalence $\theta^{\prime}(i+1)-\theta^{\prime}(i) \equiv \theta_{\pi}(i+1)-\theta_{\pi}(i) \bmod m(i \in[m-1])$, it follows that $\left\{\overline{\operatorname{Mod}}_{m}\left(\theta^{\prime}(1)-\mathbb{1}\left(\theta^{\prime}(m) \leq \theta^{\prime}(i)\right)\right): i \in[m-1]\right\}=\left\{\overline{\operatorname{Mod}}_{m}\left(\theta^{\prime}(i+1)-\theta^{\prime}(i)\right): i \in[m-1]\right\}=$ $\left\{\overline{\operatorname{Mod}}_{m}\left(\theta_{\pi}(i+1)-\theta_{\pi}(i)\right): i \in[m-1]\right\}=\{a, a+1\}$. Thus, taking the maximum of the last set of cardinality $2, \theta^{\prime}(1)=a+1$ which is $a+\theta_{\pi}(1)$ by $\theta_{\pi} \in \mathfrak{S}_{m}^{1}$. Therefore, $\theta^{\prime}=\overline{\operatorname{Mod}}_{m}\left(\theta_{\pi}(\cdot)+a\right)$, which is nothing but $\hat{\theta}$ in the line 7 .

Thus the lines 4,5,6 and 7 together compute the correct inverse image $Y=\left(\Gamma_{m} \mid \mathcal{V}_{m}\right)^{-1}\left(\left\{\theta_{\pi}\right\}\right)$ for $\theta_{\pi}=$ $\Psi_{m}^{-1}(\pi)$. Then the set $X$ in the line 10 is $\bigcup_{\pi \in \mathcal{V}_{m-1}}\left(\Gamma_{m} \mid \mathcal{V}_{m}\right)^{-1}\left(\left\{\Psi_{m}^{-1}(\pi)\right\}\right)=\left(\left.\Psi_{m}\right|_{\mathcal{Y}_{m} \cap \mathfrak{S}_{m}^{1}} \circ \Gamma_{m} \mid \mathcal{V}_{m}\right)^{-1}\left(\mathcal{V}_{m-1}\right)=$ $\mathcal{V}_{m}$.
4.3. Recursion of the lifting. In this subsection, we assume that $m \geq 2$. The composition $\mathcal{V}_{m} \xrightarrow{\Gamma_{m}} \mathcal{Y}_{m} \cap$ $\mathfrak{S}_{m}^{1} \xrightarrow{\Psi_{m}} \mathcal{V}_{m-1}$ gives rise to the projection $\mathcal{V}_{m} \rightarrow \mathcal{V}_{m-1}$ and it can be applied recursively to reach to $\mathcal{V}_{1}=\mathfrak{S}_{1}$ containing only one element $\left.\operatorname{id}\right|_{[1]} \in \mathfrak{S}_{1}$. In other words, starting with $\mathcal{V}_{1}$, inverting the arrows by the iterative application of the lifting, Procedure 1, we obtain a generation process of $\mathcal{V}_{1} \rightarrow \mathcal{V}_{2} \rightarrow \cdots \rightarrow \mathcal{V}_{m}$ for arbitrary degree $m$.


Figure 1. Recursive application of $\Psi \circ \Gamma$ or its inverse
Figure 1 shows the tree corresponding to the process for $m \leq 6$. It has two interpretations; ascending the family tree (projection) and descending the family tree (generation). Here we explain it based on the latter interpretation.

Permutations are shown using the one-line notation and a bold font. At the root of the tree, the unique element $\mathbf{1} \in \mathcal{V}_{1}$ exists. For $m=2$, the line 3 of Procedure 1 is applied to $\pi=\mathbf{1} \in \mathcal{V}_{1}$, yielding $\theta_{\pi}=\mathbf{1 2} \in \mathcal{Y}_{2} \cap \mathfrak{S}_{2}^{1}$. For this $\theta_{\pi}$, "If" statement in the line 4 is the case with $a=1$. Thus, the line 5 produces two elements $\theta_{1,0}=\mathbf{1 2}$ and $\theta_{1,1}=\mathbf{2 1} \in \mathcal{V}_{2}$. In the picture, they are denoted as $\mathbf{1 2}^{(0)}$ and $\mathbf{2 1}^{(1)}$ respectively, where the index $(b)$ indicates the second parameter $b$ of $\theta_{a, b}$. When we arrange these nodes in the tree, there are two ways, however we employ the ordering ( 0 )-first (left), (1)-second (right). Since $\theta_{\pi}=\mathbf{1 2}$ is the only element of $\mathcal{Y}_{2} \cap \mathfrak{S}_{2}^{1}$, we reach to the line 10 with $X=\{\mathbf{1 2}, \mathbf{2 1}\}$. Thus, we have constructed $\mathcal{V}_{2}=\{\mathbf{1 2}, \mathbf{2 1}\}$.

The run of Procedure 1 for $m=3$ is similar and we have $\mathcal{V}_{3}=\{\mathbf{1 2 3}, \mathbf{2 3 1}, \mathbf{2 1 3}, \mathbf{3 2 1}\}$ at the line 10 .
For $m=4$, however, for some $\theta_{\pi} \in \mathcal{Y}_{4} \cap \mathfrak{S}_{4}^{1}$, namely for $\theta_{\pi}=\mathbf{1 3 4 2}, \mathbf{1 3 2 4}$ in Fig. 1 , the line 6 is the case and for each of them, a single child $\hat{\theta} \in \mathcal{V}_{4}$ is produced ( 2413 for $\theta_{\pi}=\mathbf{1 3 4 2}$ and $\mathbf{3 1 4 2}$ for $\theta_{\pi}=\mathbf{1 3 2 4}$ ).

Iterating the lifting procedure, we finally have $\sum_{k=1}^{6} \phi(k)=1+1+2+2+4+2=12$ elements of $\mathcal{V}_{6}$.
Remark 5. By using Procedure 1 and Corollary 1, we are able to construct $\mathcal{S}_{m}^{*}$, the inverses of the Sós permutations for arbitrary degree $m$, without depending on Farey sequence. If we may depend on Farey sequence, such construction is straightforward: By Surányi's bijection [2, Sats I], the denominators of successive two fractions in $m$-th Farey sequence correspond to the 1st and $m$-th terms of a Sós permutation respectively, while a Sós permutation is uniquely determined by its 1 st and $m$-th terms, by the recurrence of Theorem A. Using this bijection, the entire $\mathcal{S}_{m}$, the set of Sós permutations of degree $m$, is constructable from $m$-th Farey sequence. We also note that even the recurrence Eq. (3) which defines the set $\mathcal{V}_{m}$, does not appear in Procedure 1 .

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