

Extremal cases of distortion risk measures with partial information

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Abstract This paper considers the best- and worst-case of a general class of distortion risk measures when only partial information regarding the underlying distributions is available. Specifically, explicit sharp lower and upper bounds for a general class of distortion risk measures are derived based on the first two moments along with some shape information, such as symmetry/unimodality property of the underlying distributions. The proposed approach provides a unified framework for extremal problems of distortion risk measures.

Key words: Best-case risk; Distortion risk measures; Extreme cases; Worst-case risk; Partial information; Symmetry/Unimodality

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1 Introduction

Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR) are two of the most widely accepted risk measures in financial and insurance industries (see, e.g., Denuit et al. (2005)). These two risk measures are unified in a more general two-parameter family of risk measures, called the Range-Value-at-Risk (RVaR). The family of RVaR was introduced by Cont et al. (2010) in the context of robustness properties of risk measures. A more general risk measure, called the distortion risk measure (DRM) developed from the research on premium principles by Denneberg (1990), has been widely used in behavioral economics and risk management as an alternative to expected utility. By choosing appropriate distortion functions, the distortion risk measure offers a unified framework for quantile-based risk measures, and it includes a range of prominent risk measures such as VaR, TVaR and RVaR as special cases. Each of VaR, TVaR, RVaR and DRM can be represented as weighted average quantiles of a random variable, and so we refer to these risk measures also as quantile-based risk measures.

To derive the quantile-based risk measures, one often requires an exact form of the distribution of the specific risk, which is usually not feasible in practice. It is not always possible to obtain full information about the distribution of a particular random variable, and sometimes we can only estimate the distribution function based on sample data. Assuming that we know the moments of a specific random variable rather than its full information provides a feasible alternative. More precisely, estimating its first two moments based on real data sets makes the model more feasible in practical applications. It is then useful to consider the worst-case and best-case estimates of risk measures that are related to moment uncertainty. This prompts researchers to develop risk measures for extreme cases that require only the first two moments. The optimization problem of deriving the worst-case bounds for risk measures, given only partial information about the underlying distribution, has been dealt with in many fields.

The problem of finding sharp bounds for risk measures under model uncertainty has received much attention in the literature. El Ghaoui et al. (2003) considered the worst-case VaR problem when the first two moments of the underlying distribution are known and then derived the closed-form

solution to it. Chen et al. (2011) obtained closed-form solutions for the worst-case TVaR when the first two moments are known. Li (2018) derived an analytical bound under mean and variance information for a broad class of risk measures, known as law-invariant coherent risk measures. Li et al. (2018) established closed-form solutions for worst-case RVaR with the first two moments and some shape information such as symmetry and unimodality. Zhu and Shao (2018) further generalized the result of Li et al. (2018) to distortion risk measures with additional symmetry information and they considered both worst-case and best-case bounds. Liu et al. (2020) determined the worst-case values of a law-invariant convex risk functional when the mean and a higher order moment such as the variance of a risk are known. Cai et al. (2023) generalized the results of Li (2018) and Liu et al. (2020) to the case of any distortion risk measure. Bernard et al. (2023) derived quasi explicit best- and worst-case values of a large class of distortion risk measures when the underlying loss distribution has a given mean and variance and lies within a $\sqrt{\varepsilon}$ -Wasserstein ball around a given reference distribution, which generalizes the results of Li (2018) and Zhu and Shao (2018) corresponding to a Wasserstein tolerance of $\varepsilon = +\infty$. Recently, Shao and Zhang (2023a, 2023b) derived closed-form solutions for distortion risk measures in extreme cases by utilizing the first two moments (or certain higher-order absolute center moments) and the symmetry of underlying distribution. Moreover, they showed that the corresponding extreme-case distributions can be characterized by the envelopes of the distortion function.

Most of the existing works only consider the worst-case law-invariant coherent risk measures such as worst-case VaR and worst-case TVaR or worst-case RVaR. Motivated by the above-mentioned works, then present work aims to study both the worst-case and best-case estimates of any distortion risk measures based on partial information about the underlying distribution. Firstly, we attempt to directly calculate the worst-case and best-case estimates of VaR under certain assumptions concerning partial moment and shape constraints. More importantly, we want to use the proposed methods to generalize the extreme estimates and the corresponding distributions of distorted risk measures by solving the optimization problems based on four types of assumptions on the underlying distribution. Specifically, the problem can be formulated as $\inf_{X \in \mathcal{V}} \rho_h[X]$ and $\sup_{X \in \mathcal{V}} \rho_h[X]$, where \mathcal{V} de-

notes the set of random variables with four types partial information settings, respectively, and $\rho_h[X]$ is the distortion risk measure of X with distortion function h . In addition, when the bounds in these optimization problems are attainable, we find the distributions for which they are attained. The main contribution of this paper is that, in the case where the first two moments of the underlying distribution are known, a unified framework is provided for extreme-case distortion risk measures, including several well-known special extreme-case risk measures such as extreme-case VaR, extreme-case TVaR, extreme-case R VaR, etc. To directly derive extreme-bounds of distortion risk measures and their corresponding distributions, we use various calculus technique such as modified Schwarz inequality. Specifically, even with shape constraints, such as symmetry, unimodality and unimodal-symmetry of the underlying distribution, we can still obtain the worst-case and best-case estimates of distortion risk measures with the knowledge of the first two moments. We further apply our results to obtain bounds on VaR, TVaR and R VaR. As a result, we obtain analytical worst- and best-case bounds that provide a unified framework for the existing bounds and thus expanding the existing results on risk measures under various shape constraints. Most of the literature on this topic uses the standard duality method in optimization theory to obtain the results under the convexity assumption of distortion functions (see, e.g., El Ghaoui et al. (2003), Popescu (2005), Li (2018)) or via a unified method combining convex order and the notion of joint mixability (see Li et al. (2018)). The present work solves the problem by using probabilistic inequalities along with some calculus techniques.

The rest of this paper is structured as follows. Section 2 introduces the necessary notation and formulates the problem. Section 3 examines the extreme-case VaR under five types of assumptions on the underlying distribution. The main results—the analytical worst-case and best-case estimates and their corresponding distributions of distortion risk measures under partial moments and shape constraints—are presented in Section 4. In Section 5, we calculate the bounds of VaR, TVaR and R VaR using the unified result obtained in the preceding section. Finally, some concluding remarks are presented in Section 6.

2 Preliminaries

Let X be a non-degenerate random variable with distribution function F_X , finite mean μ and finite positive variance σ^2 . We define the right-continuous generalized inverse of F_X as

$$F_X^{-1+}(p) = \sup\{x : F_X(x) \leq p\}, \quad 0 \leq p < 1,$$

while the left-continuous generalized inverse of F_X is defined as

$$F_X^{-1}(p) = \inf\{x : F_X(x) \geq p\}, \quad 0 < p \leq 1.$$

For any real number x and probability level p , it is evident that $x \leq F_X^{-1+}(p) \iff P(X < x) = F_X(x-) \leq p$ and $F_X^{-1}(p) \leq x \iff p \leq F_X(x)$.

The right-continuous VaR at level $\alpha \in (0, 1)$ of a risk X is defined as

$$\text{VaR}_\alpha^+[X] = F_X^{-1+}(\alpha), \quad 0 \leq \alpha < 1,$$

and the left-continuous VaR at level $\alpha \in (0, 1)$ of a risk X is defined as

$$\text{VaR}_\alpha[X] = F_X^{-1}(\alpha), \quad 0 < \alpha \leq 1.$$

It is obvious that $\text{VaR}_\alpha^+[X] \leq x \iff \alpha \leq F(x-)$.

The Tail Value-at-Risk (TVaR) of a risk X is defined as

$$\text{TVaR}_\alpha[X] = \frac{1}{1-\alpha} \int_\alpha^1 F^{-1}(p) dp, \quad 0 < \alpha < 1.$$

The Range Value-at-Risk (RVaR) is defined for $\alpha, \beta \in (0, 1)$ as

$$\text{RVaR}_{\alpha,\beta}[X] = \frac{1}{\beta-\alpha} \int_\alpha^\beta F^{-1}(p) dp, \quad 0 < \alpha < \beta < 1.$$

Clearly, RVaR includes TVaR and VaR as limiting cases.

Definition 2.1 (Distortion Risk Measures). Given a random variable X with distribution F_X , the distortion risk measure (DRM) of X is defined via the Choquet integral in the form

$$\rho_h[X] = \int_0^\infty h(\bar{F}_X(x)) dx + \int_{-\infty}^0 (h(\bar{F}_X(x)) - 1) dx,$$

whenever at least one of the two integrals is finite. The function h refers to a distortion function which is a nondecreasing function on $[0, 1]$ with $h(0) = 0$ and $h(1) = 1$.

It is well-known that DRM is a law invariant, positively homogeneous, monotone and comonotonic additive risk measure; $\rho_h[X]$ is a coherent risk measure if and only if h is concave.

Definition 2.2 (Boyd and Vandenberghe, 2004). For a distortion function h , the convex and concave envelopes of h are defined, respectively, by

$$h_* = \sup\{g|g : [0, 1] \rightarrow [0, 1] \text{ is convex and } g(p) \leq h(p), p \in [0, 1]\},$$

$$h^* = \inf\{g|g : [0, 1] \rightarrow [0, 1] \text{ is concave and } g(p) \geq h(p), p \in [0, 1]\}.$$

Note that $(-h)_* = -h^*$. Moreover, if h is convex, then $h_* = h$; if h is concave, then $h^* = h$.

For $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+$, we denote by $V(\mu, \sigma)$ the set of random variables with mean μ and variance σ^2 , and denote by $V_S(\mu, \sigma)$, $V_U(\mu, \sigma)$, $V_{US}(\mu, \sigma)$ and $V_C(\mu, \sigma)$ the sets of symmetric, unimodal, unimodal-symmetric and concave random variables in $V(\mu, \sigma)$, respectively.

For given distortion function h , we consider the following optimization problem (worst-case and best-case)

$$\sup_{X \in \mathcal{V}(\mu, \sigma)} \rho_h[X] \text{ and } \inf_{X \in \mathcal{V}(\mu, \sigma)} \rho_h[X],$$

respectively, where $\mathcal{V}(\mu, \sigma)$ denotes one of $V(\mu, \sigma)$, $V_S(\mu, \sigma)$, $V_U(\mu, \sigma)$ and $V_{US}(\mu, \sigma)$. If a random variable X_* satisfies $\sup_{X \in \mathcal{V}(\mu, \sigma)} \rho_h[X] = \rho_h[X_*]$, then we refer to F_{X_*} as a worst-case distribution. Similarly, we refer to F_{X^*} as a best-case distribution if $\inf_{X \in \mathcal{V}(\mu, \sigma)} \rho_h[X] = \rho_h[X^*]$. The values $\sup_{X \in \mathcal{V}(\mu, \sigma)} \rho_h[X]$ and $\inf_{X \in \mathcal{V}(\mu, \sigma)} \rho_h[X]$ are correspondingly the worst-case and best-case distortion risk measures, respectively.

3 VaR bounds

In this section, we consider the worst-case and best-case VaR risk measures with the first two moments, as well as symmetry and unimodality of the underlying distributions. Most of the results are not new and they are scattered in different publications; see, e.g., Li (2018), Li et al. (2018), Zhu

and Shao (2018), Bernard et al. (2020), Cai et al. (2023) and Shao and Zhang (2023a, 2023b). Here, we give a simple proof. The method we use is different from the ones in the literature, mainly using the sharp upper bounds on the tails of the distribution of X ; for more details, see Ion et al. (2023). We present the results in the front of lemmas.

3.1 Case of general distributions

Lemma 3.1. (Cantelli's inequality (Cantelli (1928))) *Let X be a random variable with mean μ and variance σ^2 . Then, for any $x \geq 0$, the inequality*

$$P(X \geq x) \leq \frac{\sigma^2}{\sigma^2 + (x - \mu)^2}$$

holds. The equality is uniquely attained by

$$P(X = x) = \frac{\sigma^2}{\sigma^2 + (x - \mu)^2}, \quad P\left(X = \mu - \frac{\sigma^2}{x - \mu}\right) = \frac{(x - \mu)^2}{\sigma^2 + (x - \mu)^2}.$$

Using Lemma 3.1, we obtain the following proposition.

Proposition 3.1. *Let X be a random variable with mean μ and variance σ^2 . Then, the following hold:*

(i) *We have, for $0 < \alpha < 1$,*

$$\sup_{X \in V(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}}, \quad (3.1)$$

and the supremum in (3.1) is attained if and only if the worst-case rv X_ is such that*

$$P\left(X_* = \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}}\right) = 1 - \alpha, \quad P\left(X_* = \mu - \sigma \sqrt{\frac{1 - \alpha}{\alpha}}\right) = \alpha;$$

(ii) *We have, for $0 < \alpha < 1$,*

$$\inf_{X \in V(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \mu - \sigma \sqrt{\frac{1 - \alpha}{\alpha}}, \quad (3.2)$$

and the infimum in (3.2) is attained if and only if the best-case rv X^ is such that*

$$P\left(X^* = \mu + \sigma \sqrt{\frac{1 - \alpha}{\alpha}}\right) = \alpha, \quad P\left(X^* = \mu - \sigma \sqrt{\frac{\alpha}{1 - \alpha}}\right) = 1 - \alpha.$$

3.2 Case of symmetric distributions

The family of symmetric distributions (about the expectation μ) is characterized by equivalent relations

$$F(x - \mu) + F(\mu - x -) = 1, \quad F^{-1}(p) + F^{-1}(1 - p -) = 2\mu$$

and

$$P(X \leq x) = P(X \geq 2\mu - x).$$

Taking symmetry constraint into account, we now establish the following result.

Lemma 3.2. (Bienaymé-Chebyshev inequality) *Let X be a symmetric random variable with mean μ and variance σ^2 . Then, for any $x \geq 0$, with $w = \max\{1, x\}$, the inequality*

$$P\left(\frac{X - \mu}{\sigma} \geq x\right) \leq \frac{1}{2w^2}$$

holds, with equality if

$$P\left(\frac{X - \mu}{\sigma} = -w\right) = P\left(\frac{X - \mu}{\sigma} = w\right) = \frac{1}{2w^2}, \quad P(X = \mu) = 1 - \frac{1}{w^2}.$$

Using Lemma 3.2, we obtain the following proposition.

Proposition 3.2. *Let X be a symmetric random variable with mean μ and variance σ^2 . Then:*

(i) $\text{VaR}_\alpha^+[X] \leq \mu$ for $\alpha < \frac{1}{2}$; For $\alpha \geq \frac{1}{2}$, we have

$$\sup_{X \in V_S(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \mu + \sigma \sqrt{\frac{1}{2(1 - \alpha)}}, \quad (3.3)$$

and the supremum in (3.3) is attained if and only if the worst-case rv X_ is such that $P(X_* = \mu) = 2\alpha - 1$ and*

$$P\left(X_* = \mu + \sigma \sqrt{\frac{1}{2(1 - \alpha)}}\right) = 1 - \alpha, \quad P\left(X_* = \mu - \sigma \sqrt{\frac{1}{2(1 - \alpha)}}\right) = 1 - \alpha;$$

(ii) For $0 < \alpha < \frac{1}{2}$, we have

$$\inf_{X \in V_S(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \mu - \sigma \sqrt{\frac{1}{2\alpha}}, \quad (3.4)$$

and the infimum in (3.4) is attained if and only if the best-case rv X^* is such that $P(X^* = \mu) = 1 - 2\alpha$ and

$$P\left(X^* = \mu + \sigma\sqrt{\frac{1}{2\alpha}}\right) = \alpha, \quad P\left(X^* = \mu - \sigma\sqrt{\frac{1}{2\alpha}}\right) = \alpha.$$

3.3 Case of unimodal distributions

Taking unimodality constraint into account, we establish the following result.

Lemma 3.3. (Vysochanskiĭ and Petunin inequality) *Let the distribution of the standardized random variable Z be unimodal. Then, for any $v \geq 0$, the inequality*

$$P(Z \geq v) \leq \begin{cases} \frac{3-v^2}{3(1+v^2)}, & \text{if } v \in \left[0, \sqrt{\frac{5}{3}}\right), \\ \frac{4}{9(1+v^2)}, & \text{if } v \in \left[\sqrt{\frac{5}{3}}, +\infty\right), \end{cases}$$

holds, with equality for $v \in \left[0, \sqrt{\frac{5}{3}}\right)$ if $P(Z = v) = (3 - v^2)/3(1 + v^2)$, $P(Z \in U[-(3 + v^2)/2v, v]) = 4v^2/3(1 + v^2)$; and with equality for $v \in \left[\sqrt{\frac{5}{3}}, +\infty\right)$ if $P(Z = -1/v) = (3v^2 - 1)/3(1 + v^2)$, $P(Z \in U[-1/v, (1 + 3v^2)/2v]) = 4/3(1 + v^2)$.

Using Lemma 3.3, we obtain the following proposition.

Proposition 3.3. *Let X be a unimodal random variable with mean μ and variance σ^2 . Then, the following statements hold:*

(i) *We have, for $0 < \alpha < 1$,*

$$\sup_{X \in V_U(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \begin{cases} \mu + \sigma\sqrt{\frac{3\alpha}{4-3\alpha}}, & \text{if } \alpha \in \left[0, \frac{5}{6}\right), \\ \mu + \sigma\sqrt{\frac{4}{9(1-\alpha)}} - 1, & \text{if } \alpha \in \left[\frac{5}{6}, 1\right), \end{cases} \quad (3.5)$$

and for $\alpha \in \left[0, \frac{5}{6}\right)$, the supremum in (3.5) is attained if and only if the worst-case rv X_* is such that $P\left(X_* = \mu + \sigma\sqrt{\frac{3\alpha}{4-3\alpha}}\right) = 1 - \alpha$ and

$$P\left(X_* \in U\left[\mu - \sigma(2 - \alpha)\sqrt{\frac{3}{\alpha(4 - 3\alpha)}}, \mu + \sigma\sqrt{\frac{3\alpha}{4 - 3\alpha}}\right]\right) = \alpha;$$

for $\alpha \in [\frac{5}{6}, 1)$, the supremum in (3.5) is attained if and only if the worst-case rv X_* is such that $P\left(X_* = \mu - \sigma\sqrt{\frac{4}{9\alpha-5} - 1}\right) = 3\alpha - 2$ and

$$P\left(X_* \in U\left[\mu - \sigma\sqrt{\frac{4}{9\alpha-5} - 1}, \mu + \sigma(3\alpha - 1)\sqrt{\frac{1}{(1-\alpha)(9\alpha-5)}}\right]\right) = 3(1-\alpha);$$

(ii) We have, for $0 < \alpha < 1$,

$$\inf_{X \in V_U(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \begin{cases} \mu - \sigma\sqrt{\frac{3(1-\alpha)}{1+3\alpha}}, & \text{if } \alpha \in [\frac{1}{6}, 1), \\ \mu - \sigma\sqrt{\frac{4}{9\alpha} - 1}, & \text{if } \alpha \in [0, \frac{1}{6}), \end{cases} \quad (3.6)$$

and for $\alpha \in [\frac{1}{6}, 1)$, the infimum in (3.6) is attained if and only if the best-case rv X^* is such that $P\left(X^* = \mu - \sigma\sqrt{\frac{3(1-\alpha)}{1+3\alpha}}\right) = \alpha$ and

$$P\left(X^* \in U\left[\mu + \sigma(1+\alpha)\sqrt{\frac{3}{(1-\alpha)(1+3\alpha)}}, \mu - \sigma\sqrt{\frac{3(1-\alpha)}{1+3\alpha}}\right]\right) = 1 - \alpha;$$

for $\alpha \in [0, \frac{1}{6})$, the infimum in (3.6) is attained if and only if the best-case rv X^* is such that $P\left(X^* = \mu + \sigma\sqrt{\frac{4}{4-9\alpha} - 1}\right) = 1 - 3\alpha$ and

$$P\left(X^* \in U\left[\mu + \sigma\sqrt{\frac{4}{4-9\alpha} - 1}, \mu - \sigma(2-3\alpha)\sqrt{\frac{1}{\alpha(4-9\alpha)}}\right]\right) = 3\alpha.$$

3.4 Case of symmetric unimodal distributions

Taking both symmetry and unimodality constraints into account, we now establish the following result.

Lemma 3.4. *If the distribution of the standardized random variable Z is symmetric and unimodal, then*

$$P(Z \geq v) \leq \begin{cases} \frac{1}{2}(1 - \frac{v}{\sqrt{3}}), & \text{if } v \in \left[0, \frac{2}{\sqrt{3}}\right), \\ \frac{4}{9} \frac{1}{2v^2}, & \text{if } v \in \left[\frac{2}{\sqrt{3}}, +\infty\right), \end{cases}$$

holds. Equality is attained by the mixture of a uniform distribution on $[-\sqrt{3} \vee (\frac{3}{2}v), \sqrt{3} \vee (\frac{3}{2}v)]$ and a distribution degenerate at 0 such that the point mass at 0 equals $[1 - \frac{4}{3v^2}] \vee 0$.

The following Proposition 3.4 is a direct consequence of Lemma 3.4.

Proposition 3.4. *Let X be a symmetric and unimodal random variable with mean μ and variance σ^2 . Then,*

(i) *for $\alpha < \frac{1}{2}$, $\text{VaR}_\alpha^+[X] \leq \mu$, and for $\alpha \geq \frac{1}{2}$, we have*

$$\sup_{X \in V_{SU}(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \begin{cases} \mu + \sigma\sqrt{3}(2\alpha - 1), & \text{if } \alpha \in [\frac{1}{2}, \frac{5}{6}), \\ \mu + \sigma\sqrt{\frac{2}{9(1-\alpha)}}, & \text{if } \alpha \in [\frac{5}{6}, 1), \end{cases} \quad (3.7)$$

and for $\alpha \in [\frac{1}{2}, \frac{5}{6})$, the supremum in (3.7) is attained if and only if the worst-case rv X_ has the uniform distribution on $[\mu - \sigma\sqrt{3}, \mu + \sigma\sqrt{3}]$; for $\alpha \in [\frac{5}{6}, 1)$, the supremum in (3.7) is attained if and only if the worst-case rv X_* is the mixture of the uniform distribution on $[\mu - \sigma\sqrt{\frac{1}{2(1-\alpha)}}, \mu + \sigma\sqrt{\frac{1}{2(1-\alpha)}}]$ and the distribution degenerate at μ such that the point mass at μ equals $6\alpha - 5$;*

(ii) *for $\alpha > \frac{1}{2}$, $\text{VaR}_\alpha^+[X] \geq \mu$, and for $\alpha \leq \frac{1}{2}$, we have*

$$\inf_{X \in V_{SU}(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \begin{cases} \mu - \sigma\sqrt{3}(1 - 2\alpha), & \text{if } \alpha \in (\frac{1}{6}, \frac{1}{2}], \\ \mu - \sigma\sqrt{\frac{2}{9\alpha}}, & \text{if } \alpha \in (0, \frac{1}{6}], \end{cases} \quad (3.8)$$

and for $\alpha \in (\frac{1}{6}, \frac{1}{2}]$, the infimum in (3.8) is attained if and only if the best-case rv X^ has the uniform distribution on $[\mu - \sigma\sqrt{3}, \mu + \sigma\sqrt{3}]$; for $\alpha \in (0, \frac{1}{6}]$, the infimum in (3.8) is attained if and only if the best-case rv X^* is the mixture of a uniform distribution on $[\mu - \sigma\sqrt{\frac{1}{2\alpha}}, \mu + \sigma\sqrt{\frac{1}{2\alpha}}]$ and a distribution degenerate at μ such that the point mass at μ equals $1 - 6\alpha$.*

3.5 Case of concave distribution functions

Lemma 3.5. *(One-sided Gauß inequality). Let the random variable Y have second moment $E(Y^2) = 1$ and its distribution function be concave on $[0, \infty)$ and 0 on $(-\infty, 0)$. Then, for all nonnegative v , the inequality*

$$P(Y \geq v) \leq \begin{cases} 1 - \frac{v}{\sqrt{3}}, & \text{if } v \in [0, \frac{2}{\sqrt{3}}), \\ \frac{4}{9} \frac{1}{v^2}, & \text{if } v \in [\frac{2}{\sqrt{3}}, +\infty), \end{cases}$$

holds. For $v \in [0, \frac{2}{\sqrt{3}})$, equality holds iff Y has an uniform distribution on $[0, \sqrt{3}]$. For $v \in [\frac{2}{\sqrt{3}}, +\infty)$, equality holds iff Y has an one-sided boundary-inflated uniform distribution on $[0, 3v/2)$ with mass $1 - 4/(3v^2)$ at 0.

Proposition 3.5. *Let the random variable Y have mean μ and variance σ^2 and its distribution function be concave on $[0, \infty)$ and 0 on $(-\infty, 0)$. Then,*

$$\sup_{Y \in V_C(\mu, \sigma)} \text{VaR}_\alpha^+[Y] = \begin{cases} \mu + \sigma\sqrt{3}\alpha, & \text{if } \alpha \in [0, \frac{2}{3}), \\ \mu + \sigma\frac{2}{3\sqrt{1-\alpha}}, & \text{if } \alpha \in [\frac{2}{3}, 1), \end{cases} \quad (3.9)$$

and for $\alpha \in [0, \frac{2}{3})$, the supremum in (3.9) is attained iff Y has an uniform distribution on $[\mu, \mu + \sqrt{3}\sigma)$; for $\alpha \in [\frac{2}{3}, 1)$, the supremum in (3.9) is attained iff Y has an one-sided boundary-inflated uniform distribution on $[\mu, \mu + \sigma\frac{1}{\sqrt{1-\alpha}})$ with mass $3\alpha - 2$ at μ .

4 Bounds of distortion risk measures

The following two lemmas are needed for subsequent developments.

Lemma 4.1. *(Dhaene et al. (2012)) For any distortion function h and $X \in L^\infty$, the distortion risk measure $\rho_h(X)$ has the following Lebesgue-Stieltjes integral representation:*

(i) *If h is right-continuous, then*

$$\rho_h[X] = \int_0^1 F_X^{-1+}(1-p)dh(p) = \int_0^1 F_X^{-1+}(p)d\tilde{h}(p);$$

(ii) *If h is left-continuous, then*

$$\rho_h[X] = \int_0^1 F_X^{-1}(1-p)dh(p) = \int_0^1 F_X^{-1}(p)d\tilde{h}(p);$$

(iii) *If h is continuous, then*

$$\rho_h[X] = \int_0^1 F_X^{-1}(1-p)dh(p) = \int_0^1 F_X^{-1}(p)d\tilde{h}(p),$$

where $\tilde{h}(p) = 1 - h(1-p)$ for all $p \in [0, 1]$.

Lemma 4.2. *(Modified Schwarz inequality (Moriguti, 1953)) Let H be a function of bounded variation in the closed interval $[a, b]$ and continuous at both ends. Then, the inequality*

$$\int_a^b x(t)dH(t) \leq \int_a^b x(t)\bar{h}(t)dt \quad (4.1)$$

holds for any nondecreasing function $x(t)$ for which the integrals exist and are finite, where $\bar{h}(t)$ is the right-hand derivative of the greatest convex minorant \bar{H} of H . The equality in (4.1) holds if and only if $x(t)$ is a constant in every interval contained in $\{t : \bar{H}(t) < \min\{H(t-0), H(t+0)\}\}$ and, at every point of discontinuity, if any, of $H(t)$, $x(t_n) = x(t_n+0)$ when $H(t_n-0) < H(t_n+0)$, $x(t_n) = x(t_n-0)$ when $H(t_n-0) > H(t_n+0)$.

4.1 General distributions

We now deal with the extreme-case distortion risk measure for the univariate case under the assumption that the first two moments and the shape factor of the underlying distribution are all known.

Proposition 4.1. *Let X be a random variable with mean μ and variance σ^2 and h be a distortion function. Then, the following statements hold:*

(i) *We have that,*

$$\sup_{X \in V(\mu, \sigma)} \rho_h[X] = \mu + \sigma \sqrt{\int_0^1 (\tilde{h}'_*(p) - 1)^2 dp}, \quad (4.2)$$

where \tilde{h}'_* denotes the right derivative function of \tilde{h}_* . Moreover, if $\tilde{h}'_*(p) = 1$ (a.e.), the supremum in (4.2) is attained by any random variable $X \in V(\mu, \sigma)$; if $\tilde{h}'_*(p) \neq 1$ (a.e.), the supremum in (4.2) is attained by the worst-case distribution of rv X_* with

$$F_{X_*}^{-1+}(p) = \mu + \sigma \frac{\tilde{h}'_*(p) - 1}{\sqrt{\int_0^1 (\tilde{h}'_*(p) - 1)^2 dp}};$$

(ii) *We have*

$$\inf_{X \in V(\mu, \sigma)} \rho_h[X] = \mu - \sigma \sqrt{\int_0^1 (h'_*(p) - 1)^2 dp}. \quad (4.3)$$

The best-case rv $X^* = \arg \inf_{X \in V(\mu, \sigma)} \rho_h[X]$ is with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu, & \text{if } h'_*(p) = 1 \text{ (a.e.)}, \\ \mu - \sigma \frac{h'_*(p) - 1}{\sqrt{\int_0^1 (h'_*(p) - 1)^2 dp}}, & \text{if } h'_*(p) \neq 1 \text{ (a.e.)}. \end{cases}$$

Proof. If $\tilde{h}'_*(p) = 1$ (a.e.), the proof is trivial. Next, we assume $\tilde{h}'_*(p) \neq 1$ (a.e.). Using Lemma 4.2, for any constant c , we have

$$\begin{aligned}
\rho_h(X) &= \mu + \int_0^1 (F_X^{-1+}(p) - \mu) d\tilde{h}(p) \leq \mu + \int_0^1 (F_X^{-1+}(p) - \mu) \tilde{h}'_*(p) dp \\
&= \mu + \int_0^1 (F_X^{-1+}(p) - \mu) (\tilde{h}'_*(p) - c) dp \\
&\leq \mu + \left(\int_0^1 (F_X^{-1+}(p) - \mu)^2 dp \right)^{\frac{1}{2}} \cdot \left(\int_0^1 (\tilde{h}'_*(p) - c)^2 dp \right)^{\frac{1}{2}} \\
&= \mu + \sigma \left(\int_0^1 (\tilde{h}'_*(p) - c)^2 dp \right)^{\frac{1}{2}}.
\end{aligned}$$

Note that

$$\inf_{c \in \mathbb{R}} \left(\int_0^1 (\tilde{h}'_*(p) - c)^2 dp \right)^{\frac{1}{2}} = \sqrt{\int_0^1 (\tilde{h}'_*(p) - 1)^2 dp}.$$

Hence,

$$\rho_h(X) \leq \mu + \sigma \sqrt{\int_0^1 (\tilde{h}'_*(p) - 1)^2 dp}$$

with equality holding iff

$$F_{X_*}^{-1+}(p) = \mu + \sigma \frac{\tilde{h}'_*(p) - 1}{\sqrt{\int_0^1 (\tilde{h}'_*(p) - 1)^2 dp}}.$$

In this case, we can readily verify that

$$\int_0^1 F_{X_*}^{-1+}(p) d\tilde{h}(p) = \int_0^1 F_{X_*}^{-1+}(p) \tilde{h}'(p) dp.$$

Hence, (i) follows. The statement (ii) can be verified directly from (i). In fact, using the property of distortion risk measures, we can write mini-

mization problems in terms of maximization problems. More specifically,

$$\begin{aligned}
\inf_{X \in V(\mu, \sigma)} \rho_h[X] &= \mu + \sigma \inf_{Z \in V(0,1)} \int_0^1 F_Z^{-1+}(p) d\tilde{h}(p) \\
&= \mu - \sigma \sup_{Z \in V(0,1)} \int_0^1 (-F_Z^{-1+}(p)) d\tilde{h}(p) \\
&= \mu - \sigma \sup_{Z \in V(0,1)} \int_0^1 F_{-Z}^{-1+}(1-p) d\tilde{h}(p) \\
&= \mu - \sigma \sup_{Z \in V(0,1)} \int_0^1 F_{-Z}^{-1+}(p) d(1 - \tilde{h}(1-p)) \\
&= \mu - \sigma \sup_{Z \in V(0,1)} \int_0^1 F_{-Z}^{-1+}(p) dh(p).
\end{aligned}$$

Note that

$$\sup_{Z \in V(0,1)} \int_0^1 F_{-Z}^{-1+}(p) dh(p) = \sqrt{\int_0^1 (h'_*(p) - 1)^2 dp}.$$

Hence,

$$\inf_{X \in V(\mu, \sigma)} \rho_h[X] = \mu - \sigma \sqrt{\int_0^1 (h'_*(p) - 1)^2 dp}.$$

The best-case rv $X^* = \arg \inf_{X \in V(\mu, \sigma)} \rho_h[X]$ is such that

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu, & \text{if } h'_*(p) = 1 \text{ (a.e.)}, \\ \mu - \sigma \frac{h'_*(p) - 1}{\sqrt{\int_0^1 (h'_*(p) - 1)^2 dp}}, & \text{if } h'_*(p) \neq 1 \text{ (a.e.)}. \end{cases}$$

This completes the proof of the proposition.

Remark 4.1. *To the best of our knowledge, Shao and Zhang (2023a) was the first study the extreme-case distortion risk measures based on the first two moments and present closed-form solutions. To prove the main theorem, they followed a logical progression consisting of three long distinct steps. The first step involves demonstrating their Proposition 2.1 holds for convex distortion functions. The second step entails proving that Theorem 2.1 extend to piecewise constant distortion functions. Finally, the third step uses approximation methods to verify the applicability of the result to general distortion*

functions. Shao and Zhang (2023b) made use of the same three steps, under certain higher-order absolute central moments, along with symmetry properties of the underlying distributions to be known. The proof presented here is much shorter and mainly relies on Lemma 4.2.

4.2 Symmetric distributions

Proposition 4.2. *Let X be a symmetric random variable with mean μ and variance σ^2 and h be a distortion function. Then, the following statements hold:*

(i) *We have*

$$\sup_{X \in V_S(\mu, \sigma)} \rho_h[X] = \mu + \frac{1}{2}\sigma \sqrt{\int_0^1 (\tilde{h}'_*(p) - \tilde{h}'_*(1-p))^2 dp}, \quad (4.4)$$

where \tilde{h}'_* denotes the right derivative function of \tilde{h}_* . Moreover, if $\tilde{h}'_*(p) - \tilde{h}'_*(1-p) = 0$ (a.e.), the supremum in (4.4) is attained by any random variable $X \in V_S(\mu, \sigma)$; if $\tilde{h}'_*(p) - \tilde{h}'_*(1-p) \neq 0$ (a.e.), the supremum in (4.4) is attained by the worst-case distribution of rv X_* with

$$F_{X_*}^{-1+}(p) = \mu + \sigma \frac{\tilde{h}'_*(p) - \tilde{h}'_*(1-p)}{\sqrt{\int_0^1 (\tilde{h}'_*(p) - \tilde{h}'_*(1-p))^2 dp}};$$

(ii) *We have*

$$\inf_{X \in V_S(\mu, \sigma)} \rho_h[X] = \mu - \frac{1}{2}\sigma \sqrt{\int_0^1 (h'_*(p) - h'_*(1-p))^2 dp}. \quad (4.5)$$

The best-case rv $X^* = \arg \inf_{X \in V_S(\mu, \sigma)} \rho_h[X]$ is with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu, & \text{if } h'_*(p) - h'_*(1-p) = 0 \text{ (a.e.)}, \\ \mu - \sigma \frac{h'_*(p) - h'_*(1-p)}{\sqrt{\int_0^1 (h'_*(p) - h'_*(1-p))^2 dp}}, & \text{if } h'_*(p) - h'_*(1-p) \neq 0 \text{ (a.e.)}. \end{cases}$$

Proof. Using $F^{-1}(p) + F^{-1}(1-p) = 2\mu$, we can express $\rho_h[X]$ as

$$\rho_h[X] = \mu + \frac{1}{2} \int_0^1 (F_X^{-1+}(p) - \mu) d(h(p) + h(1-p)),$$

from which we get the desired results by using same arguments as those for Proposition 4.2.

Remark 4.2. *Shao and Zhang (2023a) first established the extreme-case distortion risk measures based on the first two moments and symmetry constraints. To prove the main theorem, they undertook a logical progression consisting of three long distinct steps. But, the proof provided here is short and simple relying mainly on Lemma 4.2.*

4.3 Unimodal distributions

The following problem has been considered by Bernard et al. (2020) for Range Value-at-Risk. Li et al. (2018) have also discussed the extreme-case Range Value-at-Risk. Here, we proceed to study the extreme-case DRM with general distortion functions.

Proposition 4.3. *Let X be a unimodal random variable with mean μ and variance σ^2 and h be a distortion function. Then, the following statements hold:*

(i) *We have*

$$\sup_{X \in V_U(\mu, \sigma)} \rho_h[X] = \mu + \sigma \sup_{b \in [0,1]} \{\Delta_R(b), \Delta_L(b)\}, \quad (4.6)$$

where

$$\begin{aligned} \Delta_R(b) &= \frac{-(1+b^2)}{\sqrt{(1-b)^3(1/3+b)}} + \frac{2b}{\sqrt{(1-b)^3(1/3+b)}} \tilde{h}(b) \\ &\quad + \frac{2}{\sqrt{(1-b)^3(1/3+b)}} \int_b^1 p \tilde{h}'_+(p) dp \end{aligned}$$

and

$$\begin{aligned} \Delta_L(b) &= \sqrt{\frac{3b}{4-3b}} - \frac{2b\sqrt{3}}{\sqrt{b^3(4-3b)}} \tilde{h}(b) \\ &\quad + \frac{2\sqrt{3}}{\sqrt{b^3(4-3b)}} \int_0^b p \tilde{h}'_+(p) dp; \end{aligned}$$

(ii) *We have*

$$\inf_{X \in V_U(\mu, \sigma)} \rho_h[X] = \mu - \sigma \sup_{b \in [0,1]} \{\Xi_R(b), \Xi_L(b)\}, \quad (4.7)$$

where

$$\begin{aligned}\Xi_R(b) &= \frac{-(1+b^2)}{\sqrt{(1-b)^3(1/3+b)}} + \frac{2b}{\sqrt{(1-b)^3(1/3+b)}}h(b) \\ &\quad + \frac{2}{\sqrt{(1-b)^3(1/3+b)}} \int_b^1 ph'_+(p)dp\end{aligned}$$

and

$$\begin{aligned}\Xi_L(b) &= \sqrt{\frac{3b}{4-3b}} - \frac{2b\sqrt{3}}{\sqrt{b^3(4-3b)}}h(b) \\ &\quad + \frac{2\sqrt{3}}{\sqrt{b^3(4-3b)}} \int_0^b ph'_+(p)dp.\end{aligned}$$

Proof. (i) As in Bernard et al. (2020), we introduce two sets U_R and U_L as follows:

$$U_R = \{X : F_X^{-1}(p) = a \text{ for } p \in [0, b); c(p-b) + a \text{ for } p \in [b, 1]\}$$

and

$$U_L = \{X : F_X^{-1}(p) = a \text{ for } p \in [b, 1]; c(p-b) + a \text{ for } p \in [0, b)\}.$$

It is clear that

$$U_R \cap V(\mu, \sigma) \subseteq V_U(\mu, \sigma), \quad U_L \cap V(\mu, \sigma) \subseteq V_U(\mu, \sigma).$$

From the proof of Proposition 1 of Bernard et al. (2020), we see that for every $X \in V_U(\mu, \sigma)$, there exists a random variable $Y_R \in U_R \cap V(\mu, \sigma)$ such that $F_X^{-1}(p) = F_{Y_R}^{-1}(p)$, or, there exists a random variable $Y_L \in U_L \cap V(\mu, \sigma)$ such that $F_X^{-1}(p) = F_{Y_L}^{-1}(p)$. In fact, we can take

$$F_{Y_R}^{-1}(p) = \begin{cases} \mu - \sigma \sqrt{\frac{1-b}{1/3+b}}, & \text{if } p \in [0, b), \\ \mu + \sigma \frac{2p-1-b^2}{\sqrt{(1-b)^3(1/3+b)}}, & \text{if } p \in [b, 1), \end{cases}$$

and

$$F_{Y_L}^{-1}(p) = \begin{cases} \mu + \sigma \sqrt{3} \frac{2p-2b+b^2}{\sqrt{b^3(4-3b)}}, & \text{if } p \in [0, b), \\ \mu + \sigma \sqrt{\frac{3b}{4-3b}}, & \text{if } p \in [b, 1); \end{cases}$$

for details, see Bernard et al. (2020). For any distortion function h , if $F_X^{-1}(p) = F_{Y_R}^{-1}(p)$, then the distortion risk measure of X is given by

$$\begin{aligned}
\rho_h[X] &= \mu + \int_0^1 (F_X^{-1+}(p) - \mu) d\tilde{h}(p) \\
&= \mu + \int_0^1 (F_X^{-1+}(p) - \mu) \tilde{h}'_+(p) dp \\
&= \mu - \sigma \int_0^b \sqrt{\frac{1-b}{1/3+b}} \tilde{h}'_+(p) dp + \sigma \int_b^1 \frac{2p-1-b^2}{\sqrt{(1-b)^3(1/3+b)}} \tilde{h}'_+(p) dp \\
&= \mu - \frac{\sigma(1+b^2)}{\sqrt{(1-b)^3(1/3+b)}} + \frac{2b\sigma}{\sqrt{(1-b)^3(1/3+b)}} \tilde{h}(b) \\
&\quad + \frac{2\sigma}{\sqrt{(1-b)^3(1/3+b)}} \int_b^1 p \tilde{h}'_+(p) dp \\
&\equiv \mu + \sigma \Delta_R(b).
\end{aligned}$$

Similarly, if $F_X^{-1}(p) = F_{Y_L}^{-1}(p)$, then the distortion risk measure of X is given by

$$\begin{aligned}
\rho_h[X] &= \mu + \int_0^1 (F_X^{-1+}(p) - \mu) d\tilde{h}(p) \\
&= \mu + \sigma \int_b^1 \sqrt{\frac{3b}{4-3b}} \tilde{h}'_+(p) dp + \sigma \sqrt{3} \int_0^b \frac{2p-2b+b^2}{\sqrt{b^3(4-3b)}} \tilde{h}'_+(p) dp \\
&= \mu + \sigma \sqrt{\frac{3b}{4-3b}} - \frac{2b\sqrt{3}\sigma}{\sqrt{b^3(4-3b)}} \tilde{h}(b) \\
&\quad + \frac{2\sqrt{3}\sigma}{\sqrt{b^3(4-3b)}} \int_0^b p \tilde{h}'_+(p) dp \\
&\equiv \mu + \sigma \Delta_L(b).
\end{aligned}$$

Thus,

$$\sup_{X \in V_U(\mu, \sigma)} \rho_h[X] = \mu + \sigma \sup_{b \in [0,1]} \{ \Delta_R(b), \Delta_L(b) \},$$

which establishes statement (i). The statement (ii) can be verified similarly. In fact,

$$\begin{aligned}
\inf_{X \in V_U(\mu, \sigma)} \rho_h[X] &= \mu + \sigma \inf_{Z \in V_U(0,1)} \int_0^1 F_Z^{-1+}(p) d\tilde{h}(p) \\
&= \mu - \sigma \sup_{Z \in V_U(0,1)} \int_0^1 (-F_Z^{-1+}(p)) d\tilde{h}(p) \\
&= \mu - \sigma \sup_{Z \in V_U(0,1)} \int_0^1 F_{-Z}^{-1+}(1-p) d\tilde{h}(p) \\
&= \mu - \sigma \sup_{Z \in V_U(0,1)} \int_0^1 F_Z^{-1+}(p) dh(p),
\end{aligned}$$

from which the statement (ii) follows. This completes the proof of the proposition.

4.4 Symmetric unimodal distributions

The following problem has been discussed by Bernard et al. (2020) for Range Value-at-Risk. The following proposition discusses the extreme-case DRM with general distortion functions.

Proposition 4.4. *Let X be a symmetric unimodal random variable with mean μ and variance σ^2 and h be a distortion function. Further, let $\tilde{h}(p) = 1 - h(1 - p)$ for all $p \in [0, 1]$. Then, the following statements hold.*

(i) *We have*

$$\sup_{X \in V_{SU}(\mu, \sigma)} \rho_h[X] = \mu + \sigma \sup_{b \in [0,1]} \Theta(b), \quad (4.8)$$

where

$$\Theta(b) = \frac{\int_0^{1-b} p\tilde{h}'_+(p)dp - (1-b)\tilde{h}(1-b) + \int_b^1 p\tilde{h}'_+(p)dp - b(1-\tilde{h}(b))}{\sqrt{\frac{2}{3}(1-b)^3}};$$

(ii) *We have*

$$\inf_{X \in V_{SU}(\mu, \sigma)} \rho_h[X] = \mu - \sigma \sup_{b \in [0,1]} \Pi(b), \quad (4.9)$$

where

$$\Pi(b) = \frac{\int_0^{1-b} ph'_+(p)dp - (1-b)h(1-b) + \int_b^1 ph'_+(p)dp - b(1-h(b))}{\sqrt{\frac{2}{3}(1-b)^3}}.$$

Proof. (i) As in Bernard et al. (2022), we denote by $\mathcal{V}(\alpha)$ the set of random variables X with quantile functions

$$F_X^{-1}(p) = \begin{cases} a + c(p - 1 + b), & \text{if } p \in (0, 1 - b), \\ a, & \text{if } p \in [1 - b, b], \\ a + c(p - b), & \text{if } p \in (b, 1), \end{cases}$$

where

$$(a, b, c) \in \mathbb{R} \times \left[\frac{1}{2}, \max(\alpha, 1 - \alpha) \right] \times \mathbb{R}^+.$$

From the proof of Proposition 1 of Bernard et al. (2022), we see that for every $X \in V_{SU}(\mu, \sigma)$, there exists a random variable $Y \in \mathcal{V}(\alpha) \cap V(\mu, \sigma)$ such that $F_X^{-1}(p) = F_Y^{-1}(p)$. Using $E[Y] = \mu$ and $\text{var}[Y] = \sigma^2$, we are able to express a and c as functions of b, μ and σ , which lead to the following expression for $F_Y^{-1}(p)$:

$$F_Y^{-1}(p) = \begin{cases} \mu + \sigma \frac{p-1+b}{\sqrt{2(1/3-b+b^2-b^3/3)}}, & \text{if } p \in (0, 1 - b), \\ \mu, & \text{if } p \in [1 - b, b], \\ \mu + \sigma \frac{p-b}{\sqrt{2(1/3-b+b^2-b^3/3)}}, & \text{if } p \in (b, 1). \end{cases}$$

For any distortion function h , if $F_X^{-1}(p) = F_Y^{-1}(p)$, then the distortion risk measure of X is given by

$$\begin{aligned} \rho_h[X] &= \mu + \int_0^1 (F_X^{-1}(p) - \mu) d\tilde{h}(p) \\ &= \mu + \int_0^1 (F_X^{-1}(p) - \mu) \tilde{h}'_+(p) dp \\ &= \mu + c \int_0^{1-b} p \tilde{h}'_+(p) dp + c(b-1) \tilde{h}(1-b) \\ &\quad + c \int_b^1 p \tilde{h}'_+(p) dp - cb(1 - \tilde{h}(b)) \\ &\equiv \mu + \sigma \Theta(b), \end{aligned}$$

where

$$c = \frac{\sigma}{\sqrt{\frac{2}{3}(1-b)^3}}.$$

Hence,

$$\sup_{X \in V_{SU}(\mu, \sigma)} \rho_h[X] = \mu + \sigma \sup_{b \in [0, 1]} \Theta(b),$$

as required. The statement (ii) can be verified similarly. This completes the proof of the proposition.

5 Illustrative examples

In this section, we use a special distortion risk measure, RVaR (specifically, TVaR and VaR), to illustrate the propositions established as preceding section. The obtained results are consistent with existing ones. We consider the distortion function $h(p) = \min \left\{ \frac{p+\beta-1}{\beta-\alpha}, 1 \right\} 1_{\{p \geq 1-\beta\}}$ with $0 < \alpha < \beta < 1$ and $p \in [0, 1]$, with the associated distortion measure given by

$$\text{RVaR}_{\alpha, \beta}[X] = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \text{VaR}^+(p) dp, \quad 0 < \alpha < \beta < 1.$$

Example 5.1 (Case of general distributions). The worst-case and best-case values of risk measures VaR, TVaR and RVaR under constraints on the first two moments have been discussed extensively (see, e.g., Bernard et al. (2023) and the references therein) which also follow from Proposition 4.1 as follows. Letting

$$h(p) = \min \left\{ \frac{p + \beta - 1}{\beta - \alpha}, 1 \right\} 1_{\{p \geq 1 - \beta\}}$$

with $0 < \alpha < \beta < 1$ and $p \in [0, 1]$, we get

$$\tilde{h}(p) = 1 - h(1 - p) = \begin{cases} 0, & \text{if } p \in [0, \alpha), \\ \frac{p-\alpha}{\beta-\alpha}, & \text{if } p \in [\alpha, \beta), \\ 1, & \text{if } p \in [\beta, 1]. \end{cases}$$

So,

$$\tilde{h}_*(p) = \begin{cases} 0, & \text{if } p \in [0, \alpha), \\ \frac{p-\alpha}{1-\alpha}, & \text{if } p \in [\alpha, 1), \end{cases}$$

and

$$\tilde{h}'_*(p) = \begin{cases} 0, & \text{if } p \in [0, \alpha), \\ \frac{1}{1-\alpha}, & \text{if } p \in [\alpha, 1). \end{cases}$$

Upon using Proposition 4.1, we get

$$\sup_{X \in V(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \mu + \sigma \sqrt{\frac{\alpha}{1 - \alpha}}, \quad (5.1)$$

and the supremum in (5.1) is attained by the worst-case distribution of X_* with

$$F_{X_*}^{-1+}(p) = \begin{cases} \mu - \sigma \sqrt{\frac{1-\alpha}{\alpha}}, & \text{if } p \in [0, \alpha), \\ \mu + \sigma \sqrt{\frac{\alpha}{1-\alpha}}, & \text{if } p \in [\alpha, 1]. \end{cases}$$

Similarly, since

$$h(p) = \begin{cases} 0, & \text{if } p \in [0, 1 - \beta), \\ \frac{p+\beta-1}{\beta-\alpha}, & \text{if } p \in [1 - \beta, 1 - \alpha), \\ 1, & \text{if } p \in [1 - \alpha, 1], \end{cases}$$

we have

$$h_*(p) = \begin{cases} 0, & \text{if } p \in [0, 1 - \beta), \\ \frac{p+\beta-1}{\beta}, & \text{if } p \in [1 - \beta, 1], \end{cases}$$

and

$$h'_*(p) = \begin{cases} 0, & \text{if } p \in [0, 1 - \beta), \\ \frac{1}{\beta}, & \text{if } p \in [1 - \beta, 1]. \end{cases}$$

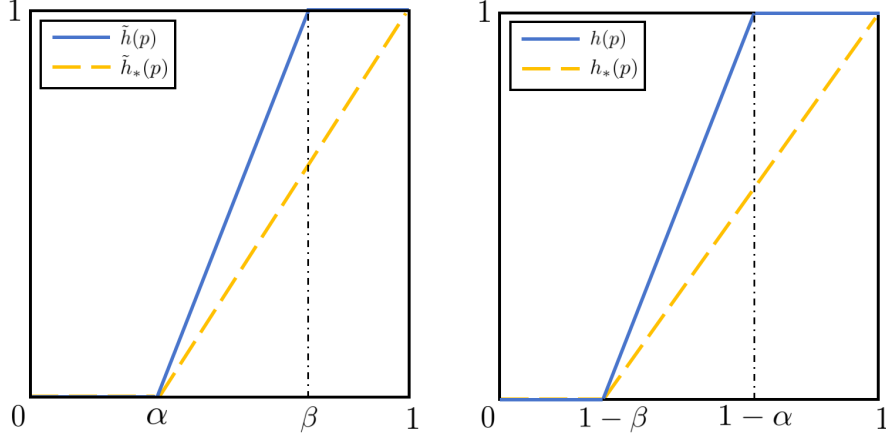
Using Proposition 4.1, we have

$$\inf_{X \in V(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \mu - \sigma \sqrt{\frac{1 - \beta}{\beta}}, \quad (5.2)$$

and the best-case rv $X^* = \arg \inf_{X \in V(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X]$ is with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu + \sigma \sqrt{\frac{\beta}{1-\beta}}, & \text{if } p \in [0, 1 - \beta), \\ \mu - \sigma \sqrt{\frac{1-\beta}{\beta}}, & \text{if } p \in [1 - \beta, 1]. \end{cases}$$

Figure 1. The left and right panels present dual distortion function and distortion function and their corresponding convex envelopes with respect to $\text{RVaR}_{\alpha, \beta}[X]$, respectively.



Because

$$\text{VaR}_\alpha^+[X] = \lim_{\beta \downarrow \alpha} \text{RVaR}_{\alpha, \beta}[X],$$

$$\text{TVaR}_\alpha[X] = \lim_{\beta \uparrow 1} \text{RVaR}_{\alpha, \beta}[X],$$

we can directly obtain the following:

For the worst-case,

$$\begin{aligned} \sup_{X \in V(\mu, \sigma)} \text{VaR}_\alpha^+[X] &= \sup_{X \in V(\mu, \sigma)} \text{TVaR}_\alpha[X] \\ &= \sup_{X \in V(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \mu + \sigma \sqrt{\frac{\alpha}{1-\alpha}}, \end{aligned}$$

and the supremum is attained by the worst-case distribution of rv X_* with

$$F_{X_*}^{-1+}(p) = \begin{cases} \mu - \sigma \sqrt{\frac{1-\alpha}{\alpha}}, & \text{if } p \in [0, \alpha), \\ \mu + \sigma \sqrt{\frac{\alpha}{1-\alpha}}, & \text{if } p \in [\alpha, 1]; \end{cases}$$

for the best-case,

$$\inf_{X \in V(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \mu - \sigma \sqrt{\frac{1-\alpha}{\alpha}},$$

and the best-case rv $X^* = \arg \inf_{X \in V(\mu, \sigma)} \text{VaR}_\alpha[X]$ is with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu + \sigma \sqrt{\frac{\alpha}{1-\alpha}}, & \text{if } p \in [0, 1-\alpha), \\ \mu - \sigma \sqrt{\frac{1-\alpha}{\alpha}}, & \text{if } p \in [1-\alpha, 1]; \end{cases}$$

Also,

$$\inf_{X \in V(\mu, \sigma)} \text{TVaR}_\alpha[X] = \mu,$$

and the best-case rv $X^* = \arg \inf_{X \in V(\mu, \sigma)} \text{TVaR}_\alpha[X]$ is characterized by $F_{X^*}^{-1+}(p) = \mu, p \in (0, 1]$.

Example 5.2 (Case of symmetric distributions). Letting $\tilde{h}(p) = \min \left\{ \frac{p-\alpha}{\beta-\alpha}, 1 \right\} 1_{\{p \geq \alpha\}}$ with $\frac{1}{2} \leq \alpha < \beta < 1$ and $p \in [0, 1]$, we get

$$\tilde{h}'_*(p) = \begin{cases} 0, & \text{if } p \in [0, \alpha), \\ \frac{1}{1-\alpha}, & \text{if } p \in [\alpha, 1), \end{cases}$$

and

$$\tilde{h}'_*(1-p) = \begin{cases} \frac{1}{1-\alpha}, & \text{if } p \in [0, 1-\alpha], \\ 0, & \text{if } p \in (1-\alpha, 1). \end{cases}$$

By using Proposition 4.2, we get

$$\sup_{X \in V_S(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \mu + \sigma \sqrt{\frac{1}{2(1-\alpha)}} \quad (5.3)$$

if $\tilde{h}'_*(p) - \tilde{h}'_*(1-p) = 0$ (a.e.), i.e., $p \in (1-\alpha, \alpha)$, the supremum in (5.3) is attained by any random variable $X \in V_S(\mu, \sigma)$; if $\tilde{h}'_*(p) - \tilde{h}'_*(1-p) \neq 0$ (a.e.), the supremum in (5.3) is attained by the worst-case distribution of rv X_* with

$$F_{X_*}^{-1+}(p) = \begin{cases} \mu - \sigma \sqrt{\frac{1}{2(1-\alpha)}}, & \text{if } p \in [0, 1-\alpha], \\ \mu + \sigma \sqrt{\frac{1}{2(1-\alpha)}}, & \text{if } p \in [\alpha, 1). \end{cases}$$

Similarly,

$$h'_*(p) = \begin{cases} 0, & \text{if } p \in [0, 1-\beta), \\ \frac{1}{\beta}, & \text{if } p \in [1-\beta, 1], \end{cases}$$

and

$$h'_*(1-p) = \begin{cases} \frac{1}{\beta}, & \text{if } p \in [0, \beta], \\ 0, & \text{if } p \in (\beta, 1]. \end{cases}$$

By using Proposition 4.2, we get

$$\inf_{X \in V_S(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \mu - \sigma \sqrt{\frac{1-\beta}{2\beta^2}}, \quad (5.4)$$

the best-case rv $X^* = \arg \inf_{X \in V_S(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X]$ is with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu + \sigma \sqrt{\frac{1}{2(1-\beta)}}, & \text{if } p \in [0, 1 - \beta), \\ \mu, & \text{if } p \in [1 - \beta, \beta], \\ \mu - \sigma \sqrt{\frac{1}{2(1-\beta)}}, & \text{if } p \in (\beta, 1). \end{cases}$$

For $\frac{1}{2} \leq \alpha < \beta < 1$, we can directly obtain the following results:

(i) For the worst-case, we have

$$\begin{aligned} \sup_{X \in V_S(\mu, \sigma)} \text{VaR}_{\alpha}^+[X] &= \sup_{X \in V_S(\mu, \sigma)} \text{TVaR}_{\alpha}[X] \\ &= \sup_{X \in V_S(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \mu + \sigma \sqrt{\frac{1}{2(1-\alpha)}} \end{aligned}$$

if $\tilde{h}'_*(p) - \tilde{h}'_*(1-p) = 0$ (a.e.), i.e., $p \in (1-\alpha, \alpha)$, the supremum is attained by any random variable $X \in V_S(\mu, \sigma)$; if $\tilde{h}'_*(p) - \tilde{h}'_*(1-p) \neq 0$ (a.e.), the supremum is attained by the worst-case distribution of rv X_* with

$$F_{X_*}^{-1+}(p) = \begin{cases} \mu - \sigma \sqrt{\frac{1}{2(1-\alpha)}}, & \text{if } p \in [0, 1 - \alpha], \\ \mu + \sigma \sqrt{\frac{1}{2(1-\alpha)}}, & \text{if } p \in [\alpha, 1]; \end{cases}$$

(ii) For the best-case, we have

$$\inf_{X \in V_S(\mu, \sigma)} \text{VaR}_{\alpha}[X] = \mu - \sigma \sqrt{\frac{1-\alpha}{2\alpha^2}},$$

and the best-case rv $X^* = \arg \inf_{X \in V_S(\mu, \sigma)} \text{VaR}_{\alpha}[X]$ is with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu + \sigma \sqrt{\frac{1}{2(1-\alpha)}}, & \text{if } p \in [0, 1 - \alpha), \\ \mu, & \text{if } p \in [1 - \alpha, \alpha], \\ \mu - \sigma \sqrt{\frac{1}{2(1-\alpha)}}, & \text{if } p \in (\alpha, 1), \end{cases}$$

and

$$\inf_{X \in V_S(\mu, \sigma)} \text{TVaR}_{\alpha}[X] = \mu,$$

with the best-case rv $X^* = \arg \inf_{X \in V_S(\mu, \sigma)} \text{TVaR}_{\alpha}[X]$ being with $F_{X^*}^{-1+}(p) = \mu, p \in (0, 1)$.

Likewise, for $\tilde{h}(p) = \min \left\{ \frac{p-\alpha}{\beta-\alpha}, 1 \right\} 1\{p \geq \alpha\}$ with $0 < \alpha < \beta < \frac{1}{2}$ and $p \in [0, 1]$, by using Proposition 4.2, we have

$$\sup_{X \in V_S(\mu, \sigma)} \text{RVar}_{\alpha, \beta}[X] = \mu, \quad (5.5)$$

and the supremum in (5.5) is attained by any random variable $X_* \in V_S(\mu, \sigma)$, and

$$\inf_{X \in V_S(\mu, \sigma)} \text{RVar}_{\alpha, \beta}[X] = \mu - \sigma \sqrt{\frac{1}{2\beta}}, \quad (5.6)$$

with the best-case rv $X^* = \arg \inf_{X \in V_S(\mu, \sigma)} \text{RVar}_{\alpha, \beta}[X]$ being with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu + \sigma \sqrt{\frac{1}{2\beta}}, & \text{if } p \in [0, 1 - \beta), \\ \mu, & \text{if } p \in [1 - \beta, \beta], \\ \mu - \sigma \sqrt{\frac{1}{2\beta}}, & \text{if } p \in (\beta, 1]. \end{cases}$$

Based on the above, for $0 < \alpha < \beta < \frac{1}{2}$, we directly obtain the following result:

(a) For the worst-case,

$$\sup_{X \in V_S(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \sup_{X \in V_S(\mu, \sigma)} \text{TVaR}_\alpha[X] = \sup_{X \in V_S(\mu, \sigma)} \text{RVar}_{\alpha, \beta}[X] = \mu,$$

and the supremum is attained by any random variable $X_* \in V_S(\mu, \sigma)$;

(b) For the best-case,

$$\inf_{X \in V_S(\mu, \sigma)} \text{VaR}_\alpha^+[X] = \mu - \sigma \sqrt{\frac{1}{2\alpha}},$$

and the best-case rv $X^* = \arg \inf_{X \in V_S(\mu, \sigma)} \text{VaR}_\alpha[X]$ is with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu + \sigma \sqrt{\frac{1}{2\alpha}}, & \text{if } p \in [0, 1 - \alpha), \\ \mu, & \text{if } p \in [1 - \alpha, \alpha], \\ \mu - \sigma \sqrt{\frac{1}{2\alpha}}, & \text{if } p \in (\alpha, 1], \end{cases}$$

and

$$\inf_{X \in V_S(\mu, \sigma)} \text{TVaR}_\alpha[X] = \mu - \sigma \sqrt{\frac{1}{2}},$$

with the best-case rv $X^* = \arg \inf_{X \in V_S(\mu, \sigma)} \text{TVaR}_\alpha[X]$ being with

$$F_{X^*}^{-1+}(p) = \begin{cases} \mu + \sigma\sqrt{\frac{1}{2}}, & \text{if } p = 0, \\ \mu, & \text{if } p \in (0, 1), \\ \mu - \sigma\sqrt{\frac{1}{2}}, & \text{if } p = 1. \end{cases}$$

Example 5.3 (Case of unimodal distributions). For $0 < \alpha < \beta < 1$, letting

$$\tilde{h}(p) = 1 - h(1 - p) = \begin{cases} 0, & \text{if } p \in [0, \alpha), \\ \frac{p-\alpha}{\beta-\alpha}, & \text{if } p \in [\alpha, \beta), \\ 1, & \text{if } p \in [\beta, 1] \end{cases}$$

in (4.6), we obtain

$$\Delta_R(b) = \begin{cases} \frac{\beta-1+\alpha-b^2}{\sqrt{(1-b)^3(1/3+b)}}, & \text{if } b < \alpha, \\ \frac{(\beta-1)(\beta-b^2)+\alpha(b-1)^2}{(\beta-\alpha)\sqrt{(1-b)^3(1/3+b)}}, & \text{if } \alpha \leq b < \beta, \\ -\sqrt{\frac{1-b}{1/3+b}}, & \text{if } b \geq \beta, \end{cases}$$

and

$$\Delta_L(b) = \begin{cases} \sqrt{\frac{3b}{4-3b}}, & \text{if } b < \alpha, \\ \sqrt{\frac{3b}{4-3b}} - \frac{\sqrt{3}(b-\alpha)^2}{(\beta-\alpha)\sqrt{b^3(4-3b)}}, & \text{if } \alpha \leq b < \beta, \\ \frac{\sqrt{3}[(b-1)^2-1+\beta+\alpha]}{\sqrt{b^3(4-3b)}}. & \text{if } b \geq \beta. \end{cases}$$

We immediately find

$$\begin{aligned} & \max \left\{ \sup_{b \in [0,1]} \Delta_R(b), \sup_{b \in [0,1]} \Delta_L(b) \right\} \\ &= \begin{cases} M, & \text{if } 0 \leq \alpha < \frac{1}{2}, \alpha < \beta < 1, \\ \max \left\{ \sqrt{\frac{8}{9(2-(\alpha+\beta))}} - 1, M \right\}, & \text{if } \frac{1}{2} \leq \alpha < \frac{5}{6}, \tilde{\beta} < \beta < 1, \\ \sqrt{\frac{8}{9(2-(\alpha+\beta))}} - 1, & \text{if } \frac{5}{6} \leq \alpha < \beta < 1, \end{cases} \end{aligned}$$

where

$$M = \frac{(\beta - \alpha - 1)c^2 + 2\alpha c - \alpha^2}{c^2(\beta - \alpha)} \sqrt{\frac{3c}{4 - 3c}},$$

and for $\frac{1}{2} \leq \alpha < \frac{5}{6}$, $\tilde{\beta}$ is the admissible root of the equation

$$-\frac{14}{9}\beta^4 + \left(\frac{4}{3}\alpha + \frac{95}{27}\right)\beta^3 + (2\alpha^2 - 5\alpha - 2)\beta^2 + (4 - \alpha)\alpha\beta + (\alpha - 2)\alpha^2 = 0.$$

Here,

$$c = \frac{\alpha[3\alpha + 2 - \sqrt{(3\alpha - 2)^2 + 12(1 - \beta)}]}{4\alpha + 2(\beta - 1)}.$$

We thus obtain

$$\begin{aligned} & \sup_{X \in V_U(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] \\ &= \begin{cases} \mu + \sigma M, & \text{if } 0 \leq \alpha < \frac{1}{2}, \alpha < \beta < 1, \\ \mu + \sigma \max \left\{ \sqrt{\frac{8}{9(2 - (\alpha + \beta))}} - 1, M \right\}, & \text{if } \frac{1}{2} \leq \alpha < \frac{5}{6}, \tilde{\beta} < \beta < 1, \\ \mu + \sigma \sqrt{\frac{8}{9(2 - (\alpha + \beta))}} - 1, & \text{if } \frac{5}{6} \leq \alpha < \beta < 1. \end{cases} \end{aligned} \quad (5.7)$$

Similarly, we obtain

$$\begin{aligned} \Xi_R(b) &= \begin{cases} \frac{1 - \beta - \alpha - b^2}{\sqrt{(1 - b)^3(1/3 + b)}}, & \text{if } b \leq 1 - \beta, \\ \frac{(1 - \beta)(b - 1)^2 + \alpha(b^2 - 1 + \alpha)}{(\beta - \alpha)\sqrt{(1 - b)^3(1/3 + b)}}, & \text{if } 1 - \beta < b \leq 1 - \alpha, \\ -\sqrt{\frac{1 - b}{1/3 + b}}, & \text{if } b > 1 - \alpha, \end{cases} \\ \Xi_L(b) &= \begin{cases} \sqrt{\frac{3b}{4 - 3b}}, & \text{if } b \leq 1 - \beta, \\ \sqrt{\frac{3b}{4 - 3b}} - \frac{\sqrt{3}(1 - \beta - b)^2}{(\beta - \alpha)\sqrt{b^3(4 - 3b)}}, & \text{if } 1 - \beta < b \leq 1 - \alpha, \\ \frac{\sqrt{3}[(b - 1)^2 + 1 - \beta - \alpha]}{\sqrt{b^3(4 - 3b)}}, & \text{if } b > 1 - \alpha. \end{cases} \end{aligned}$$

We then immediately find

$$\begin{aligned} & \max \left\{ \sup_{b \in [0, 1]} \Xi_R(b), \sup_{b \in [0, 1]} \Xi_L(b) \right\} \\ &= \begin{cases} \sqrt{\frac{8}{9(\alpha + \beta)}} - 1, & \text{if } 0 \leq \alpha < \beta < \frac{1}{6}, \\ \min \left\{ \sqrt{\frac{8}{9(\alpha + \beta)}} - 1, N \right\}, & \text{if } \frac{1}{6} \leq \beta < \frac{1}{2}, 0 \leq \alpha < \tilde{\alpha}, \\ N, & \text{if } \frac{1}{2} \leq \beta < 1, 0 \leq \alpha < \beta, \end{cases} \end{aligned}$$

where

$$N = \frac{(\beta - \alpha - 1)c_1^2 + 2(1 - \beta)c_1 - (1 - \beta)^2}{c_1^2(\beta - \alpha)} \sqrt{\frac{3c_1}{4 - 3c_1}},$$

and for $\frac{1}{6} \leq \beta < \frac{1}{2}$, $\tilde{\alpha}$ is the admissible root of the equation

$$-3\beta^3 + 3(2\alpha^2 - 3\alpha + 2)\beta^2 + (4\alpha^3 - 9\alpha^2 + 12\alpha - 4)\beta - \frac{\alpha}{9}(42\alpha^3 - 37\alpha^2 - 6\alpha + 36) + \frac{8}{9} = 0.$$

Here,

$$c_1 = \frac{(1 - \beta)[5 - 3\beta - \sqrt{(1 - 3\beta)^2 + 12\alpha}]}{4(1 - \beta) - 2\alpha}.$$

We thus obtain

$$\begin{aligned} & \inf_{X \in V_U(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] \\ &= \begin{cases} \mu - \sigma \sqrt{\frac{8}{9(\alpha + \beta)} - 1}, & \text{if } 0 \leq \alpha < \beta < \frac{1}{6}, \\ \mu - \sigma \min \left\{ \sqrt{\frac{8}{9(\alpha + \beta)} - 1}, N \right\}, & \text{if } \frac{1}{6} \leq \beta < \frac{1}{2}, 0 \leq \alpha < \tilde{\alpha}, \\ \mu - \sigma N, & \text{if } \frac{1}{2} \leq \beta < 1, 0 \leq \alpha < \beta. \end{cases} \quad (5.8) \end{aligned}$$

In particular, letting $\beta \rightarrow 1$ in (5.7) and (5.8), respectively, we obtain

$$\sup_{X \in V_U(\mu, \sigma)} \text{TVaR}_{\alpha}[X] = \begin{cases} \mu + \sigma \frac{\sqrt{\alpha(8/9 - \alpha)}}{1 - \alpha}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ \mu + \sigma \sqrt{\frac{8}{9(1 - \alpha)} - 1}, & \text{if } \frac{1}{2} \leq \alpha < 1, \end{cases}$$

and

$$\inf_{X \in V_U(\mu, \sigma)} \text{TVaR}_{\alpha}[X] = \mu.$$

Example 5.4 (Case of symmetric unimodal distributions). For $0 < \alpha < \beta < 1$, letting

$$\tilde{h}(b) = 1 - h(1 - b) = \begin{cases} 0, & \text{if } b \in [0, \alpha), \\ \frac{b - \alpha}{\beta - \alpha}, & \text{if } b \in [\alpha, \beta), \\ 1, & \text{if } b \in [\beta, 1], \end{cases}$$

$$\tilde{h}(1 - b) = \begin{cases} 0, & \text{if } 1 - b \in [0, \alpha) \text{ i.e. } b \in (1 - \alpha, 1], \\ \frac{1 - b - \alpha}{\beta - \alpha}, & \text{if } 1 - b \in [\alpha, \beta) \text{ i.e. } b \in (1 - \beta, 1 - \alpha], \\ 1, & \text{if } 1 - b \in [\beta, 1) \text{ i.e. } b \in (0, 1 - \beta], \end{cases}$$

and

$$\tilde{h}'(p) = \begin{cases} 0, & \text{if } p \in [0, \alpha) \cup [\beta, 1], \\ \frac{1}{\beta - \alpha}, & \text{if } p \in [\alpha, \beta), \end{cases}$$

in (4.8), we obtain

$$\Theta(b) = \begin{cases} \frac{\beta+\alpha-2b}{2\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } 1-\alpha < b < \alpha, \\ \frac{\beta(\beta-2b)+b^2}{2(\beta-\alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \max\{\alpha, 1-\alpha\} < b < \beta, \\ 0, & \text{if } \max\{\beta, 1-\alpha\} < b < 1, \\ \frac{-(b-1)^2-2\alpha(\alpha-1)+\beta(\beta-2b)}{2(\beta-\alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } 1-\beta < b < \min\{\alpha, 1-\alpha\}, \\ \frac{(\beta+\alpha)(\beta-\alpha-2b)+2(\alpha+b)-1}{2(\beta-\alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \max\{\alpha, 1-\beta\} < b < \min\{\beta, 1-\alpha\}, \\ \frac{(1-b)(2\alpha-1+b)-\alpha^2}{2(\beta-\alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \max\{\beta, 1-\beta\} < b < 1-\alpha, \\ \frac{\beta+\alpha-1}{\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \frac{1}{2} < b < \min\{\alpha, 1-\beta\}, \\ \frac{2\beta(\beta-1)-\alpha(\alpha-2)+b(b-2\alpha)}{2(\beta-\alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \alpha < b < \min\{\beta, 1-\beta\}, \\ \frac{\beta+\alpha+2(b-1)}{2\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \beta < b < 1-\beta. \end{cases}$$

We then immediately find that

$$\sup_{b \in [0,1]} \Theta(b) = \begin{cases} 0, & \text{if } (\alpha; \beta) \in M_1, \\ \frac{2(\alpha+\beta-1)}{3(\beta-\alpha)\sqrt{\alpha-\beta+1}}, & \text{if } (\alpha; \beta) \in M_2, \\ \sqrt{3}(\alpha + \beta - 1), & \text{if } (\alpha; \beta) \in M_3, \\ \sqrt{\frac{4}{9(2-(\alpha+\beta))}}, & \text{if } (\alpha; \beta) \in M_4, \end{cases}$$

where

$$M_1 = \left(0, \frac{1}{2}\right) \times (\alpha, 1-\alpha), M_2 = \left(0, \frac{1}{3}\right) \times \left[\max\left\{1-\alpha, \frac{3\alpha+2}{3}\right\}, 1\right),$$

$$M_3 = \left[\frac{1}{6}, \frac{5}{6}\right) \times \left[\max\{1-\alpha, \alpha\}, \min\left\{\frac{3\alpha+2}{3}, 1, \frac{5-3\alpha}{3}\right\}\right),$$

$$M_4 = \left[\frac{2}{3}, 1\right) \times \left[\max\left\{\frac{5-3\alpha}{3}, \alpha\right\}, 1\right).$$

Thus, we obtain

$$\sup_{X \in V_{SV}(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \begin{cases} \mu, & \text{if } (\alpha; \beta) \in M_1, \\ \mu + \sigma \frac{2(\alpha+\beta-1)}{3(\beta-\alpha)\sqrt{\alpha-\beta+1}}, & \text{if } (\alpha; \beta) \in M_2, \\ \mu + \sigma \sqrt{3}(\alpha + \beta - 1), & \text{if } (\alpha; \beta) \in M_3, \\ \mu + \sigma \sqrt{\frac{4}{9(2-(\alpha+\beta))}}, & \text{if } (\alpha; \beta) \in M_4. \end{cases} \quad (5.9)$$

In particular, letting $\beta \rightarrow 1$ in (5.9), we obtain

$$\sup_{X \in V_{SU}(\mu, \sigma)} \text{TVaR}_\alpha[X] = \begin{cases} \mu + \sigma \frac{2\sqrt{\alpha}}{3(1-\alpha)}, & \text{if } \alpha \in (0, \frac{1}{3}), \\ \mu + \sigma \sqrt{3\alpha}, & \text{if } \alpha \in [\frac{1}{3}, \frac{2}{3}), \\ \mu + \sigma \sqrt{\frac{4}{9(1-\alpha)}}, & \text{if } \alpha \in [\frac{2}{3}, 1). \end{cases} \quad (5.10)$$

For $0 < \alpha < \beta < 1$, letting

$$h(b) = \begin{cases} 0, & \text{if } b \in [0, 1 - \beta), \\ \frac{b + \beta - 1}{\beta - \alpha}, & \text{if } b \in [1 - \beta, 1 - \alpha), \\ 1, & \text{if } b \in [1 - \alpha, 1], \end{cases}$$

$$h(1 - b) = \begin{cases} 0, & \text{if } 1 - b \in [0, 1 - \beta) \text{ i.e. } b \in (\beta, 1], \\ \frac{\beta - b}{\beta - \alpha}, & \text{if } 1 - b \in [1 - \beta, 1 - \alpha) \text{ i.e. } b \in (\alpha, \beta], \\ 1, & \text{if } 1 - b \in [1 - \alpha, 1) \text{ i.e. } b \in (0, \alpha], \end{cases}$$

and

$$h'(p) = \begin{cases} 0, & \text{if } p \in [0, 1 - \beta) \cup [1 - \alpha, 1], \\ \frac{1}{\beta - \alpha}, & \text{if } p \in [1 - \beta, 1 - \alpha), \end{cases}$$

in (4.9), we obtain

$$\Pi(b) = \begin{cases} \frac{(\alpha - \beta)(\alpha + \beta - 2b)}{2(\beta - \alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } 1 - \alpha < b < \alpha, \\ \frac{(\alpha - 1)^2 + \alpha^2 + \beta(2b - \beta) + b(b - 2)}{2(\beta - \alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } 1 - \beta < b < \min\{\alpha, 1 - \alpha\}, \\ \frac{\alpha(\alpha - 1) - \beta(\beta - 1)}{(\beta - \alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \frac{1}{2} < b < 1 - \beta, \\ \frac{(b - 1)^2 - (1 - \beta)^2 - 2(1 - b)(\beta - b)}{2(\beta - \alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \max\{\alpha, 1 - \alpha\} < b < \beta, \\ \frac{(1 - b + \beta)(1 + b - \beta) + (b + \alpha)(b + \alpha - 2)}{2(\beta - \alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \max\{\alpha, 1 - \beta\} < b < \min\{\beta, 1 - \alpha\}, \\ \frac{2\beta(1 - \beta) + \alpha(\alpha + 2b - 2) - b^2}{2(\beta - \alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \alpha < b < \min\{\beta, 1 - \beta\}, \\ 0, & \text{if } 1 - \alpha < b < 1, \\ \frac{(1 - \alpha)^2 - b^2 - 2b(1 - \alpha - b)}{2(\beta - \alpha)\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \max\{\beta, 1 - \beta\} < b < 1 - \alpha, \\ \frac{-b}{\sqrt{\frac{2}{3}(1-b)^3}}, & \text{if } \beta < b < 1 - \beta. \end{cases}$$

We then immediately find that

$$\sup_{b \in [0, 1]} \Pi(b) = \begin{cases} \sqrt{\frac{4}{9(\alpha + \beta)}}, & \text{if } (\alpha; \beta) \in N_1, \\ \sqrt{3}(1 - \alpha - \beta), & \text{if } (\alpha; \beta) \in N_2, \\ \frac{2(1 - \alpha - \beta)}{3(\beta - \alpha)\sqrt{1 + \alpha - \beta}}, & \text{if } (\alpha; \beta) \in N_3, \\ 0, & \text{if } (\alpha; \beta) \in N_4, \end{cases}$$

where

$$\begin{aligned} N_1 &= \left(0, \frac{1}{6}\right) \times \left(\alpha, \frac{1-3\alpha}{3}\right), \\ N_2 &= \left(0, \frac{1}{2}\right) \times \left[\max\left\{\frac{1-3\alpha}{3}, \alpha\right\}, \min\left\{\frac{3\alpha+2}{3}, 1-\alpha\right\}\right), \\ N_3 &= \left(0, \frac{1}{6}\right) \times \left[\frac{3\alpha+2}{3}, 1-\alpha\right), N_4 = (0, 1) \times [\max\{1-\alpha, \alpha\}, 1). \end{aligned}$$

Thus, we obtain

$$\inf_{X \in V_{SU}(\mu, \sigma)} \text{RVaR}_{\alpha, \beta}[X] = \begin{cases} \mu - \sigma \sqrt{\frac{4}{9(\alpha+\beta)}}, & \text{if } (\alpha; \beta) \in N_1, \\ \mu - \sigma \sqrt{3(1-\alpha-\beta)}, & \text{if } (\alpha; \beta) \in N_2, \\ \mu - \sigma \frac{2(1-\alpha-\beta)}{3(\beta-\alpha)\sqrt{1+\alpha-\beta}}, & \text{if } (\alpha; \beta) \in N_3, \\ \mu, & \text{if } (\alpha; \beta) \in N_4. \end{cases} \quad (5.11)$$

In particular, letting $\beta \rightarrow 1$ in (5.11) we obtain

$$\inf_{X \in V_{SU}(\mu, \sigma)} \text{TVaR}_{\alpha}[X] = \mu.$$

6 Conclusions

In this work, we have obtained closed-form solutions for the extreme distortion risk measures, both worst-case and best-case, based on only the first two moments and shape information such as the symmetry/unimodality property of the underlying distribution. In addition, we have showed that the corresponding extreme-case distributions can be characterized by the envelopes of the distortion functions. Our results generalize several well-known extreme-case risk measures with closed-form solutions. Further, most of the established results can be generalized to signed Choquet integrals (see Wang et al (2020)) which we plan to carry out in our future research. We also plan to study the lower and upper bounds on DRM when, in addition to the moment constraints, the distributions in the uncertainty set lie within an $\sqrt{\varepsilon}$ -Wasserstein ball of the reference distribution F (see, e.g., Bernard et al. (2023)). We are currently working on those problem of interest and hope to report the finding in a future paper.

CRedit authorship contribution statement

Mengshuo Zhao: Investigation, Methodology, Writing–original draft.
Narayanaswamy Balakrishnan: Supervision, Methodology, Writing–review & editing.
Chuancun Yin: Conceptualization, Investigation, Methodology, Validation, Writing–original draft, Writing–review & editing.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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