Intuitionistic Quantum Logic Perspective: A New Approach to Natural Revision Theory*

Heng Zhou[a], Yongjun Wang[a,*], Baoshan Wang[a], Jian Yan[a], Xiaoyang Wang[b]

[a] *School of Mathematical Science, Beihang University, 37 Xueyuan Road, Haidian District, Beijing, 100191, China*
[b] *School of Philosophy, Beijing Normal University, Beijing, China*

Abstract

The classical belief revision framework proposed by Alchourron, Gardenfors, and Makinson involves the revision of a theory based on eight postulates. This paper focuses on exploring the revision theory based on quantum mechanics, known as the natural revision theory. In quantum systems, there are two reasoning modes: static intuitionistic reasoning, which incorporates contextuality, and dynamic reasoning, which is achieved through projection measurement.

We combine the advantages of the two intuitionistic quantum logics proposed by Doering and Coecke respectively. We aim to provide a truth-value assignment for intuitionistic quantum logic that not only aligns with the characteristics of quantum mechanics but also allows for truth-value reasoning. We investigate the natural revision theory based on this approach.

We introduce two types of revision operators corresponding to the two reasoning modes in quantum systems: object-level revision and operator-level revision, and we highlight the distinctions between these two operators. Unlike classical revision, we consider the revision of consequence relations in intuitionistic quantum logic. We demonstrate that, within the framework of the natural revision theory, both types of revision operators work together

*This work was supported by the National Natural Science Foundation of China(11871083).

*Corresponding author

Email addresses: zhouheng@buaa.edu.cn (Heng Zhou), wangyj@buaa.edu.cn (Yongjun Wang), bwang@buaa.edu.cn (Baoshan Wang), sy1809143math@buaa.edu.cn (Jian Yan), wangxiaoyang@bnu.edu.cn (Xiaoyang Wang)

on the reasoning system of consequence relations. The outcomes of revision process are influenced by the order in which the interweaved operators are applied.

Keywords: intuitionistic quantum logic, contextuality, projection measurement, truth-value reasoning, natural revision

1. Introduction

Classical belief revision was first proposed by Alchourron, Gardenfors, and Makinson in 1985[1]. A theory is contracted or revised in terms of six elementary and two supplementary postulates. They mainly researched how to appropriately select the subset consistent with the new information in the original theory according to the new one. Such selection criteria are usually subjective. In this paper, we mainly study the revision theory called natural revision, based on the objective laws of quantum mechanics.

The logical characterization of quantum mechanics was first introduced by Birkhoff and von Neumann in 1936[2]. They showed that a quantum system can be described via Hilbert space. There is a one-to-one correspondence between projection operators of a quantum system and closed subspaces of Hilbert space. The operations between closed subspaces satisfy orthomodular law. Quantum logic represented by closed subspaces forming an orthomodular lattice is called standard quantum logic. It is still a hot spot in the field of quantum computing.

The reasoning of quantum logic is non-monotonic reasoning based on features of quantum mechanics. Monotonicity means that new information added to the reasoning does not affect the result of the implication of old information. In a quantum system, when reasoning is revised by projection measurement as new information, some results implied from the old ones are no longer valid because the projection measurement changes the state of the system. We intend to investigate the nature of reasoning in quantum logic and thus provide a theory of natural revision determined by the objective laws of quantum mechanics.

Standard quantum logic takes closed subspaces of Hilbert space as elements, the intersection of closed subspaces as the meet operation, the span of closed subspaces as the join operation, and the orthogonal complement of a closed subspace as the negation operation, which forms an orthomodular lattice called the property lattice. There is no implicative connective in the property lattice, so truth-value reasoning cannot be carried out.

P.D. Finch[3, 4] defined the Sasaki hook $\rightsquigarrow_s$ and it is used as the implicative connective in property lattice. The left adjoint of Sasaki hook, called Sasaki projection $\wedge_s$, characterizes projection measurements in a quantum system. Roman and Zuazua[5] discussed that the Sasaki adjoint is some kind of deduction theorem. Engesser and Gabbay[6] inspected the link between non-monotonic consequence relations and belief revision. They argued that reasoning in quantum logic is a non-monotonic consequence relation, and the Sasaki projection is the revision operation of consequence relations. The adjoint operation Sasaki hook is the internalizing connective. A consequence revision system (CRS) is constructed by using formulas as operators. They examined the consequence relations based on truth-value reasoning in the property lattice, and regard a, b, c in $a \wedge_s b \leq c$ as elements from the same level. With $\wedge_s$ as the revision operation, formulas a and b can be used as operators to revise a consequence relation. Sasaki adjoint can operate in the form of modus ponens, i.e., $a \wedge_s (a \rightsquigarrow_s b) \leq b$. However, because of the different meanings of elements in the operation, Sasaki adjoint cannot play the role of truth-value reasoning.

Bob Coecke[7] indicated that Sasaki hook is not a static implicative connective but induces a dynamic one with a parameter in dynamic operational quantum logic. Define $\varphi_a^*(b) := a \wedge_s b$ as well as $\varphi_{a,*}(b) := a \rightsquigarrow_s b$, and $\varphi_a^*(b) \leq c \Leftrightarrow b \leq \varphi_{a,*}(c) \Leftrightarrow b \overset{\varphi_a}{\rightsquigarrow} c$ is satisfied, in which b and c are variables while a is a parameter. Obviously, they are elements from different levels. $\varphi_a^*(b) \leq c$ represents that with space a as the projection space, the result of projecting b space to a is contained in c space. In [6], considering only the revision form based on the projection operation is not enough to characterize quantum reasoning. We study the natural revision theory based on quantum logic by investigating the truth-value reasoning of quantum logic.

In our study, we consider the application of topos theory for the truth assignment of properties in quantum systems. Isham et al.[8, 9] characterized physical quantities of a quantum system by presheaf structure. They emphasized that contextuality is a crucial characteristic of quantum systems, and properties in a quantum system should be assigned local truth values within specific contexts. Doering et al.[10, 11] defined a mapping called daseinisation to approximate a property across all contexts. They suggested that the global truth value of properties can be determined based on the outer daseinisation. Doering[12, 13] showed that elements corresponding to the outer daseinisations of properties generate a bi-Heyting algebra.

However, upon investigating the implicative connective in this bi-Heyting algebra, we find that the implicative connective in the sense of Heyting algebra is too trivial, while the one in the sense of co-Heyting algebra is appropriately defined. Therefore, the elements selected by Doering constitute a co-Heyting algebra that satisfies the law of excluded middle, whereas ordinary truth-value reasoning should rely on a Heyting algebra that satisfies the law of non-contradiction as the reasoning structure.

Coecke[14] constructed an intuitionistic quantum logic based on Heyting algebra using the downsets of elements in the property lattice. However, Coecke's structure is only consistent with ordinary truth-value reasoning, which does not involve contextuality, and is insufficient to describe the characteristics of quantum mechanics. Thus, we propose an intuitionistic quantum logic that combines the respective advantages of the two intuitionistic quantum logics mentioned above. This logic satisfies the characteristics of quantum mechanics and allows for truth-value reasoning.

In our approach, we propose selecting elements according to the inner daseinisation for the global truth value of properties. This selection ensures that the chosen elements constitute a Heyting algebra. Truth-value reasoning within a Heyting algebra is static reasoning. However, projection measurement, as an external action, transforms one Heyting algebra into another, and the reasoning generated by the Sasaki adjoint represents dynamic truth-value reasoning between Heyting algebras.

We believe that a consequence relation occurs within a Heyting algebra. There are two types of revision operations for a consequence relation. The first is to add new assumptions into the Heyting algebra as new information, corresponding to the $\wedge$ operation as revision, which we refer to as object-level revision. The second is the action of projection measurement between Heyting algebras as new information, corresponding to the $\wedge_s$ operation as revision, which we refer to as operator-level revision. Both types of revision use formulas as operators to revise a consequence relation. The difference lies in the level of the revision operator and the method of revising the antecedents of a consequence relation. Object-level revision changes the consequence relation within a Heyting algebra to another one within the same Heyting algebra, while operator-level revision changes the consequence relation within a Heyting algebra to one within another Heyting algebra.

Engesser et al. utilized the orthomodular lattice to represent the natural revision theory. However, the orthomodular lattice fails to differentiate between the two levels of operations, resulting in a single reasoning mode. Consequently, their work [6] only includes one revision operator, which corresponds to the "expansion" in the revision theory. In contrast, intuitionistic quantum logic allows for the distinction between static reasoning and dynamic one. In our approach, we propose two levels of revision operators that correspond to the "expansion" and "revision" in the revision theory, respectively. The outcomes of revision process are influenced by the order in which the interweaved operators are applied. These operators enable the description of the natural revision theory following the fundamental characteristics of quantum mechanics.

In section 2, we provide the basic knowledge and background relevant to this paper. In section 3, we review the existing revision theory, discuss the revision of consequence relation based on the operations in the orthomodular lattice by Engesser et al., and highlight the shortcomings of this method. We then investigate the truth assignment of properties in a quantum system based on topos theory, analyze the role of the law of excluded middle and the law of non-contradiction in the structure proposed by Doering et al., and present a more suitable method for truth-value reasoning in section 4. In section 5, we discuss the static and dynamic reasoning of quantum logic based on the provided truth assignment, consider the two types of revision operators for a consequence relation, and present the natural revision theory based on the truth-value reasoning of quantum logic. Finally, we summarize the paper and propose prospects for research.

2. Preliminaries

2.1. Standard quantum logic

A quantum physical system is represented by a Hilbert space $\mathcal{H}$ over complex field $\mathbb{C}$, where a physical quantity is represented by a self-adjoint operator $\hat{A}$ in $\mathcal{H}$. The value of a physical quantity $\hat{A}$ is taken in a Borel set Δ, called $\hat{A} \in \Delta$. A projective operator(or projector in short) $\hat{P}$ represents the event of assigning values to a physical quantity, which is denoted by $\hat{P} = E[\hat{A} \in \Delta]$. Projectors are special self-adjoint operators. There is a one-to-one correspondence between projectors and closed subspaces of a Hilbert space. A closed subspace corresponds to a property of a quantum system whereas a one-dimension closed subspace corresponds to a state of the system. We mainly investigate projectors while studying quantum systems.

In other words, we do not care what value a physical quantity gets, but whether an event that a quantity gets a certain value is true.

Definition 1. *Define $\langle L, \leq, \wedge, \vee, \neg\rangle$ be an ortholattice if $\langle L, \leq, \wedge, \vee\rangle$ is bounded, and a unary operation $\neg$ defined in the lattice satisfies for any $a, b \in L$, there are:*

- $\neg(\neg a) = a$
- $a \leq b \Leftrightarrow \neg b \leq \neg a$
- $a \wedge \neg a = 0$
- $a \vee \neg a = 1$

We make no distinction between lattice $\langle L, \leq, \wedge, \vee\rangle$ and set L of elements of the lattice.

Definition 2. *An ortholattice L is called an orthomodular lattice if, for any $a, b \in L, a \leq b$, the Orthomodular Law is satisfied.*

$$a \vee (\neg a \wedge b) = b$$

Given a Hilbert space $\mathcal{H}$, denote a collection of its closed subspaces $Sub(\mathcal{H})$. Define partial relation $\leq$ to be the inclusion of closed subspaces. $\wedge$ operation is the intersection of two closed subspaces. $\vee$ is the span of two closed subspaces. $\neg$ is the complement of a closed subspace. $\langle Sub(\mathcal{H}), \leq, \wedge, \vee, \neg\rangle$ is an orthomodular lattice. There is a one-to-one correspondence between closed subspaces of a Hilbert space and projectors in a quantum system, and a projector represents a property in the system. Hence the orthomodular lattice is called a property lattice. Without causing ambiguity, we denote the property lattice by L.

Definition 3. *Given a pair of mappings $f : M \to N$ and $g : N \to M$. Define f and g be a pair of Galois adjoint, denoted by $f \dashv g$, if $f(a) \leq b \Leftrightarrow a \leq g(b)$ is satisfied for any $a \in M, b \in N$. Call f a left adjoint of g, and g a right adjoint of f.*

The internal operations of the property lattice are not sufficient to characterize quantum mechanics, so a pair of external operations $\wedge_s$ and $\rightsquigarrow_s$ are introduced into the property lattice. Furthermore, the mappings $a \wedge_s \_$ and

$a \rightsquigarrow_s \_$ form a pair of adjoints as well, called Sasaki adjoint, which is denoted by $a \wedge_s \_ \dashv a \rightsquigarrow_s \_$. In which

$$a \wedge_s b := a \wedge (\neg a \vee b)$$

expresses the projection of b onto a, or projecting b by a. We notice that there is no commutative law for $\wedge_s$. The right adjoint

$$a \rightsquigarrow_s b := \neg a \vee (a \wedge b)$$

reflects the reasoning ability of the property lattice. The relation between Sasaki adjoint

$$a \wedge_s b \leq c \Leftrightarrow b \leq a \rightsquigarrow_s c$$

is analogous to Modus Ponens in logical reasoning mechanism and to obtain

$$a \wedge_s (a \rightsquigarrow_s c) \leq c$$

Introducing Sasaki adjoint into property lattice L as a mathematical tool to characterize quantum logic, which is called standard quantum logic.

2.2. Topos quantum theory

An alternative mathematical characterization of quantum logic is given by Isham and Butterfield et al.[8, 9] using topos structure $Set^{\mathcal{V}(\mathcal{H})^{op}}$. They emphasize the crucial role of contextuality in depicting quantum logic. They indicate that self-adjoint operators in a Hilbert space $\mathcal{H}$ constitute a von Neumann algebra $N(\mathcal{H})$, and every commutative von Neumann subalgebra V in $N(\mathcal{H})$ represents a context. The collection of all commutative von Neumann subalgebras is denoted by $\mathcal{V}(N(\mathcal{H}))$, and by $\mathcal{V}(\mathcal{H})$ for short. For more knowledge about topos quantum theory, please refer to [15].

Definition 4. *A context is a n-dimensional commutative von Neumann subalgebra V. The set of generators of V is denoted by $\mathcal{F}_V = \{\hat{P}_1, \hat{P}_2, ..., \hat{P}_n\}$. Every component of the spectral decomposition of self-adjoint operators in V is an element in $\mathcal{F}_V$. Every self-adjoint operator in V is a linear combination of the generators above. If we only consider all projectors in V, which is denoted by $P(V)$, then $P(V)$ constitutes a Boolean subalgebra.*

Definition 5. *The set* $\sigma(V) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ *is defined to be the spectrum of an n-dimensional commutative von Neumann algebra* V*, where every* λ_i *maps a self-adjoint operator in* V *to the coefficient of* $\hat{P}_i$ *component of spectral decomposition of the operator.* λ_i *is of one-to-one correspondence to* $\hat{P}_i$*.*

Definition 6. *The spectral presheaf in* $Set^{\mathcal{V}(\mathcal{H})^{op}}$ *is a contravariant functor* $\underline{\Sigma}\colon \mathcal{V}(\mathcal{H})^{op} \to Set$*, which*

- *for objects:* $\underline{\Sigma}(V) \coloneqq \sigma(V)$
- *for morphisms: if* $V_2 \subseteq V_1$*, i.e. there is a morphism* $i_{V_2V_1}\colon V_2 \to V_1$ *in* $\mathcal{V}(\mathcal{H})$*, then* $\underline{\Sigma}(i_{V_2V_1})\colon \sigma(V_1) \to \sigma(V_2)$ *is a restriction mapping.*

Definition 7. *For any context* V*, define an isomorphism between projectors and subsets of the spectral presheaf by*

$$\alpha_V\colon P(V) \to Cl(\underline{\Sigma}_V)$$
$$\hat{P} \mapsto \{\lambda \in \underline{\Sigma}_V | \lambda(\hat{P}) = 1\}$$

in which $Cl(\underline{\Sigma}_V)$ *represents clopen subsets of* $\underline{\Sigma}_V$*.*

A property does not belong to every context in general. Doering[10, 11] indicates that for the projector corresponding to a property, one can find an approximation of the projector in every context.

Definition 8. *Given a projector* $\hat{P}$ *corresponding to a property, define two approximations of* $\hat{P}$ *in context* V *by*

- $\delta^o(\hat{P})_V = \bigwedge\{\hat{Q} \in P(V) | \hat{P} \leq \hat{Q}\}$ *and*
- $\delta^i(\hat{P})_V = \bigvee\{\hat{Q} \in P(V) | \hat{Q} \leq \hat{P}\}$

where δ^o *is called outer daseinisation, which means approximate from above; and* δ^i *is called inner daseinisation, which means approximate from below.*

Notice that Doering uses only the outer daseinisation when approximating a projector via daseinisation. Without causing confusion, Doering denotes outer daseinisation by daseinisation briefly. However, in this paper, we need to distinguish outer and inner daseinisation.

2.3. Consequence relation

According to [6], given a class Fml of formulas closed under the connectives $\neg, \wedge, \vee$ and containing $\top$ and $\perp$ for truth and falsity respectively. A consequence relation $\vdash \subseteq Fml \times Fml$ should satisfy the following conditions.

- Reflexivity

$$\alpha \vdash \alpha$$

- Cut

$$\frac{\alpha \wedge \beta \vdash \gamma, \alpha \vdash \beta}{\alpha \vdash \gamma}$$

- Restricted Monotonicity

$$\frac{\alpha \vdash \beta, \alpha \vdash \gamma}{\alpha \wedge \beta \vdash \gamma}$$

The relation between general non-monotonic consequence relations and revision operators is as follows. Let Δ be a theory, $\circ$ a revision operation. Define $\vdash_\Delta$ by

$$A \vdash_\Delta B \qquad iff \qquad B \in \Delta \circ A.$$

Given a $\vdash$, $\circ$ is obtained by

$$\Delta \circ A = \{B | \Delta, A \vdash B\}$$

3. Revision theory in standard quantum logic

In their work on standard quantum logic, Engesser et al.[6] propose a revision theory based on the Sasaki adjoint operation of property lattice and treat all the elements in the operation indistinctively. However, Bob Coecke[7] showed that the roles of elements in Sasaki adjoint are different. Hence the revision theory defined by standard quantum logic may not conform to the nature of quantum mechanics and not reflect its features precisely.

Definition 9. *Let $\vdash$ be a consequence relation. Define a internalizing connective $\rightsquigarrow$ of a consequence relation $\vdash$ as follows: For any formulas α and β, $\alpha \vdash \beta$ if and only if $\vdash \alpha \rightsquigarrow \beta$.*

It is worth noting that in classical logic, the material implication $\rightarrow$ is the internalizing connective of classical consequence relation, and the definition above corresponds to the deduction theorem of classical logic.

By analogy with operations in classical logic, Engesser et al. try to give a consequence relation and internalizing connective satisfying quantum logic reasoning. Inspired by classical logic, treating formulas as operators, they define a consequence revision system (CRS).

Definition 10. *Let Fml be a class of formulas, and $\mathcal{C}$ be a class of consequence relations on $Fml \times Fml$. Let function F be $F : Fml \times \mathcal{C} \rightarrow \mathcal{C}$. Call F an action on $\mathcal{C}$, if for any consistent consequence relations $\vdash \in \mathcal{C}$ and $\alpha, \beta \in Fml$ the following conditions are satisfied.*

- $F(\top, \vdash) = \vdash$;
- $F(\alpha, \vdash) = 0 \quad iff \quad \vdash \neg\alpha$;
- $F(\beta, F(\alpha, \vdash)) = F(\alpha, \vdash) \quad iff \quad \alpha \vdash \beta$.

If F is an action on $\mathcal{C}$, then call $\langle \mathcal{C}, F \rangle$ a consequence revision system CRS.[6]

Every formula $\alpha \in Fml$ corresponds to a revision operator $\bar{\alpha} : \mathcal{C} \rightarrow \mathcal{C}$ on consequence relations, which is defined by $\bar{\alpha} \vdash := F(\alpha, \vdash)$. We denote $\bar{\alpha} \vdash$ by $\vdash_\alpha$.

Treat an action of a formula to a consequence relation as the formula operates on the formulas in the hypothesis set of the consequence relation. Define $*$ be the revision operation on the set of formulas if the following conditions are satisfied.

- $1 * a = a * 1 = a$
- there exists an adjoint $a * \_ \dashv a \rightsquigarrow \_$
- $a * b \leq a$
- $a \perp b := a \leq \neg b \quad$ iff $\quad a * b = 0$
- $b \leq a \quad$ iff $\quad a * b = b$

Then, the action of function $F : Fml \times \mathcal{C} \to \mathcal{C}$ on a consequence relation is $F(a, \vdash_b) = \vdash_{a*b}$.

Regarding the operation $\wedge_s$ in property lattice L as the revision operation, partial relation on property lattice as consequence relation, the connective $\rightsquigarrow_s$ as the internalizing connective of consequence relation, it is easy to verify that $\wedge_s$ satisfies the conditions of revision operation, and for any $x \in L$, $\vdash_x$ satisfies the conditions of consequence relation.

Now in property lattice L, there is

$$a \wedge_s b \vdash c \Leftrightarrow a \vdash_b c \Leftrightarrow \vdash_{a \wedge_s b} c \Leftrightarrow \vdash_b a \rightsquigarrow_s c \Leftrightarrow b \vdash a \rightsquigarrow_s c$$

Notice that for $a \wedge_s b \vdash c \Leftrightarrow a \vdash_b c$, the consequence relation $\vdash$ becomes the consequence relation $\vdash_b$ when revised by formula b. That is $F(b, \vdash) = \vdash_b$. By the definition of revision operation, let $\vdash = \vdash_1$, there is $F(b, \vdash) = F(b, \vdash_1) = \vdash_{b \wedge_s 1}$. Furthermore, for $a \wedge_s b \vdash c \Leftrightarrow a \vdash_b c \Leftrightarrow \vdash_{a \wedge_s b} c$, there is $F(a, F(b, \vdash_1)) = F(a, \vdash_b) = \vdash_{a \wedge_s b}$. Firstly, b revises the consequence $\vdash = \vdash_1$, and then a revises the result from the previous step.

This approach is to consider the operations of property lattice as purely mathematical, which ignores the physical meaning of the operations. For equation $a \wedge_s b = a \wedge (\neg a \vee b) = c$, this approach treats the variables a, b, c as indistinctive variables. However, when considering the physical interpretation of operation $\wedge_s$, the equation $a \wedge_s b = c$ should be interpreted as taking space a as a projection space, projecting space b onto space a, and obtaining space c as the result. Moreover, Bob Coecke showed that the Sasaki hook is not a static implicative connective but induces a dynamic one with a parameter. Define $\varphi_a^*(b) := a \wedge_s b$ and $\varphi_{a,*}(b) := a \rightsquigarrow_s b$, and then $\varphi_a^*(b) \leq c \Leftrightarrow b \leq \varphi_{a,*}(c) \Leftrightarrow b \overset{\varphi_a}{\rightsquigarrow} c$ is obtained, where b and c are variables but a is a parameter, which means they are elements from different levels. As external operations of property lattice L, the projection operation and its adjoint do not carry out inside a property lattice, but between two property lattices L_1 and L_2. Specifically, the element b assigned true in property lattice L_1 is acted upon by a projection operation φ_a corresponding to space a, resulting in the element c in property lattice L_2 being assigned true. Therefore, the consequence relation system(CRS) with internalizing connective based on standard quantum logic may not reflect the features of truth-value operations in quantum logic.

The revision theory based on standard quantum logic may not accurately reflect the nature of quantum mechanics. To address this problem, we attempt to define an alternative natural revision theory conforming to the characteristics of quantum mechanics. To achieve this, the logical formalism of quantum systems and the truth-value assignment of quantum logic need to be reconsidered.

4. Truth assignment in topos quantum theory

According to the K-S theorem, all physical quantities cannot be assigned simultaneously and satisfy the functional relation between physical quantities. Isham et al. indicate that contextuality is the core concept to depict a quantum system. Investigating the assignment of physical quantities should examine the assignment of quantities in all contexts. Hence Isham et al. intend to characterize quantum systems with topos theory and inspect the truth-value assignment therein.

4.1. Bi-Heyting algebra structure

A subobject $\underline{S}$ of $\underline{\Sigma}$ such that the components $\underline{S}_V$ are clopen sets for all V is called a clopen subobject.

Definition 11. *Let $N(\mathcal{H})$ be the von Neumann algebra constituted by self-adjoint operators in Hilbert space $\mathcal{H}$. Let L be the lattice constituted by projection operators in the von Neumann algebra. The mapping*

$$\begin{aligned} \underline{\delta}^o \colon L &\to Sub_{cl}(\underline{\Sigma}) \\ \hat{P} &\mapsto \underline{\delta^o(\hat{P})} := (\alpha_V(\delta^o(\hat{P})_V))_{V \in N(\mathcal{H})} \end{aligned}$$

is called the outer daseinisation of projections. Here, $\underline{\delta^o(\hat{P})} \in Sub_{cl}(\underline{\Sigma})$ represents the subobject of the spectral presheaf $\underline{\Sigma}$.

Doering shows that the mapping $\underline{\delta}^o$ preserves all joins and is an order-preserving injection, but not a surjection. It is evident that $\underline{\delta^o(\hat{0})} = \underline{0}$, the empty subobject, and $\underline{\delta^o(\hat{1})} = \underline{\Sigma}$. For joins, we have

$$\forall \hat{P}, \hat{Q} \in L \colon \underline{\delta^o(\hat{P} \vee \hat{Q})} = \underline{\delta^o(\hat{P})} \vee \underline{\delta^o(\hat{Q})}.$$

However, for meets, we have

$$\forall \hat{P}, \hat{Q} \in L \colon \underline{\delta^o(\hat{P} \wedge \hat{Q})} \leq \underline{\delta^o(\hat{P})} \wedge \underline{\delta^o(\hat{Q})}.$$

In general, $\underline{\delta^o(\hat{P})} \wedge \underline{\delta^o(\hat{Q})}$ is not of the form $\underline{\delta^o(\hat{R})}$ for any projection $\hat{R} \in L$.

Doering argues that the elements of $Sub_{cl}(\underline{\Sigma})$ form a complete bi-Heyting algebra. Let $\underline{S}, \underline{T} \in Sub_{cl}(\underline{\Sigma})$ be two clopen subobjects. We have the following equations:

$$\forall V \in \mathcal{V}(\mathcal{H})\colon\ (\underline{S} \wedge \underline{T})_V = \underline{S}_V \cap \underline{T}_V$$
$$(\underline{S} \vee \underline{T})_V = \underline{S}_V \cup \underline{T}_V.$$

For each $\underline{S} \in Sub_{cl}(\underline{\Sigma})$, the functor

$$\underline{S} \wedge \_\colon Sub_{cl}(\underline{\Sigma}) \to Sub_{cl}(\underline{\Sigma})$$

has a right adjoint given by

$$\underline{S} \to \_\colon Sub_{cl}(\underline{\Sigma}) \to Sub_{cl}(\underline{\Sigma}).$$

The Heyting implication is determined by the adjunction:

$$\underline{S} \wedge \underline{R} \leq \underline{T} \quad iff \quad \underline{R} \leq (\underline{S} \to \underline{T}).$$

This implies that

$$(\underline{S} \to \underline{T}) = \bigvee \{\underline{R} \in Sub_{cl}(\underline{\Sigma}) | \underline{S} \wedge \underline{R} \leq \underline{T}\}.$$

The contextwise definition is as follows: for every $V \in \mathcal{V}(\mathcal{H})$,

$$(\underline{S} \to \underline{T})_V = \{\lambda \in \underline{\Sigma}_V | \forall V' \subseteq V\colon if\ \lambda|_{V'} \in \underline{S}_{V'}, then\ \lambda|_{V'} \in \underline{T}_{V'}\}.$$

The Heyting negation $\neg$ for each $\underline{S} \in Sub_{cl}(\underline{\Sigma})$ is defined as

$$\neg \underline{S} := (\underline{S} \to \underline{0}).$$

The contextwise expression for $\neg \underline{S}$ is given by

$$(\neg \underline{S})_V = \{\lambda \in \underline{\Sigma} | \forall V' \subseteq V\colon \lambda|_{V'} \notin \underline{S}_{V'}\}.$$

We have slightly modified Doering's definition of the operations on co-Heyting algebra. For any $\underline{S} \in Sub_{cl}(\underline{\Sigma})$, the functor

$$\_ \vee \underline{S}\colon Sub_{cl}(\underline{\Sigma}) \to Sub_{cl}(\underline{\Sigma})$$

has a left adjoint

$$\_ \leftarrow \underline{S}\colon Sub_{cl}(\underline{\Sigma}) \to Sub_{cl}(\underline{\Sigma})$$

which is referred to as co-Heyting implication. It is characterized by the adjunction

$$\underline{T} \leftarrow \underline{S} \leq \underline{R} \quad iff \quad \underline{T} \leq (\underline{R} \vee \underline{S}),$$

hence

$$(\underline{T} \leftarrow \underline{S}) = \bigwedge \{\underline{R} \in Sub_{cl}(\underline{\Sigma}) | \underline{T} \leq \underline{R} \vee \underline{S}\}.$$

We also define a co-Heyting negation $\sim$ for each $\underline{S} \in Sub_{cl}(\underline{\Sigma})$ by

$$\sim \underline{S} := (\underline{\Sigma} \leftarrow \underline{S}).$$

Theorem 1. *The definition of Heyting implication connective is too trivial for truth-value reasoning, while the definition of co-Heyting implication connective is suitable.*

Proof. We first examine the definition of Heyting negation $\neg \underline{S} := (\underline{S} \rightarrow \underline{0})$. Given that $\underline{S}_V = \{\lambda_i\}_{i \in I}$, we consider $\neg \underline{S}_V$. For every $\lambda_j \in \underline{\Sigma}_V$, if $\lambda_j \in \{\lambda_i\}_{i \in I}$, then λ_j does not satisfy the constraint of $\neg \underline{S}_V$ in V. If $\lambda_j \notin \{\lambda_i\}_{i \in I}$, then one can always choose a $V' \subseteq V$ satisfying $\lambda_i|_{V'} = \lambda_j|_{V'}$ such that λ_j does not satisfy the constraint of $\neg \underline{S}_V$ in V'. Hence $\neg \underline{S}_V$ could only be the empty set for every V.

Furthermore, we investigate the definition of $\underline{S} \rightarrow \underline{T}$. Without loss of generality, let $\underline{S}_V = \{\lambda_i\}_{i \in I}$ and $\underline{T}_V = \{\lambda_j\}_{j \in J}$ (where I, J for index set). For all $\lambda \in \underline{T}_V$, λ satisfies the constraint of $(\underline{S} \rightarrow \underline{T})_V$ obviously. For all $\lambda \notin \underline{T}_V$, if $\lambda \in \underline{S}_V$, λ does not satisfy the constraint of $(\underline{S} \rightarrow \underline{T})_V$ in V. If $\lambda \notin \underline{S}_V$, one can always find a $V' \subseteq V$ and some $\lambda_i \in \underline{S}_V$ satisfying $\lambda|_{V'} = \lambda_i|_{V'}$ such that λ does not satisfy the constraint of $(\underline{S} \rightarrow \underline{T})_V$ in V'. Therefore $(\underline{S} \rightarrow \underline{T})_V = \underline{T}_V$ for every V.

Meanwhile, consider the definition of co-Heyting negation

$$\sim \underline{S} = \bigwedge \{\underline{R} \in Sub_{cl}(\underline{\Sigma}) | \underline{\Sigma} \leq \underline{R} \vee \underline{S}\} = \bigwedge \{\underline{R} \in Sub_{cl}(\underline{\Sigma}) | \underline{\Sigma} = \underline{R} \vee \underline{S}\}.$$

Provided that $\underline{S}_V = \{\lambda_i\}_{i \in I}$, we have $\sim \underline{S}_V = \underline{\Sigma}_V \backslash \{\lambda_i\}_{i \in I} := \{\lambda_j\}_{j \in J}$. For any $V' \subseteq V$, the constraint $\underline{\Sigma}_{V'} = \sim \underline{S}_{V'} \vee \underline{S}_{V'}$ will always be satisfied.

Similarly, consider the definition of co-Heyting implication $(\underline{T} \leftarrow \underline{S})$. Suppose $\underline{S}_V = \{\lambda_i\}_{i \in I}$ and $\underline{T}_V = \{\lambda_j\}_{j \in J}$. Take $\{\lambda_k\}_{k \in K} = \{\lambda_j\}_{j \in J} \backslash \{\lambda_i\}_{i \in I}$, and we have $\{\lambda_j\}|_{V'} \subseteq \{\lambda_k\}|_{V'} \cup \{\lambda_i\}|_{V'}$ for every $V' \subseteq V$. Hence $(\underline{T} \leftarrow \underline{S}) = \bigwedge \{\underline{R} \in Sub_{cl}(\underline{\Sigma}) | \forall V,\ \underline{T}_V \backslash \underline{S}_V \subseteq \underline{R}_V\}$. $\square$

We note that $\neg \underline{S}$, as well as Heyting implication, is defined by satisfying the law of non-contradiction $\underline{S} \wedge \neg \underline{S} = 0$, while $\sim \underline{S}$, as well as co-Heyting implication, is defined by satisfying the law of excluded middle $\underline{S} \vee \sim \underline{S} = 1 = \underline{\Sigma}$.

Given that $\underline{S}_V \wedge \underline{R}_V \subseteq \underline{T}_V$ for some V. When considering some $V' \subseteq V$, $\underline{R}_{V'}$ restricts some elements in $\underline{S}_V$ but not in $\underline{T}_V$ into $\underline{R}_{V'}$. Thus we may have $\underline{S}_{V'} \wedge \underline{R}_{V'} \nsubseteq \underline{T}_{V'}$ in some V'. Therefore, the set $\underline{R}_V$ has to be taken "smaller" and be small enough to satisfy the partial order after being amplified by any V'. It must be $\underline{R} = \underline{T}$ eventually.

Correspondingly, given that $\underline{T}_V \subseteq \underline{R}_V \vee \underline{S}_V$ for some V, it is easy to verify that the partial order is satisfied for any $V' \subseteq V$. This is because the elements in $Sub_{cl}(\underline{\Sigma})$ preserve order and join.

Consider the Heyting algebra constituted by elements in $Sub_{cl}(\underline{\Sigma})$. The implicative connective $(\underline{S} \to \underline{T}) = \underline{T}$ is too trivial to apply to truth-value reasoning in the sense of Heyting algebra. But truth-value reasoning in the sense of co-Heyting algebra is suitable. We notice that in every context, the outer daseinisations of elements of orthogonal complement $\hat{P}, \hat{P}^{\perp}$ in property lattice L satisfy the law of excluded middle but do not satisfy the law of non-contradiction.

In this paper, we aim to construct the truth-value reasoning that is suitable for Heyting algebra. To achieve this, we intend to select elements in contexts based on the law of non-contradiction. By doing so, we can choose elements that satisfy the logical connectives in Heyting algebra.

4.2. Heyting algebra structure

In our perspective, for any projection $\hat{P}$, the role of daseinisation is to find the "approximation" of $\hat{P}$ in every context. For a given context V, if $\hat{P} \in V$, the approximation of $\hat{P}$ in the context is $\hat{P}$ itself. If $\hat{P} \notin V$, then the approximation of $\hat{P}$ is the element closest to $\hat{P}$ in the context. The outer daseinisation is the element just larger than $\hat{P}$, namely the one that can just be implied by $\hat{P}$. Similarly, the inner daseinisation is the element just smaller than $\hat{P}$, namely the one that can just imply $\hat{P}$. We notice that elements selected by the outer daseinisation in every context satisfy the constraints of morphisms in topos $Set^{\mathcal{V}(\mathcal{H})^{op}}$. Therefore, we regard the outer daseinisation of $\hat{P}$ as a "reflection" of $\hat{P}$ in every context.

Proposition 1. *For any $\hat{P} \in L$ and any $V \in \mathcal{V}(\mathcal{H})$, we have $\delta^i(\hat{P})_V \perp \delta^o(\hat{P}^{\perp})_V$, i.e., $\delta^i(\hat{P}) \perp \delta^o(\hat{P}^{\perp})$ holds for every context (namely contextwisely).*

Proof. For any $V \in \mathcal{V}(\mathcal{H})$. Let us define $\tilde{P} := \delta^i(\hat{P})_V$. We have $\tilde{P} \in V$ and $\tilde{P} \leq \hat{P}$. Furthermore, for any $\hat{P}_V \in V$, if $\hat{P}_V \leq \hat{P}$, then $\hat{P}_V \leq \tilde{P}$.

Now, let's define $\tilde{Q} := \delta^o(\hat{P}^\perp)_V$. We have $\tilde{Q} \in V$ and $\tilde{Q} \geq \hat{P}^\perp$. Similarly, for any $\hat{Q}_V \in V$, if $\hat{Q}_V \geq \hat{P}^\perp$, then $\hat{Q}_V \geq \tilde{Q}$.

Since $\tilde{P}, \tilde{Q} \in V$, we have $\tilde{P} \perp \tilde{Q}$ if and only if $\tilde{P} \wedge \tilde{Q} = 0, \tilde{P} \vee \tilde{Q} = 1$.

Assume that $\tilde{P} \wedge \tilde{Q} = R > 0$. Then we have $R^\perp = \tilde{P}^\perp \vee \tilde{Q}^\perp$. Since $R \leq \tilde{P}$ and $\tilde{P} \leq \hat{P}$. We can imply that $R \leq \hat{P}$, and $R^\perp \geq \hat{P}^\perp$. Hence $R^\perp \geq \tilde{Q}$.

Since $R^\perp = \tilde{P}^\perp \vee \tilde{Q}^\perp$, we have $R^\perp \wedge \tilde{Q} = (\tilde{P}^\perp \vee \tilde{Q}^\perp) \wedge \tilde{Q} = \tilde{P}^\perp \wedge \tilde{Q}$. Since $R^\perp \geq \hat{P}^\perp, \tilde{Q} \geq \hat{P}^\perp$. We can conclude that $R^\perp \wedge \tilde{Q} \geq \hat{P}^\perp$ and $R^\perp \wedge \tilde{Q} \geq \tilde{Q}$. Consequently, $\tilde{P}^\perp \wedge \tilde{Q} \geq \tilde{Q}$. Since $\tilde{P}^\perp \wedge \tilde{Q} \leq \tilde{Q}$, we have $\tilde{P}^\perp \wedge \tilde{Q} = \tilde{Q}$. Therefore $\tilde{Q} \leq \tilde{P}^\perp$. As a result, we can imply that $\tilde{P} \wedge \tilde{Q} \leq \tilde{P} \wedge \tilde{P}^\perp = 0$. This contradicts the assumption that $\tilde{P} \wedge \tilde{Q} = R > 0$. Therefore $\tilde{P} \wedge \tilde{Q} = 0$ holds.

Similarly, $\tilde{P} \vee \tilde{Q} = 1$ holds. We can imply that $\tilde{P} \perp \tilde{Q}$. □

Proposition 2. *$\delta^i(\hat{P})_V \leq \delta^o(\hat{P})_V$, i.e. $\delta^i(\hat{P}) \leq \delta^o(\hat{P})$ holds for every context.*

Proof. This proposition can be easily verified by the definition of daseinisation. □

Consider two properties $\hat{P}$ and $\hat{P}^\perp$ being orthogonal complements. By the law of non-contradiction, when we assign $\hat{P}$ to true, $\hat{P}^\perp$ must be assigned false. When assigning truth values in every context V, the truth assignment must make the "reflection" of $\hat{P}$ true, and that of $\hat{P}^\perp$ false. According to the above two propositions, assigning true to $\delta^i(\hat{P})_V$ in context V meets the requirement. Thus, we propose assigning $\delta^i(\hat{P})_V$ to true in every context V if we would like to assign $\hat{P}$ to true.

Proposition 3. *In the sense of sets, we have $\delta^o(\hat{P}) = \uparrow \hat{P}$, and $\delta^i(\hat{P}) = \downarrow \hat{P}$.*

Proof. $\forall V \in \mathcal{V}(\mathcal{H})$, we have $\hat{P} \leq \delta^o(\hat{P})_V$, hence $\delta^o(\hat{P})_V \in \uparrow \hat{P}$. It follows that $\delta^o(\hat{P}) \subseteq \uparrow \hat{P}$.
$\forall \hat{(Q)} \in \uparrow \hat{P}$, we choose $\{\hat{Q}, \hat{Q}^\perp\}$ as the generators to construct the context $V_{\hat{Q}}$ such that $\delta^o(\hat{P})_{V_{\hat{Q}}} = \hat{Q}$. It follows that $\uparrow \hat{P} \subseteq \delta^o(\hat{P})$. Therefore $\delta^o(\hat{P}) = \uparrow \hat{P}$. Similarly, we have $\delta^i(\hat{P}) = \downarrow \hat{P}$. □

Given a property lattice L, we start with constructing a lattice $I(L)$ by taking the order ideal of every element in L. It is well-known that L

is isomorphic to $I(L)$. Next, we add disjunctive elements to $I(L)$ through MacNeille completion, obtaining the lattice $D(L)$, where the elements are downsets of one or more elements in L. We have an embedding $I(L) \hookrightarrow D(L)$, and the added disjunctive elements are neither redundant nor missing. Any element in $D(L)$ that is not in the form of an ideal is a union of ideals, and all unions of different ideals are in $D(L)$.

Inspired by Dan Marsden[16], we can consider "globality" and "contextuality" separately. For outer daseinisation, we denoted by $\uparrow \hat{P}$ the set of all true properties that $\hat{P}$ being true implies. These properties are bound to be true whenever they appear in any context. $\uparrow \hat{P} \cap P(V)$ represents all true properties in context V (when $\hat{P}$ is assigned true), where $P(V)$ denotes all projections in V. $\bigwedge\{\uparrow \hat{P} \cap P(V)\} = \delta^o(\hat{P})_V$ holds.

For inner daseinisation, we denoted by $\downarrow \hat{P}$ the set of all properties that imply $\hat{P}$ to be true. These properties are true in specific contexts, but not in all contexts. $\downarrow \hat{P} \cap P(V)$ represents a set of properties that imply $\hat{P}$ to be true in context V. $\bigvee\{\downarrow \hat{P} \cap P(V)\} = \delta^i(\hat{P})_V$ holds. That is, in context V, we assign the set $\downarrow \hat{P} \cap P(V)$ to true. Note that we only know that the top element $\delta^i(\hat{P})_V$ of the set $\downarrow \hat{P} \cap P(V)$ is true, but we do not know the truth value of other elements.

Definition 12. *Let L be a property lattice, and define the mapping*

$$\begin{aligned} \overline{\delta^i} &: L \to D(L) \\ \hat{P} &\mapsto \delta^i(\hat{P}) = \downarrow \hat{P} \end{aligned}$$

where $\delta^i(\hat{P}) = \downarrow \hat{P}$ is understood in terms of sets.

Similar to $\underline{\delta}^o$, $\overline{\delta^i}$ preserves all meets and is an order-preserving injection, but not a surjection. It is evident that $\overline{\delta^i(\hat{0})} = \downarrow \hat{0} = \{\hat{0}\}$, and $\overline{\delta^i(\hat{1})} = \downarrow \hat{1} = L$. For meets, we have

$$\forall \hat{P}, \hat{Q} \in L : \overline{\delta^i(\hat{P})} \wedge \overline{\delta^i(\hat{Q})} = \overline{\delta^i(\hat{P} \wedge \hat{Q})}.$$

However, for joins, we have

$$\forall \hat{P}, \hat{Q} \in L : \overline{\delta^i(\hat{P})} \vee \overline{\delta^i(\hat{Q})} \leq \overline{\delta^i(\hat{P} \vee \hat{Q})}.$$

In general, $\overline{\delta^i(\hat{P})} \vee \overline{\delta^i(\hat{Q})}$ does not have the form of $\overline{\delta^i(\hat{R})}$ for any projection $\hat{R} \in L$.

Next, we show that the elements in $D(L)$ constitute a complete Heyting algebra. For a downset $\overline{S} \in D(L)$, we define $\overline{S}_V := \alpha_V(\bigvee\{\overline{S} \cap P(V)\})$. Let $\overline{S}, \overline{T} \in D(L)$ be two downsets, then

$$\begin{aligned} \forall V \in \mathcal{V}(\mathcal{H}) : (\overline{S} \wedge \overline{T})_V &= \overline{S}_V \cap \overline{T}_V \\ (\overline{S} \vee \overline{T})_V &= \overline{S}_V \cup \overline{T}_V. \end{aligned}$$

For each $\overline{S} \in D(L)$, the functor

$$\overline{S} \wedge \_ : D(L) \to D(L)$$

has a right adjoint

$$\overline{S} \to \_ : D(L) \to D(L).$$

The Heyting implication is given by the adjunction

$$\overline{S} \wedge \overline{R} \leq \overline{T} \quad iff \quad \overline{R} \leq (\overline{S} \to \overline{T}).$$

This implies that

$$\begin{aligned} (\overline{S} \to \overline{T}) &= \bigvee\{\overline{R} \in D(L) | \overline{S} \wedge \overline{R} \leq \overline{T}\} \\ &= \{r \in L | \forall a \in \overline{S}, a \wedge r \in \overline{T}\}. \end{aligned}$$

The contextwise definition is: for every $V \in \mathcal{V}(\mathcal{H})$,

$$(\overline{S} \to \overline{T})_V = \bigvee\{(\overline{S} \to \overline{T}) \cap P(V)\}.$$

The Heyting negation $\neg$ is defined for every $\overline{S} \in D(L)$ as follows:

$$\neg\overline{S} := (\overline{S} \to \overline{0}).$$

Therefore, we propose the following theorem.

Theorem 2. *The elements of $D(L)$ constitute a complete Heyting algebra, referred to as the propositional lattice denoted by H. Each $\overline{S} \in D(L)$ is called a "truth-value global element".*

Proof. For any $\overline{R}, \overline{S}, \overline{T} \in D(L)$ and any $V \in \mathcal{V}(\mathcal{H})$, we have

$$(\overline{R} \wedge (\overline{S} \wedge \overline{T}))_V = \overline{R}_V \cap (\overline{S}_V \cap \overline{T}_V) = (\overline{R}_V \cap \overline{S}_V) \cap \overline{T}_V = ((\overline{R} \wedge \overline{S}) \wedge \overline{T})_V$$

and

$$(\overline{R} \vee (\overline{S} \vee \overline{T}))_V = \overline{R}_V \cup (\overline{S}_V \cup \overline{T}_V) = (\overline{R}_V \cup \overline{S}_V) \cup \overline{T}_V = ((\overline{R} \vee \overline{S}) \vee \overline{T})_V.$$

Thus, the commutative and associative laws hold in $D(L)$.

In $D(L)$, there exist the maximum element $1 :=\downarrow \hat{1}$ and the minimum element $0 :=\downarrow \hat{0}$, where $\downarrow \hat{1}$ represents the set of all properties in the property lattice L and $\downarrow \hat{0}$ represents the set of the bottom element $\hat{0}$ of L.

The Heyting implication is given by the above adjunction:

$$\overline{S} \wedge \overline{R} \leq \overline{T} \quad iff \quad \overline{R} \leq (\overline{S} \rightarrow \overline{T}).$$

Thus, the elements of $D(L)$ constitute a complete Heyting algebra. □

It is worth noting that this Heyting algebra coincides with the one defined by Bob Coecke in [14]. The Heyting algebra constructed by Coecke indeed corresponds to inner daseinisation. A downset $\downarrow \hat{P}$ is associated with an actuality set[14], and the Heyting algebra generated by all downsets is the injective hull of the property lattice.

In outer daseinisation, we assign true to every element in $\uparrow \hat{P}$. Specifically, in each context V, every element in the set $\uparrow \hat{P} \cap P(V)$ is assigned as true. The conjunction $\bigwedge\{\uparrow \hat{P} \cap P(V)\}$ is definitely assigned as true. In inner daseinisation, we assign the set $\downarrow \hat{P}$ as true. In each context V, the set $\downarrow \hat{P} \cap P(V)$ is assigned as true. However, only the element $\bigvee\{\downarrow \hat{P} \cap P(V)\}$ is determined to be true in the current context.

Each property in the property lattice, as a generator of a certain context, corresponds to a linear functional in a one-to-one manner. When a property is not one-dimensional, its splitting is usually not unique. The splitting of a property represents the join of generators of properties in a particular context, declaring the context in which the property exists. The Heyting algebra $D(L)$ is generated by downsets of elements of the property lattice, where the additional disjunctive elements represent the declaration that a property is true only in those specific contexts, rather than in all contexts.

5. Natural revision in topos quantum logic

We propose that truth-value reasoning in quantum logic occurs both within a single propositional lattice, as well as between different propositional lattices. For internal truth-value reasoning of an individual element of a propositional lattice, we use the adjoint operation of Heyting algebra. This type of truth-value reasoning that occurs within one propositional lattice is referred to as static quantum logic. The standard quantum logic mentioned in this paper refers to the truth-value reasoning that occurs between two propositional lattices, using external information (projection measurement action) as a parameter to map elements assigned to true of one lattice to those of another. That is, the true-valued elements in the first lattice are mapped to those in the second lattice through Sasaki adjoint operations that use projection space as a parameter. This type of truth-value reasoning that occurs between two propositional lattices is referred to as dynamic quantum logic.

A consequence relation $\vdash \subseteq Fml \times Fml$ is determined by a binary relation obtained from antecedents and consequents. In a propositional lattice, we take the elements that are assigned to true as the antecedents of a consequence relation, and the elements that are inferred to be true according to the partial order relation based on the true-valued elements as the consequents. A truth-value assignment in the propositional lattice corresponds to the antecedents that are associated with a consequence relation. By adding new information, the antecedents of a consequence relation are changed, which is equivalent to revising one consequence relation to another.

When examining natural revision based on quantum logic, we should consider operations in the two types of reasoning - dynamic and static - as two revision operators for consequence relations. We suppose that a consequence relation $\vdash \subseteq Fml \times Fml$ occurs within a propositional lattice, where the formula set Fml represents elements in the propositional lattice, i.e., $Fml = H$. There are two revision operators for revising the consequence relation: one is to add new information using the truth-value reasoning of static quantum logic, and the other is to add new information using that of dynamic quantum logic.

Lemma 1. *The $\wedge$ operation within a Heyting algebra can be seen as a type of revision operation, referred to as object-level revision. Every formula $\alpha \in H$*

induces an operator on $\mathcal{C}$

$$\bar{\alpha} : \mathcal{C} \to \mathcal{C}$$

via

$$\bar{\alpha}(\vdash_1) = \vdash_2 \quad \Leftrightarrow \quad \alpha \wedge A_1 = A_2$$

in which A_1 *and* A_2 *represents antecedents of* $\vdash_1$ *and* $\vdash_2$ *respectively.*

Proof. For any $\overline{S} \in H$, we have $\overline{S} \subseteq L$ and $1 = L$. Hence, $1 \wedge \overline{S} = \overline{S} \wedge 1 = \overline{S}$ holds.

By the definition of Heyting implication, $\overline{S} \wedge \_ \dashv \overline{S} \to \_$ holds.

For any $\overline{S}, \overline{T} \in H$ and any $V \in \mathcal{V}(\mathcal{H})$, $(\overline{S} \wedge \overline{T})_V = \overline{S}_V \cap \overline{T}_V \subseteq \overline{S}_V$. So $\overline{S} \wedge \overline{T} \leq \overline{S}$ holds.

If $\overline{S} \leq \neg\overline{T}$, then for any $x \in \overline{S}$, we have $x \in \neg\overline{T}$. $\neg\overline{T} = (\overline{T} \to \overline{0}) = \{r \in L | \forall a \in \overline{T}, a \wedge r = 0\}$. So, for any $x \in \overline{S}$ and any $a \in \overline{T}$, we have $a \wedge x = 0$. Hence, $\overline{S} \wedge \overline{T} = 0$ holds. Conversely, if $\overline{S} \wedge \overline{T} = 0$, then for any $x \in \overline{S}$ and any $a \in \overline{T}$, we have $x \wedge a = 0$. So, $x \in \neg\overline{T}$ holds for any $x \in \overline{S}$. Hence, $\overline{S} \leq \neg\overline{T}$ holds.

$\overline{T} \leq \overline{S} \Leftrightarrow \forall V \in \mathcal{V}(\mathcal{H}), \overline{T}_V \subseteq \overline{S}_V \Leftrightarrow \forall V \in \mathcal{V}(\mathcal{H}), \overline{S}_V \cap \overline{T}_V = \overline{T}_V \Leftrightarrow \overline{S} \wedge \overline{T} = \overline{T}$. □

This revision operation takes the elements within a propositional lattice as operators and performs the $\wedge$ operation of Heyting algebra on new information and the antecedents of the original consequence relation in the lattice to obtain new antecedents within the same lattice. These new antecedents correspond to a new consequence relation.

Lemma 2. *The* $\wedge_s$ *operation outside a Heyting algebra can be viewed as another type of revision operation, referred to as operator-level revision. Every formula corresponding to a closed space* $a \in L$ *induces an operator on* $\mathcal{C}$

$$\bar{\varphi_a^*} : \mathcal{C} \to \mathcal{C}$$

via

$$\bar{\varphi_a^*}(\vdash_{H_1}) = \vdash_{H_2} \quad \Leftrightarrow \quad a \wedge_s A_{H_1} = A_{H_2}$$

in which A_{H_1} *and* A_{H_2} *represents antecedents of* $\vdash_{H_1}$ *and* $\vdash_{H_2}$ *respectively.*

Proof. For any $a \in L$, we have $1 \wedge_s a = 1 \wedge (\neg 1 \vee a) = 1 \wedge (0 \vee a) = 1 \wedge a = a$ and $a \wedge_s 1 = a \wedge (\neg a \vee 1) = a \wedge 1 = a$.

By the definition of Sasaki adjoint, $a \wedge_s \_ \dashv a \rightsquigarrow_s \_$ holds.

For any $a, b \in L$, $a \wedge_s b = a \wedge (\neg a \vee b) \leq a$ holds.

If $a \leq \neg b$, then $a \leq \neg b \Rightarrow b \leq \neg a \Rightarrow \neg a \vee b = \neg a \Rightarrow a \wedge_s b = a \wedge (\neg a \vee b) = a \wedge \neg a = 0$. Conversely, if $a \wedge_s b = a \wedge (\neg a \vee b) = 0$, then $\neg a = \neg a \vee 0 = \neg a \vee (a \wedge (\neg a \vee b))$. Clearly $\neg a \leq \neg a \vee b$. By the orthomodular law, we have $\neg a \vee (a \wedge (\neg a \vee b)) = \neg a \vee b$. Thus, $\neg a = \neg a \vee b \Rightarrow b \leq \neg a \Rightarrow a \leq \neg b$.

If $b \leq a$, then by the dual orthomodular law, $a \wedge (\neg a \vee b) = b$. Thus, $a \wedge_s b = a \wedge (\neg a \vee b) = b$. Conversely, if $a \wedge_s b = a \wedge (\neg a \vee b) = b$, then $a \wedge (\neg a \vee b) \leq a$ holds. Thus $b \leq a$ holds. □

This revision operation takes the elements in a property lattice as operators and performs the Sasaki projection φ_a^* on new information and the antecedents of the original consequence relation in a propositional lattice to obtain new antecedents within another propositional lattice. These new antecedents also correspond to a new consequence relation.

The elements in the two propositional lattices are identical, but the truth-value assignments of the elements may differ.

Theorem 3. *Given a consequence relation $\vdash$ within a propositional lattice H. Let $\alpha \in H$ and $a \in L$ be formulas. There are two types of revision operators, called object-level revision and operator-level revision respectively, to revise the consequence relation. The former, denoted by $\bar{\alpha}$, maps the consequence relation to another one within the same lattice H. The latter, denoted by $\bar{\varphi_a^*}$, maps the consequence relation to another one within a different propositional lattice.*

Proof. It is straightforward to demonstrate by using the above two lemmas. □

When we use formulas as operators to revise the consequence relation, there are two sources of formulas: those from within the propositional lattice and those from outside. When a formula from within the propositional lattice is used as an operator to revise the consequence relation, the revision operator is applied to the antecedents of the consequence relation based on the $\wedge$ operation within Heyting algebra. This results in a new consequence relation within the same lattice. When a formula from outside the propositional lattice is used as an operator, the selected formula corresponds to a closed subspace, namely a projection operation, and is used as a parameter with Sasaki projection φ_a^* operation to revise the antecedents of the original consequence relation. This results in a new consequence relation within another propositional lattice.

For example, consider the two equations below.

$$\varphi_a^*(B \wedge C) = D \qquad B \wedge \varphi_a^*(C) = E$$

In a propositional lattice, there exists an element C that is assigned to true, which corresponds to a consequence relation $\vdash_C$ based on an antecedent $\{C\}$. For the former equation, upon adding new information "element B is assigned to true", an object-level revision operation $\wedge$ is performed on consequence relation $\vdash_C$ to obtain antecedents $\{B, C\}$, which correspond to consequence relation $\vdash_{B\wedge C}$. Then, upon adding new information "projecting onto space a", an operator-level revision operation $\wedge_s$ is employed to revise consequence relation $\vdash_{B\wedge C}$, resulting in an antecedent $\{\varphi_a^*(B \wedge C) = D\}$ and corresponding to consequence relation $\vdash_D$. For the latter, firstly upon adding new information "projecting onto space a", an operator-level revision operation $\wedge_s$ is employed to revise consequence relation $\vdash_C$, resulting in an antecedent $\{\varphi_a^*(C)\}$ and corresponding to consequence relation $\vdash_{\varphi_a^*(C)}$. Then, upon adding new information "element B is assigned to true", an object-level revision operation $\wedge$ is performed on consequence relation $\vdash_{\varphi_a^*(C)}$ to obtain an antecedent $\{B \wedge \varphi_a^*(C) = E\}$, which corresponds to consequence relation $\vdash_E$. As shown in the Figure 1.

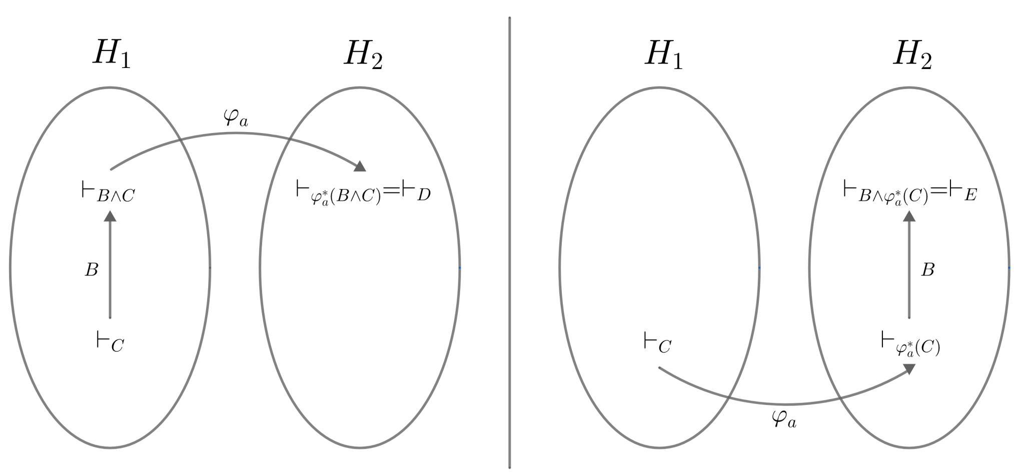

Figure 1: Natural revision

Both object-level revision and operator-level revision involve changing the truth values of elements in the propositional lattice. The difference lies in that object-level revision directly adds a new element assigned to be true, and performs $\wedge$ operation on the original elements to change their truth values. On the other side, although it is known from $a \wedge_s b \leq a$ that the new element a is assigned to true, for operator-level revision, the new element is applied to elements of the original lattice by $\wedge_s$ operation to obtain elements of another lattice which is assigned to true.

If we regard natural revision as a consequence revision system (CRS), this system should have two revision operations, $\wedge$ and $\wedge_s$, which correspond to two internalizing connectives, $\rightarrow$ and $\rightsquigarrow_s$ respectively. However, for the $a \wedge_s \_$ operation, since the element a is an external operator and should be a parameter of this operation, it is denoted as $\varphi_a^*(\_)$. Correspondingly, the a in the internalizing connective $a \rightsquigarrow_s \_$ should also be a parameter, denoted as $\varphi_{a,*}(\_)$. In this manner, the revision operator and so-called internalizing connective obtained by Sasaki adjoint are not suitable for the formalism of internalizing connectives. That is, the modus ponens obtained by Sasaki adjoint in standard quantum logic is not suitable for ordinary truth-value reasoning. The consequence relation only occurs within the propositional lattice, namely the formula set Fml in $\vdash \subseteq Fml \times Fml$ needs to be selected from the elements in the same propositional lattice. The propositional lattice has the $\wedge$ operation of Heyting algebra, and new information can transform the formula set Fml_1 into another formula set Fml_2 in the same lattice. On the other hand, the Sasaki projection operation is an external operation of the propositional lattice, and new information can transform one formula set Fml_{H_1} in the lattice H_1 into another formula set Fml_{H_2} in another lattice H_2.

In summary, truth-value reasoning in quantum logic occurs both within a single propositional lattice and between different propositional lattices. The internal truth-value reasoning within a single lattice is referred to as static quantum logic, while the truth-value reasoning between two propositional lattices is referred to as dynamic quantum logic. Consequence relations can be revised using two revision operators: object-level revision within a propositional lattice and operator-level revision between propositional lattices. The former uses the $\wedge$ operation of Heyting algebra, while the latter uses the Sasaki projection φ_a^* operation of Sasaki adjoint. By employing these revision operations, we can add new information to the antecedents of a consequence relation and revise it accordingly, leading to new consequence relations within the same or different propositional lattices.

Conclusion

In this paper, we point out that the revision theory based on standard quantum logic cannot correspond to truth-value reasoning in quantum logic. The reasoning rules generated by Sasaki adjoint in standard quantum logic are not truth-value reasoning within the system, but truth-value reasoning obtained by taking external elements as parameters of the operation. To accurately describe truth-value reasoning in quantum logic, it is necessary to reconsider truth-value assignment in quantum logic. In this paper, we use topos quantum theory to characterize the truth-value assignment of quantum logic and indicate that the truth-value assignment proposed by Isham et al. in topos quantum theory does not apply to truth-value reasoning in the Heyting algebra sense. To give truth-value reasoning that satisfies Heyting algebra, we redefine truth-value assignment in topos quantum theory by taking the downset of a property as its inner daseinisation in all contexts, construct a Heyting algebra with downsets as elements, and assign truth values to downsets satisfies the law of non-contradiction. In this formalism, truth-value reasoning in the sense of Heyting algebra is carried out. We call such Heyting algebra a propositional lattice. Considering Sasaki adjoint in the sense of the propositional lattice, we regard the Sasaki adjoint as an operation that selects a property from the property lattice as a parameter and maps elements in one propositional lattice to those in another propositional lattice. By using internal and external operations in Heyting algebra, the reasoning rules of quantum logic can be characterized. Quantum logic is determined by both static and dynamic reasoning.

After giving truth-value assignment and reasoning rules for quantum logic, we consider giving a natural revision theory based on quantum logic. We correspond the static and dynamic reasoning in quantum logic to two revision operations, which are object-level revision and operator-level revision, respectively. The static reasoning corresponds to object-level revision, which is a revision based on the $\wedge$ operation inside the Heyting algebra. The dynamic reasoning corresponds to operator-level revision, which is a revision based on the $\wedge_s$ operation outside the Heyting algebra. The consequence relation $\vdash \subseteq Fml \times Fml$ arises within a propositional lattice and the set of formulas Fml is constituted by elements in the propositional lattice. Object-level revision revises the consequence relation to another consequence relation in the same propositional lattice, while operator-level revision revises the consequence relation within one propositional lattice to another consequence relation in another propositional lattice with the same elements but different truth-value assignments.

The study of truth-value assignment and truth-value reasoning in intuitionistic quantum logic can be extended to mixed states of quantum mechanics. The investigation of how entanglement and superposition phenomena in quantum mechanics manifest in quantum logic is worth studying. In quantum computation, unitary operators represent evolution operations and can also serve as dynamic operators. This further expands the natural revision theory and provides a theoretical foundation for quantum computation.

References

[1] C. Alchourron, P. Gardenfors, D. Makinson, On the Logic of Theory Change, Journal of Symbolic Logic 50 (1985) 510–530.

[2] G. Birkhoff, J. Von Neumann, The Logic of Quantum Mechanics, Annals of mathematics (1936) 823–843.

[3] P. Finch, Sasaki projections on orthocomplemented posets, Bulletin of the Australian Mathematical Society 1 (3) (1969) 319–324. doi:10.1017/S0004972700042192.

[4] P. D. Finch, Quantum logic as an implication algebra, Bulletin of the Australian Mathematical Society 2 (1) (1970) 101–106.

[5] L. Roman, R. E. Zuazua, Quantum implication, International journal of theoretical physics 38 (3) (1999) 793–797.

[6] K. Engesser, D. M. Gabbay, Quantum logic, Hilbert space, revision theory, Artificial Intelligence 136 (1) (2002) 61–100.

[7] B. Coecke, S. Smets, The sasaki hook is not a [static] implicative connective but induces a backward [in time] dynamic one that assigns causes, International Journal of Theoretical Physics 43 (7-8) (2004) 1705–1736.

[8] C. J. Isham, J. Butterfield, Topos perspective on the Kochen-Specker theorem: I. Quantum states as generalized valuations, International journal of theoretical physics 37 (11) (1998) 2669–2733.

[9] J. Hamilton, C. J. Isham, J. Butterfield, Topos perspective on the Kochen-Specker theorem: III. von Neumann algebras as the base category, International Journal of Theoretical Physics 39 (2000) 1413–1436.

[10] A. Döring, C. J. Isham, A topos foundation for theories of physics: I. Formal languages for physics, Journal of Mathematical Physics 49 (5) (2008) 053515.

[11] A. Döring, C. J. Isham, A topos foundation for theories of physics: II. Daseinisation and the liberation of quantum theory, Journal of Mathematical Physics 49 (5) (2008) 053516.

[12] A. Döring, Topos quantum logic and mixed states, Electronic Notes in Theoretical Computer Science 270 (2) (2011) 59–77.

[13] A. Döring, J. Chubb, A. Eskandarian, V. Harizanov, Topos-based logic for quantum systems and bi-heyting algebras, Logic and Algebraic Structures in Quantum Computing 45 (2016) 151–173.

[14] B. Coecke, Quantum logic in intuitionistic perspective, Studia Logica (2002) 411–440.

[15] C. Flori, Lectures on topos quantum theory, arXiv preprint arXiv:1207.1744 (2012).

[16] D. Marsden, Comparing intuitionistic quantum logics, Ph.D. thesis, University of Oxford. URl: http://www. cs. ox. ac. uk/people/bob. coecke/Marsden. pdf (2010).

[17] K. H. E. Gudrun, Orthomodular Lattices, Orthomodular Lattices, 1983.

[18] M. A. Nielsen, I. Chuang, Quantum computation and quantum information (2002).

[19] Y. Kitajima, Negations and Meets in Topos Quantum Theory, Foundations of Physics 52 (1) (2022) 12.

[20] M. L. Dalla Chiara, R. Giuntini, et al., Quantum logics, Handbook of philosophical logic 6 (2001) 129–228.