# SELF-CONSISTENCY, RELATIVISM AND MANY-PARTICLE SYSTEM 

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#### Abstract

The interrelation between the concepts of self-consistency, relativism and many-particle systems is considered within the framework of a unified consideration of classical and quantum physics based on the first principle of the probability conservation law. The probability conservation law underlies the Vlasov equation chain. From the first Vlasov equation, the Schrödinger equation, the Hamilton-Jacobi equation, the equation of motion of a charged particle in an electromagnetic field, the Maxwell equations, the Pauli equation and the Dirac equation are constructed. The paper shows with mathematical rigor that quantum systems with a time independent function of quasi-density probability in phase space are not capable to emit electromagnetic radiation.

It is shown that at the micro-level a quantum object may be considered rather as an «extended» object than a point one. And the hydrodynamic description of continuum mechanics is applicable for such an object.

A number of exact solutions of quantum and classical model systems is considered, demonstrating a new insight at the quantum mechanics representation.


Key words: quantum mechanics, Dirac equation, Pauli equation, Schrödinger equation, electromagnetic radiation, Maxwell equations, Vlasov equation, rigorous result

## Introduction

The historical retrospective of the paper's topic goes back to time when the Maxwell system of equations was written:

$$
\begin{align*}
& \square \varphi=\frac{\rho}{\varepsilon_{0}}, \square \overrightarrow{\mathrm{~A}}=\mu_{0} \overrightarrow{\mathrm{~J}},  \tag{i.1}\\
& \operatorname{div}_{r} \overrightarrow{\mathrm{~A}}+\frac{1}{c^{2}} \frac{\partial \varphi}{\partial t}=0, \tag{i.2}
\end{align*}
$$

where $\quad=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}-\Delta_{r}, \mu_{0} \varepsilon_{0}=1 / c^{2} ; \rho$ is the density of electric charges; $\overrightarrow{\mathrm{J}}$ is the density of electric currents; $\varphi, \overrightarrow{\mathrm{A}}$ are the scalar and vector potential of the electromagnetic field $\overrightarrow{\mathrm{E}}, \overrightarrow{\mathrm{B}}$ respectively:

$$
\begin{equation*}
\overrightarrow{\mathrm{E}}=-\frac{\partial \overrightarrow{\mathrm{A}}}{\partial t}-\nabla_{r} \varphi, \overrightarrow{\mathrm{~B}}=\operatorname{curl}_{r} \overrightarrow{\mathrm{~A}} . \tag{i.3}
\end{equation*}
$$

The Maxwell equations have the form (i.1) subject to the condition (i.2). Since there is some freedom in the choice of potentials $\varphi, \vec{A}$, the Lorentz gauge condition (i.2) is acceptable. In
the limiting case $c \rightarrow \infty$, the condition (i.2) goes into the Coulomb gauge $\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}=0$, and the d'Alembert's operator transforms into the Laplace operator $\square \rightarrow-\Delta$.

The natural question of the invariance of the Maxwell equations when transforming a coordinate system led to an update of the very concept of a coordinate system. Time $t$ had to be introduced as coordinate axis $c t$ (the Minkowski space-time). The Lorentz transformation was the simplest linear transformation of the coordinates in 4-dimensional space-time $x^{\mu}$, conserving the invariance of the d'Alembert's operator $\square=\partial^{\mu} \partial_{\mu}$. In fact, the Maxwell equations (i.1) can be represented as a single oscillation equation in which the d'Alembert's operator $\square=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}-\Delta$ contains a non-Euclidean metric $g_{\mu \nu} \mapsto(+,-,-,-)$. The replacement of the Galilean transformations with the Lorentz transformations served as the starting point for the construction of the special theory of relativity (STR), in which Einstein postulated the maximum velocity $c$ (the velocity of light in a vacuum) of the propagation of interaction in the Universe.

The finiteness of the velocity of interaction propagation directly follows from the Maxwell equations (i.1). Indeed, the solution to the equations (i.1) can be constructed from the solutions to the Poisson equation, where the change of variable $t_{\text {ret }}=t-\left|\vec{r}-\vec{r}^{\prime}\right| / c$ is made. Time $t_{\text {ret }}$ takes the retarded time $\left|\vec{r}-\vec{r}^{\prime}\right| / c$ of the interaction into account. At $c \rightarrow \infty$, there is no retarded time and the interaction propagates instantly. Taking into account the retarded time leads to the Lienard-Wiechert potentials for $\varphi(\vec{r}, t)$ and $\overrightarrow{\mathrm{A}}(\vec{r}, t)$. It turned out that timedependent electric $\overrightarrow{\mathrm{E}}(\vec{r}, t)$ and magnetic $\overrightarrow{\mathrm{B}}(\vec{r}, t)$ fields arise when a point particle (using the Dirac $\delta$-function representation) with charge $q$ moves along a classical trajectory, determined at each moment of time $t_{\text {ret }}$ by the radius vector $\vec{r}_{s}\left(t_{\text {ret }}\right)$ and velocity $\vec{v}_{s}\left(t_{\text {ret }}\right)$ :
$\overrightarrow{\mathrm{E}}(\vec{r}, t)=\frac{q}{4 \pi \varepsilon_{0}} \frac{1}{\left|\vec{r}-\vec{r}_{s}\right|^{2}\left(1-\vec{n}_{s} \vec{\beta}_{s}\right)^{3}}\left[\left(\vec{n}_{s}-\vec{\beta}_{s}\right)\left(1-\beta_{s}^{2}+\frac{\left|\vec{r}-\vec{r}_{s}\right|}{c} \vec{n}_{s} \cdot \dot{\vec{\beta}}_{s}\right)-\dot{\vec{\beta}} s \frac{\left|\vec{r}-\vec{r}_{s}\right|}{c} \vec{n}_{s} \cdot\left(\vec{n}_{s}-\vec{\beta}_{s}\right)\right]$,
$\overrightarrow{\mathrm{B}}(\vec{r}, t)=\frac{1}{c} \vec{n}_{s} \times \overrightarrow{\mathrm{E}}(\vec{r}, t)$,
where $\vec{\beta}_{s}=\vec{v}_{s} / c, \vec{n}_{s}=(\vec{r}-\vec{r}) /\left|\vec{r}-\vec{r}_{s}\right|$. Note that the information on coordinate $\vec{r}_{s}\left(t_{r e t}\right)$ is included in scalar potential $\varphi(\vec{r}, t)$, and the information on velocity $\vec{v}_{s}\left(t_{\text {ret }}\right)$ is included in vector potential $\overrightarrow{\mathrm{A}}(\vec{r}, t)$. Knowing potentials $\varphi(\vec{r}, t)$ and $\overrightarrow{\mathrm{A}}(\vec{r}, t)$, using the formulas (i.3) one can obtain expressions (i.4).

The terms in expression (i.4) can be divided into two groups: with coefficients $1 /\left|\vec{r}-\vec{r}_{s}\right|^{2}$ and $1 /\left|\vec{r}-\vec{r}_{s}\right|$. At large distances $\left|\vec{r}-\vec{r}_{s}\right| \gg 1$, the main contribution to expression (i.4) will be made by terms with the factor $1 /\left|\vec{r}-\vec{r}_{s}\right|$. In the nonrelativistic approximation $\left(\beta_{s} \ll 1\right)$, expression (i.4) for field $\overrightarrow{\mathrm{E}}(\vec{r}, t)$ will take the form $\overrightarrow{\mathrm{E}}=\frac{q}{4 \pi \varepsilon_{0} c} \frac{1}{\left|\vec{r}-\vec{r}_{s}\right|} \vec{n}_{s} \times\left(\vec{n}_{s} \times \dot{\vec{\beta}}_{s}\right)$, which indicates orthogonality $\overrightarrow{\mathrm{E}} \perp \vec{n}_{s}$. Thus, the vectors $\overrightarrow{\mathrm{B}}, \overrightarrow{\mathrm{E}}$ and $\vec{n}_{s}$ are mutually orthogonal. The particle will emit electromagnetic energy, characterized by the Poynting vector $\overrightarrow{\mathrm{S}}=\frac{c}{4 \pi} \overrightarrow{\mathrm{E}} \times \overrightarrow{\mathrm{B}}$, which in the indicated approximations will take the form

$$
\begin{equation*}
\overrightarrow{\mathrm{S}}=\frac{q^{2} \vec{n}_{s}}{4 \pi c\left|\vec{r}-\vec{r}_{s}\right|}\left|\vec{n}_{s} \times\left(\vec{n}_{s} \times \dot{\vec{\beta}}_{s}\right)\right|^{2} . \tag{i.5}
\end{equation*}
$$

This result is known as the Larmor formula, which relates the power of electromagnetic radiation to particle acceleration $\dot{\vec{\beta}}_{s}$. The Larmor formula turned out to be the «cornerstone» in the construction of the theory of the atom. After the works of Rutherford, which showed the presence of a nucleus in an atom, a question arose about the stability of the movement of an electron around the atomic nucleus. According to the Larmor formula (especially in the general (i.4) case), an electron moving with acceleration should radiate electromagnetic energy. The electron's loss of energy would lead to its «fall» onto the atomic nucleus. However, experimental data indicate the stability of the atom.

Bohr first attempted to «save» the classical model of the atom. His approach made it possible to describe the spectrum of the hydrogen atom, but the spectra of the following atoms from the Mendeleev table caused serious difficulties.

The second attempt was the formulation of a probability model of the micro-world laws, which led to the introduction of the concept of the wave function $\Psi$ and the Schrödinger equation for $i$ t. The Heisenberg uncertainty principle $\Delta x_{k} \Delta p_{k} \geq \hbar / 2, k=1 . .3$ removed the very concept of a trajectory in the micro world from consideration. The absence of the concept of a deterministic classical trajectory $\left(\vec{r}_{s}, \vec{v}_{s}\right)$ in the micro world made the acceleration $\dot{\vec{\beta}}_{s}$ and electromagnetic radiation in the Larmor formula (i.5) unnecessary to be considered.

The Larmor formula (i.5), expressions (i.4) and the Maxwell equations themselves (i.1) were a macroscopic description. The very concept of charge density $\rho$, current density $\overrightarrow{\mathrm{J}}$, included in the Maxwell equations, are macroscopic quantities. Charge density $\rho$ requires the introduction of the operation of averaging the charge over a certain small volume $\delta \omega$. A similar procedure is required for current density $\overrightarrow{\mathrm{J}}=\rho \vec{v}$. Quantum mechanics describes micro-world systems and the averaging operation must be introduced in a different way. Another problem is the Heisenberg uncertainty principle, according to which it is impossible to measure the coordinate and velocity simultaneously with proper accuracy. From the standpoint of the Maxwell equations, the coordinate corresponds to the macroscopic parameter of density $\rho$, and the momentum (velocity) corresponds to current density $\overrightarrow{\mathrm{J}}=\rho \vec{v}$. Thus, the Heisenberg uncertainty principle leads to the fundamental impossibility of solving Maxwell equations (i.1) for quantum systems, since the input data $\rho, \overrightarrow{\mathrm{J}}$ are both immeasurable/non-existent.

Note that at the micro level it is impossible to describe a particle through a $\delta$-function, since in this case its coordinate and momentum would be strictly defined, which would contradict the uncertainty principle and the presence of a wave function, which indicates the probability distribution of coordinates and momentum.

Thus, in the micro world a particle is rather an «extended» object than a point one, having uncertainty by the coordinate and momentum. In a sense, the particle becomes a «macro» object. For example, an electron in an atom may be represented by an $s$-orbital which is the probability density $|\Psi|^{2}$ of observing the electron at certain points in space. Mass $m_{q}$ and charge $q$ of the electron can be represented as the density of mass/charge $m|\Psi|^{2} / q|\Psi|^{2}$ distributed in coordinate space. The question naturally arises about the electric field of such a macro object. From the position of the Maxwell macroscopic equations, electric field $\overrightarrow{\mathrm{E}}_{\Psi}$ must correspond to charge density $q|\Psi|^{2}$. But density $q|\Psi|^{2}$ is not the classical charge density. Density $q|\Psi|^{2}$ is a
probabilistic/statistical quantity. In macroscopic electrodynamics, charge density $\rho$ is formed by a large number of particles. Electric field $\vec{E}_{\rho}$ of such a system is a superposition of the fields of all charged particles from distribution $\rho$. At the micro level, the statistical density of distribution $q|\Psi|^{2}$ is the density of the same particle, the coordinate of which is not determined.

In a sense, due to uncertainty $\delta r_{q}$ of the electron coordinate $\vec{r}_{q} \pm \delta r_{q}$, value $\operatorname{div}_{r} \overrightarrow{\mathrm{E}}_{\Psi}=0$ at some point $\vec{r}$. Indeed, $\varepsilon_{0} \operatorname{div}_{r} \overrightarrow{\mathrm{E}}_{\Psi}=q \delta\left[\vec{r}-\left(\vec{r}_{q} \pm \delta r_{q}\right)\right]$ and the probability that we accidentally hit the electron itself, which has a «zero» size [1, 2], by choosing an arbitrary point $\vec{r}$ is zero. Let us clarify that the electron is a point particle, but at the micro level it is described by a wave function, and the measurement of its coordinate is subject to the Heisenberg uncertainty principle. Speaking about an electron as a macro object we mean rather its probabilistic size - for example, $\pm 2 \sigma_{r}\left(\sigma_{r}\right.$ is the standard deviation of function $\left.|\Psi|^{2}\right)$ than its geometric size (in this case it is infinitely small).

The eigenfield $\overrightarrow{\mathrm{E}}_{\Psi}$ of such a macro object does not have to obey the Maxwell macroscopic equations. As a result, there is a need to construct a statistical analogue of the Maxwell equations at the micro level, in which the values of the density of charge $\rho$ and current $\vec{J}$ are of a statistical/probabilistic nature ( $\rho_{\Psi}=q|\Psi|^{2}, \overrightarrow{\mathrm{~J}}_{\Psi}=\rho_{\Psi}\langle\vec{v}\rangle$ ). The procedure for averaging the Maxwell statistical equations should lead to the known macroscopic Maxwell equations.

From the above it follows that the question of the existence or absence of electromagnetic radiation in an atom cannot be resolved from the standpoint of the Maxwell macroscopic equations and classical electrodynamics. The answer to this question requires the construction of field equations on probability principles.

In this paper, the construction and analysis of statistical field equations is carried out based on the first principle - the probability conservation law. The probability conservation law underlies the infinite self-linking chain of the Vlasov equations for probability density function $f_{\infty}(\vec{r}, \vec{v}, \dot{\vec{v}}, ., t)$ in infinite-dimensional generalized phase space $\Omega_{\infty}$ (GPS) [3].

The work has the following structure. Section 1 briefly provides basic information on distribution functions, mean kinematical values and the Vlasov equation chain. The laws of conservation of motion and energy are written for the first two equations of the Vlasov chain. In $\S 2$ we consider the first and second Vlasov equations. From the first equation of the chain the Schrödinger equation taking into account the Lorentz gauge, the equation of motion of a charged particle in an electromagnetic field as well as the Hamilton-Jacobi equation are obtained. The concept of the self-consistent system is introduced, which makes it possible to construct the Maxwell equations both for external fields and for probabilistic (quantum) fields of a particle. The connection is shown between the second Vlasov equation and the Moyal equation for the Wigner function of the quasi-density probabilities of a quantum system in phase space. Paragraph 3 is devoted to obtaining the Pauli equation and the Dirac equation from the first Vlasov equation by increasing the amount of kinematic information about the quantum system. The basis is the Helmholtz decomposition (§1) for the vector field of probability flow, in which the transition is made from a complex function to a spinor and a bispinor. An analogue of the Helmholtz theorem for a vector field defined in 4D space-time is considered. In $\S 4$ we consider self-consistent and many-particle systems and the methods of their description in the relativistic and non-relativistic limit. A theorem is proven about the absence of electromagnetic radiation in a self-consistent, time-independent (in phase space) quantum system. The exact solutions of selfconsistent, non-relativistic quantum and classical model systems are analyzed in detail. The interrelation between the self-consistency and the many-particle system is described. The Conclusions paragraph contains a description of the main work results. The proofs of the theorems are given in Appendix A, B and C.

## §1 First principle

In the middle of the 20th century Vlasov invented the first principle - the probability conservation law in the generalized phase space $\Omega_{\infty}[4]$ :

$$
\begin{equation*}
\frac{\partial}{\partial t} f_{\infty}+\operatorname{div}_{\xi}\left(f_{\infty} \vec{u}_{\xi}\right)=0, \tag{1.1}
\end{equation*}
$$

where $\operatorname{div}_{\xi} \stackrel{\text { det }}{=} \operatorname{div}_{r}+\operatorname{div}_{v}+\operatorname{div}_{\dot{v}}+\operatorname{div}_{i v}+\ldots ; f_{\infty}(\vec{\xi}, t)$ is a distribution function or probability density function; $\vec{\xi}=\{\vec{r}, \vec{v}, \dot{\vec{v}}, \ddot{\vec{v}}, \dddot{\vec{v}}, \ldots\}^{T} \in \Omega_{\infty} ; \vec{u}_{\xi}{ }^{\text {det }}\{\vec{v}, \dot{\vec{v}}, \ddot{\vec{v}}, \ldots\}^{T}$ is a generalized velocity. For each point $\vec{\xi}_{0}=\left\{\vec{r}_{0}, \vec{v}_{0}, \dot{\vec{v}}_{0}, \ldots\right\}^{T} \in \Omega_{\infty}$ there is a corresponding generalized phase trajectory [5]:

$$
\begin{gather*}
\vec{\xi}(t)=e^{t \hat{D}} \vec{\xi}_{0}=M_{\infty}(t) \vec{\xi}_{0},  \tag{1.2}\\
M_{N_{a}}(t) \stackrel{\vec{u}_{\xi}=e^{t \hat{D}} \vec{u}_{\xi_{0}}, \dot{\vec{u}}_{\xi}=e^{t \hat{\rightharpoonup}} \dot{\vec{u}}_{\xi_{0}}, \ddot{\vec{u}}_{\xi}=e^{t \hat{D}} \ddot{\vec{u}}_{\xi_{0}}, \ldots}{\left(\begin{array}{ccccc}
1 & t & t^{2} / 2 & t^{3} / 3!\ldots \\
0 & 1 & t & t^{2} / 2 \ldots \\
0 & 0 & 1 & t \ldots \\
\ldots & \ldots & \ldots & \ldots . \\
0 & \ldots & \ldots & 0 & 1
\end{array}\right), \quad M_{\infty}=\lim _{N_{a} \rightarrow+\infty} M_{N_{a}}, \operatorname{det} M_{\infty}=1,} \tag{1.3}
\end{gather*}
$$

where $M_{\infty}$ is the Taylor evolutional matrix in GPS, and $M_{N_{a}}$ is an evolutional matrix $N_{a} \times N_{a}$ with order of approximation or averaging $N_{a} ; \hat{D}$ is the differentiation operator along the generalized phase trajectory $\vec{u}_{\xi} \stackrel{\text { det }}{=} \hat{D} \stackrel{\text { det }}{=}\{\vec{v}, \dot{\vec{v}}, \ddot{\vec{v}}, \ldots\}^{T}$. Note that point $\vec{\xi}_{0} \in \Omega_{\infty}$ defines the oneparameter group of the Lie transformations.

Continual integration of equation (1.1) over phase subspaces leads to the infinite selflinking chain of the Vlasov equations:

$$
\begin{align*}
& \frac{\partial}{\partial t} f^{1}+\operatorname{div}_{r}\left(f^{1}\langle\vec{v}\rangle_{1}\right)=0, \\
& \frac{\partial}{\partial t} f^{1,2}+\operatorname{div}_{r}\left(f^{1,2} \vec{v}\right)+\operatorname{div}_{v}\left(f^{1,2}\langle\dot{\vec{v}}\rangle_{1,2}\right)=0,  \tag{1.5}\\
& \frac{\partial}{\partial t} f^{1,2,3}+\operatorname{div}_{r}\left(f^{1,2,3} \vec{v}\right)+\operatorname{div}_{v}\left(f^{1,2,3} \dot{\vec{v}}\right)+\operatorname{div}_{\dot{v}}\left(f^{1,2,3}\langle\ddot{\vec{v}}\rangle_{1,2,3}\right)=0,
\end{align*}
$$

where
$f^{0}=N(t)=\int_{\Omega^{1}} f^{1}(\vec{r}, t) d^{3} r=\int_{\Omega^{1}} \int_{\Omega^{2}} f^{1,2}(\vec{r}, \vec{v}, t) d^{3} r d^{3} v=\int_{\Omega^{1}} \int_{\Omega^{2}} \int_{\Omega^{3}} f^{1,2,3}(\vec{r}, \vec{v}, \dot{\vec{v}}, t) d^{3} r d^{3} v d^{3} \dot{v}=\ldots$

$$
\begin{align*}
& f^{1}(\vec{r}, t)\langle\vec{v}\rangle_{1}(\vec{r}, t)=\int_{\Omega^{2}} f^{1,2}(\vec{r}, \vec{v}, t) \vec{v} d^{3} v, \\
& f^{1,2}(\vec{r}, \vec{v}, t)\langle\dot{\vec{v}}\rangle_{1,2}(\vec{r}, \vec{v}, t)=\int_{\Omega^{3}} f^{1,2,3}(\vec{r}, \vec{v}, \dot{\vec{v}}, t) \dot{\vec{v}} d^{3} \dot{v},  \tag{1.6}\\
& f^{1}(\vec{r}, t)\langle\dot{\vec{v}}\rangle_{1}(\vec{r}, t)=\int_{\Omega^{2}} f^{1,2}\langle\dot{\vec{v}}\rangle_{1,2} d^{3} v, \ldots
\end{align*}
$$

Integration in (1.6) is carried out over the corresponding phase subspaces $\Omega_{\infty}=\Omega^{1} \times \Omega^{2} \times \ldots$ Value $N(t)$ corresponds to the number of particles in the system when $f$ stands for the distribution function and it corresponds to a normalization factor when $f$ is the probability density. Depending on the interpretation of function $f$, the average vector field $\langle\vec{v}\rangle_{1}(\vec{r}, t)$ determines the velocity of the continuous medium in point $\vec{r}$ at time $t$ or alternatively the velocity of the probability flow. Similarly, $\langle\dot{\vec{v}}\rangle_{1,2}$ is the acceleration field, and $\langle\ddot{\vec{v}}\rangle_{1,2,3}$ is the second-order acceleration field. Note that the integration of the third Vlasov equation (1.5) over acceleration space $\int d^{3} \dot{v}$ will give the second equation (1.5) for the function $f^{1,2}$, and integration of the second equation over velocity space $\int d^{3} v$ will lead to the first equation for function $f^{1}$.

If one multiplies the second Vlasov equation by the velocity $\vec{v}$ and integrate over space $\Omega^{2}$, we obtain the equation of motion in the hydrodynamic approximation:

$$
\begin{gather*}
\hat{\pi}_{1}\left\langle v_{k}\right\rangle=\left(\frac{\partial}{\partial t}+\left\langle v_{\lambda}\right\rangle \frac{\partial}{\partial x^{k}}\right)\left\langle v_{k}\right\rangle=-\frac{1}{f_{1}} \frac{\partial P_{k \lambda}}{\partial x^{k}}+\left\langle\dot{v}_{k}\right\rangle_{1},  \tag{1.7}\\
P_{k \lambda}=\int_{\Omega^{2}} f^{1,2}\left(v_{k}-\left\langle v_{k}\right\rangle\right)\left(v_{\lambda}-\left\langle v_{\lambda}\right\rangle\right) d^{3} v, \tag{1.8}
\end{gather*}
$$

where $P_{k \lambda}$ is a pressure tensor. Multiplying the second Vlasov equation by $v^{2}$, and integrating it over velocity space $\Omega^{2}$, we obtain the law of conservation of energy for continuum mechanics:

$$
\begin{gather*}
\frac{\partial}{\partial t}\left[\frac{f^{1}}{2}|\langle\vec{v}\rangle|^{2}+\frac{1}{2} \operatorname{Tr} P_{k k}\right]+\frac{\partial}{\partial x^{2}}\left[\frac{f^{1}}{2}|\langle\vec{v}\rangle|^{2}\left\langle v_{\lambda}\right\rangle+\frac{1}{2}\left\langle v_{\lambda}\right\rangle \operatorname{Tr} P_{k k}+\left\langle v_{k}\right\rangle P_{k \lambda}+\frac{1}{2} \operatorname{Tr} P_{k k \lambda}\right]=\int_{\Omega^{2}} f^{1,2}\left\langle\dot{v}_{k}\right\rangle v_{k} d^{3} v, \\
P_{k \lambda s}=\int_{\Omega^{2}}^{\operatorname{det}}\left(v_{k}-\left\langle v_{k}\right\rangle_{1}\right)\left(v_{\lambda}-\left\langle v_{\lambda}\right\rangle_{1}\right)\left(v_{s}-\left\langle v_{s}\right\rangle_{1}\right) f^{1,2} d^{3} v . \tag{1.9}
\end{gather*}
$$

The first term in (1.9) determines the change in energy density over time, the second term corresponds to the divergence of the energy current density, and the right-hand side represents the work of external forces.

The chain of equations (1.5) can be written in a compact and physically clear form [6]:

$$
\begin{equation*}
\hat{\pi}_{1 . \ldots n} S^{1 . \ldots n}=-Q_{1 . \ldots n}, n \in \mathbb{N}, \tag{1.10}
\end{equation*}
$$

where

$$
\begin{align*}
& \hat{\pi}_{1 . \ldots n}=\frac{\partial}{\partial t}+\vec{v} \nabla_{r}+\dot{\vec{v}} \nabla_{v}+\ldots+\left\langle\vec{\xi}^{n+1}\right\rangle_{1 \ldots n} \nabla_{\xi^{n}},  \tag{1.11}\\
& S^{1, \ldots n}=\operatorname{det} f^{1, \ldots n}, Q_{1, \ldots, n}=\operatorname{div}_{\xi^{n}}\left\langle\vec{\xi}^{n+1}\right\rangle_{1, \ldots, n} . \tag{1.12}
\end{align*}
$$

Operator (1.11) corresponds to the total derivative with respect to time in phase subspace of $n$ kinematical values. Values $Q_{1, \ldots, n}(1.12)$ determine the density of dissipation sources. As a result, from equations (1.10) it follows that the change in the probability density along the phase trajectory (1.2) is equal to the sources of dissipation. The first Vlasov equation $(n=1)$ is known as the continuity equation in continuum mechanics and the field theory. If there are no sources of dissipation $Q_{1} \stackrel{\text { det }}{=} \operatorname{div}_{r}\langle\vec{v}\rangle_{1}=0$, then the probability density $f^{1}=$ const along the trajectory of motion (1.2) in the coordinate space. The second equation $(n=2)$ is used in plasma physics, astrophysics, solid state physics, statistical physics [7-13] and is known as the Vlasov equation. If there are no sources of dissipation $Q_{1,2}=\operatorname{div}_{v}\langle\dot{\vec{v}}\rangle_{1,2}, Q_{1,2}=0$, then the second Vlasov equation turns into the Liouville equation [14]. Therefore, density function $f^{1,2}=$ const along the phase trajectory (1.2).

The chain of Vlasov equations (1.10) can be written via the Boltzmann $H^{1, \ldots n}$-functions:

$$
\begin{equation*}
H^{1, \ldots, n}(t) \stackrel{\operatorname{det}}{=} \frac{1}{f^{0}} \int_{\Omega^{1}} \ldots \int_{\Omega^{n}} f^{1, \ldots, n}\left(\vec{\xi}^{1, \ldots, n}, t\right) S^{1, \ldots, n} \prod_{s=1}^{n} d^{3} \xi^{k_{s}}=\left\langle S^{1, \ldots,{ }_{n}}\right\rangle_{0}(t), \tag{1.13}
\end{equation*}
$$

which satisfy the evolutionary equations:

$$
\begin{equation*}
\hat{\pi}_{0}\left[f^{0} H^{1 \ldots n}\right]=-f^{0}\left\langle Q_{1 \ldots n}\right\rangle_{0}, n \in \mathbb{N}, \tag{1.14}
\end{equation*}
$$

where operator $\hat{\pi}_{0}=d / d t$. In statistical physics, the most known is the Boltzmann $H$-function corresponding to (1.13) $H^{1,2}$. According to equation (1.14), the evolution of Boltzmann $H^{1, \ldots, n}$ function is determined by the average sources of dissipation $\left\langle Q_{1 \ldots n}\right\rangle_{0}$.

## §2 Schrödinger equation

The Schrödinger equation was known to be originally obtained using a phenomenological method [15]. In [16] the Schrödinger equation, the Hamilton-Jacobi equation, the equation of motion of a charged particle in an electromagnetic field and the Maxwell system of equations are obtained from the first Vlasov equation (1.10), which is based on the first principle of the probability conservation law (1.1).

When obtaining the Schrödinger equation for a scalar particle in an electromagnetic field [16], the Coulomb gauge was used $\operatorname{div}_{r} \vec{A}=0$. Note that an analogue result may be obtained for the Lorentz gauge $\operatorname{div}_{r} \vec{A}=\chi$, where $q c^{2} \chi=-\partial U / \partial t$. For clarity of notation, we omit the indices of functions and fields indicating the type of phase subspace.

The first Vlasov equation (1.5) contains two unknown functions $f^{1}=f(\vec{r}, t)$ and $\langle\vec{v}\rangle_{1}=\langle\vec{v}\rangle(r, t)$. Such an equation is possible to be solved in two cases: either function $\langle\vec{v}\rangle$ or $f$ is known, or there is a mathematical relationship between functions $\langle\vec{v}\rangle$ and $f$. From a physical point of view, the Vlasov equation describes a certain physical object, therefore functions $\langle\vec{v}\rangle$ and $f$ must correspond to the same object. Therefore, it is logical to consider the second case, when there is a mathematical relationship between $\langle\vec{v}\rangle$ and $f$ in addition to the Vlasov equation. As the simplest mathematical object that combines two quantities, one can take a complex function. A complex function is known to have two components: a real and an
imaginary part. On the complex plane it is convenient to work with the Euler representation of a complex number in the form of its modulus and phase.

Based on the above, we will use complex function $\Psi \in \mathbb{C}$ as a representation that combines two real functions $f$ and $\langle\vec{v}\rangle$ included in the first Vlasov equation. Let us set the modulus of complex number $\Psi$ equal to $|\Psi|^{2}=f \geq 0$. The positivity property of probability density $f$ will be satisfied automatically. We relate scalar phase $\varphi=\arg \Psi$ with vector field $\langle\vec{v}\rangle$ via the Helmholtz theorem [17] on the decomposition of the vector field (1.6) into vortex $\overrightarrow{\mathrm{A}}_{\Psi}$ and potential $\nabla_{r} \Phi$ fields:

$$
\begin{equation*}
\langle\vec{v}\rangle(\vec{r}, t)=-\alpha \nabla_{r} \Phi(\vec{r}, t)+\gamma \overrightarrow{\mathrm{A}}_{\Psi}(\vec{r}, t), \operatorname{div}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}=\chi \tag{2.1}
\end{equation*}
$$

where $\alpha, \gamma$ are some constant values, and $\chi$ is some free function. Let us transform expansion (2.1)

$$
\begin{array}{r}
\langle\vec{v}\rangle=i^{2} \alpha \nabla_{r} \Phi+\gamma \overrightarrow{\mathrm{A}}_{\Psi}=i \alpha \nabla_{r}(0+i \Phi)+\gamma \overrightarrow{\mathrm{A}}_{\Psi}=i \alpha \nabla_{r}\left(\ln \left|\frac{\Psi}{\Psi^{*}}\right|+i \Phi\right)+\gamma \overrightarrow{\mathrm{A}}_{\Psi}, \\
\frac{\Psi}{\Psi^{*}}=e^{i 2 \varphi} \Rightarrow \operatorname{Arg}\left[\frac{\Psi}{\Psi^{*}}\right]=2 \varphi(\vec{r}, t)+2 \pi k \stackrel{\operatorname{det}}{=} \Phi(\vec{r}, t), k \in \mathbb{Z} . \tag{2.3}
\end{array}
$$

Taking into account definition (2.3), the representation (2.2) of average velocity takes the form:

$$
\langle\vec{v}\rangle(\vec{r}, t)=i \alpha \nabla_{r}\left(\ln \left|\frac{\Psi}{\Psi^{*}}\right|+i \operatorname{Arg}\left[\frac{\Psi}{\Psi^{*}}\right]\right)+\gamma \overrightarrow{\mathrm{A}}_{\Psi}=i \alpha \nabla_{r} \operatorname{Ln}\left[\frac{\Psi}{\Psi^{*}}\right]+\gamma \overrightarrow{\mathrm{A}}_{\Psi},
$$

or

$$
\begin{gather*}
\overrightarrow{\mathrm{J}}(\vec{r}, t) \stackrel{\operatorname{det}}{=} f(\vec{r}, t)\langle\vec{v}\rangle(\vec{r}, t)=i \alpha\left[\Psi^{*} \nabla_{r} \Psi-\Psi \nabla_{r} \Psi^{*}\right]+\gamma \Psi^{*} \overrightarrow{\mathrm{~A}}_{\Psi} \Psi, \\
\mathrm{J}^{k}=-\alpha \Psi^{*} \partial^{k} \Phi \Psi+\gamma \Psi^{*} \mathrm{~A}_{\Psi}^{k} \Psi, k=1 \ldots 3, \tag{2.4}
\end{gather*}
$$

where $\overrightarrow{\mathbf{J}}(\vec{r}, t)$ is the current density and it is taking into account that $|\Psi|^{2}=f$.
Definition 1 Representation (2.4) of vector field $\langle\vec{v}\rangle(\vec{r}, t)$ or field $\overrightarrow{\mathrm{J}}(\vec{r}, t)$ for $\Psi \in \mathbb{C}$ will be called the Helmholtz $\Psi \mathbb{C}$-decomposition with gauge $\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}=\chi$.

Theorem 1 If the vector field $\langle\vec{v}\rangle(\vec{r}, t)$ of the probability flow admits a Helmholtz $\Psi \mathbb{C}-$ decomposition with gauge $\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}=\chi$, then the first Vlasov equation (1.5) for function $f^{1}=\Psi \Psi^{*}$ becomes the $\Psi$-equation:

$$
\begin{equation*}
\frac{i}{\beta} \frac{\partial \Psi}{\partial t}=-\alpha \beta\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2} \Psi+U \Psi \tag{2.5}
\end{equation*}
$$

where $\hat{\mathrm{p}} \stackrel{\text { det }}{=}-\frac{i}{\beta} \nabla_{r}$ and $\beta \neq 0, \beta \in \mathbb{R}$, is a constant value; $U(\vec{r}, t) \in \mathbb{R}$ is some function.

The proof of Theorem 1 is given in Appendix A.
If constant values $\alpha, \beta, \gamma$ are chosen as:

$$
\begin{equation*}
\alpha=-\frac{\hbar}{2 m}, \beta=\frac{1}{\hbar}, \gamma=-\frac{q}{m}, \tag{2.6}
\end{equation*}
$$

then equation (2.5) will take the form of the Schrödinger equation for a scalar particle in an electromagnetic field

$$
\begin{equation*}
i \hbar \frac{\partial \Psi}{\partial t}=\frac{1}{2 m}\left(\hat{\mathrm{p}}-q \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2} \Psi+U \Psi \tag{2.7}
\end{equation*}
$$

where $\hat{\mathrm{p}}=-i \hbar \nabla_{r}$. corresponds to the momentum operator. Note that, despite the presence of free function $\chi$, the equation $(2.5) /(2.7)$ does not depend on it explicitly. Let us show that function $U$ corresponds to the potential energy.

Theorem 2 Function $U(\vec{r}, t) \in \mathbb{R}$ from equation (2.5) corresponds the equation:

$$
\begin{equation*}
-\frac{1}{\beta} \frac{\partial \varphi}{\partial t}=-\frac{1}{4 \alpha \beta}|\langle\vec{v}\rangle|^{2}+\mathrm{V} \stackrel{\operatorname{det}}{=} \mathrm{H}, \tag{2.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{V}=U+\mathrm{Q}, \quad \mathrm{Q}=\frac{\alpha}{\beta} \frac{\Delta_{r}|\Psi|}{|\Psi|}, \tag{2.9}
\end{equation*}
$$

and vector field $\langle\vec{v}\rangle(\vec{r}, t)$ of the probability flow satisfies the equation of motion:

$$
\begin{equation*}
\hat{\pi}_{1}\langle\vec{v}\rangle=\frac{d}{d t}\langle\vec{v}\rangle=-\gamma\left(\overrightarrow{\mathrm{E}}_{\Psi}+\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}_{\Psi}\right), \tag{2.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\overrightarrow{\mathrm{E}}_{\Psi} \stackrel{\text { det }}{=}-\frac{\partial \overrightarrow{\mathrm{A}}_{\Psi}}{\partial t}-\frac{2 \alpha \beta}{\gamma} \nabla_{r} \mathrm{~V}, \quad \overrightarrow{\mathrm{~B}}_{\Psi} \stackrel{\operatorname{det}}{=} \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}_{\Psi} . \tag{2.11}
\end{equation*}
$$

The proof of Theorem 2 is given in Appendix A.
When replacing (2.6), equation (2.8) becomes the Hamilton-Jacobi equation $-\hbar \frac{\partial \varphi}{\partial t}=\frac{m}{2}|\langle\vec{v}\rangle|^{2}+\mathrm{V}=\mathrm{H}$, and expression (2.9) determines the quantum potential $\mathrm{Q}=-\frac{\hbar^{2}}{2 m} \frac{\Delta_{r}|\Psi|}{|\Psi|}$ from the de Broglie-Bohm «pilot-wave» theory [18-20]. Note that potential V in the Hamilton function H differs from potential $U$ in the Schrödinger equation by the value of the quantum potential Q . In classical limit $\hbar \ll 1$ we can assume that $\mathrm{V} \approx U$. Value $\hbar \varphi=\mathrm{S}$ determines the action.

Definition 2 We will call the relationship between vortex field $\overrightarrow{\mathrm{A}}_{\Psi}$ component (2.4) and scalar potential V (2.9) using the equation (2.9)

$$
\begin{equation*}
\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}+\frac{2 \alpha \beta}{\gamma} \frac{1}{c^{2}} \frac{\partial \mathrm{~V}}{\partial t}=0 \tag{2.12}
\end{equation*}
$$

## Lorentz $\Psi$-gauge.

Without loss of generality, vortex field $\overrightarrow{\mathrm{A}}_{\Psi}$ can be represented as the sum of classical vector potential $\overrightarrow{\mathrm{A}}$ and some «quantum» vector potential $\overrightarrow{\mathrm{A}}_{Q}$, that is, (2.11)

$$
\begin{equation*}
\overrightarrow{\mathrm{A}}_{\Psi} \stackrel{\operatorname{det}}{=} \overrightarrow{\mathrm{A}}+\overrightarrow{\mathrm{A}}_{Q}, \overrightarrow{\mathrm{~B}}_{\Psi}=\operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}+\operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}_{Q}=\overrightarrow{\mathrm{B}}+\overrightarrow{\mathrm{B}}_{Q} . \tag{2.13}
\end{equation*}
$$

In the classical approximation $(\hbar \ll 1)$ for $\mathrm{Q}=0$ and $\overrightarrow{\mathrm{A}}_{Q}=\vec{\theta}$ according to (2.11), (2.12) and (2.6), the Lorentz $\Psi$-gauge (2.12) turns into the usual Lorentz gauge

$$
\begin{equation*}
\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}+\frac{1}{q c^{2}} \frac{\partial U}{\partial t}=0 \tag{2.14}
\end{equation*}
$$

or

$$
\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}+\frac{1}{c^{2}} \frac{\partial \varphi}{\partial t}=0
$$

In the general case, when taking into account quantum potential Q and the vector potential $\overrightarrow{\mathrm{A}}_{Q}, \Psi$-gauge (2.12) splits into Lorentz gauge (2.14) and quantum gauge

$$
\begin{equation*}
\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}_{Q}+\frac{1}{q c^{2}} \frac{\partial \mathrm{Q}}{\partial t}=0 . \tag{2.15}
\end{equation*}
$$

Electric field $\overrightarrow{\mathrm{E}}_{\Psi}$ (2.11) takes the form:

$$
\begin{equation*}
\overrightarrow{\mathrm{E}}_{\Psi}=-\frac{\partial \overrightarrow{\mathrm{A}}}{\partial t}-\frac{1}{q} \nabla_{r} U-\frac{\partial \overrightarrow{\mathrm{A}}_{Q}}{\partial t}-\frac{1}{q} \nabla_{r} \mathrm{Q}=\overrightarrow{\mathrm{E}}+\overrightarrow{\mathrm{E}}_{Q} \tag{2.16}
\end{equation*}
$$

where $\overrightarrow{\mathrm{E}}, \overrightarrow{\mathrm{B}}$ are classical electromagnetic fields; and $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$ are electromagnetic fields caused by the probability distributions of the quantum system. The charge $\rho$ and current $\overrightarrow{\mathrm{J}}$ densities are determined for classical fields $\vec{E}, \vec{B}$. For a quantum system located in external classical fields, according to the uncertainty principle, charge density $q f$ (related to the coordinate) and current density $q f \vec{v}$ (related to velocity $\vec{v}$ ) are not determined. The indicated densities are not classical electrodynamics characteristics, they are probabilistic by nature. For example, field $\overrightarrow{\mathrm{E}}_{Q}$ (2.16) includes the quantum potential Q , expressed in terms of density $\sqrt{f}=|\Psi|$ as (2.9). Dependence (2.9), in the general case, does not agree with the Coulomb representation $\Delta_{r} \mathrm{Q}=-q f / \varepsilon_{0}$.

Definition 3 We will say that the system is self-consistent if the condition (2.17) is satisfied for $\rho_{\Psi} \stackrel{\text { det }}{=} \varepsilon_{0} \operatorname{div}_{r} \overrightarrow{\mathrm{E}}_{\Psi} \stackrel{\text { det }}{=} \operatorname{div}_{r} \overrightarrow{\mathrm{D}}_{\Psi}$

$$
\begin{equation*}
\rho_{\Psi}=q f . \tag{2.17}
\end{equation*}
$$

Lemma 1 For a self-consistent system, the following field equations are valid:

$$
\begin{gather*}
\operatorname{div}_{r} \overrightarrow{\mathrm{D}}_{\Psi}=\rho_{\Psi}, \quad \operatorname{div}_{r} \overrightarrow{\mathrm{~B}}_{\Psi}=0,  \tag{2.18}\\
\operatorname{curl}_{r} \overrightarrow{\mathrm{E}}_{\Psi}=-\frac{\partial \overrightarrow{\mathrm{B}}_{\Psi}}{\partial t}, \quad \frac{\partial}{\partial t} \overrightarrow{\mathrm{D}}_{\Psi}+\overrightarrow{\mathrm{J}}_{\Psi}=\operatorname{curl}_{r} \overrightarrow{\mathrm{H}}_{\Psi}, \tag{2.19}
\end{gather*}
$$

where $\overrightarrow{\mathrm{J}}_{\Psi} \stackrel{\text { det }}{=} \rho_{\Psi}\langle\vec{v}\rangle$, a $\overrightarrow{\mathrm{H}}_{\Psi}$ are some field, for example, $\mu_{0} \overrightarrow{\mathrm{H}}_{\Psi} \stackrel{\text { det }}{=} \overrightarrow{\mathrm{B}}_{\Psi}$. In this case, the Lorentz $\Psi$-gauge condition (2.12) brings the system of equations (2.18)-(2.19) to the form:

$$
\begin{align*}
& \square \varphi_{\Psi}=\frac{\rho_{\Psi}}{\varepsilon_{0}}, \quad \square \overrightarrow{\mathrm{~A}}_{\Psi}=\mu_{0} \overrightarrow{\mathrm{~J}}_{\Psi},  \tag{2.20}\\
& \varphi_{\Psi}=\varphi+\varphi_{Q}, \quad \overrightarrow{\mathrm{~J}}_{\Psi}=\overrightarrow{\mathrm{J}}+\overrightarrow{\mathrm{J}}_{Q},  \tag{2.21}\\
& \mathrm{~V} \stackrel{\operatorname{det}}{=} q \varphi_{\Psi}, \mathrm{Q} \stackrel{\text { det }}{=} q \varphi_{Q}, U=q \varphi, \quad \rho_{\Psi}=\rho+\rho_{Q}, \tag{2.22}
\end{align*}
$$

where $\rho_{Q} \stackrel{\text { det }}{=} \varepsilon_{0} \operatorname{div}_{r} \overrightarrow{\mathrm{E}}_{Q} \stackrel{\text { det }}{=} \operatorname{div}_{r} \overrightarrow{\mathrm{D}}_{Q}$.
The proof of Lemma 1 is given in Appendix A.
Remark 1 In the classical limit $(\hbar \ll 1)$ at $\mathrm{Q}=0$ and $\overrightarrow{\mathrm{A}}_{Q}=\vec{\theta}$ equations (2.18)-(2.19) for a selfconsistent system transform into the well-known Maxwell equations for classical electromagnetic fields $\vec{E}, \vec{B}$ :

$$
\begin{gather*}
\operatorname{div}_{r} \overrightarrow{\mathrm{D}}=\rho, \operatorname{div}_{r} \overrightarrow{\mathrm{~B}}=0,  \tag{2.23}\\
\operatorname{curl}_{r} \overrightarrow{\mathrm{E}}=-\frac{\partial \overrightarrow{\mathrm{B}}}{\partial t}, \quad \frac{\partial}{\partial t} \overrightarrow{\mathrm{D}}+\overrightarrow{\mathrm{J}}=\operatorname{curl}_{r} \overrightarrow{\mathrm{H}}, \tag{2.24}
\end{gather*}
$$

where $\overrightarrow{\mathrm{J}}=\rho\langle\vec{v}\rangle$ and $\mu_{0} \overrightarrow{\mathrm{H}}=\overrightarrow{\mathrm{B}}$. In this case, the Lorentz gauge (2.14) can be performed. In the general case, when taking into account «quantum» fields $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$, and having classical Maxwell equations (2.23)-(2.24) being true for fields $\overrightarrow{\mathrm{E}}, \overrightarrow{\mathrm{B}}$, we obtain an analogue of the Maxwell equations for fields $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$ :

$$
\begin{gather*}
\operatorname{div}_{r} \overrightarrow{\mathrm{D}}_{Q}=\rho_{Q}, \quad \operatorname{div}_{r} \overrightarrow{\mathrm{~B}}_{Q}=0,  \tag{2.25}\\
\operatorname{curl}_{r} \overrightarrow{\mathrm{E}}_{Q}=-\frac{\partial \overrightarrow{\mathrm{B}}_{Q}}{\partial t}, \quad \frac{\partial}{\partial t} \overrightarrow{\mathrm{D}}_{Q}+\overrightarrow{\mathrm{J}}_{Q}=\operatorname{curl}_{r} \overrightarrow{\mathrm{H}}_{Q}, \tag{2.26}
\end{gather*}
$$

where $\overrightarrow{\mathrm{J}}_{Q}=\rho_{Q}\langle\vec{v}\rangle$ and $\mu_{0} \overrightarrow{\mathrm{H}}_{Q}=\overrightarrow{\mathrm{B}}_{Q}$. In this case, for equations (2.25)-(2.26) quantum gauge (2.15) can be performed.

Remark 2 Note that field equations (2.18)-(2.19) for self-consistent systems are obtained only on the basis of the first Vlasov equation (1.5), which is a consequence of the first principle - the probability conservation law. Equations (2.18)-(2.19) in the form (2.20) are representable through 4-potential $A^{\mu}, \mu=0 \ldots 3$

$$
\begin{gather*}
\square A^{\mu}=\mu_{0} J^{\mu},  \tag{2.27}\\
A^{\mu}=\left(\frac{\varphi_{\Psi}}{c}, \overrightarrow{\mathrm{~A}}_{\Psi}\right), \quad J^{\mu}=\left(c \rho_{\Psi}, \overrightarrow{\mathrm{J}}_{\Psi}\right),
\end{gather*}
$$

where $\square=\partial^{\mu} \partial_{\mu}, \partial_{\mu}=\frac{\partial}{\partial x^{\mu}}, \partial^{\mu}=g^{\mu \nu} \partial_{v}$. Metric tensor $g^{\mu \nu}=(+,-,-,-)$ corresponds to the d'Alembert operator (2.27). Equation (2.27) is invariant under the Lorentz transformation, thus it is a relativistic equation. The problem of integration (averaging) over the velocity space (1.3) is solved rather by momentum than velocity. Indeed, with $v \rightarrow c$ momentum $p \rightarrow \infty$. In this case, equation (2.27) is uniquely related to the «Schrödinger equation» (2.7). Note that equation (2.7) is not invariant under Lorentz transformations, since it contains derivatives of different orders. The question arises: «How can a relativistic theory follow from the non-relativistic Schrödinger equation?» The answer to this question has two points.

First, equation (2.27) was obtained for self-consistent systems (2.17). Condition (2.17) is very rigorous. As it is shown below in $\S 4$, nonrelativistic quantum systems, as a rule, do not satisfy this condition, although there are exceptions (depending on the type of gauge). The problem is that density $\rho_{\Psi}$ contains both classical charge density $\rho$ and quantum (probability) density $\rho_{Q}$. Density $\rho_{Q}$ is not the charge density in the usual sense. It is the classical «equivalent» of the charge density that induces field $\overrightarrow{\mathrm{E}}_{Q}$ (2.16). Field $\overrightarrow{\mathrm{E}}_{Q}$ is determined by quantum potential $-\nabla_{r} \mathrm{Q}$, which is related to probability density $f=|\Psi|^{2}$ not by the Coulomb law $\int \frac{f\left(\vec{r}^{\prime}, t\right)}{\left|\vec{r}-\vec{r}^{\prime}\right|} d^{3} r^{\prime}$, but by expression (2.9) $\frac{\Delta_{r}|\Psi|}{|\Psi|}$. A somewhat similar situation arises in the Dirac equation (see $\S 3,4$ ).

Secondly, equation (2.7) is not exactly the Schrödinger equation. In the general case, there is a vector potential $\overrightarrow{\mathrm{A}}_{\Psi}$ that differs, according to (2.13), from the classical vector potential $\overrightarrow{\mathrm{A}}$. A similar remark concerns the field equation itself (2.27), in which, in addition to vector potential $\overrightarrow{\mathrm{A}}_{\Psi}$, there is a scalar potential $\varphi_{\Psi}(2.21)$ associated with the quantum potential Q (2.9). In the nonrelativistic approximation, only external fields $\vec{E}, \vec{B}$ that satisfy the Maxwell equations remain, and the relation with particle fields (quantum fields) $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$ disappears. The classical Schrödinger equation does not contain the particle's own fields. There are only external potentials $\overrightarrow{\mathrm{A}}$ and $U=q \varphi$. The influence of the particle's own «field», for example, in the form of spin, is manifested in the equations of Pauli and Dirac (see §3).

Let us consider in more detail the physical meaning of probabilistic («quantum») fields, $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$. For this we will compare the two equations of motion (1.7) and (2.10). Equation (1.7) is obtained from the second Vlasov equation (see § 1). A special case of the second Vlasov
equation (1.5) for function $f^{1,2}$ is the Moyal equation for the Wigner function $W(\vec{r}, \vec{p}, t)$ of a quantum system in phase space [21, 22]

$$
\begin{equation*}
W(\vec{r}, \vec{p}, t)=\frac{1}{(2 \pi \hbar)^{3}} \int_{\mathbb{R}^{3}} \Psi^{*}\left(\vec{r}-\frac{\vec{s}}{2}, t\right) \Psi\left(\vec{r}+\frac{\vec{s}}{2}, t\right) e^{-\frac{i}{\hbar} \vec{s} \cdot \vec{p}} d^{3} S, \tag{2.28}
\end{equation*}
$$

where function $W(\vec{r}, \vec{p}, t)$ is written for the coherent state. The chain of equations (1.5) is selflinking, so to solve it a cut-off at some equation is necessary. The chain can be cut off by dynamic approximation of the average kinematical value (1.6). In [23, 24] the Vlasov-Moyal approximation was proposed

$$
\begin{equation*}
f^{1,2}\left\langle\dot{v}_{k}\right\rangle_{1,2}=\sum_{l=0}^{+\infty} \frac{(-1)^{l+1}(\hbar / 2)^{2 l}}{m^{2 l+1}(2 l+1)!} \frac{\partial U}{\partial x^{k}}\left(\bar{\nabla}_{r} \cdot \vec{\nabla}_{v}\right)^{2 l} f^{1,2} \tag{2.29}
\end{equation*}
$$

where $f^{1,2}(\vec{r}, \vec{v}, t)=m^{3} W(\vec{r}, \vec{p}, t)$. Substituting approximation (2.29) into the second Vlasov equation (1.5), we obtain the Moyal equation for the quasi-density probabilities [25]

$$
\begin{equation*}
\frac{\partial W}{\partial t}+\frac{1}{m} \vec{p} \cdot \nabla_{r} W-\nabla_{r} U \cdot \nabla_{p} W=\sum_{l=1}^{+\infty} \frac{(-1)^{l}(\hbar / 2)^{2 l}}{(2 l+1)!} U\left(\bar{\nabla}_{r} \cdot \vec{\nabla}_{p}\right)^{2 l+1} W . \tag{2.30}
\end{equation*}
$$

Note that the probability conservation law (1.1) does not impose conditions on the sign of function $f$. Consequently, negative values of the Wigner function $W(\vec{r}, \vec{p}, t)$ do not contradict the presented approach.

Averaging the Vlasov-Moyal approximation (2.29) over the space of velocities leads to an analogue of Newton's second law, which is present in the hydrodynamic equation of motion (1.7):

$$
\begin{equation*}
\left\langle\dot{v}_{k}\right\rangle_{1}=-\frac{1}{m} \frac{\partial U}{\partial x^{k}} . \tag{2.31}
\end{equation*}
$$

Approximation (2.31) was originally used in a phenomenological manner by Vlasov in his theory. Here it is obtained using the quantum mechanics in the phase space. The peculiarity of expression (2.31) is the presence of Schrödinger potential $U$ instead of V (2.9) from the Hamilton-Jacobi equation. Let us discuss this peculiarity in more detail. Since the Moyal equation (2.30) was obtained without taking into account the magnetic field, the corresponding equation of motion (2.10) will take the form

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+\left\langle v_{\lambda}\right\rangle \frac{\partial}{\partial x_{\lambda}}\right)\left\langle v_{k}\right\rangle=-\frac{1}{m} \frac{\partial \mathrm{~V}}{\partial x^{k}}=-\frac{1}{m} \frac{\partial U}{\partial x^{k}}-\frac{1}{m} \frac{\partial \mathrm{Q}}{\partial x^{k}} . \tag{2.32}
\end{equation*}
$$

Equation (2.32) is an analogue of the equation of motion (1.7), therefore, taking into account (2.31), we obtain

$$
\begin{equation*}
\frac{1}{m} \frac{\partial \mathrm{Q}}{\partial x^{k}}=\frac{1}{f_{1}} \frac{\partial P_{k \lambda}}{\partial x^{\lambda}} \tag{2.33}
\end{equation*}
$$

Expression (2.33) shows that quantum potential Q creates quantum pressure. That is, a quantum system allows for a «macroscopic» description.

Indeed, pressure tensor $P_{k \lambda}$ determines the covariance matrix when averaged over velocity space (1.8). On the one hand, a quantum system is a microscopic description, and on the other hand, it is a macroscopic description when averaged over higher kinematical values.

If we assume that function $f$ corresponds to the distribution function of particles of the same type, then in the macroscopic description $q f$ this is the charge density $\rho$ and equations (2.18)-(2.19) transform into the classical Maxwell equations (2.23)-(2.24). Let us illustrate this statement using the example of single-velocity distribution function $f^{1,2}$ :

$$
\begin{equation*}
f^{1,2}(\vec{r}, \vec{v}, t)=\rho(r, t) \delta[\vec{v}-\vec{u}(\vec{r}, t)], \tag{2.33}
\end{equation*}
$$

where $\vec{u}(\vec{r}, t)$ is the vector velocity field of the continuous medium. Substituting (2.33) into (1.6), we obtain $\vec{u}=\langle\vec{v}\rangle_{1}$ and $f^{1}=\rho$. Pressure tensor (1.8) $P_{k \lambda}=0$ corresponds to distribution function (2.33), that is, pressure force $-\frac{1}{f_{1}} \frac{\partial P_{k \lambda}}{\partial x^{k}}$ in equation (1.7) is absent and the quantum pressure according to (2.33) is equal to zero $\mathrm{Q}=0$. The absence of quantum potentials Q and $\overrightarrow{\mathrm{A}}_{Q}$, according to equations (2.13) and (2.16), gives a trivial solution to equations (2.25)-(2.26) for quantum fields $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$. As a result, the hydrodynamic equation of motion (1.7) transforms into the usual Euler equation, in which there is only an external force (2.31). Such a system is self-consistent (2.17) and corresponds to the Maxwell electrodynamics (2.23)-(2.24), but each particle is a point particle, which is acted upon by an external force (2.31). The intrinsic microscopic nature of the particle in the form of quantum pressure (2.33) is absent here.

Let us consider distribution (2.33) in more detail. If a phase volume of the phase space $\{\vec{r}, \vec{p}=m \vec{v}\}$ corresponds to the physical system, then for each point with coordinate $\vec{r}$ there is an uncertainty by velocity $\vec{v}$, described by the probability density function $f^{2}(\vec{v}, t)=\int_{\Omega^{1}} f^{1,2}(\vec{r}, \vec{v}, t) d^{3} r$. The same consideration is true for GPS, in which the infinite set of kinematical values $\vec{u}_{\xi}=\{\vec{v}, \dot{\vec{v}}, \ddot{\vec{v}}, \ldots\}^{T}$ corresponds to point $\vec{r}$, and these kinematical values determine the form of the generalized phase trajectory (1.2). The uncertainty by velocity $\vec{u}_{\xi}$ (due to the probability density) leads to the uncertainty by trajectory $\vec{\xi}(t)(1.2)$, which is consistent with the Heisenberg uncertainty principle. Due to the uncertainty by velocity, for some small volume $\delta \omega$ (macroscopic description) there is a spread of velocities (determined by $f^{2}$ ), leading to pressure $P_{k \lambda} \neq 0$ and $\mathrm{Q} \neq 0$.

If the physical system is represented by a phase hypersurface (not by a phase volume) in the phase space then each point $\vec{r}$ has its certain velocity $\vec{v}$. This situation is described by distribution (2.33). In this case the system consists of streams of particles interacting through the self-consistent potential $U$.

Remark 3 It should be noted that distribution (2.33) in the general case is not a solution to the Moyal equation (2.30). At $\rho \geq 0$ (or $\rho \leq 0$ ), the distribution (2.33) will be either positive or negative, which contradicts the Hudson theorem for the Wigner function (2.28). According to the Hudson theorem and its generalization to the 3D case [26, 27], the only positive (negative) Wigner function is the Gaussian distribution in the phase space. Nevertheless, it is possible to
construct a limiting transition from the Wigner function, which contains information about quantum pressure, to distribution (2.33), in which this information no longer exists. As an example, let us take the Wigner function for the ground state of a quantum harmonic oscillator with frequency $\omega$ [28]:

$$
\begin{gather*}
W_{0}(x, p)=\frac{1}{\pi \hbar} e^{-\varepsilon(x, p)}, \varepsilon(x, p)=\frac{p^{2}}{2 m^{2} \sigma_{v}^{2}}+\frac{x^{2}}{2 \sigma_{x}^{2}},  \tag{2.34}\\
\sigma_{x} \sigma_{v}=\frac{\hbar}{2 m}, \frac{\sigma_{v}}{\sigma_{x}}=\omega . \tag{2.35}
\end{gather*}
$$

Expression (2.35) corresponds to the Heisenberg uncertainty principle. In the case of (2.34), potential $U=m \omega^{2} x^{2} / 2$ in the Vlasov-Moyal approximation (2.29) gives only one nonzero term, that is $\langle\dot{v}\rangle_{1,2}=-\frac{1}{m} \frac{\partial U}{\partial x}$. As a result, the Moyal equation (2.30) and the second Vlasov equation (1.5) will take the form:

$$
\begin{equation*}
\left(\partial_{t}+\frac{p}{m} \partial_{x}-\partial_{x} U \partial_{p}\right) W_{0}=0 \tag{2.36}
\end{equation*}
$$

where $f^{1,2}(x, v)=m W(x, m v)$. According to (2.33) and (1.8), the quantum pressure has the form

$$
\begin{gather*}
P(x)=\sigma_{v}^{2} f^{1}(x),\langle v\rangle=0, \mathrm{Q}(x)=-\frac{\hbar^{2}}{8 m \sigma_{x}^{4}} x^{2}+\mathcal{E}_{0},  \tag{2.37}\\
f^{1}(x)=\frac{1}{\sqrt{2 \pi} \sigma_{x}} e^{-\frac{x^{2}}{2 \sigma_{x}^{2}}},
\end{gather*}
$$

where $\mathcal{E}_{0}=\hbar \omega / 2$ is the energy of the ground state. Equation of motion (2.32) corresponds to the equilibrium system:

$$
\begin{equation*}
-\frac{\partial U}{\partial x}-\frac{\partial \mathrm{Q}}{\partial x}=0 \tag{2.38}
\end{equation*}
$$

that is, external force $-\frac{\partial U}{\partial x}$ is balanced by force of quantum pressure $\frac{\partial \mathrm{Q}}{\partial x}$. In terms of hydrodynamics, a quantum particle is not a point particle. This particle is an object which is determined by a certain probability distribution density $f^{1}$, or a certain medium. Compression of this medium by an external field in the form $-\frac{\partial U}{\partial x}$ leads to a reaction of this medium in the form of quantum pressure $\frac{\partial \mathrm{Q}}{\partial x}$.

The limiting transition of distribution (2.34) to distribution (2.33) is possible at $\sigma_{p} \rightarrow 0$. Value $\sigma_{p}$ characterizes the uncertainty by momentum. The reduction of this uncertainty by momentum $\sigma_{p} \rightarrow 0$ results in transition to a single-velocity distribution (2.33):

$$
\begin{gather*}
\lim _{\sigma_{p} \rightarrow 0} W_{0}(x, p)=\frac{1}{2 \pi} e^{-\frac{x^{2}}{2 \sigma_{x}^{2}}} \lim _{\sigma_{p} \rightarrow 0} \frac{m}{\sigma_{x} \sigma_{p}} e^{-\frac{p^{2}}{2 \sigma_{p}^{2}}}=f^{1,2}(x, v)=\rho(x) \delta(v),  \tag{2.39}\\
\rho(x)=\lim _{\sigma_{x} \rightarrow+\infty} \frac{m}{\sqrt{2 \pi} \sigma_{x}} e^{-\frac{x^{2}}{2 \sigma_{x}^{2}}}=0, \tag{2.40}
\end{gather*}
$$

where the Heisenberg uncertainty principle (2.35) plays the role of normalizing the probability density function. Passage to the limit $\sigma_{p} \rightarrow 0$ leads to an increase in $\sigma_{x} \rightarrow+\infty$, which is known as the dispersion of the wave packet (2.40). For a harmonic oscillator, this means that its frequency (2.35) $\omega=\sigma_{v} / \sigma_{x} \rightarrow 0$. Quantum potential Q (2.37) and quantum pressure $\frac{\partial \mathrm{Q}}{\partial x}=-m \omega^{2} x$ also tend to zero. Indeed, passage to the limit $\sigma_{p} \rightarrow 0$ leads to a weakening of the external potential $U \rightarrow 0$, which localized the particle inside the potential well. As a result, according to (2.38), the force of quantum pressure decreases. From the position of continuum mechanics, passage to the limit $\sigma_{p} \rightarrow 0$ allows the following interpretation. The gas, localized in the pressure vessel, was released outside, which led to its distribution throughout the space (dispersion of the wave packet $\sigma_{x} \rightarrow+\infty$ ) and a drop in pressure $P(x)$ (2.37).

Now let us move from the physical interpretation of quantum potential $Q$ to quantum (probabilistic) vector potential $\overrightarrow{\mathrm{A}}_{Q}$ included in the «Schrödinger equation» (2.5). As mentioned above, equation (2.5) differs from the classical Schrödinger equation by the presence of term $\overrightarrow{\mathrm{A}}_{Q}$ (2.13). If probabilistic vector-potential $\overrightarrow{\mathrm{A}}_{Q}$ can be neglected, then $\overrightarrow{\mathrm{A}}_{\Psi}=\overrightarrow{\mathrm{A}}$ and equation (2.5) transforms into the usual Schrödinger equation for a scalar particle in an electromagnetic field. Such an equation does not take into account the particle’s own «fields», such as spin effects. Note that the original equation (2.5) for self-consistent systems leads to the relativistic Lorentz transformations (see Lemma 1) and the Maxwell equations. Consequently, the presence of probabilistic potentials $Q$ and $\overrightarrow{\mathrm{A}}_{Q}$ may contain the interpretation of «relativistic» effects, for example, spin (note that the Pauli equation is not invariant).

Indeed, equation of motion (2.10) contains an analogue of the Lorentz force for a particle in a magnetic field $q\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}_{\Psi}$. According to (2.13), field $\overrightarrow{\mathrm{B}}_{\Psi}$ is a superposition of fields $\overrightarrow{\mathrm{B}}_{\Psi}=\overrightarrow{\mathrm{B}}+\overrightarrow{\mathrm{B}}_{Q}$, where $\overrightarrow{\mathrm{B}}$ is an external field, and $\overrightarrow{\mathrm{B}}_{Q}$ is a probabilistic field of a particle. Consequently, in the well-known experiment of Gerlach and Stern [29-31] on detecting the electron spin, in addition to the Lorentz force from external field $q\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}$ according to (2.10), there will be additional force $q\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}_{Q}$ determined by the probabilistic vortex potential $\overrightarrow{\mathrm{A}}_{Q}$. Depending on vortex potential $\overrightarrow{\mathrm{A}}_{Q}$, there will be a different type of solution to equation (2.5), that is, the possibility of taking the spin effect into consideration. Thus, the right-hand side of equation of motion (2.10) can be represented in the form of a classical $\overrightarrow{\mathrm{F}}$ and probabilistic (quantum) $\overrightarrow{\mathrm{F}}_{Q}$ force acting on the particle

$$
\begin{gather*}
\overrightarrow{\mathrm{F}}_{\Psi}=\overrightarrow{\mathrm{F}}+\overrightarrow{\mathrm{F}}_{Q},  \tag{2.41}\\
\overrightarrow{\mathrm{~F}}=q(\overrightarrow{\mathrm{E}}+\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}), \overrightarrow{\mathrm{F}}_{Q}=q\left(\overrightarrow{\mathrm{E}}_{Q}+\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}_{Q}\right) .
\end{gather*}
$$

Classical force $\overrightarrow{\mathrm{F}}$ gives a non-probabilistic trajectory, and the probabilistic force $\overrightarrow{\mathrm{F}}_{Q}$ specifies the quantum uncertainty of the trajectory in the Heisenberg principle. Term $\overrightarrow{\mathrm{F}}_{Q}$ gives a probabilistic trajectory, that is, the probability that the force has such a value. As a result, there is an infinite number of trajectories along which movement can occur, but each has its own probability. This interpretation is similar to the well-known formulation of quantum mechanics through the path integral by Feynman.

Recall that according to the Helmholtz $\Psi \mathbb{C}$-decomposition (2.4), the velocity of probability flow $\langle\vec{v}\rangle$ is related to phase $\varphi$ of the wave function $\Psi$. Probabilistic fields $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$ provide probabilistic solutions to equation of motion (2.10) for flow $\langle\vec{v}\rangle$, and therefore also for phase $\varphi$, the incursion of which is present in Feynman's path integrals.

## §3 Pauli and Dirac equations

Let us consider a possible option for taking into account additional kinematic information (vortex field $\overrightarrow{\mathrm{A}}_{Q}$ ) of the particle itself located in external fields. When obtaining an analogue of the Schrödinger equation (2.7), the system was described by a complex function $\Psi \in \mathbb{C}$. Complex function $\Psi=|\Psi| \exp (i \varphi)$ contains two free parameters: modulus $|\Psi|$ and phase $\varphi$. These two parameters were used to describe two unknown functions in the Vlasov equation (1.5): probability density $f=|\Psi|^{2}$ and probability flow velocity $\langle\vec{v}\rangle=-2 \alpha \nabla_{r} \varphi+\gamma \overrightarrow{\mathrm{A}}_{\Psi}$. At the same time, in order to describe the vortex field, it was necessary to introduce another free parameter $\overrightarrow{\mathrm{A}}_{\Psi}$, which, according to (2.13), in addition to the vector-potential $\overrightarrow{\mathrm{A}}$ (external magnetic field), contains the probabilistic vortex field of the particle $\overrightarrow{\mathrm{A}}_{Q}$. From equation (2.7) it is impossible to explicitly obtain the form of field $\overrightarrow{\mathrm{A}}_{Q}$, which is included there as a parameter. Of course, field $\overrightarrow{\mathrm{A}}_{Q}$ satisfies the gauge expression (2.15), but according to the Helmholtz theorem, the field cannot be reconstructed from divergence ( $\operatorname{div}_{r}$ ) only [23]. Field $\overrightarrow{\mathrm{A}}_{Q}$ can be found from the system of equations (2.25)-(2.26), but for this it is necessary to know probability density $\rho_{Q}$ and current $\overrightarrow{\mathrm{J}}_{Q}$, which is expressed through field $\overrightarrow{\mathrm{A}}_{Q}$ itself.

Thus, usual complex function $\Psi \in \mathbb{C}$ does not have a sufficient number of free parameters to describe the particle's own kinematic characteristics. To solve this problem, we increase the number of free parameters by considering a 2D complex space $\mathbb{C}^{2}$ with elements $\psi$ (spinors):

$$
\psi=\binom{\psi_{1}}{\psi_{2}} \in \mathbb{C}^{2}, \psi^{\dagger}=\left(\begin{array}{ll}
\psi_{1}^{*} & \psi_{2}^{*} \tag{3.1}
\end{array}\right)=\left(\psi^{*}\right)^{T},|\psi|^{2}=\psi^{\dagger} \psi, \quad \psi_{1}, \psi_{2} \in \mathbb{C} .
$$

Spinor $\psi$, unlike complex function $\Psi$, has four instead of two free parameters. By analogy with the Helmholtz $\Psi \mathbb{C}$-decomposition (2.4) for the function $\Psi \in \mathbb{C}$, we define the decomposition for spinor $\psi \in \mathbb{C}^{2}$.

Definition 4 Let us denote the representation of current density field $\overrightarrow{\mathbf{J}}(\vec{r}, t)$ for $\psi \in \mathbb{C}^{2}$

$$
\begin{equation*}
\overrightarrow{\mathrm{J}}=i \alpha\left(\psi^{\dagger} \nabla_{r} \psi-\psi^{T} \nabla_{r} \psi^{*}\right)+\gamma \psi^{\dagger} \overrightarrow{\mathrm{A}} \psi \tag{3.2}
\end{equation*}
$$

as the Helmholtz $\psi \mathbb{C}^{2}$ - decomposition with gauge $\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}=\chi$.

Theorem 3 Let the vector field of the current probability density $\overrightarrow{\mathbf{J}}$ admit a Helmholtz $\psi \mathbb{C}^{2}-$ decomposition, then the first Vlasov equation (1.5) will take the form:

$$
\begin{equation*}
\frac{i}{\beta} \mathrm{I} \frac{\partial \psi}{\partial t}=-\alpha \beta\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2} \psi+\mathrm{I} U \psi \tag{3.3}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{i}{\beta} \mathrm{I} \frac{\partial \psi}{\partial t}=-\alpha \beta \mathrm{I}\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)^{2} \psi+\mathrm{I} U \psi+\frac{\gamma}{2 \beta}(\vec{\sigma} \cdot \overrightarrow{\mathrm{~B}}) \psi, \tag{3.4}
\end{equation*}
$$

where $|\psi|^{2}=f^{1} ; \vec{\sigma}=\left(\sigma^{1}, \sigma^{2}, \sigma^{3}\right)$ are the Pauli matrices; $\sigma^{0}=\mathrm{I}$ is an identity matrix. $\overrightarrow{\mathrm{B}} \stackrel{\text { det }}{=} \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}} ; \hat{\mathrm{p}}=\frac{i}{\beta} \nabla_{r}, U \in \mathbb{R}$ is some function, and $\alpha, \beta, \gamma$ are some constant values.

The proof of Theorem 3 is given in Appendix B.
When choosing constant values $\alpha, \beta, \gamma$ according to (2.6), equation (3.3)/(3.4) transforms into the well-known non-relativistic Pauli equation for a particle with spin in an external electromagnetic field [32].

$$
\begin{equation*}
i \hbar \mathrm{I} \frac{\partial \psi}{\partial t}=\frac{1}{2 m} \mathrm{I}(\hat{\mathrm{p}}-q \overrightarrow{\mathrm{~A}})^{2} \psi+\mathrm{I} U \psi-\frac{q \hbar}{2 m}(\vec{\sigma} \cdot \overrightarrow{\mathrm{~B}}) \psi . \tag{3.5}
\end{equation*}
$$

From a comparison of equations (2.7) and (3.5) one can see the appearance of a new term $\frac{q \hbar}{2 m}(\vec{\sigma} \cdot \overrightarrow{\mathrm{~B}})$ responsible for the spin of the particle.

Equations (2.5) and (3.3) were obtained from the first Vlasov equation (1.5) using Helmholtz $\Psi \mathbb{C}$ - and $\psi \mathbb{C}^{2}$ - decomposition, respectively. Both equations (2.5) and (3.3) are not relativistic, since they do not have invariance under the Lorentz transformation. Note that the original Vlasov equation (1.5) is invariant. The first problem in invariance violation appears when using the Helmholtz decomposition for 3D vector field $\langle\vec{v}\rangle$ or in 3D coordinate space $\vec{J}$. Indeed, time $t$ is included in decompositions (2.1), (2.4) and (3.2) as a parameter. In the Helmholtz decomposition there are derivatives only along coordinate axes $\partial_{k}$, and there are no derivatives along the time axis $\partial_{0}$. To solve this problem, it is necessary to extend the Helmholtz decomposition to 4D space-time.

Lemma 2 Let $3 D$ vector field $\vec{F}\left(x^{\mu}\right)$ be defined on a 4D space-time $x^{\mu}=\left(\begin{array}{ll}\text { ct } & \vec{r}\end{array}\right)^{T}=\left(\begin{array}{ll}x^{0} & \vec{x}\end{array}\right)^{T}$ and admit the representation

$$
\begin{equation*}
\vec{F}\left(x^{\mu}\right)=-\nabla_{r} \Pi+\operatorname{curl}_{r} \vec{S}-\frac{\partial}{\partial t} \vec{R} \tag{3.6}
\end{equation*}
$$

then functions $\Pi, \vec{R}$ and $\vec{S}$ satisfy the equations:

$$
\begin{equation*}
-\frac{1}{c^{2}} \frac{\partial \vec{F}}{\partial t}=\square \vec{R}, \operatorname{div}_{r} \vec{F}=\square \Pi, \operatorname{curl}_{r} \vec{F}=\square \vec{S}, \tag{3.7}
\end{equation*}
$$

with gauges:

$$
\begin{equation*}
\frac{1}{c^{2}} \frac{\partial \Pi}{\partial t}+\operatorname{div}_{r} \vec{R}=0, \frac{1}{c^{2}} \frac{\partial \vec{S}}{\partial t}+\operatorname{curl}_{r} \vec{R}=0, \operatorname{div}_{r} \vec{S}=0 \tag{3.8}
\end{equation*}
$$

The proof of Lemma 2 is given in Appendix B.
In the nonrelativistic limit at $c \rightarrow \infty$, gauge conditions (3.8) give $\operatorname{div}_{r} \vec{R}=0, \operatorname{curl}_{r} \vec{R}=0$, and (3.7) leads to $\square \vec{R}=\vec{\theta}$. Thus, field $\vec{R}$ is constant, that is, the term $\partial_{0} \vec{R}=\vec{\theta}$ in decomposition (3.6). As a result, expansion (3.6) transforms into the Helmholtz decomposition. A similar transformation occurs if we assume that the fields are time independent. In this case, all derivatives with respect to time are equal to zero.

Instead of two vector fields $\vec{R}$ and $\vec{S}$ we can introduce field $\overrightarrow{\mathcal{J}} \stackrel{\text { det }}{=} \vec{S}-c \vec{R}$. According to (3.7), field $\overrightarrow{\mathcal{J}}$ is represented in terms of quaternions:

$$
\begin{equation*}
\square \overrightarrow{\mathcal{J}}=\frac{1}{c} \frac{\partial \vec{F}}{\partial t}+\operatorname{curl}_{r} \vec{F}=\left(\partial_{0} ; \partial_{k}\right)(0 ; \vec{F}) . \tag{3.9}
\end{equation*}
$$

Scalar potential $\Pi$ and field $\overrightarrow{\mathcal{J}}$ define four-vector $\mathcal{A}^{\mu}=(\Pi, \overrightarrow{\mathcal{J}})$ satisfying the equation

$$
\begin{equation*}
\square \mathcal{A}^{\mu}=\left(\operatorname{div}_{r} \vec{F}, \frac{1}{c} \frac{\partial \vec{F}}{\partial t}+\operatorname{curl}_{r} \vec{F}\right) . \tag{3.10}
\end{equation*}
$$

Equation (3.10) is a 4D form of the Lemma 1. Knowing the right-hand side, we can write the solution to the d'Alembert equation (3.10) in terms of the Lienard-Wiechert potentials and find four-vector potential $\mathcal{A}^{\mu}$ from which it is possible to restore field $\vec{F}$ according to (3.6). In the non-relativistic case at $c \rightarrow \infty$, the right-hand side of equation (3.10) contains only $\operatorname{div}_{r} \vec{F}$ and $\operatorname{curl}_{r} \vec{F}$, and operator $\square \rightarrow-\Delta$, which reduces to the Helmholtz theorem for the 3D region.

Note that the right-hand side of decomposition (3.10) is similar to the Maxwell equations through the Riemann-Silberstein vector (RS-vector) $\overrightarrow{\mathrm{F}}=\sqrt{\varepsilon_{0}} \overrightarrow{\mathrm{E}}+i \sqrt{\mu_{0}} \overrightarrow{\mathrm{H}}$ :

$$
\begin{equation*}
\sqrt{\mu_{0}}(c \rho,-\overrightarrow{\mathrm{J}})=\left(\operatorname{div}_{r} \overrightarrow{\mathrm{~F}}, \frac{1}{c} \frac{\partial \overrightarrow{\mathrm{~F}}}{\partial t}+i \operatorname{curl}_{r} \overrightarrow{\mathrm{~F}}\right) . \tag{3.11}
\end{equation*}
$$

On the one hand, decomposition (3.6) gives hope for a transition to relativism, but on the other hand, when substituting it into the Vlasov equation, a second problem arises: different orders of derivatives with respect to coordinate and time. There are two ways to solve this problem. The first is to increase the order of the derivatives in term $\partial_{0} f$ from the Vlasov equation. This approach is actually implemented in the Klein-Gordon equation. The second way is to lower the order of derivatives in the first term of decomposition (2.4):

$$
\begin{equation*}
\mathrm{J}^{k}=\frac{\hbar}{m} \Psi^{*} \partial^{k} \varphi \Psi \tag{3.12}
\end{equation*}
$$

This expression (3.12) takes the relation (2.3) into account for the phase of wave function $\varphi$. The Helmholtz $\Psi \mathbb{C}$-decomposition (2.4) used a complex wave function $\Psi=|\Psi| \exp (i \varphi)$ characterized by modulus $|\Psi|$ and phase $\varphi$. For $\Psi \in \mathbb{C}$ the phase $\varphi$ specifies the angle of rotation in the complex plane. When transiting to the Helmholtz $\psi \mathbb{C}^{2}$-decomposition (3.2), spinors $\psi \in \mathbb{C}^{2}$ were considered instead of $\Psi \in \mathbb{C}$. Spinors $\psi$ can be acted upon by complex matrices $M$ of size $2 \times 2$, that is $M \psi$. The indicated matrices can be decomposed into basis matrices - the Pauli matrices $\sigma^{\mu}, \mu=0 \ldots 3$. The Pauli matrices correspond to the four basis elements $1, i, j, k$ in the quaternion decomposition $\mathrm{q}=(a ; \vec{u})=a+i u_{1}+j u_{2}+k u_{3}$ :

$$
1 \mapsto \sigma^{0}=\left(\begin{array}{ll}
1 & 0  \tag{3.13}\\
0 & 1
\end{array}\right), i \mapsto i \sigma^{3}=i\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right), j \mapsto i \sigma^{2}=i\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), k \mapsto i \sigma^{1}=i\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) .
$$

Quaternion algebra ( $\mathbb{Q}$ ) was originally constructed by Hamilton and further developed by Clifford in the form of hypercomplex numbers [33], for example, octaves $(\mathbb{O})$ and sedenions $(\mathbb{S})$. Quaternions can be represented in the form of complex matrices $2 \times 2$ (3.13), or in the form of real matrices $4 \times 4$. Each 3D vector $\vec{r}=\left(x_{1}, x_{2}, x_{3}\right)$ in Euclidean space can be associated with a $2 \times 2$ matrix according to the rule

$$
\vec{\sigma} \cdot \vec{r}=\sigma^{k} x_{k}=\left(\begin{array}{cc}
x_{3} & x_{1}-i x_{2}  \tag{3.14}\\
x_{1}+i x_{2} & -x_{3}
\end{array}\right) .
$$

Representation (3.14) is used in the Pauli equation (3.3) for the momentum operator. According to spinor algebra, matrix (3.14) specifies the rotation of the spinor. Rotating a spinor from one position to another is an ambiguous operation. There are two possible rotating operations, differing only in sign. Thus, it is logical to take rotation matrices $\vec{\sigma} \cdot \vec{r}$ as phase $\varphi$. Since it is necessary to take all axes of 4 D space-time $x^{\mu}=\left(\begin{array}{ll}x^{0} & \vec{x}\end{array}\right)^{T}$ into account we define rotation matrix $\hat{\varphi}$ for bispinor $\psi \in \mathbb{C}^{4}$ (two directions of rotation) as follows:

$$
\begin{gather*}
\left(\begin{array}{cc}
\sigma^{0} x_{0} & \sigma^{s} x_{s} \\
\sigma^{s} x_{s} & \sigma^{0} x_{0}
\end{array}\right)=\left(\begin{array}{cc}
\mathrm{I}_{2} & 0 \\
0 & -\mathrm{I}_{2}
\end{array}\right)\left(\begin{array}{cc}
\sigma^{0} x_{0} & \sigma^{s} x_{s} \\
-\sigma^{s} x_{s} & -\sigma^{0} x_{0}
\end{array}\right)=\gamma^{0} \hat{\varphi}\left(x_{\mu}\right),  \tag{3.15}\\
\hat{\varphi}\left(x^{\mu}\right)=\left(\begin{array}{cc}
\sigma^{0} x_{0} & -\vec{\sigma} \cdot \vec{x} \\
\vec{\sigma} \cdot \vec{x} & -\sigma^{0} x_{0}
\end{array}\right)
\end{gather*}
$$

The rotation phase matrix (3.15) is defined at points of 4D space-time, that is, each point $x^{\mu}$ has its own rotation (phase). Note that from the Hamilton-Jacobi equation (2.8) the phase is related to the concept of action $\hbar \varphi=\mathrm{S}$. From a geometric point of view, action S corresponds to the length of the trajectory (in the case of quantum mechanics - the Feynman path integrals) in curved space with the indicatrix $\mathcal{L}=1$, where $\mathcal{L}$ is the Lagrange [34]. Thus, for each point in 4D space-time the «trajectory length» S is determined by rotation matrix (3.15).

The Vlasov equation in 4-component form will take the form:

$$
\begin{equation*}
\frac{\partial}{\partial c t} c f+\frac{\partial}{\partial x^{k}}\left[f\left\langle v^{k}\right\rangle\right]=0 \Rightarrow \partial_{\mu} J^{\mu}=0 \tag{3.16}
\end{equation*}
$$

where

$$
J^{\mu}=\left(\begin{array}{ll}
c f & f\langle\vec{v}\rangle)^{T}=\left(\begin{array}{ll}
c|\psi|^{2} & \overrightarrow{\mathrm{~J}}
\end{array}\right)^{T},|\psi|^{2}=\psi^{\dagger} \psi .
\end{array}\right.
$$

Let us transform representation of the current density (3.12) for bispinor $\psi \in \mathbb{C}^{4}$ and the phase matrix (3.15):

$$
\begin{align*}
& J^{0}=\frac{\hbar}{m} \Psi^{*} \Psi \mapsto c \psi^{\dagger} \psi,  \tag{3.17}\\
& J^{k}=\frac{\hbar}{m} \Psi^{*} \partial^{k} \varphi \Psi \mapsto c \psi^{\dagger} \gamma^{0} \partial^{k} \hat{\varphi} \psi,
\end{align*}
$$

where, in contrast to (2.4), density $f=|\psi|^{2}$ is a dimensionless value. Let us calculate expression $\partial^{k} \hat{\varphi}$ from (3.17). The following expression will be obtained:

$$
\begin{gather*}
\partial^{k} \hat{\varphi}=\left(\begin{array}{cc}
0 & \sigma^{k} \\
-\sigma^{k} & 0
\end{array}\right), \partial^{0} \hat{\varphi}=\left(\begin{array}{cc}
\mathrm{I}_{2} & 0 \\
0 & -\mathrm{I}_{2}
\end{array}\right), \\
\partial^{\mu} \hat{\varphi}=\gamma^{\mu} \tag{3.18}
\end{gather*}
$$

where $\gamma^{\mu}$ are the Dirac matrices. From a physical point of view, expression (3.18) can have the following interpretation. The partial derivative of the phase (action) with respect to time $\partial^{0} \hat{\varphi}$ in accordance with the Hamilton-Jacobi equation (2.8) determines the energy of the system. From expression (3.18) it follows that there are positive and negative values of the system energy (a particle and an antiparticle). In accordance with the Helmholtz decomposition (2.1) the derivative with respect to coordinate space from the phase (action) $\partial^{k} \hat{\varphi}$ is associated with the potential component of the momentum $\langle\vec{p}\rangle=\hbar \nabla_{r} \varphi$. From expression (3.18) it follows that the momentum is related to the Pauli matrices $\sigma^{k}$, which corresponds to the Pauli equation (3.3) $\vec{\sigma} \cdot \hat{p}$.

Taking expression (3.18) into account the representation (3.17) for current density $J^{\mu}$ will take the form:

$$
\begin{equation*}
J^{\mu}=c \bar{\psi} \gamma^{\mu} \psi, \tag{3.19}
\end{equation*}
$$

where $\bar{\psi} \stackrel{\text { det }}{=} \psi^{\dagger} \gamma^{0}$ is a dual wave function and $\gamma^{0} \gamma^{0}=\mathrm{I}$ is an identity matrix.

Theorem 4 Let the probability current density $J^{\mu}$ admit the (3.19) representation, then the Vlasov equation (3.16) will take the form:

$$
\begin{equation*}
i \hbar c \partial_{0} \psi=\gamma^{0}\left[c \gamma^{k}\left(\hat{\mathrm{p}}_{k}-q \mathrm{~A}_{k}\right)+m c^{2} \mathrm{I}\right] \psi+q \varphi \psi, \tag{3.20}
\end{equation*}
$$

or

$$
\left[\gamma^{\mu}\left(i \hbar \partial_{\mu}-q A_{\mu}\right)-m c \mathrm{I}\right] \psi=0
$$

where $m, q, \hbar$ are constant values, $\hat{\mathrm{p}}_{k}=-i \hbar \partial_{k}$, and $A^{\mu}=(\varphi / c, \overrightarrow{\mathrm{~A}})$ is some four-vector field.
The proof of Theorem 4 is given in Appendix B.
Equation (3.20) is known as the relativistic Dirac equation for a particle (and an antiparticle) in an electromagnetic field [35].

## §4 Self-consistency, relativism and many-particle system

Let us consider the relationship between the concepts of self-consistency, relativism and many-particle system. We start with the concepts of self-consistency and many-particle system. Firstly, we should make it clear which systems satisfy and unsatisfy the condition (2.17). There are two types of systems described by the equations of mathematical physics. The first type is when there are external fields acting on the «particle», and the action of the «particle» on the external fields is negligible. The second type of the system is when there is self-consistency between the particle's fields and external fields. In this case the equations describing the system contain the action of the entire system on itself.

Lemma 3 For a self-consistent system with external potentials $U=q \varphi, \overrightarrow{\mathrm{~A}}$, gauges (2.12), (2.14), (2.15) the solution to equation (2.7) can be represented as $\Psi(\vec{r}, t)=|\Psi(\vec{r}, t)| \exp [i \varphi(\vec{r}, t)]$, where $|\Psi|, \varphi$ and $\overrightarrow{\mathrm{A}}_{Q}$ satisfy the equations:

$$
\begin{gather*}
\frac{\lambda_{C}}{2} \frac{\Delta_{r} r(\Psi(\vec{r}, t) \mid}{|\Psi(\vec{r}, t)|}+\alpha \int \frac{\left|\Psi\left(\vec{r}^{\prime}, t_{r}\right)\right|^{2} d^{3} r^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|}=\frac{1}{\hbar c} U(\vec{r}, t),  \tag{4.1}\\
q \square \overrightarrow{\mathrm{~A}}_{Q}=\frac{1}{m c^{2}}\left[\hbar \nabla_{r} \varphi-q\left(\overrightarrow{\mathrm{~A}}+\overrightarrow{\mathrm{A}}_{Q}\right)\right] \square \mathrm{Q}, \tag{4.2}
\end{gather*}
$$

where $t_{r}=t-\left|\vec{r}-\vec{r}^{\prime}\right| / c ; \lambda_{C}=\hbar / m c$ is a reduced Compton wavelength, and $\alpha=q^{2} /\left(4 \pi \varepsilon_{0} \hbar c\right)$ is the fine structure constant.

The proof of Lemma 3 is given in Appendix C.
The solution to equation (4.1) is $|\Psi|$, by which Q is defined. Equation (4.2) contains two unknown quantities: phase $\varphi$ and probabilistic potential $\overrightarrow{\mathrm{A}}_{Q}$, so it cannot be solved. If we make some assumptions about the type of phase, then equation (4.2) can be written with one unknown value $\overrightarrow{\mathrm{A}}_{Q}$. Let us consider one of these options. Let the phase admit the following representation

$$
\begin{equation*}
\varphi=-\frac{\mathcal{E}}{\hbar} t, \tag{4.3}
\end{equation*}
$$

where $\mathcal{E}$ is some constant value, then equation (4.2) will take the form:

$$
\begin{equation*}
m c^{2} \square \overrightarrow{\mathrm{~A}}_{Q}+\left(\overrightarrow{\mathrm{A}}+\overrightarrow{\mathrm{A}}_{Q}\right) \square \mathrm{Q}=0 \tag{4.4}
\end{equation*}
$$

Vector equation (4.4) breaks down into a system of three partial-differential linear equations with three unknown functions $\mathrm{A}_{Q}^{(x)}, \mathrm{A}_{Q}^{(y)}, \mathrm{A}_{Q}^{(z)}$. The Hamilton-Jacobi equation (2.8) under condition (4.3) allows us to find the value $\mathcal{E}$ :

$$
\begin{equation*}
q^{2}\left|\overrightarrow{\mathrm{~A}}+\overrightarrow{\mathrm{A}}_{Q}\right|^{2}=2 m(\mathrm{~V}-\mathcal{E}) \Rightarrow \mathcal{E}=\mathrm{V}-\frac{m c^{4}}{2}\left|\frac{\square \overrightarrow{\mathrm{~A}}_{Q}}{\square \mathrm{Q}}\right|^{2} . \tag{4.5}
\end{equation*}
$$

Let us note once again that expressions (4.4) and (4.5) are valid under assumption (4.3), which in the general case is not satisfied.

Theorem 5 Let $\partial_{0} f^{1,2}=0$, then for a self-consistent system $\partial_{0} \overrightarrow{\mathrm{~A}}_{Q}=0$ and $\partial_{0} \mathrm{Q}=0$.
The proof of Theorem 5 is given in Appendix C.
It should be noted that the Wigner function (2.28) can be used for function $f^{1,2}$.
When the condition of Theorem 5 is met, it follows that "the system does not emit an electromagnetic wave". Indeed, the time-independent nature of quantum potentials $\mathrm{Q}(\vec{r})$ and $\overrightarrow{\mathrm{A}}_{Q}(\vec{r})$ leads to solutions of the Maxwell equations (2.25)-(2.26) that do not contain radiation. For non-self-consistent systems the question of the presence of radiation disappears, since due to the failure of condition (2.17) it is impossible to construct a system of the Maxwell equations (2.20-2.22), allowing a solution in the form of an electromagnetic wave.

Note that for self-consistent systems with the Coulomb gauge, equation (2.20) becomes the Poisson equation $\Delta_{r} \varphi_{\Psi}=\frac{1}{\varepsilon_{0}} q|\Psi|^{2}$, where $q|\Psi|^{2}$ can be interpreted as the charge density. The use of charge density terminology leads to a connection with many-particle systems. We can give an example of a self-consistent system that has both a classical many-particle solution and a quantum one-particle solution.

Let us consider a space charge problem for the initial charge density distribution $\rho_{0}(r)=\rho(\vec{r}, 0)$. In the Coulomb gauge the evolution of such a system is described by the equation [36]:

$$
\begin{align*}
& \rho[R(r, t), t]=\frac{1}{P^{2}[\varsigma(r) t]}  \tag{4.7}\\
& \rho[\varsigma(r) t]-t \sqrt{\frac{P[\varsigma(r) t]-1}{P[\varsigma(r) t]}}\left[\frac{\gamma \rho_{0}(r)}{\varepsilon_{0} \varsigma(r)}+\frac{3}{2} \varsigma(r)\right]
\end{align*},
$$

Function $P$ is the inverse of function $G$

$$
\begin{equation*}
G(x)=\sqrt{x(x-1)}+\operatorname{arcosh} \sqrt{x} . \tag{4.8}
\end{equation*}
$$

The solution (4.7) is constructed under the assumption that the charged spherical layers do not intersect. The equation of motion of the charged layer is described by the characteristic equation

$$
\begin{equation*}
R(r, t)=r P[\varsigma(r) t], \tag{4.9}
\end{equation*}
$$

where $P(0)=1$. The charge is conserved inside a sphere with radius $R(r, t)$. Depending on the type of initial distribution $\rho_{0}(r)$, the characteristics (4.9) may intersect over time. The intersection of the characteristics (4.9) leads to a shock wave [37, 38] (the Coulomb explosion). Fig. 1 shows the characteristic equations for two types of initial distribution:

$$
\begin{gather*}
\rho_{0}(r)=\text { const },  \tag{4.10}\\
\rho_{0}(r)=\frac{q_{\text {total }}}{2 \pi r^{2}} \rho_{n}(2 r), \quad \rho_{n}(r)=\frac{1}{\sqrt{2 \pi} \sigma_{r} r} \exp \left\{-\frac{1}{2 \sigma_{r}^{2}}\left[\ln (r)-\mu_{r}\right]^{2}\right\}, \tag{4.11}
\end{gather*}
$$

where $\mu_{r}, \sigma_{r}$ are the mean value and the standard deviation of the lognormal distribution (4.11) respectively. Value $q_{\text {total }}$ specifies the total charge of the system. On the left, Fig. 1 shows the characteristics for distribution (4.10), and on the right - for the distribution (4.11). The characteristics equations have the form:

$$
\begin{gather*}
\varsigma_{\ell}\left(R_{0}\right) t=G[\kappa(t)], \ell=1,2,  \tag{4.12}\\
\varsigma_{1}(r)=\sqrt{-\frac{2 \gamma \rho_{0}}{3 \varepsilon_{0}}}, \quad \varsigma_{2}(r)=\sqrt{-\frac{\gamma q_{\text {total }}}{4 \pi \varepsilon_{0} r^{3}}\left\{1+\operatorname{erf}\left[\frac{\ln (2 r)-\mu_{r}}{\sigma_{r} \sqrt{2}}\right]\right\}}, \kappa(t)=\sqrt{R\left(R_{0}, t\right) / R_{0}},
\end{gather*}
$$

where $R_{0}=R\left(R_{0}, 0\right)$. In Fig. 1 it is seen that the characteristics for the distribution (4.10) do not intersect, but for the distribution (4.11) an intersection occurs at the moment of time $t_{c}$, indicating a shock wave.


Fig. 1 Characteristic equations for the two types of initial distributions (4.10)-(4.11)

Indeed, Fig. 2 shows graphs of the evolution of density (4.7) at successive times $t_{k}, k=0 \ldots 4$ for two initial distributions (4.10) and (4.11). In Fig. 2, on the right at moment of time $t_{c}$ a «gradient catastrophe» occurs (the derivative of the solution goes to infinity). On the left in Fig. 2, the distribution density remains constant along the radius. For density $\rho_{1}$ corresponding to (4.10), solution (4.7) has a simple form.

$$
\begin{equation*}
\rho_{1}[R(t), t]=\frac{\rho_{0}}{P^{3}\left(\varsigma_{1} t\right)} . \tag{4.13}
\end{equation*}
$$



Fig. 2 Evolution of charge density for the two types of initial distributions (4.10)-(4.11)

Fig. 3 shows graphs of potential energy $U_{2}(\vec{r}, t)$ from the Schrödinger equation (2.7) and graphs of quantum potential $\mathrm{Q}_{2}(\vec{r}, t)$ (2.9). It can be seen that the quantum system is timedependent and a shock wave arises in it. The similarity of the graphs of potentials $U_{2}$ and $\mathrm{Q}_{2}$ in Fig. 3 has a clear physical interpretation. A quantum particle is in external potential $U_{2}$, which exerts a force on it. Earlier (2.33) it was noted that a quantum particle, described by a wave function $\Psi_{2}(\vec{r}, t)=\sqrt{\rho_{2}(\vec{r}, t)} \exp \left[i \varphi_{2}(\vec{r}, t)\right]$, creates quantum pressure $\mathrm{Q}_{2}$ that counteracts external potential $U_{2}$. Quantum pressure force $-\nabla_{r} \mathrm{Q}$, according to (2.33), counteracts external force $-\nabla_{r} U_{2}$.


Fig. 3 Evolution of potentials $U_{2}(r, t)$ and $\mathrm{Q}_{2}(r, t)$

At initial moments of time $t_{0}$ and $t_{1}$ the distance between the characteristics remains unchanged (see Fig. 1), therefore the quantum pressure is close to zero (see Fig. 3 on the right). At moment of time $t_{2}$ (and subsequent ones $t_{3}, t_{4}$ ), the distance between the characteristics decreases sharply (see Fig. 1 on the right), which leads to an increase in density (see Fig. 2 on the right) and force of the external influence $-\nabla_{r} U_{2}$ on the particle (see Fig. 3 on the left). The particle tries to compensate for the pressure exerted on it by counteracting in the form of quantum pressure $-\nabla_{r} \mathrm{Q}_{2}$ (see Fig. 3 on the right).


Fig. 4 Evolution of resultant force $-\nabla_{r} \mathrm{~V}_{2}$ скорости $\langle\vec{v}\rangle_{1}(\vec{r}, t)$

In Fig. 3, it can be seen that derivatives $-\nabla_{r} U_{2}$ and $-\nabla_{r} \mathrm{Q}_{2}$ have the opposite signs on the interval $(0.6 ; 0.9)$. The interval $(0.6 ; 0.9)$ clearly shows an abrupt jump in the derivatives at times $t_{3}, t_{4}$. These instants of time are close to instant of time $t_{c}$ of the shock wave, so the force of external $-\nabla_{r} U_{2}$ and quantum pressure $-\nabla_{r} \mathrm{Q}_{2}$ is great.

Quantum pressure force $-\nabla_{r} \mathrm{Q}_{2}$ does not completely compensate for external force $-\nabla_{r} U$, therefore resultant force $-\nabla_{r} \mathrm{~V}_{2}=-\nabla_{r}\left(U_{2}+\mathrm{Q}_{2}\right)=\vec{F}_{2}$ is not zero (see Fig. 4 on the left) and the system expands with radial velocity $\langle v\rangle$ (see Fig. 4 on the right) according to equation of motion (1.7) or (2.10). Fig. 4, the right side, shows that instant of time $t_{4}$ is close to the gradient catastrophe, indicating a shock wave.

Despite the fact that the problem statement was originally written for the space charge problem, its solution (4.7), due to the results of $\S 2$, is also a solution to the time-dependent Schrödinger equation (2.7). Phase $\varphi_{2}$ of the wave function has a more complex form than (4.3), since it contains information about the probability flow velocity (charge) $\langle\vec{v}\rangle_{1}=\frac{\hbar}{m} \nabla_{r} \varphi_{2}(\vec{r}, t)$. Moreover, the energy of this system is not a constant value, that is $\hbar c \partial_{0} \varphi \neq$ const, according to the Hamilton-Jacobi equation (2.8). The considered system is self-consistent (condition (2.17) is satisfied for it).

The simplest case (4.13) with a constant initial density (4.10) will have zero quantum pressure $\mathrm{Q}_{1}=0$ inside the ball, since $\rho_{1}$ only depends on time (4.13). In this case, the external potential has a quadratic dependence

$$
\begin{equation*}
U_{1}(\vec{r}, t)=-\frac{q}{6 \varepsilon_{0}} \rho_{1}(t) r^{2} \tag{4.14}
\end{equation*}
$$

and the phase

$$
\begin{equation*}
\varphi_{1}(r, t)=\frac{m}{2 \hbar} g(t) r^{2}+\text { const }, \quad \dot{g}+g^{2}=\frac{q}{3 m \varepsilon_{0}} \rho_{1} . \tag{4.15}
\end{equation*}
$$

Wave function $\Psi_{1}(r, t)=\sqrt{\rho_{1}(t)} \exp \left[i \varphi_{1}(r, t)\right]$ is a solution to the time-dependent Schrödinger equation (2.7) with the external potential (4.14) and zero vector potential $\overrightarrow{\mathrm{A}}_{\Psi} \equiv \overrightarrow{\boldsymbol{\theta}}$.

Note that solution (4.7) corresponds to a non-relativistic system, since equation of characteristics (4.8)-(4.9) reduces to the Cauchy problem for Newton's second law:

$$
\begin{equation*}
R^{2} \ddot{R}=v=\frac{q}{4 \pi \varepsilon_{0}} q_{\text {total }}\left(R_{0}\right), R(0)=R_{0}, \dot{R}(0)=0 . \tag{4.16}
\end{equation*}
$$

In the relativistic case, the sphere expansion rate will be less, since radial currents $\vec{J}$ are attracted to each other under the action of the Ampere force. The Ampere force acts as a surface tension force on the sphere, slowing down its Coulomb expansion.

Since solution (4.7) for case (4.11) has non-zero quantum potential $\mathrm{Q}_{2}$, then according to (2.22) it will have probabilistic (quantum) charges $\rho_{Q}=\varepsilon_{0} \operatorname{div}_{r} \overrightarrow{\mathrm{E}}_{Q}=-\frac{\varepsilon_{0}}{q} \Delta_{r} \mathrm{Q}_{2}$, which induces a quantum pressure force counteracting the external pressure force induced by external charges $\rho=\varepsilon_{0} \operatorname{div}_{r} \overrightarrow{\mathrm{E}}=-\frac{\varepsilon_{0}}{q} \Delta_{r} U_{2}$. The emergence of quantum pressure is caused by the inhomogeneity of the initial density distribution (4.11), leading to an increase in the pressure of one layer on another and, as a consequence, quantum pressure.

Solution (4.13) has a uniform initial distribution (4.10), which persists over time. The spherical layers do not press on each other and quantum pressure does not arise $\mathrm{Q}_{1}=0$. The density of probabilistic (quantum) charges is zero $\rho_{Q}=0$, and there are only external charges $\rho=-\frac{\varepsilon_{0}}{q} \Delta_{r} U_{1}=\rho_{1}(t)$ (4.14), which determine the probability density $|\Psi|^{2}$ for a quantum system.

A specific feature of a self-consistent system is its many-particle nature. For a quantum system consisting of one particle in an external field, self-consistency is possible if the external field "feels" the field of the particle itself. In the example considered, external potential $U$ is the potential of a many-particle system. Such a many-particle system consists of particles of the same type, and each contributes to total potential $U$. Each particle "perceives" potential $U$ as external. If a self-consistent system consists of only one particle, then there are no external fields. Indeed, any external field has its sources - particles, which means such a system does not consist of one particle. Consequently, a self-consistent system of one particle is a free particle. As a rule, such a system resembles the system considered within the problem of the wave packet dispersion (see §2).

Note that solution (4.13) is related to a non-self-consistent quantum system - a harmonic oscillator. Let us consider a one-dimensional quantum harmonic oscillator with variable frequency [39] $\Omega(t)$ and external potential

$$
\begin{equation*}
U_{3}(x, t)=\frac{m \Omega^{2}(t) x^{2}}{2} . \tag{4.17}
\end{equation*}
$$

Potential (4.17) differs from potential (4.14) in sign. Indeed, external potential $U_{1}$ «expanded» the system, while potential (4.17), on the contrary, tries to keep the quantum system in the vicinity of the origin. The following wave function corresponds to potential (4.17)

$$
\begin{gather*}
\Psi_{n}(x, t)=\frac{1}{\sqrt{2^{n} n!}} \frac{1}{\sqrt{\sqrt{2 \pi} \sigma(t)}} \exp \left(-\frac{x^{2}}{4 \sigma^{2}(t)}-i \frac{\dot{\sigma}(t)}{4 \alpha \sigma(t)} x^{2}-i \beta \mathcal{E}_{n}(t)\right) H_{n}\left(\frac{x}{\sqrt{2} \sigma(t)}\right), \\
\Omega^{2}(t)=\frac{\text { det }}{=} \frac{1}{\sigma}\left(\frac{\alpha^{2}}{\sigma^{3}}-\ddot{\sigma}\right), \quad \dot{\mathcal{E}}_{n}(t) \stackrel{\operatorname{det}}{=}-\frac{\alpha}{\beta \sigma^{2}(t)}\left(n+\frac{1}{2}\right), \tag{4.18}
\end{gather*}
$$

where $n$ is the number of the quantum state, $H_{n}$ are the Hermitian polynomials. Function $\sigma(t)$ defines the mean-square deviation for probability density $f_{n}^{1}=\left|\Psi_{n}\right|^{2}$. Equation of motion (2.32) for this system has the form:

$$
\begin{equation*}
\frac{d}{d t}\langle v\rangle_{| | n}=\frac{\ddot{\sigma}}{\sigma} x=-\frac{1}{m} \frac{\partial}{\partial x}\left(U_{3}+\mathrm{Q}_{n}\right), \mathrm{Q}_{n}(x, t)=-\frac{\hbar^{2}}{8 m \sigma^{4}}\left[x^{2}-2 \sigma^{2}(1+2 n)\right] . \tag{4.19}
\end{equation*}
$$

From (4.19) it follows that the system is not self-consistent, since $\square \mathrm{V}_{3} \neq \frac{q^{2}}{\varepsilon_{0}}\left|\Psi_{n}\right|^{2}$. In the nonrelativistic limit, source density $\rho_{\Psi}$ can be represented as:

$$
\begin{equation*}
\rho_{\Psi}(t)=-\frac{\varepsilon_{0}}{q} \frac{\partial^{2}}{\partial x^{2}} \mathrm{~V}_{3}=\frac{m \varepsilon_{0}}{q} \frac{\ddot{\sigma}}{\sigma} . \tag{4.20}
\end{equation*}
$$

Charge density (4.20) is an analogue of density (4.13). Both functions determine a density that is constant along the coordinate, the value of which changes with time. It turns out that the density of external sources (4.20) inducing field $\overrightarrow{\mathrm{E}}_{\Psi}$ in which the oscillator (4.19) is located is the same as the density in the self-consistent problem (4.13).

Note that the self-consistency (2.17) allows us to write the system of the Maxwell equations (2.20)-(2.22), which contains the Lorentz transformations for the relativistic theory. Moreover, equation (2.7) itself is not invariant. However, equation (2.7) contains additional information in the form of quantum density $\rho_{Q}$ and potential $\overrightarrow{\mathrm{A}}_{Q}$. For time-independent nonrelativistic self-consistent quantum systems, there is no radiation (see Theorem 5). Timedependent self-consistent quantum systems in the Coulomb gauge also cannot contain electromagnetic radiation. If the system is not self-consistent, then the system of the Maxwell equations (2.25)-(2.26) is not satisfied for probabilistic fields $\overrightarrow{\mathrm{E}}_{Q}, \overrightarrow{\mathrm{~B}}_{Q}$. The Maxwell equations are satisfied only for the external fields (2.23)-(2.24).

As was seen in $\S 3$, the invariance of the equation for the wave function is possible with an increase in the volume of kinematical information about the system. In the non-relativistic approximation, a quantum system (due to the Heisenberg uncertainty principle) does not have the concept of a trajectory, without which it is impossible to talk about producing electromagnetic radiation by the electric current. When transitioning to relativism, the Heisenberg uncertainty principle takes the form $\Delta x \sim \frac{\hbar c}{\varepsilon}$ (or $\Delta x \sim \frac{\hbar}{p}=\lambda$ is the de Broglie
wavelength, $\varepsilon=p c$ ), where $\varepsilon$ is the particle energy. As a result, at high energy the uncertainty along the trajectory decreases and we can talk about a trajectory (Dirac electric current) that produces electromagnetic radiation. In this case, current $J^{\mu}$ (3.19) in the Dirac equation becomes close to the electric current, since it has a smaller probabilistic contribution. The problem of describing compound particles by the Dirac equation is partly contained in the violation of the self-consistency of the system.

## Conclusions

Quantum mechanics, like no other branch of physics, contains a large number of postulates, correspondence rules and principles that were introduced in a phenomenological manner. This construction of quantum theory is caused by the strong «dissimilarity» of the behavior of the micro and macro worlds. Historical attempts to "understand" the quantum world are well illustrated by the examples of dialogues between Einstein and Bohr, the EPR paradox, the theory with hidden parameters, and many others. The fatigue from fruitless attempts to understand the micro world nature called into existence Feynman's famous phrase «I think I can safely say: no one understands quantum mechanics».

Over the last century, quantum theory has made significant progress, but despite this, the physical nature of the micro world remains unclear. From the position of theoretical physics, it is important not only to know the rules, laws and equations, but also to build a mathematical model, to find some first principle which can explain the behavior of the system. For example, Newton's second law was obtained phenomenologically, but the principle of least action (PLA) made it possible to interpret it as the shortest distance between two positions of a system in some curved space, where the curvature is caused by an external influence. Feynman's path integral gave us a classical interpretation of the quantum dynamics of a system from the perspective of PLA.

The beauty of a theory is determined by the number of postulates, rules, and axioms contained in it. The more rules, the more hidden the principle of the physical system is. In this paper, an attempt is made to outline a method for constructing some equations of classical and quantum mechanics based on one single first principle - the probability conservation law. The central object of this approach is the function of distribution or density of probabilities that satisfies the probability conservation law. An expanded version of the theory presented is given in [40-43], herein its application to the description of quantum systems and their relation with classical systems is shown.

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## Appendix A

## Proof of Theorem 1

Let us make intermediate transformations

$$
\begin{equation*}
\Psi^{*} \frac{\partial \Psi}{\partial t}+\Psi \frac{\partial \Psi^{*}}{\partial t}+i \alpha \operatorname{div}_{r}\left[\Psi^{*} \nabla \Psi-\Psi \nabla \Psi^{*}-i \frac{\gamma}{\alpha} \Psi \Psi^{*} \overrightarrow{\mathrm{~A}}_{\Psi}\right]=0 . \tag{A.1}
\end{equation*}
$$

taking into account that

$$
\operatorname{div}_{r}\left[\Psi^{*} \nabla_{r} \Psi-\Psi \nabla_{r} \Psi^{*}\right]=\Psi^{*} \Delta_{r} \Psi-\Psi \Delta_{r} \Psi^{*}
$$

$$
\begin{equation*}
\operatorname{div}_{r}\left[\Psi \Psi^{*} \overrightarrow{\mathrm{~A}}_{\Psi}\right]=\Psi^{*} \overrightarrow{\mathrm{~A}}_{\Psi} \cdot \nabla_{r} \Psi+\Psi \overrightarrow{\mathrm{A}}_{\Psi} \cdot \nabla_{r} \Psi^{*}+\Psi \Psi^{*} \chi \tag{A.2}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\Psi^{*}\left[\frac{\partial}{\partial t}+i \alpha \Delta+\gamma \overrightarrow{\mathrm{A}}_{\Psi} \cdot \nabla+\frac{\gamma}{2} \chi\right] \Psi+\Psi\left[\frac{\partial}{\partial t}-i \alpha \Delta+\gamma \overrightarrow{\mathrm{A}}_{\Psi} \cdot \nabla+\frac{\gamma}{2} \chi\right] \Psi^{*}=0 \tag{A.3}
\end{equation*}
$$

or

$$
\begin{align*}
& \Psi^{*}\left[\frac{1}{\beta} \frac{\partial}{\partial t}-i \alpha \beta\left(\hat{\mathrm{p}}^{2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}}_{\Psi} \cdot \hat{\mathrm{p}}\right)+\frac{\gamma}{2 \beta} \chi\right] \Psi+  \tag{A.4}\\
& +\Psi\left[\frac{1}{\beta} \frac{\partial}{\partial t}+i \alpha \beta\left(\hat{\mathrm{p}}^{* 2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}}_{\Psi} \cdot \overline{\hat{\mathrm{p}}}\right)+\frac{\gamma}{2 \beta} \chi\right] \Psi^{*}=0
\end{align*}
$$

We take into account the ratios:

$$
\begin{align*}
& \hat{\mathrm{p}}^{2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}}_{\Psi} \cdot \hat{\mathrm{p}}=\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2}-\frac{\gamma^{2}}{4 \alpha^{2} \beta^{2}}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2}-i \frac{\gamma}{2 \alpha \beta^{2}} \chi,  \tag{A.5}\\
& \hat{\mathrm{p}}^{* 2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}}_{\Psi} \cdot \hat{\mathrm{p}}^{*}=\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2}-\frac{\gamma^{2}}{4 \alpha^{2} \beta^{2}}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2}+i \frac{\gamma}{2 \alpha \beta^{2}} \chi .
\end{align*}
$$

Substituting (A.5) into equation (A.4), we obtain

$$
\begin{gather*}
\Psi^{*}\left[\frac{1}{\beta} \frac{\partial}{\partial t}-i \alpha \beta\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2}+i \frac{\gamma^{2}}{4 \alpha \beta}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2}-\frac{\gamma}{2 \beta} \chi+\frac{\gamma}{2 \beta} \chi\right] \Psi+ \\
+\Psi\left[\frac{1}{\beta} \frac{\partial}{\partial t}+i \alpha \beta\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2}-i \frac{\gamma^{2}}{4 \alpha \beta}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2}-\frac{\gamma}{2 \beta} \chi+\frac{\gamma}{2 \beta} \chi\right] \Psi^{*}=0, \\
\Psi^{*}\left[\frac{1}{\beta} \frac{\partial}{\partial t}-i \alpha \beta\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2}\right] \Psi+\Psi\left[\frac{1}{\beta} \frac{\partial}{\partial t}+i \alpha \beta\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2}\right] \Psi^{*}=0, \tag{A.6}
\end{gather*}
$$

For convenience, we introduce operator $L$ :

$$
\begin{equation*}
L=\frac{1}{\beta} \frac{\partial}{\partial t}-i \alpha \beta\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2}, L^{*}=\frac{1}{\beta} \frac{\partial}{\partial t}+i \alpha \beta\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2} . \tag{A.7}
\end{equation*}
$$

Let us rewrite equation (A.6) using the operator (A.7):

$$
\begin{equation*}
\Psi^{*} L \Psi+\Psi L^{*} \Psi^{*}=0 \Rightarrow \Lambda+\Lambda^{*}=0, \Lambda=\Psi^{*} L \Psi . \tag{A.8}
\end{equation*}
$$

from here

$$
\begin{equation*}
\operatorname{Re} \Lambda=0 \Rightarrow \Lambda=i u, u \in \mathbb{R} \tag{A.9}
\end{equation*}
$$

From expressions (A.8) and (A.9), we obtain the equation

$$
\Psi^{*} L \Psi=i u \Rightarrow L \Psi=i \frac{u}{\bar{\Psi}}=i \frac{u}{\bar{\Psi}} \frac{\Psi}{\Psi}=i \frac{u}{|\Psi|^{2}} \Psi=-i U \Psi, U \in \mathbb{R},
$$

$$
\begin{equation*}
L \Psi=-i U \Psi . \tag{A.10}
\end{equation*}
$$

Substituting the form of operator (A.7) into (A.10), we obtain the equation:

$$
\begin{equation*}
\frac{i}{\beta} \frac{\partial \Psi}{\partial t}=-\alpha \beta\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}_{\Psi}\right)^{2} \Psi+U \Psi \tag{A.11}
\end{equation*}
$$

Theorem 1 is proved

## Proof of Theorem 2

Using (A.11) and (A.5), we obtain an expression for function $U$ :

$$
\begin{equation*}
U \Psi=\frac{i}{\beta} \frac{\partial \Psi}{\partial t}-\frac{\alpha}{\beta} \Delta_{r} \Psi+i \frac{\gamma}{\beta} \overrightarrow{\mathrm{~A}} \cdot \nabla_{r} \Psi+\frac{\gamma^{2}}{4 \alpha \beta}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2} \Psi+i \frac{\gamma}{2 \beta} \chi \Psi . \tag{A.12}
\end{equation*}
$$

Let us calculate $\Delta_{r} \Psi$

$$
\begin{equation*}
\Delta_{r} \Psi=e^{i \varphi}\left[\Delta_{r}|\Psi|-|\Psi|\left|\nabla_{r} \varphi\right|^{2}+i\left(2 \nabla_{r} \varphi \cdot \nabla_{r}|\Psi|+|\Psi| \Delta_{r} \varphi\right)\right] \tag{A.13}
\end{equation*}
$$

We substitute (A.13) into (A.12), we obtain

$$
\begin{align*}
U= & \frac{i}{\Psi \beta} \frac{\partial \Psi}{\partial t}-\frac{\alpha}{\beta} \frac{\Delta_{r} \Psi}{\Psi}+i \frac{\gamma}{\Psi \beta} \overrightarrow{\mathrm{~A}}_{\Psi} \cdot \nabla_{r} \Psi+\frac{\gamma^{2}}{4 \alpha \beta}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2}+i \frac{\gamma}{2 \beta} \chi=\frac{i}{\beta|\Psi|}\left(\frac{\partial|\Psi|}{\partial t}+i|\Psi| \frac{\partial \varphi}{\partial t}\right)- \\
& -\frac{\alpha}{\beta|\Psi|}\left[\Delta_{r}|\Psi|-|\Psi|\left|\nabla_{r} \varphi\right|^{2}+i\left(2 \nabla_{r} \varphi \cdot \nabla_{r}|\Psi|+|\Psi| \Delta_{r} \varphi\right)\right]+i \frac{\gamma}{\beta|\Psi|} \overrightarrow{\mathrm{A}}_{\Psi} \cdot\left(\nabla_{r}|\Psi|+i|\Psi| \nabla_{r} \varphi\right)+ \\
& +\frac{\gamma^{2}}{4 \alpha \beta}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2}+i \frac{\gamma}{2 \beta} \chi=\frac{i}{\beta|\Psi|} \frac{\partial|\Psi|}{\partial t}-i \frac{\alpha}{\beta}\left(\frac{2}{|\Psi|} \nabla_{r} \varphi \cdot \nabla_{r}|\Psi|+\Delta_{r} \varphi\right)+i \frac{\gamma}{|\Psi| \beta} \overrightarrow{\mathrm{A}}_{\Psi} \cdot \nabla_{r}|\Psi|+ \\
& +i \frac{\gamma}{2 \beta} \chi-\frac{1}{\beta} \frac{\partial \varphi}{\partial t}-\frac{\alpha}{\beta} \frac{\Delta_{r}|\Psi|}{|\Psi|}+\frac{\alpha}{\beta}\left|\nabla_{r} \varphi\right|^{2}-\frac{\gamma}{\beta} \overrightarrow{\mathrm{A}}_{\Psi} \cdot \nabla_{r} \varphi+\frac{\gamma^{2}}{4 \alpha \beta}\left|\overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2} . \tag{A.14}
\end{align*}
$$

From expression (A.10) it follows that the imaginary part (A.14) must be equal to zero. Indeed, expression (A.14) takes the form:

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} \ln f-\alpha \nabla_{r} \Phi \cdot \nabla_{r} \ln f-\alpha \Delta_{r} \Phi+\gamma \overrightarrow{\mathrm{A}}_{\Psi} \cdot \nabla_{r} \ln f+\gamma \chi=0, \\
-\frac{1}{\beta} \frac{\partial \varphi}{\partial t}-\frac{\alpha}{\beta} \frac{\Delta_{r}|\Psi|}{|\Psi|}+\frac{1}{4 \alpha \beta}\left|\alpha \nabla_{r} \Phi\right|^{2}-\frac{1}{2 \alpha \beta} \alpha \nabla \Phi \cdot \gamma \overrightarrow{\mathrm{~A}}+\frac{1}{4 \alpha \beta}\left|\gamma \overrightarrow{\mathrm{~A}}_{\Psi}\right|^{2}-U=0,
\end{array}\right.
$$

or

$$
\left\{\begin{array}{l}
\frac{\partial f}{\partial t}+\left(-\alpha \nabla_{r} \Phi+\gamma \overrightarrow{\mathrm{A}}_{\Psi}\right) \cdot \nabla_{r} f+f\left(-\alpha \Delta \Delta_{r} \Phi+\gamma \mathrm{div}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}\right)=0, \\
-\frac{1}{\beta} \frac{\partial \varphi}{\partial t}-\mathrm{Q}+\frac{1}{4 \alpha \beta}\left(\left|\alpha \nabla_{r} \Phi\right|^{2}-2 \alpha \nabla_{r} \Phi \cdot \gamma \overrightarrow{\mathrm{~A}}_{\Psi}+\left|\gamma \overrightarrow{\mathrm{A}}_{\Psi}\right|^{2}\right)-U=0 .
\end{array}\right.
$$

from here

$$
\left\{\begin{array}{l}
\frac{\partial f}{\partial t}+\langle\vec{v}\rangle \cdot \nabla_{r} f+f \operatorname{div}_{r}\langle\vec{v}\rangle=0  \tag{A.15}\\
-\frac{1}{\beta} \frac{\partial \varphi}{\partial t}=-\frac{1}{4 \alpha \beta}|\langle\vec{v}\rangle|^{2}+U+\mathrm{Q}
\end{array}\right.
$$

The first equation in (A.15) coincides with the first Vlasov equation, and the second leads to the Hamilton-Jacobi equation.

$$
\begin{align*}
& -\frac{1}{\beta} \frac{\partial \varphi}{\partial t}=-\frac{1}{4 \alpha \beta}|\langle\vec{v}\rangle|^{2}+\mathrm{V}=\mathrm{H}  \tag{A.16}\\
& \mathrm{~V}=U+\mathrm{Q}, \quad \mathrm{Q}=\frac{\alpha}{\beta} \frac{\Delta_{r}|\Psi|}{|\Psi|} \tag{A.17}
\end{align*}
$$

We obtain the equation of motion from equation (A.16):

$$
\begin{equation*}
\frac{1}{2 \alpha \beta} \frac{\partial}{\partial t}\left(-\alpha \nabla_{r} \Phi\right)=\frac{1}{2 \alpha \beta} \frac{\partial}{\partial t}\langle\vec{v}\rangle-\frac{\gamma}{2 \alpha \beta} \frac{\partial}{\partial t} \overrightarrow{\mathrm{~A}}_{\Psi}=-\frac{1}{4 \alpha \beta} \nabla_{r}|\langle\vec{v}\rangle|^{2}+\nabla_{r} \mathrm{~V}, \tag{A.18}
\end{equation*}
$$

taking into account that $\left(\langle\vec{v}\rangle \cdot \nabla_{r}\right)\langle\vec{v}\rangle=\frac{1}{2} \nabla_{r}|\langle\vec{v}\rangle|^{2}-\gamma\langle\vec{v}\rangle \times \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}$, then (A.18) takes the form:

$$
\begin{aligned}
& \frac{\partial}{\partial t}\langle\vec{v}\rangle+\left(\langle\vec{v}\rangle \cdot \nabla_{r}\right)\langle\vec{v}\rangle=\gamma \frac{\partial \overrightarrow{\mathrm{A}}_{\Psi}}{\partial t}-\gamma\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}_{\Psi}+2 \alpha \beta \nabla_{r} \mathrm{~V}= \\
&=-\gamma\left(-\frac{\partial \overrightarrow{\mathrm{A}}_{\Psi}}{\partial t}-\frac{2 \alpha \beta}{\gamma} \nabla_{r} \mathrm{~V}+\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}_{\Psi}\right), \\
& \frac{d}{d t}\langle\vec{v}\rangle=-\gamma\left(\overrightarrow{\mathrm{E}}_{\Psi}+\langle\vec{v}\rangle \times \overrightarrow{\mathrm{B}}_{\Psi}\right), \overrightarrow{\mathrm{E}}_{\Psi}=-\frac{\partial \overrightarrow{\mathrm{A}}_{\Psi}}{\partial t}-\frac{2 \alpha \beta}{\gamma} \nabla_{r} \mathrm{~V} .
\end{aligned}
$$

Theorem 2 is proved.

## Proof of Lemma 1

From the condition (2.17) and the first Vlasov equation (1.5) it follows that:

$$
\begin{gather*}
\frac{\partial}{\partial t} \operatorname{div}_{r} \vec{D}_{\Psi}+\operatorname{div}_{r}\left(\rho_{\Psi}\langle\vec{v}\rangle\right)=0 \Rightarrow \operatorname{div}_{r}\left(\frac{\partial}{\partial t} \vec{D}_{\Psi}+\overrightarrow{\mathrm{J}}_{\Psi}\right)=0, \\
\frac{\partial}{\partial t} \vec{D}_{\Psi}+\overrightarrow{\mathrm{J}}_{\Psi}=\operatorname{curl}_{r} \overrightarrow{\mathrm{H}}_{\Psi} \tag{A.19}
\end{gather*}
$$

where $\overrightarrow{\mathrm{H}}_{\Psi}$ is some field, since $\operatorname{div}_{r} \operatorname{curl}_{r} \overrightarrow{\mathrm{H}}_{\Psi}=0$. The definition of vortex field $\overrightarrow{\mathrm{B}}_{\Psi}$ (2.11) leads to the equation

$$
\begin{equation*}
\operatorname{div}_{r} \overrightarrow{\mathrm{~B}}_{\Psi}=\operatorname{div}_{r} \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}=0 . \tag{A.20}
\end{equation*}
$$

Calculating operator curl ${ }_{r}$ from expression (2.11) $\overrightarrow{\mathrm{E}}_{\Psi}$ gives (2.17):

$$
\begin{equation*}
\operatorname{curl}_{r} \overrightarrow{\mathrm{E}}_{\Psi}=-\frac{\partial}{\partial t} \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}=-\frac{\partial}{\partial t} \overrightarrow{\mathrm{~B}}_{\Psi} . \tag{A.21}
\end{equation*}
$$

Resulting equations (A.19)-(A.21) prove the validity of equations (2.16)-(2.17). We obtain equations (2.18)-(2.20) under the condition of Lorentz $\Psi$-gauge. Calculating the operator $\operatorname{div}_{r}$ from expression (2.11) and taking into account the gauge condition (2.12), we obtain

$$
\begin{equation*}
\operatorname{div}_{r} \overrightarrow{\mathrm{E}}_{\Psi}=\frac{\rho_{\Psi}}{\varepsilon_{0}}=-\frac{\partial}{\partial t} \operatorname{div}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}-\frac{2 \alpha \beta}{\gamma} \Delta_{r} \mathrm{~V}=\frac{2 \alpha \beta}{\gamma}\left(\frac{1}{c^{2}} \frac{\partial^{2} \mathrm{~V}}{\partial t^{2}}-\Delta_{r} \mathrm{~V}\right)=\frac{1}{q} \square \mathrm{~V} \tag{A.22}
\end{equation*}
$$

Using representations (2.11) in equation (2.17), we obtain the equation:

$$
\begin{gather*}
-\varepsilon_{0} \frac{\partial^{2}}{\partial t^{2}} \overrightarrow{\mathrm{~A}}_{\Psi}-\varepsilon_{0} \nabla_{r} \frac{\partial}{\partial t} \mathrm{~V}+\overrightarrow{\mathrm{J}}_{\Psi}=\frac{1}{\mu_{0}} \operatorname{curl}_{r} \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}_{\Psi} \\
-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \overrightarrow{\mathrm{~A}}_{\Psi}-\nabla_{r}\left(\frac{1}{c^{2}} \frac{\partial}{\partial t} \mathrm{~V}+\operatorname{div}_{r} \overrightarrow{\mathrm{~A}}_{\Psi}\right)+\Delta_{r} \overrightarrow{\mathrm{~A}}_{\Psi}=-\mu_{0} \overrightarrow{\mathrm{~J}}_{\Psi} \\
\left(\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}-\Delta_{r}\right) \overrightarrow{\mathrm{A}}_{\Psi}=\mu_{0} \overrightarrow{\mathrm{~J}}_{\Psi} \tag{A.23}
\end{gather*}
$$

Expressions (A.22) and (A.23) are consistent with equations (2.18)-(2.20). Lemma 1 is completely proved.

## Appendix B

## Proof of Theorem 3

Substituting the Helmholtz $\psi \mathbb{C}^{2}$-decomposition (3.2) into the first Vlasov equation (1.5), we obtain

$$
\psi^{\dagger} \frac{\partial \psi}{\partial t}+\frac{\partial \psi^{\dagger}}{\partial t} \psi+i \alpha \operatorname{div}_{r}\left[\psi^{\dagger} \mathrm{I} \nabla_{r} \psi-\psi^{T} \mathrm{I} \nabla_{r} \psi^{*}-i \frac{\gamma}{\alpha} \psi^{\dagger} \vec{A} \psi\right]=0, \quad \mathrm{I} \nabla_{r} \stackrel{\operatorname{det}}{=}\left(\begin{array}{cc}
\nabla_{r} & 0  \tag{B.1}\\
0 & \nabla_{r}
\end{array}\right) .
$$

Let us perform intermediate transformations:

$$
\begin{align*}
& \psi^{\dagger} \frac{\partial \psi}{\partial t}+\frac{\partial \psi^{\dagger}}{\partial t} \psi=\psi_{1}^{*} \partial_{t} \psi_{1}+\psi_{2}^{*} \partial_{t} \psi_{2}+\psi_{1} \partial_{t} \psi_{1}^{*}+\psi_{2} \partial_{t} \psi_{2}^{*}= \\
& =\left(\begin{array}{ll}
\psi_{1}^{*} & \psi_{2}^{*}
\end{array}\right)\left(\begin{array}{cc}
\partial_{t} & 0 \\
0 & \partial_{t}
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}+\left(\begin{array}{ll}
\psi_{1} & \psi_{2}
\end{array}\right)\left(\begin{array}{cc}
\partial_{t} & 0 \\
0 & \partial_{t}
\end{array}\right)\binom{\psi_{1}^{*}}{\psi_{2}^{*}}, \\
& \psi^{\dagger} \frac{\partial \psi}{\partial t}+\frac{\partial \psi^{\dagger}}{\partial t} \psi=\psi^{\dagger} \mathrm{I} \partial_{t} \psi+\psi^{T} \mathrm{I} \partial_{t} \psi^{*},  \tag{B.2}\\
& \psi^{\dagger} \mathrm{I} \nabla_{r} \psi-\psi^{T} \mathrm{I} \nabla_{r} \psi^{*}=\left(\begin{array}{ll}
\psi_{1}^{*} & \psi_{2}^{*}
\end{array}\right)\left(\begin{array}{cc}
\nabla_{r} & 0 \\
0 & \nabla_{r}
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}-\left(\begin{array}{ll}
\psi_{1} & \psi_{2}
\end{array}\right)\left(\begin{array}{cc}
\nabla_{r} & 0 \\
0 & \nabla_{r}
\end{array}\right)\binom{\psi_{1}^{*}}{\psi_{2}^{*}}, \\
& \psi^{\dagger} \mathrm{I} \nabla_{r} \psi-\psi^{T} \mathrm{I} \nabla_{r} \psi^{*}=\psi_{1}^{*} \nabla_{r} \psi_{1}+\psi_{2}^{*} \nabla_{r} \psi_{2}-\psi_{1} \nabla_{r} \psi_{1}^{*}-\psi_{2} \nabla_{r} \psi_{2}^{*}, \tag{B.3}
\end{align*}
$$

$$
\begin{align*}
& \operatorname{div}_{r}\left[\psi^{\dagger} \mathrm{I} \nabla_{r} \psi-\psi^{T} \mathrm{I} \nabla_{r} \psi^{*}\right]=\nabla_{r} \psi_{1}^{*} \nabla_{r} \psi_{1}+\psi_{1}^{*} \Delta_{r} \psi_{1}+\nabla_{r} \psi_{2}^{*} \nabla_{r} \psi_{2}+\psi_{2}^{*} \Delta_{r} \psi_{2}- \\
& -\nabla_{r} \psi_{1} \nabla_{r} \psi_{1}^{*}-\psi_{1} \Delta_{r} \psi_{1}^{*}-\nabla_{r} \psi_{2} \nabla_{r} \psi_{2}^{*}-\psi_{2} \Delta_{r} \psi_{2}^{*}=\psi_{1}^{*} \Delta_{r} \psi_{1}+\psi_{2}^{*} \Delta_{r} \psi_{2}-\psi_{1} \Delta_{r} \psi_{1}^{*}-\psi_{2} \Delta_{r} \psi_{2}^{*}= \\
& =\left(\begin{array}{ll}
\psi_{1}^{*} & \psi_{2}^{*}
\end{array}\right)\left(\begin{array}{cc}
\Delta_{r} & 0 \\
0 & \Delta_{r}
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}-\left(\begin{array}{ll}
\psi_{1} & \psi_{2}
\end{array}\right)\left(\begin{array}{cc}
\Delta_{r} & 0 \\
0 & \Delta_{r}
\end{array}\right)\binom{\psi_{1}^{*}}{\psi_{2}^{*}}=\psi^{\dagger} \Delta_{r} \psi-\psi^{T} \Delta_{r} \psi^{*}, \\
& \operatorname{div}_{r}\left(\psi^{\dagger} \mathrm{I} \nabla_{r} \psi-\psi^{T} \mathrm{I} \nabla_{r} \psi^{*}\right)=\psi^{\dagger} \mathrm{I} \Delta_{r} \psi-\psi^{T} \mathrm{I} \Delta_{r} \psi^{*},  \tag{B.4}\\
& \operatorname{div}_{r}\left(\psi^{\dagger} \overrightarrow{\mathrm{A}} \psi\right)=|\psi|^{2} \operatorname{div}_{r} \overrightarrow{\mathrm{~A}}+\overrightarrow{\mathrm{A}} \cdot\left(\psi_{1} \nabla_{r} \psi_{1}^{*}+\psi_{1}^{*} \nabla_{r} \psi_{1}+\psi_{2} \nabla_{r} \psi_{2}^{*}+\psi_{2}^{*} \nabla_{r} \psi_{2}\right)=\psi^{\dagger} \chi \psi+ \\
& +\left(\begin{array}{ll}
\psi_{1} & \psi_{2}
\end{array}\right)\left(\begin{array}{cc}
\overrightarrow{\mathrm{A}} \cdot \nabla_{r} & 0 \\
0 & \overrightarrow{\mathrm{~A}} \cdot \nabla_{r}
\end{array}\right)\binom{\psi_{1}^{*}}{\psi_{2}^{*}}+\left(\begin{array}{ll}
\psi_{1}^{*} & \psi_{2}^{*}
\end{array}\right)\left(\begin{array}{cc}
\overrightarrow{\mathrm{A}} \cdot \nabla_{r} & 0 \\
0 & \overrightarrow{\mathrm{~A}} \cdot \nabla_{r}
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}, \\
& \operatorname{div}_{r}\left(\psi^{\dagger} \overrightarrow{\mathrm{A}} \psi\right)=\psi^{\dagger} \chi \psi+\psi^{T} \mathrm{I} \overrightarrow{\mathrm{~A}} \cdot \nabla_{r} \psi^{*}+\psi^{\dagger} \mathrm{I} \overrightarrow{\mathrm{~A}} \cdot \nabla_{r} \psi \text {. } \tag{B.5}
\end{align*}
$$

Substituting expressions (B.2)-(B.4) into equation (B.1), we obtain:

$$
\begin{gathered}
\psi^{\dagger} \mathrm{I} \partial_{t} \psi+\psi^{T} \mathrm{I} \partial_{t} \psi^{*}+i \alpha \psi^{\dagger} \mathrm{I} \Delta_{r} \psi-i \alpha \psi^{T} \mathrm{I} \Delta_{r} \psi^{*}+\gamma \psi^{\dagger} \chi \psi+\gamma \psi^{T} \mathrm{I} \overrightarrow{\mathrm{~A}} \cdot \nabla_{r} \psi^{*}+\gamma \psi^{\dagger} \mathrm{I} \overrightarrow{\mathrm{~A}} \cdot \nabla_{r} \psi=0 \\
\psi^{\dagger} \mathrm{I}\left(\partial_{t}+i \alpha \Delta_{r}+\gamma \overrightarrow{\mathrm{A}} \cdot \nabla_{r}+\frac{\gamma}{2} \chi\right) \psi+\psi^{T} \mathrm{I}\left(\partial_{t}-i \alpha \Delta_{r}+\gamma \overrightarrow{\mathrm{A}} \cdot \nabla_{r}+\frac{\gamma}{2} \chi\right) \psi^{*}=0
\end{gathered}
$$

or

$$
\begin{equation*}
\psi^{\dagger} \mathrm{I}\left[\frac{1}{\beta} \frac{\partial}{\partial t}-i \alpha \beta\left(\hat{\mathrm{p}}^{2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}} \cdot \hat{\mathrm{p}}\right)+\frac{\gamma}{2 \beta} \chi\right] \psi+\psi^{T} \mathrm{I}\left[\frac{1}{\beta} \frac{\partial}{\partial t}+i \alpha \beta\left(\hat{\mathrm{p}}^{* 2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}} \cdot \hat{\mathrm{p}}^{*}\right)+\frac{\gamma}{2 \beta} \chi\right] \psi^{*}=0 \tag{B.6}
\end{equation*}
$$

Let us transform the terms in equation (B.5):

$$
\begin{align*}
\mathrm{I}\left(\hat{\mathrm{p}}^{2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}} \cdot \hat{\mathrm{p}}\right) & =\mathrm{I}\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)^{2}-\mathrm{I} \frac{\gamma^{2}}{4 \alpha^{2} \beta^{2}}|\overrightarrow{\mathrm{~A}}|^{2}-i \mathrm{I} \frac{\gamma}{2 \alpha \beta^{2}} \chi= \\
& =\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}+\frac{\gamma}{2 \alpha \beta^{2}}\left(\vec{\sigma} \cdot \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}-\mathrm{I} i \chi\right)-\mathrm{I} \frac{\gamma^{2}}{4 \alpha^{2} \beta^{2}}|\overrightarrow{\mathrm{~A}}|^{2}  \tag{B.7}\\
\mathrm{I}\left(\hat{\mathrm{p}}^{* 2}-\frac{\gamma}{\alpha \beta} \overrightarrow{\mathrm{A}} \cdot \hat{\mathrm{p}}^{*}\right) & =\left[\vec{\sigma}^{*} \cdot\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}+\frac{\gamma}{2 \alpha \beta^{2}}\left(\vec{\sigma}^{*} \cdot \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}}+\mathrm{I} i \chi\right)-\mathrm{I} \frac{\gamma^{2}}{4 \alpha^{2} \beta^{2}}|\overrightarrow{\mathrm{~A}}|^{2}
\end{align*}
$$

where, in accordance with expression $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b})=\mathrm{I}(\vec{a} \cdot \vec{b})+i \vec{\sigma} \cdot(\vec{a} \times \vec{b})$, it is taken into account that

$$
\begin{equation*}
\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}=\mathrm{I}\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)^{2}-\frac{\gamma}{2 \alpha \beta^{2}} \vec{\sigma} \cdot \operatorname{curl}_{r} \overrightarrow{\mathrm{~A}} . \tag{B.8}
\end{equation*}
$$

Substituting (B.7) into equation (B.6), we obtain

$$
\begin{aligned}
& \psi^{\dagger}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}-i \alpha \beta\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}-i \frac{\gamma}{2 \beta} \vec{\sigma} \cdot \overrightarrow{\mathrm{~B}}+i \mathrm{I} \frac{\gamma^{2}}{4 \alpha \beta}|\overrightarrow{\mathrm{~A}}|^{2}\right) \psi+ \\
& +\psi^{T}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}+i \alpha \beta\left[\vec{\sigma}^{*} \cdot\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}+i \frac{\gamma}{2 \beta} \vec{\sigma}^{*} \cdot \overrightarrow{\mathrm{~B}}-i \mathrm{I} \frac{\gamma^{2}}{4 \alpha \beta}|\overrightarrow{\mathrm{~A}}|^{2}\right) \psi^{*}=0,
\end{aligned}
$$

or

$$
\begin{align*}
& \psi^{\dagger}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}-i \alpha \beta\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}+i \mathrm{I} \frac{\gamma^{2}}{4 \alpha \beta}|\overrightarrow{\mathrm{~A}}|^{2}\right) \psi+  \tag{B.9}\\
& +\psi^{T}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}+i \alpha \beta\left[\vec{\sigma}^{*} \cdot\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}-i \mathrm{I} \frac{\gamma^{2}}{4 \alpha \beta}|\overrightarrow{\mathrm{~A}}|^{2}\right) \psi^{*}+i \frac{\gamma}{2 \beta}\left[\psi^{T}\left(\vec{\sigma}^{*} \cdot \overrightarrow{\mathrm{~B}}\right) \psi^{*}-\psi^{\dagger}(\vec{\sigma} \cdot \overrightarrow{\mathrm{B}}) \psi\right]=0 .
\end{align*}
$$

Let us simplify the third term in equation (B.9):

$$
\begin{align*}
& \vec{\sigma} \cdot \overrightarrow{\mathrm{B}}=\sigma^{1} \mathrm{~B}_{x}+\sigma^{2} \mathrm{~B}_{y}+\sigma^{3} \mathrm{~B}_{z}=\left(\begin{array}{cc}
\mathrm{B}_{z} & \mathrm{~B}_{x}-i \mathrm{~B}_{y} \\
\mathrm{~B}_{x}+i \mathrm{~B}_{y} & -\mathrm{B}_{z}
\end{array}\right), \vec{\sigma}^{*} \cdot \overrightarrow{\mathrm{~B}}=\left(\begin{array}{cc}
\mathrm{B}_{z} & \mathrm{~B}_{x}+i \mathrm{~B}_{y} \\
\mathrm{~B}_{x}-i \mathrm{~B}_{y} & -\mathrm{B}_{z}
\end{array}\right), \\
& \psi^{T}\left(\vec{\sigma}^{*} \cdot \overrightarrow{\mathrm{~B}}\right) \psi^{*}-\psi^{\dagger}(\vec{\sigma} \cdot \overrightarrow{\mathrm{B}}) \psi=\left(\begin{array}{ll}
\psi_{1} & \psi_{2}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{B}_{z} & \mathrm{~B}_{x}+i \mathrm{~B}_{y} \\
\mathrm{~B}_{x}-i \mathrm{~B}_{y} & -\mathrm{B}_{z}
\end{array}\right)\binom{\psi_{1}^{*}}{\psi_{2}^{*}}- \\
& -\left(\begin{array}{ll}
\psi_{1}^{*} & \psi_{1}^{*}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{B}_{z} & \mathrm{~B}_{x}-i \mathrm{~B}_{y} \\
\mathrm{~B}_{x}+i \mathrm{~B}_{y} & -\mathrm{B}_{z}
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}=\left(\begin{array}{ll}
\psi_{1} & \psi_{2}
\end{array}\right)\binom{\mathrm{B}_{z} \psi_{1}^{*}+\left(\mathrm{B}_{x}+i \mathrm{~B}_{y}\right) \psi_{2}^{*}}{\left(\mathrm{~B}_{x}-i \mathrm{~B}_{y}\right) \psi_{1}^{*}-\mathrm{B}_{z} \psi_{1}^{*}}- \\
& -\left(\begin{array}{ll}
\psi_{1}^{*} & \left.\psi_{2}^{*}\right)\binom{\mathrm{B}_{z} \psi_{1}+\left(\mathrm{B}_{x}-i \mathrm{~B}_{y}\right) \psi_{2}}{\left(\mathrm{~B}_{x}+i \mathrm{~B}_{y}\right) \psi_{1}-\mathrm{B}_{z} \psi_{2}}=\psi_{1}\left[\mathrm{~B}_{z} \psi_{1}^{*}+\left(\mathrm{B}_{x}+i \mathrm{~B}_{y}\right) \psi_{2}^{*}\right]+\psi_{2}\left[\left(\mathrm{~B}_{x}-i \mathrm{~B}_{y}\right) \psi_{1}^{*}-\mathrm{B}_{z} \psi_{2}^{*}\right]- \\
-\psi_{1}^{*}\left[\begin{array}{l}
\left.\mathrm{B}_{z} \psi_{1}+\left(\mathrm{B}_{x}-i \mathrm{~B}_{y}\right) \psi_{2}\right]-\psi_{2}^{*}\left[\left(\mathrm{~B}_{x}+i \mathrm{~B}_{y}\right) \psi_{1}-\mathrm{B}_{z} \psi_{2}\right]=\psi_{1} \mathrm{~B}_{z} \psi_{1}^{*}-\psi_{1}^{*} \mathrm{~B}_{z} \psi_{1}-\psi_{2} \mathrm{~B}_{z} \psi_{2}^{*}+\psi_{2}^{*} \mathrm{~B}_{z} \psi_{2}+ \\
+\psi_{1} \psi_{2}^{*}\left(\mathrm{~B}_{x}+i \mathrm{~B}_{y}\right)-\psi_{1} \psi_{2}^{*}\left(\mathrm{~B}_{x}+i \mathrm{~B}_{y}\right)+\psi_{1}^{*} \psi_{2}\left(\mathrm{~B}_{x}-i \mathrm{~B}_{y}\right)-\psi_{1}^{*} \psi_{2}\left(\mathrm{~B}_{x}-i \mathrm{~B}_{y}\right), \\
\psi^{T}\left(\vec{\sigma}^{*} \cdot \overrightarrow{\mathrm{~B}}\right) \psi^{*}-\psi^{\dagger}(\vec{\sigma} \cdot \overrightarrow{\mathrm{B}}) \psi=0 .
\end{array}\right.
\end{array} . \begin{array}{l}
\mathrm{B} .10)
\end{array}\right.
\end{align*}
$$

Taking into account expression (B.10), equation (B.9) will take the form

$$
\begin{align*}
& \psi^{\dagger}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}-i \alpha \beta\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}\right) \psi+i \frac{\gamma^{2}}{4 \alpha \beta}|\overrightarrow{\mathrm{~A}}|^{2}\left(\psi^{\dagger} \mathrm{I} \psi-\psi^{T} \mathrm{I} \psi^{*}\right)+ \\
& +\psi^{T}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}+i \alpha \beta\left[\vec{\sigma}^{*} \cdot\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}\right) \psi^{*}=0 . \tag{B.11}
\end{align*}
$$

The second term in equation (B.11) is equal to zero, indeed

$$
\psi^{\dagger} \mathrm{I} \psi-\psi^{T} \mathrm{I} \psi^{*}=\psi^{\dagger} \psi-\psi^{T} \psi^{*}=|\psi|^{2}-|\psi|^{2}=0,
$$

from here

$$
\begin{equation*}
\psi^{\dagger}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}-i \alpha \beta\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}\right) \psi+\psi^{T}\left(\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}+i \alpha \beta\left[\vec{\sigma}^{*} \cdot\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}\right) \psi^{*}=0 . \tag{B.12}
\end{equation*}
$$

For convenience, we introduce operator $L$ :

$$
\begin{equation*}
L \stackrel{\text { det }}{=} \frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}-i \alpha \beta\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2}, L^{*}=\frac{1}{\beta} \mathrm{I} \frac{\partial}{\partial t}+i \alpha \beta\left[\vec{\sigma}^{*} \cdot\left(\hat{\mathrm{p}}^{*}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2} . \tag{B.13}
\end{equation*}
$$

Let us rewrite equation (B.12) using operator (B.13)

$$
\begin{equation*}
\psi^{\dagger} L \psi+\psi^{T} L^{*} \psi^{*}=0 \Rightarrow \Lambda+\Lambda^{*}=0, \Lambda \stackrel{\operatorname{det}}{=} \psi^{\dagger} L \psi . \tag{B.14}
\end{equation*}
$$

Since $\Lambda \in \mathbb{C}^{[2 \times 2]}$, from here $\Lambda+\Lambda^{*}=2 \operatorname{Re} \Lambda$

$$
\begin{equation*}
\operatorname{Re} \Lambda=0 \Rightarrow \Lambda=i u, u \in \mathbb{R} . \tag{B.15}
\end{equation*}
$$

From expressions (B.14) and (B.15) we obtain the equation

$$
\begin{gather*}
\psi^{\dagger} L \psi=-i|\psi|^{2} U=-i \psi^{\dagger} \mathrm{I} U \psi \Rightarrow \psi^{\dagger} \cdot(L \psi+i \mathrm{I} U \psi)=0, U \in \mathbb{R}, \\
L \psi=-i \mathrm{I} U \psi . \tag{B.16}
\end{gather*}
$$

Substituting the form of operator (B.13) into (B.16) we arrive at the equation:

$$
\begin{equation*}
\frac{i}{\beta} \mathrm{I} \frac{\partial \psi}{\partial t}=-\alpha \beta\left[\vec{\sigma} \cdot\left(\hat{\mathrm{p}}-\frac{\gamma}{2 \alpha \beta} \overrightarrow{\mathrm{~A}}\right)\right]^{2} \psi+\mathrm{I} U \psi . \tag{B.17}
\end{equation*}
$$

Theorem 3 is proved.

## Proof of Lemma 2

Let us calculate the derivatives with respect to time and coordinates from expression (3.6) and take into account gauges in (3.8), we obtain:

$$
\begin{gather*}
\operatorname{div}_{r} \vec{F}=-\Delta_{r} \Pi-\frac{\partial}{\partial t} \operatorname{div}_{r} \vec{R}=\frac{1}{c^{2}} \frac{\partial \Pi}{\partial t}-\Delta_{r} \Pi=\square \Pi,  \tag{B.18}\\
\operatorname{curl}_{r} \vec{F}=\operatorname{curl}_{r} \operatorname{curl}_{r} \vec{S}-\frac{\partial}{\partial t} \operatorname{curl}_{r} \vec{R}=\nabla_{r} \operatorname{div}_{r} \vec{S}-\Delta_{r} \vec{S}+\frac{1}{c^{2}} \frac{\partial^{2} \vec{S}}{\partial t^{2}}=\square \vec{S}, \tag{B.19}
\end{gather*}
$$

From the first gauge in (3.8), we calculate $\nabla_{r}$, and from the second - curl $_{r}$ :

$$
\left\{\begin{array}{l}
\frac{1}{c^{2}} \frac{\partial}{\partial t} \nabla_{r} \Pi+\nabla_{r} \operatorname{div}_{r} \vec{R}=0, \\
\frac{1}{c^{2}} \frac{\partial}{\partial t} \operatorname{curl}_{r} \vec{S}+\nabla_{r} \operatorname{div}_{r} \vec{R}-\Delta_{r} \vec{R}=0
\end{array} \Rightarrow \frac{1}{c^{2}} \frac{\partial}{\partial t}\left(\operatorname{curl}_{r} \vec{S}-\nabla_{r} \Pi\right)-\Delta_{r} \vec{R}=0\right.
$$

$$
\begin{equation*}
\frac{1}{c^{2}} \frac{\partial}{\partial t}\left(\vec{F}+\frac{\partial \vec{R}}{\partial t}\right)-\Delta_{r} \vec{R}=0 \Rightarrow-\frac{1}{c^{2}} \frac{\partial \vec{F}}{\partial t}=\square \vec{R}, \tag{B.20}
\end{equation*}
$$

where the last equation takes into account expansion (3.6). Lemma 2 is proved.

## Proof of Theorem 4

Bispinor $\psi$ can be represented in terms of two spinors $\vartheta$ and $\eta$ :

$$
\begin{gather*}
\psi=\left(\begin{array}{ll}
\vartheta & \eta)^{T}=\left(\begin{array}{lll}
\psi_{1} & \psi_{2} & \psi_{3}
\end{array} \psi_{4}\right.
\end{array}\right)^{T}, \vartheta=\left(\begin{array}{ll}
\psi_{1} & \psi_{2}
\end{array}\right)^{T}, \eta=\left(\begin{array}{ll}
\psi_{3} & \psi_{4}
\end{array}\right)^{T}, \\
|\psi|^{2}=\psi^{\dagger} \psi=|\vartheta|^{2}+|\eta|^{2},  \tag{B.21}\\
\bar{\psi}=\psi^{\dagger} \gamma^{0}=\left(\begin{array}{ll}
\vartheta^{\dagger} & \eta^{\dagger}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{I}_{2} & 0 \\
0 & -\mathrm{I}_{2}
\end{array}\right)=\left(\begin{array}{ll}
\vartheta^{\dagger} & -\eta^{\dagger}
\end{array}\right)=\left(\begin{array}{llll}
\psi_{1}^{*} & \psi_{2}^{*} & -\psi_{3}^{*} & -\psi_{4}^{*}
\end{array}\right) .
\end{gather*}
$$

Let us write expressions for density of current $J^{\mu}$ :

$$
\begin{gather*}
J^{0}=c \bar{\psi} \gamma^{0} \psi=c \psi^{\dagger} \gamma^{0} \gamma^{0} \psi=c \psi^{\dagger} \psi=c\left(\psi_{1}^{*} \psi_{1}+\psi_{2}^{*} \psi_{2}+\psi_{3}^{*} \psi_{3}+\psi_{4}^{*} \psi_{4}\right),  \tag{B.22}\\
\mathrm{J}^{k}=c \psi^{\dagger} \gamma^{0} \gamma^{k} \psi=c\left(\begin{array}{ll}
\vartheta^{\dagger} & -\eta^{\dagger}
\end{array}\right)\left(\begin{array}{cc}
0 & \sigma^{k} \\
-\sigma^{k} & 0
\end{array}\right)\binom{\vartheta}{\eta}=c\left(\begin{array}{ll}
\vartheta^{\dagger} & -\eta^{\dagger}
\end{array}\right)\binom{\sigma^{k} \eta}{-\sigma^{k} \vartheta}, \\
\mathrm{~J}^{k}=c\left(\vartheta^{\dagger} \sigma^{k} \eta+\eta^{\dagger} \sigma^{k} \vartheta\right) . \tag{B.23}
\end{gather*}
$$

Hence,

$$
\begin{align*}
& \mathrm{J}^{1}=c\left(\vartheta^{\dagger} \sigma^{1} \eta+\eta^{\dagger} \sigma^{1} \vartheta\right)=c\left(\psi_{1}^{*} \quad \psi_{2}^{*}\right)\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\binom{\psi_{3}}{\psi_{4}}+c\left(\begin{array}{ll}
\psi_{3}^{*} & \psi_{4}^{*}
\end{array}\right)\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}= \\
& =c\left(\psi_{1}^{*} \psi_{4}+\psi_{4}^{*} \psi_{1}+\psi_{2}^{*} \psi_{3}+\psi_{3}^{*} \psi_{2}\right)=2 c \operatorname{Re}\left(\psi_{1}^{*} \psi_{4}+\psi_{2}^{*} \psi_{3}\right) \text {, } \\
& \mathrm{J}^{2}=c\left(\vartheta^{\dagger} \sigma^{2} \eta+\eta^{\dagger} \sigma^{2} \vartheta\right)=c\left(\psi_{1}^{*} \quad \psi_{2}^{*}\right)\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right)\binom{\psi_{3}}{\psi_{4}}+c\left(\begin{array}{ll}
\psi_{3}^{*} & \psi_{4}^{*}
\end{array}\right)\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}=  \tag{B.24}\\
& =i c\left(\psi_{2}^{*} \psi_{3}-\psi_{1}^{*} \psi_{4}+\psi_{4}^{*} \psi_{1}-\psi_{3}^{*} \psi_{2}\right)=-2 c \operatorname{Im}\left(\psi_{2}^{*} \psi_{3}-\psi_{1}^{*} \psi_{4}\right) \text {, } \\
& \mathrm{J}^{3}=c\left(\vartheta^{\dagger} \sigma^{3} \eta+\eta^{\dagger} \sigma^{3} \vartheta\right)=c\left(\begin{array}{ll}
\psi_{1}^{*} & \psi_{2}^{*}
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)\binom{\psi_{3}}{\psi_{4}}+c\left(\begin{array}{ll}
\psi_{3}^{*} & \psi_{4}^{*}
\end{array}\right)\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)\binom{\psi_{1}}{\psi_{2}}= \\
& =c\left(\psi_{1}^{*} \psi_{3}-\psi_{4}^{*} \psi_{2}+\psi_{3}^{*} \psi_{1}-\psi_{2}^{*} \psi_{4}\right)=2 c \operatorname{Re}\left(\psi_{1}^{*} \psi_{3}-\psi_{4}^{*} \psi_{2}\right) .
\end{align*}
$$

Let us substitute the expressions for bispinors (B.22) and currents (B.24) into the first Vlasov equation (3.16)

$$
\begin{aligned}
& \psi_{1} \partial_{0} \psi_{1}^{*}+\psi_{1}^{*} \partial_{0} \psi_{1}+\psi_{2} \partial_{0} \psi_{2}^{*}+\psi_{2}^{*} \partial_{0} \psi_{2}+\psi_{3} \partial_{0} \psi_{3}^{*}+\psi_{3}^{*} \partial_{0} \psi_{3}+\psi_{4} \partial_{0} \psi_{4}^{*}+\psi_{4}^{*} \partial_{0} \psi_{4}+ \\
& +\partial_{x}\left(\psi_{1}^{*} \psi_{4}+\psi_{4}^{*} \psi_{1}+\psi_{2}^{*} \psi_{3}+\psi_{3}^{*} \psi_{2}\right)+i \partial_{y}\left(\psi_{2}^{*} \psi_{3}-\psi_{1}^{*} \psi_{4}+\psi_{4}^{*} \psi_{1}-\psi_{3}^{*} \psi_{2}\right)+ \\
& +\partial_{z}\left(\psi_{1}^{*} \psi_{3}-\psi_{4}^{*} \psi_{2}+\psi_{3}^{*} \psi_{1}-\psi_{2}^{*} \psi_{4}\right)=0,
\end{aligned}
$$

or

$$
\begin{align*}
& \psi_{1} \partial_{0} \psi_{1}^{*}+\psi_{1}^{*} \partial_{0} \psi_{1}+\psi_{2} \partial_{0} \psi_{2}^{*}+\psi_{2}^{*} \partial_{0} \psi_{2}+\psi_{3} \partial_{0} \psi_{3}^{*}+\psi_{3}^{*} \partial_{0} \psi_{3}+\psi_{4} \partial_{0} \psi_{4}^{*}+\psi_{4}^{*} \partial_{0} \psi_{4}+ \\
& +\psi_{1}^{*} \partial_{x} \psi_{4}+\psi_{4} \partial_{x} \psi_{1}^{*}+\psi_{4}^{*} \partial_{x} \psi_{1}+\psi_{1} \partial_{x} \psi_{4}^{*}+\psi_{2}^{*} \partial_{x} \psi_{3}+\psi_{3} \partial_{x} \psi_{2}^{*}+\psi_{3}^{*} \partial_{x} \psi_{2}+\psi_{2} \partial_{x} \psi_{3}^{*} \\
& +i \psi_{2}^{*} \partial_{y} \psi_{3}+i \psi_{3} \partial_{y} \psi_{2}^{*}-i \psi_{1}^{*} \partial_{y} \psi_{4}-i \psi_{4} \partial_{y} \psi_{1}^{*}+i \psi_{4}^{*} \partial_{y} \psi_{1}+i \psi_{1} \partial_{y} \psi_{4}^{*}-i \psi_{3}^{*} \partial_{y} \psi_{2}-i \psi_{2} \partial_{y} \psi_{3}^{*}+ \\
& +\psi_{1}^{*} \partial_{z} \psi_{3}+\psi_{3} \partial_{z} \psi_{1}^{*}-\psi_{4}^{*} \partial_{z} \psi_{2}-\psi_{2} \partial_{z} \psi_{4}^{*}+\psi_{3}^{*} \partial_{z} \psi_{1}+\psi_{1} \partial_{z} \psi_{3}^{*}-\psi_{2}^{*} \partial_{z} \psi_{4}-\psi_{4} \partial_{z} \psi_{2}^{*}=0, \\
& \psi_{1}\left(\partial_{0} \psi_{1}^{*}+\partial_{x} \psi_{4}^{*}+i \partial_{y} \psi_{4}^{*}+\partial_{z} \psi_{3}^{*}\right)+\psi_{1}^{*}\left(\partial_{0} \psi_{1}+\partial_{x} \psi_{4}-i \partial_{y} \psi_{4}+\partial_{z} \psi_{3}\right) \\
& +\psi_{2}\left(\partial_{0} \psi_{2}^{*}+\partial_{x} \psi_{3}^{*}-i \partial_{y} \psi_{3}^{*}-\partial_{z} \psi_{4}^{*}\right)+\psi_{2}^{*}\left(\partial_{0} \psi_{2}+\partial_{x} \psi_{3}+i \partial_{y} \psi_{3}-\partial_{z} \psi_{4}\right)  \tag{B.25}\\
& +\psi_{3}\left(\partial_{0} \psi_{3}^{*}+\partial_{x} \psi_{2}^{*}+i \partial_{y} \psi_{2}^{*}+\partial_{z} \psi_{1}^{*}\right)+\psi_{3}^{*}\left(\partial_{0} \psi_{3}+\partial_{x} \psi_{2}-i \partial_{y} \psi_{2}+\partial_{z} \psi_{1}\right) \\
& +\psi_{4}\left(\partial_{0} \psi_{4}^{*}+\partial_{x} \psi_{1}^{*}-i \partial_{y} \psi_{1}^{*}-\partial_{z} \psi_{2}^{*}\right)+\psi_{4}^{*}\left(\partial_{0} \psi_{4}+\partial_{x} \psi_{1}+i \partial_{y} \psi_{1}-\partial_{z} \psi_{2}\right)=0 .
\end{align*}
$$

We take into account that

$$
\begin{gather*}
\mathrm{J}^{k} \mathrm{~A}_{k}-\mathrm{J}^{k} \mathrm{~A}_{k}=0 \\
\frac{1}{c} \mathrm{~J}^{k} \mathrm{~A}_{k}=\psi_{1}^{*}\left(\mathrm{~A}_{x} \psi_{4}-i \mathrm{~A}_{y} \psi_{4}+\mathrm{A}_{z} \psi_{3}\right)+\psi_{2}^{*}\left(\mathrm{~A}_{x} \psi_{3}+i \mathrm{~A}_{y} \psi_{3}-\mathrm{A}_{z} \psi_{4}\right)+  \tag{B.26}\\
+\psi_{3}^{*}\left(\mathrm{~A}_{x} \psi_{2}-i \mathrm{~A}_{y} \psi_{2}+\mathrm{A}_{z} \psi_{1}\right)+\psi_{4}^{*}\left(\mathrm{~A}_{x} \psi_{1}+i \mathrm{~A}_{y} \psi_{1}-\mathrm{A}_{z} \psi_{2}\right)
\end{gather*}
$$

where $\mathrm{A}_{k}$ is some vector field. Substituting (B.26) into equation (B.25), we obtain

$$
\begin{align*}
& \psi_{1}\left(\partial_{0} \psi_{1}^{*}+\partial_{x} \psi_{4}^{*}+i \partial_{y} \psi_{4}^{*}+\partial_{z} \psi_{3}^{*}\right)+\psi_{1}^{*}\left(\partial_{0} \psi_{1}+\partial_{x} \psi_{4}-i \partial_{y} \psi_{4}+\partial_{z} \psi_{3}-i \kappa \mathrm{~A}_{x} \psi_{4}-\kappa \mathrm{A}_{y} \psi_{4}-i \kappa \mathrm{~A}_{z} \psi_{3}\right) \\
& +\psi_{2}\left(\partial_{0} \psi_{2}^{*}+\partial_{x} \psi_{3}^{*}-i \partial_{y} \psi_{3}^{*}-\partial_{z} \psi_{4}^{*}\right)+\psi_{2}^{*}\left(\partial_{0} \psi_{2}+\partial_{x} \psi_{3}+i \partial_{y} \psi_{3}-\partial_{z} \psi_{4}-i \kappa \mathrm{~A}_{x} \psi_{3}+\kappa \mathrm{A}_{y} \psi_{3}+i \kappa \mathrm{~A}_{z} \psi_{4}\right) \\
& +\psi_{3}\left(\partial_{0} \psi_{3}^{*}+\partial_{x} \psi_{2}^{*}+i \partial_{y} \psi_{2}^{*}+\partial_{z} \psi_{1}^{*}\right)+\psi_{3}^{*}\left(\partial_{0} \psi_{3}+\partial_{x} \psi_{2}-i \partial_{y} \psi_{2}+\partial_{z} \psi_{1}-i \kappa \mathrm{~A}_{x} \psi_{2}-\kappa \mathrm{A}_{y} \psi_{2}-i \kappa \mathrm{~A}_{z} \psi_{1}\right) \\
& +\psi_{4}\left(\partial_{0} \psi_{4}^{*}+\partial_{x} \psi_{1}^{*}-i \partial_{y} \psi_{1}^{*}-\partial_{z} \psi_{2}^{*}\right)+\psi_{4}^{*}\left(\partial_{0} \psi_{4}+\partial_{x} \psi_{1}+i \partial_{y} \psi_{1}-\partial_{z} \psi_{2}-i \kappa \mathrm{~A}_{x} \psi_{1}+\kappa \mathrm{A}_{y} \psi_{1}+i \kappa \mathrm{~A}_{z} \psi_{2}\right) \\
& +i \kappa \psi_{1}^{*}\left(\mathrm{~A}_{x} \psi_{4}-i \mathrm{~A}_{y} \psi_{4}+A_{z} \psi_{3}\right)+i \kappa \psi_{2}^{*}\left(\mathrm{~A}_{x} \psi_{3}+i \mathrm{~A}_{y} \psi_{3}-\mathrm{A}_{z} \psi_{4}\right)+i \kappa \psi_{3}^{*}\left(\mathrm{~A}_{x} \psi_{2}-i \mathrm{~A}_{y} \psi_{2}+\mathrm{A}_{z} \psi_{1}\right)+ \\
& +i \kappa \psi_{4}^{*}\left(\mathrm{~A}_{x} \psi_{1}+i \mathrm{~A}_{y} \psi_{1}-\mathrm{A}_{z} \psi_{2}\right)=0, \tag{B.27}
\end{align*}
$$

where $i \kappa$ is some coefficient. Let us rearrange the terms in expression (B.26)

$$
\begin{align*}
\frac{1}{c} \mathrm{~J}^{k} \mathrm{~A}_{k} & =\left(i \mathrm{~A}_{y} \psi_{4}^{*}+\mathrm{A}_{z} \psi_{3}^{*}+\mathrm{A}_{x} \psi_{4}^{*}\right) \psi_{1}+\left(\mathrm{A}_{x} \psi_{3}^{*}-i \mathrm{~A}_{y} \psi_{3}^{*}-\mathrm{A}_{z} \psi_{4}^{*}\right) \psi_{2}+  \tag{B.28}\\
& +\left(\mathrm{A}_{z} \psi_{1}^{*}+i \mathrm{~A}_{y} \psi_{2}^{*}+\mathrm{A}_{x} \psi_{2}^{*}\right) \psi_{3}+\left(\mathrm{A}_{x} \psi_{1}^{*}-i \mathrm{~A}_{y} \psi_{1}^{*}-\mathrm{A}_{z} \psi_{2}^{*}\right) \psi_{4} .
\end{align*}
$$

We substitute expression (B.28) into equation (B.27)

$$
\begin{aligned}
& \psi_{1}\left(\partial_{0} \psi_{1}^{*}+\partial_{x} \psi_{4}^{*}+i \partial_{y} \psi_{4}^{*}+\partial_{z} \psi_{3}^{*}+i \kappa \mathrm{~A}_{x} \psi_{4}^{*}-\kappa \mathrm{A}_{y} \psi_{4}^{*}+i \kappa \mathrm{~A}_{z} \psi_{3}^{*}\right)+ \\
& +\psi_{2}\left(\partial_{0} \psi_{2}^{*}+\partial_{x} \psi_{3}^{*}-i \partial_{y} \psi_{3}^{*}-\partial_{z} \psi_{4}^{*}+i \kappa \mathrm{~A}_{x} \psi_{3}^{*}+\kappa \mathrm{A}_{y} \psi_{3}^{*}-i \kappa \mathrm{~A}_{z} \psi_{4}^{*}\right)+ \\
& +\psi_{3}\left(\partial_{0} \psi_{3}^{*}+\partial_{x} \psi_{2}^{*}+i \partial_{y} \psi_{2}^{*}+\partial_{z} \psi_{1}^{*}+i \kappa \mathrm{~A}_{z} \psi_{1}^{*}-\kappa \mathrm{A}_{y} \psi_{2}^{*}+i \kappa \mathrm{~A}_{x} \psi_{2}^{*}\right)+ \\
& +\psi_{4}\left(\partial_{0} \psi_{4}^{*}+\partial_{x} \psi_{1}^{*}-i \partial_{y} \psi_{1}^{*}-\partial_{z} \psi_{2}^{*}+i \kappa \mathrm{~A}_{x} \psi_{1}^{*}+\kappa \mathrm{A}_{y} \psi_{1}^{*}-i \kappa \mathrm{~A}_{z} \psi_{2}^{*}\right)+
\end{aligned}
$$

$+\psi_{1}^{*}\left(\partial_{0} \psi_{1}+\partial_{x} \psi_{4}-i \partial_{y} \psi_{4}+\partial_{z} \psi_{3}-i \kappa \mathrm{~A}_{x} \psi_{4}-\kappa \mathrm{A}_{y} \psi_{4}-i \kappa \mathrm{~A}_{z} \psi_{3}\right)+$
$+\psi_{2}^{*}\left(\partial_{0} \psi_{2}+\partial_{x} \psi_{3}+i \partial_{y} \psi_{3}-\partial_{z} \psi_{4}-i \kappa \mathrm{~A}_{x} \psi_{3}+\kappa \mathrm{A}_{y} \psi_{3}+i \kappa \mathrm{~A}_{z} \psi_{4}\right)+$
$+\psi_{3}^{*}\left(\partial_{0} \psi_{3}+\partial_{x} \psi_{2}-i \partial_{y} \psi_{2}+\partial_{z} \psi_{1}-i \kappa \mathrm{~A}_{x} \psi_{2}-\kappa \mathrm{A}_{y} \psi_{2}-i \kappa \mathrm{~A}_{z} \psi_{1}\right)+$
$+\psi_{4}^{*}\left(\partial_{0} \psi_{4}+\partial_{x} \psi_{1}+i \partial_{y} \psi_{1}-\partial_{z} \psi_{2}-i \kappa \mathrm{~A}_{x} \psi_{1}+\kappa \mathrm{A}_{y} \psi_{1}+i \kappa \mathrm{~A}_{z} \psi_{2}\right)=0$,

Let us write equation (B.29) in a quadratic form. The first four terms of equation (B.29) have the form

$$
\begin{align*}
& \psi_{1}\left(\partial_{0} \psi_{1}^{*}+\partial_{x} \psi_{4}^{*}+i \partial_{y} \psi_{4}^{*}+\partial_{z} \psi_{3}^{*}+i \kappa \mathrm{~A}_{x} \psi_{4}^{*}-\kappa \mathrm{A}_{y} \psi_{4}^{*}+i \kappa \mathrm{~A}_{z} \psi_{3}^{*}\right)+ \\
& +\psi_{2}\left(\partial_{0} \psi_{2}^{*}+\partial_{x} \psi_{3}^{*}-i \partial_{y} \psi_{3}^{*}-\partial_{z} \psi_{4}^{*}+i \kappa \mathrm{~A}_{x} \psi_{3}^{*}+\kappa \mathrm{A}_{y} \psi_{3}^{*}-i \kappa \mathrm{~A}_{z} \psi_{4}^{*}\right)+ \\
& +\psi_{3}\left(\partial_{0} \psi_{3}^{*}+\partial_{x} \psi_{2}^{*}+i \partial_{y} \psi_{2}^{*}+\partial_{z} \psi_{1}^{*}+i \kappa \mathrm{~A}_{z} \psi_{1}^{*}-\kappa \mathrm{A}_{y} \psi_{2}^{*}+i \kappa \mathrm{~A}_{x} \psi_{2}^{*}\right)+ \\
& +\psi_{4}\left(\partial_{0} \psi_{4}^{*}+\partial_{x} \psi_{1}^{*}-i \partial_{y} \psi_{1}^{*}-\partial_{z} \psi_{2}^{*}+i \kappa \mathrm{~A}_{x} \psi_{1}^{*}+\kappa \mathrm{A}_{y} \psi_{1}^{*}-i \kappa \mathrm{~A}_{z} \psi_{2}^{*}\right)= \\
& =\left(\begin{array}{ll}
\vartheta^{T} & \eta^{T}
\end{array}\right) \partial_{0}\binom{\vartheta^{*}}{\eta^{*}}+\left(\begin{array}{ll}
\vartheta^{T} & \eta^{T}
\end{array}\right)\binom{\sigma^{k^{*}} \partial_{k} \eta^{*}}{\sigma^{k^{*}} \partial_{k} \vartheta^{*}}+i \kappa\left(\begin{array}{ll}
\vartheta^{T} & \eta^{T}
\end{array}\right)\binom{\sigma^{k^{*}} \mathrm{~A}_{k} \eta^{*}}{\sigma^{k^{*}} \mathrm{~A}_{k} \vartheta^{*}}= \\
& =\left(\begin{array}{ll}
\vartheta^{T} & \eta^{T}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k^{*}} \partial_{k}+i \kappa \sigma^{k^{*}} \mathrm{~A}_{k} \\
\sigma^{k^{*} \partial_{k}+i \kappa \sigma^{k^{*}} \mathrm{~A}_{k}} & \mathrm{I}_{2} \partial_{0}
\end{array}\right)\binom{\vartheta^{*}}{\eta^{*}} \text {. } \tag{B.30}
\end{align*}
$$

Let us transform the second four terms of expression (B.29), we obtain

$$
\begin{align*}
& \psi_{1}^{*}\left(\partial_{0} \psi_{1}+\partial_{x} \psi_{4}-i \partial_{y} \psi_{4}+\partial_{z} \psi_{3}-i \kappa \mathrm{~A}_{x} \psi_{4}-\kappa \mathrm{A}_{y} \psi_{4}-i \kappa \mathrm{~A}_{z} \psi_{3}\right)+ \\
& +\psi_{2}^{*}\left(\partial_{0} \psi_{2}+\partial_{x} \psi_{3}+i \partial_{y} \psi_{3}-\partial_{z} \psi_{4}-i \kappa \mathrm{~A}_{x} \psi_{3}+\kappa \mathrm{A}_{y} \psi_{3}+i \kappa \mathrm{~A}_{z} \psi_{4}\right)+ \\
& +\psi_{3}^{*}\left(\partial_{0} \psi_{3}+\partial_{x} \psi_{2}-i \partial_{y} \psi_{2}+\partial_{z} \psi_{1}-i \kappa \mathrm{~A}_{x} \psi_{2}-\kappa \mathrm{A}_{y} \psi_{2}-i \kappa \mathrm{~A}_{z} \psi_{1}\right)+ \\
& +\psi_{4}^{*}\left(\partial_{0} \psi_{4}+\partial_{x} \psi_{1}+i \partial_{y} \psi_{1}-\partial_{z} \psi_{2}-i \kappa \mathrm{~A}_{x} \psi_{1}+\kappa \mathrm{A}_{y} \psi_{1}+i \kappa \mathrm{~A}_{z} \psi_{2}\right)= \\
& =\left(\begin{array}{ll}
\vartheta^{\dagger} & \eta^{\dagger}
\end{array}\right) \partial_{0}\binom{\vartheta}{\eta}+\left(\begin{array}{ll}
\vartheta^{\dagger} & \eta^{\dagger}
\end{array}\right)\binom{\sigma^{k} \partial_{k} \eta}{\sigma^{k} \partial_{k} \vartheta}-i \kappa\left(\vartheta^{\dagger} \quad \eta^{\dagger}\right)\binom{\sigma^{k} \mathrm{~A}_{k} \eta}{\sigma^{k} \mathrm{~A}_{k} \vartheta}=  \tag{B.31}\\
& =\left(\begin{array}{ll}
\vartheta^{\dagger} & \eta^{\dagger}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} \\
\sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} & \mathrm{I}_{2} \partial_{0}
\end{array}\right)\binom{\vartheta}{\eta} \text {. }
\end{align*}
$$

Taking into account expressions (B.30) and (B.31), equation (B.29) takes the form

$$
\begin{align*}
& \left(\begin{array}{ll}
\vartheta^{T} & \eta^{T}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k^{*}} \partial_{k}+i \kappa \sigma^{k^{*}} \mathrm{~A}_{k} \\
\sigma^{k^{*}} \partial_{k}+i \kappa \sigma^{k^{*}} \mathrm{~A}_{k} & \mathrm{I}_{2} \partial_{0}
\end{array}\right)\binom{\vartheta^{*}}{\eta^{*}}+  \tag{B.32}\\
& +\left(\begin{array}{ll}
\vartheta^{\dagger} & \eta^{\dagger}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} \\
\sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} & \mathrm{I}_{2} \partial_{0}
\end{array}\right)\binom{\vartheta}{\eta}=0 .
\end{align*}
$$

Let us introduce the matrix operator

$$
\begin{align*}
L & \stackrel{\operatorname{det}}{=}\left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} \\
\sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} & \mathrm{I}_{2} \partial_{0}
\end{array}\right)=  \tag{B.33}\\
& =\left(\begin{array}{cc}
\mathrm{I}_{2} & 0 \\
0 & -\mathrm{I}_{2}
\end{array}\right)\left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} \\
-\sigma^{k} \partial_{k}+i \kappa \sigma^{k} \mathrm{~A}_{k} & -\mathrm{I}_{2} \partial_{0}
\end{array}\right)=\gamma^{0} \mathrm{~L} .
\end{align*}
$$

Let us write equation (B.32) through operator (B.33)

$$
\begin{gather*}
\psi^{T} L^{*} \psi^{*}+\psi^{\dagger} L \psi=0, \\
\Lambda \stackrel{\operatorname{det}}{=} \psi^{\dagger} L \psi, \Lambda^{*}+\Lambda=0 \Rightarrow \operatorname{Re} \Lambda=0 \Rightarrow \Lambda=i u, u \in \mathbb{R} . \tag{B.34}
\end{gather*}
$$

Real function $u$ can be represented as a quadratic form

$$
u \stackrel{\operatorname{det}}{=} \psi^{\dagger} V \psi=\left(\begin{array}{ll}
\vartheta^{\dagger} & \eta^{\dagger}
\end{array}\right)\left(\begin{array}{cc}
\lambda_{1} \mathrm{I}_{2} & 0  \tag{B.35}\\
0 & \lambda_{2} \mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta}=\left(\begin{array}{ll}
\vartheta^{\dagger} & \eta^{\dagger}
\end{array}\right)\binom{\lambda_{1} \vartheta}{\lambda_{2} \eta}=\lambda_{1}|\vartheta|^{2}+\lambda_{2}|\eta|^{2} \in \mathbb{R},
$$

where $\lambda_{1,2} \in \mathbb{R}$ are some functions. Substituting representation (B.35) into equation (B.34), we obtain

$$
\begin{gather*}
\psi^{\dagger} L \psi=-i u=-i \psi{ }^{\dagger} V \psi \Rightarrow \psi^{\dagger} \cdot(L \psi+i V \psi)=0, \\
L \psi=-i V \psi . \tag{B.36}
\end{gather*}
$$

Taking into account (B.35), (B.33), equation (B.36) in the matrix form takes the form:

$$
\begin{aligned}
& \gamma^{0}\left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} \\
-\sigma^{k} \partial_{k}+i \kappa \sigma^{k} \mathrm{~A}_{k} & -\mathrm{I}_{2} \partial_{0}
\end{array}\right)\binom{\vartheta}{\eta}=-i\left(\begin{array}{cc}
\lambda_{1} \mathrm{I}_{2} & 0 \\
0 & \lambda_{2} \mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta}, \\
& \left(\begin{array}{cc}
\mathrm{I}_{2} \partial_{0} & \sigma^{k} \partial_{k}-i \kappa \sigma^{k} \mathrm{~A}_{k} \\
-\sigma^{k} \partial_{k}+i \kappa \sigma^{k} \mathrm{~A}_{k} & -\mathrm{I}_{2} \partial_{0}
\end{array}\right)\binom{\vartheta}{\eta}=-i\left(\begin{array}{cc}
\lambda_{1} \mathrm{I}_{2} & 0 \\
0 & -\lambda_{2} \mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta},
\end{aligned}
$$

or

$$
\begin{gather*}
\binom{\partial_{0} \vartheta+\sigma^{k} \partial_{k} \eta-i \kappa \sigma^{k} \mathrm{~A}_{k} \eta+i \lambda_{1} \vartheta}{-\partial_{0} \eta-\sigma^{k} \partial_{k} \vartheta+i \kappa \sigma^{k} \mathrm{~A}_{k} \vartheta-i \lambda_{2} \eta}=\binom{0}{0}, \\
\partial_{0}\left(\begin{array}{cc}
\mathrm{I}_{2} & 0 \\
0 & -\mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta}+\left(\begin{array}{cc}
0 & \sigma^{k} \partial_{k} \\
-\sigma^{k} \partial_{k} & 0
\end{array}\right)\binom{\vartheta}{\eta}-i \kappa\left(\begin{array}{cc}
0 & \sigma^{k} \mathrm{~A}_{k} \\
-\sigma^{k} \mathrm{~A}_{k} & 0
\end{array}\right)\binom{\vartheta}{\eta}+i\left(\begin{array}{cc}
\lambda_{1} \mathrm{I}_{2} & 0 \\
0 & -\lambda_{2} \mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta}=\binom{0}{0} \tag{B.37}
\end{gather*}
$$

Let us make a change of variables

$$
\begin{equation*}
\lambda_{1,2} \stackrel{\text { det }}{=} \frac{1}{\hbar c}\left(U \pm m c^{2}\right), \tag{B.38}
\end{equation*}
$$

where $\hbar, c, m$ are some constant values, and $U \in \mathbb{R}$ is a function. Substituting (B.38) into equation (B.37), we

$$
\left(\begin{array}{cc}
\lambda_{1} \mathrm{I}_{2} & 0 \\
0 & -\lambda_{1} \mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta}=\frac{U}{\hbar c}\left(\begin{array}{cc}
\mathrm{I}_{2} & 0 \\
0 & -\mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta}+\frac{m c^{2}}{\hbar c}\left(\begin{array}{cc}
\mathrm{I}_{2} & 0 \\
0 & \mathrm{I}_{2}
\end{array}\right)\binom{\vartheta}{\eta}=\frac{U}{\hbar c} \gamma^{0} \psi+\frac{m c^{2}}{\hbar c} \mathrm{I} \psi
$$

or

$$
\begin{gather*}
i \hbar c \partial_{0} \gamma^{0} \psi+i \hbar c \gamma^{k} \partial_{k} \psi+\hbar c \kappa \gamma^{k} \mathrm{~A}_{k} \psi-U \gamma^{0} \psi-m c^{2} \mathrm{I} \psi=0, \\
i \hbar c \gamma^{0} \partial_{0} \psi=\left[c \gamma^{k}\left(\hat{\mathrm{p}}_{k}-\hbar \kappa \mathrm{A}_{k}\right)+U \gamma^{0}+m c^{2} \mathrm{I}\right] \psi . \tag{B.39}
\end{gather*}
$$

If $U=q \varphi$ and $\hbar \kappa=q$, then the equation coincides with (B.39) and will take the form

$$
\begin{equation*}
i \hbar c \partial_{0} \psi=\gamma^{0}\left[c \gamma^{k}\left(\hat{\mathrm{p}}_{k}-q \mathrm{~A}_{k}\right)+m c^{2} \mathrm{I}\right] \psi+q \varphi \psi . \tag{B.40}
\end{equation*}
$$

Let us write equation (B.40) in another form

$$
\begin{gather*}
i \hbar c\left(\gamma^{0} \partial_{0}+\gamma^{k} \partial_{k}\right) \psi-q c\left(\gamma^{0} \frac{\varphi}{c}-\gamma^{k} \mathrm{~A}_{k}\right) \psi-m c^{2} \mathrm{I} \psi=0, \\
i \hbar c \gamma^{\mu} \partial_{\mu} \psi-q c\left(\gamma^{0} A_{0}+\gamma^{k} A_{k}\right) \psi-m c^{2} \mathrm{I} \psi=0, \\
{\left[\gamma^{\mu}\left(i \hbar \partial_{\mu}-q A_{\mu}\right)-m c \mathrm{I}\right] \psi=0} \tag{B.41}
\end{gather*}
$$

where the relation $A_{k}=-\mathrm{A}_{k}$ between the covariant components of four-vector $A_{k}$ and the components of 3 D vector $\mathrm{A}_{k}$ is taken into account. Theorem 4 is proved.

## Appendix C

## Proof of Lemma 3

Let the system be self-consistent, that is, condition (2.17) is satisfied, then it is possible for it to construct the Maxwell equation system for external (2.23)-(2.24) and probabilistic eigenfields (2.25)-(2.26). The self-consistency condition for equation (2.7) means (2.20)-(2.22):

$$
\begin{align*}
& \square \varphi_{\Psi}=\frac{1}{q} \square \mathrm{~V}=\frac{\rho_{\Psi}}{\varepsilon_{0}}=q \frac{f}{\varepsilon_{0}} \Rightarrow \square \mathrm{~V}=\frac{q^{2}}{\varepsilon_{0}}|\Psi|^{2},  \tag{C.1}\\
& \frac{\hbar^{2}}{2 m} \frac{\Delta_{r}|\Psi(\vec{r}, t)|}{|\Psi(\vec{r}, t)|}+\frac{q^{2}}{4 \pi \varepsilon_{0}} \int \frac{\left|\Psi\left(\vec{r}^{\prime}, t_{r}\right)\right|^{2} d^{3} r^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|}=U(\vec{r}, t), \tag{C.2}
\end{align*}
$$

where $t_{r}=t-\left|\vec{r}-\vec{r}^{\prime}\right| / c$ and the representation for the Lienard-Wiechert potential is used. Expression (C.2) is an integro-differential equation for the modulus of wave function $|\Psi|$. Knowing $|\Psi|$, one can find quantum potential Q and potential V (2.9). From equations (2.20)(2.21) probability density $\rho_{Q}=\frac{\varepsilon_{0}}{q} \square \mathrm{Q}$ is determined, as well as vector potential $\square \overrightarrow{\mathrm{A}}_{Q}=\mu_{0} \rho_{Q}\langle\vec{v}\rangle$ during calibration (2.15):

$$
\begin{equation*}
q \square \overrightarrow{\mathrm{~A}}_{Q}=\frac{1}{m c^{2}}\left[\hbar \nabla_{r} \varphi-q\left(\overrightarrow{\mathrm{~A}}+\overrightarrow{\mathrm{A}}_{Q}\right)\right] \square \mathrm{Q}, \tag{C.3}
\end{equation*}
$$

where Helmholtz decomposition (2.1) is taken into account. Lemma 3 is proved.

## Proof of Theorem 5

According to (1.6), it follows from the condition $\partial_{0} f^{1,2}=0$ that

$$
\begin{gather*}
f^{1}(\vec{r})=\int_{\Omega^{2}} f^{1,2}(\vec{r}, \vec{v}) d^{3} v,  \tag{C.4}\\
\langle\vec{v}\rangle_{1}(\vec{r})=\frac{1}{f^{1}(\vec{r})} \int_{\Omega^{2}} f^{1,2}(\vec{r}, \vec{v}) \vec{v} d^{3} v . \tag{C.5}
\end{gather*}
$$

Since $f^{1}(\vec{r})=|\Psi|^{2}$, then from expression (C.4), we obtain the stationarity of the quantum potential (2.9):

$$
\begin{equation*}
\partial_{0} \mathrm{Q}(\vec{r})=0 . \tag{C.6}
\end{equation*}
$$

From expressions (2.20)-(2.22) and (C.6) it follows that

$$
\begin{equation*}
\square \mathrm{Q}(\vec{r})=-\Delta_{r} \mathrm{Q}(\vec{r})=\frac{q}{\varepsilon_{0}} \rho_{Q} \Rightarrow \rho_{Q}=\rho_{Q}(\vec{r}) . \tag{C.7}
\end{equation*}
$$

In Helmholtz decomposition (2.1) for the field (C.5), fields $-\alpha \nabla_{r} \Phi(\vec{r}, t)$ and $\gamma \overrightarrow{\mathrm{A}}_{\Psi}(\vec{r}, t)$ are independent, since they correspond to the potential and vortex components. Hence,

$$
\begin{equation*}
\langle\vec{v}\rangle(\vec{r})=-\alpha \nabla_{r} \Phi(\vec{r}, t)+\gamma \overrightarrow{\mathrm{A}}_{\Psi}(\vec{r}), \quad \varphi(\vec{r}, t)=g(\vec{r})-\frac{\mathcal{E}}{\hbar} t \tag{C.8}
\end{equation*}
$$

where $\mathcal{E}$ is a constant value, and $g(\vec{r})$ is some function. According to expressions (C.4) and (C.8), wave function has the form

$$
\begin{equation*}
\Psi(\vec{r}, t)=\sqrt{f^{1}(\vec{r})} \exp \left[i g(\vec{r})-i \frac{\mathcal{E}}{\hbar} t\right] \tag{C.9}
\end{equation*}
$$

Taking into account (C.5)-(C.6) and (C.8), we write Hamilton-Jacobi equation (2.8)-(2.9)

$$
\begin{equation*}
\mathcal{E}=\frac{m}{2}\left|\langle\vec{v}\rangle_{1}(\vec{r})\right|^{2}+U+\mathrm{Q}(\vec{r}) \Rightarrow U=U(\vec{r}) \tag{C.10}
\end{equation*}
$$

Current density (2.26) according to (C.5) and (C.7) is also stationary:

$$
\overrightarrow{\mathrm{J}}_{Q}(\vec{r})=\rho_{Q}(\vec{r})\langle\vec{v}\rangle_{1}(\vec{r}),
$$

hence (2.27),

$$
\begin{equation*}
\square \overrightarrow{\mathrm{A}}_{Q}=\mu_{0} \overrightarrow{\mathrm{~J}}_{Q} \Rightarrow \overrightarrow{\mathrm{~A}}_{Q}=\frac{\mu_{0}}{4 \pi} \int \frac{\overrightarrow{\mathrm{~J}}_{Q}\left(\vec{r}^{\prime}\right) d^{3} r^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|} \Rightarrow \overrightarrow{\mathrm{A}}_{Q}=\overrightarrow{\mathrm{A}}_{Q}(\vec{r}) . \tag{C.11}
\end{equation*}
$$

Theorem 5 is proved.

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