SURFACES WITH CONCENTRIC OR PARALLEL K-CONTOURS

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Abstract. Surfaces with concentric K-contours and parallel K-contours in Euclidean 3-space are defined. Crucial examples are presented and characterization of them are given.

1. Introduction

The contours of the Gaussian curvature function K on the graph surface

$$z = \frac{x}{x^2 + y^2} \tag{1.1}$$

in the Euclidean 3-space $(\mathbb{R}^3; x, y, z)$ map to concentric circles on the xy-plane by orthogonal projection, so it would be permissible to say that the surface (1.1) has weak symmetry in some sense. We will refer to this property by saying a surface has *concentric K-contours*. We can immediately note that helicoidal surfaces have the same property. (Here a *helicoidal surface* is, by definition, a surface in $\mathbb{R}^3$ which is invariant under a one-parameter group of rigid screw motions; it is a generalization of both surfaces of revolution and right helicoids. A helicoidal surface is also called a *generalized helicoid* (cf. [1])). We also found that the surface called a *monkey saddle* has the same property. (See Section 22.2 in [2], where the monkey saddle appears as an example for which the converse of Gauss' Theorema Egregium does not hold.) In view of these circumstances, simple questions come to mind:

(i) Are there any surfaces with concentric K-contours other than (1.1), helicoidal surfaces or the monkey saddle?
(ii) Can we find all surfaces with concentric K-contours?

The authors searched the literature, but failed to find research on this.

One of our purposes is to provide a family of examples, denoted by $\boldsymbol{x}_{m,c}$ in this paper, which includes both (1.1) and the monkey saddle. Another purpose is to give a partial answer to the question (ii). In fact, under a certain assumption, any surface with concentric K-contours must be a surface $\boldsymbol{x}_{m,c}$ or a helicoidal surface (Theorem 2.4).

On the other hand, it has been an interesting problem to understand how much the behavior of the Gauss map determines the surface. For instance, Kenmotsu [4] showed a representation theorem for an arbitrary surface in $\mathbb{R}^3$ in terms of the Gauss map and the mean curvature function of the surface. In addition to this, Hoffman, Osserman and Schoen [3] proved that for a complete oriented surface of constant mean curvature in $\mathbb{R}^3$, if its Gauss image lies in some open hemisphere,

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then it is a plane; if the Gauss image lies in a closed hemisphere, then it is a plane or a right circular cylinder. In this paper, we will show that a behavior of the Gauss map, called *semi-rotational equivariance*, characterizes the surfaces $\boldsymbol{x}_{m,c}$ (Theorem 2.5).

This paper also reports on the case where concentric circles are replaced by parallel straight lines. We say that a surface has *parallel K-contours* if the contours of the Gaussian curvature function K produce parallel straight lines on a plane by orthogonal projection.

We refer to standard textbooks [1], [5], [6], etc, for fundamental facts about surface theory.

2. Surfaces with concentric K-contours

Throughout this paper, we shall use the following notation and assumption: M denotes a connected, smooth 2-manifold and $\boldsymbol{x}\colon M \to \mathbb{R}^3$ a smooth immersion. K denotes the Gaussian curvature function on M. We set $M_k := \{p \in M \mid K(p) = k\}$ for a real number k, and consider the family $\mathcal{C} := \{M_k\}_{k\in\mathbb{R}}$. It is always assumed that M *has no open subset where* $\operatorname{grad} K = 0$ because we wish to study the case where $\mathcal{C}$ is formed by a family of curves.

Definition 2.1. We say that $\boldsymbol{x}\colon M \to \mathbb{R}^3$ has *concentric K-contours* if there exists a plane in $\mathbb{R}^3$ such that the orthogonal projection $\pi\colon \mathbb{R}^3 \to P$ maps $\mathcal{C}$ to a family of concentric circles on P.

It is obvious that helicoidal surfaces have concentric K-contours.

2.1. A non-helicoidal example.

Let m be an integer not equal to $0, 1$, and let c be a non-zero real number. Consider a graph surface

$$\boldsymbol{x}_{m,c}(z) = (\operatorname{Re} z, \operatorname{Im} z, c\operatorname{Re}(z^m)) = \left(x, y, \frac{c}{2}\left\{(x+iy)^m + (x-iy)^m\right\}\right) \tag{2.1}$$

for $z = x + iy$. Note that $\boldsymbol{x}_{-1,1}$ and $\boldsymbol{x}_{3,1}$ coincide with the surface (1.1) and the monkey saddle, respectively. In terms of the polar coordinates $z = re^{i\theta}$, $\boldsymbol{x}_{m,c}$ is expressed as

$$\boldsymbol{x}_{m,c}(r,\theta) = (r\cos\theta, r\sin\theta, cr^m\cos m\theta). \tag{2.2}$$

See Figures 1 and 2. The first and second fundamental forms I, $\mathrm{I\!I}$ and a unit normal $\boldsymbol{n}$ are as follows:

$$\begin{aligned}
\mathrm{I} &= (1 + c^2m^2r^{2m-2}\cos^2 m\theta)dr^2 + 2(-c^2m^2r^{2m-1}\cos m\theta\sin m\theta)drd\theta \\
&\quad + (r^2 + c^2m^2r^{2m}\sin^2 m\theta)d\theta^2, \\
\boldsymbol{n}(r,\theta) &= \frac{1}{r\sqrt{1+c^2m^2r^{2m-2}}}\left(-cmr^m\cos(m-1)\theta, cmr^m\sin(m-1)\theta, r\right), \qquad (2.3)\\
\mathrm{I\!I} &= \frac{cm(m-1)}{r\sqrt{1+c^2m^2r^{2m-2}}}\left\{r^{m-1}\cos m\theta dr^2 - 2r^m\sin m\theta drd\theta - r^{m+1}\cos m\theta d\theta^2\right\}.
\end{aligned}$$

From these, the Gaussian curvature K and the mean curvature H are

$$\begin{aligned}
K = K(r) &= -\frac{c^2m^2(m-1)^2r^{2m-4}}{(1+c^2m^2r^{2m-2})^2}, \qquad (2.4)\\
H = H(r,\theta) &= -\frac{c^3m^3(m-1)r^{3m-4}\cos m\theta}{2(1+c^2m^2r^{2m-2})^{3/2}}.
\end{aligned}$$

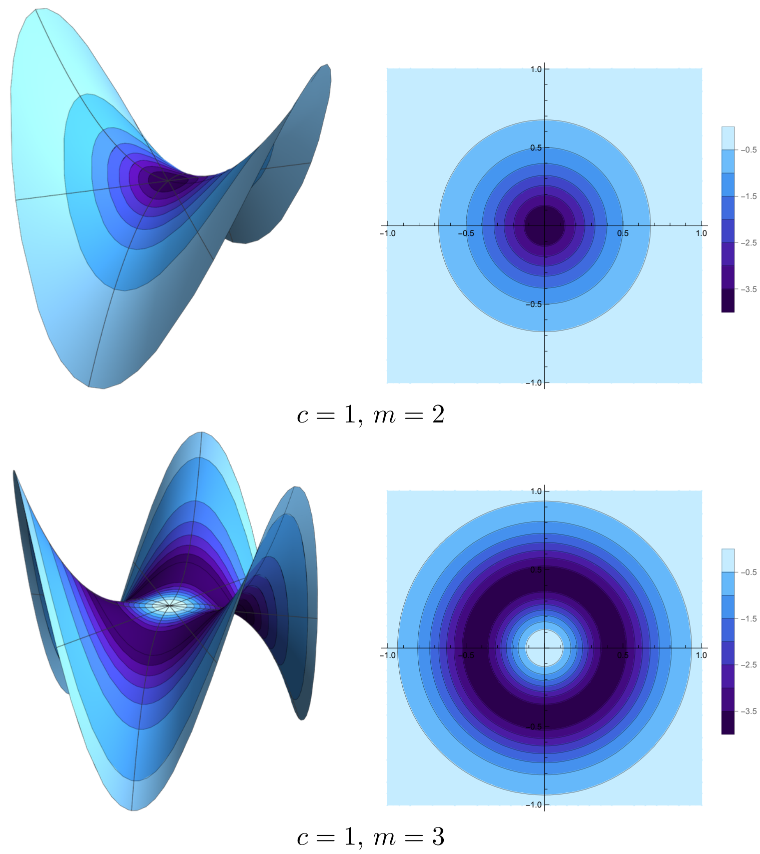


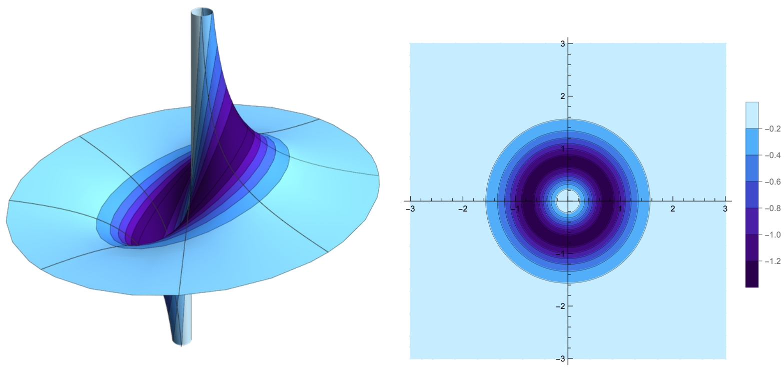
FIGURE 1. The surfaces $\boldsymbol{x}_{m,c}$ (left) and their projections to the xy-plane (right) with positive integers m. Each surface is gray-scaled by its Gaussian curvature.

It follows directly from (2.4) that $\boldsymbol{x}_{m,c}$ has concentric K-contours with respect to the xy-plane. Note that the first fundamental form I does not have rotational symmetry but the Gaussian curvature K does.

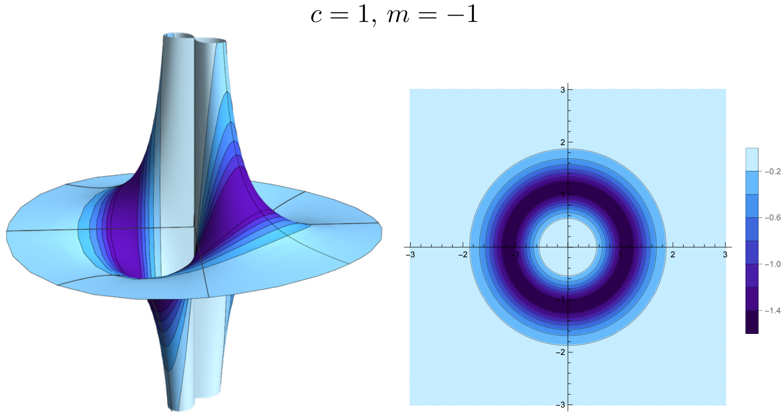
Remark 2.2. (1) It follows from (2.1) that $\boldsymbol{x}_{m,c}$ is an entire graph over the xy-plane if m is a positive integer. In particular, $\boldsymbol{x}_{m,c}$ is a hyperbolic paraboloid if $m = 2$ and a monkey saddle if $m = 3$. In the case where m is a negative integer, $\boldsymbol{x}_{m,c}$ is a graph punctured at the origin.

(2) Although $\boldsymbol{x}_{m,c}$ can be defined for $m = 0, 1$ or $c = 0$, it is a plane hence has constant Gaussian curvature zero. Therefore we exclude the case $m = 0, 1$ and the case $c = 0$.

It follows from (2.2) that $\boldsymbol{x}_{m,c}$ can be defined even if m is a non-integer as a multi-valued graph over $\mathbb{R}^2 \setminus \{(0,0)\}$ or a surface defined on the universal cover. See Figure 3. From now on, we assume that *the number m for $\boldsymbol{x}_{m,c}$ does not have to be an integer, that is, $m \in \mathbb{R} \setminus \{0, 1\}$.*



$c = 1,\ m = -1$



$c = 1,\ m = -2$

FIGURE 2. The surfaces $\boldsymbol{x}_{m,c}$ (left) and their projections to the xy-plane (right) with negative integers m. Each surface is gray-scaled by its Gaussian curvature.

2.2. **Semi-rotational equivariance.** We also call the unit normal (2.3) the *Gauss map* of $\boldsymbol{x}_{m,c}$ according to custom. One can see from (2.3) that

$$\boldsymbol{n}(r, \theta + \alpha) = \mathcal{R}_{(1-m)\alpha} \circ \boldsymbol{n}(r, \theta),$$

where $\mathcal{R}_{(1-m)\alpha}$ denotes the rotation of angle $(1-m)\alpha$ with respect to the z-axis. Focusing on this property, we give the following definition:

Definition 2.3. A surface $\boldsymbol{x}\colon M \to \mathbb{R}^3$ is said to have *semi-rotational Gauss map* if there exist a straight line $l \subset \mathbb{R}^3$, a plane $P \subset \mathbb{R}^3$, and a 1-parameter group $\{\phi_t\}$ of diffeomorphisms of M such that

(1) l is orthogonal to P,
(2) $\pi \circ \boldsymbol{x} \circ \phi_t = R_t \circ \pi \circ \boldsymbol{x}$, and
(3) $\boldsymbol{n} \circ \phi_t = \mathcal{R}_{kt} \circ \boldsymbol{n}$ for some constant k

with a suitable choice of orientations of l and P, where $\pi\colon \mathbb{R}^3 \to P$ is orthogonal projection, R_t denotes a rotation on P of angle t with the center $P \cap l$, and $\mathcal{R}_{kt}$ denotes a rotation in $\mathbb{R}^3$ of angle kt with respect to the axis l.

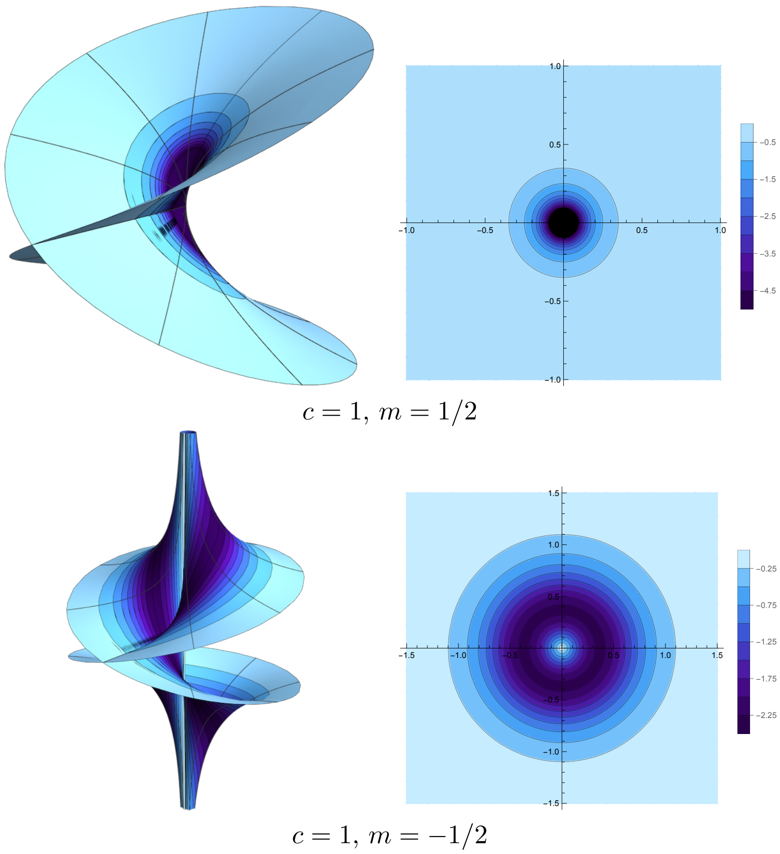


FIGURE 3. The surfaces $\boldsymbol{x}_{m,c}$ (left) and their projections to the xy-plane (right) with non integers m. Each surface is gray-scaled by its Gaussian curvature.

Note that a helicoidal surface has semi-rotational Gauss map with $k = 1$, which should be said to have *rotational* Gauss map. So we shall use the term *'strictly semi-rotational'* in the sense of 'semi-rotational but not rotational'.

2.3. Characterizations of the surface $\boldsymbol{x}_{m,c}$.

Theorem 2.4. *Let $\boldsymbol{x}\colon M \to \mathbb{R}^3$ be a surface with concentric K-contours. If the area element dA is invariant along each K-contour, then $\boldsymbol{x}$ is a helicoidal surface or locally congruent to a surface $\boldsymbol{x}_{m,c}$ for some m, c.*

Theorem 2.5. *Let a surface $\boldsymbol{x}\colon M \to \mathbb{R}^3$ have semi-rotational Gauss map. Then $\boldsymbol{x}$ is a helicoidal surface or locally congruent to a surface $\boldsymbol{x}_{m,c}$ for some m, c.*

Corollary 2.6. *Let a surface $\boldsymbol{x}\colon M \to \mathbb{R}^3$ have strictly semi-rotational Gauss map. Then $\boldsymbol{x}$ is locally congruent to a surface $\boldsymbol{x}_{m,c}$ for some m, c.*

Before proving the theorems above, we write down formulas for the area element dA, the Gaussian curvature K and the unit normal field $\boldsymbol{n}$ for a surface $\boldsymbol{x}(r, \theta) =$

$(r\cos\theta, r\sin\theta, F(r,\theta))$:

$$dA = \varDelta\, dr \wedge d\theta, \tag{2.5}$$

$$\boldsymbol{n} = \frac{1}{\varDelta}\left(F_\theta \sin\theta - rF_r\cos\theta,\ -rF_r\sin\theta - F_\theta\cos\theta,\ r\right), \tag{2.6}$$

$$K = \frac{1}{\varDelta^4}\left\{r^2F_{rr}(rF_r + F_{\theta\theta}) - (F_\theta - rF_{r\theta})^2\right\}, \tag{2.7}$$

where

$$\varDelta = \sqrt{r^2 + r^2F_r^2 + F_\theta^2}. \tag{2.8}$$

Proof of Theorem 2.4. Considering a rigid motion in $\mathbb{R}^3$, we may assume that the plane P is the xy-plane and K-contours draw concentric circles with the center $(0,0)$ in the xy-plane. $\boldsymbol{x}$ is at least locally re-parameterized as $\boldsymbol{x}(r,\theta) = (r\cos\theta, r\sin\theta, F(r,\theta))$. The function $\varDelta$ is of one variable r because of (2.5) and the assumption of invariance of dA. It follows from (2.8) that $r^2F_r^2 + F_\theta^2$ is also a function of one variable r. Therefore, there exist functions $\alpha = \alpha(r)$, $\beta = \beta(r,\theta)$ such that

$$rF_r = \alpha\cos\beta,\ \ F_\theta = \alpha\sin\beta. \tag{2.9}$$

By differentiating (2.9), we have

$$r^2F_{rr} = (r\alpha' - \alpha)\cos\beta - r\alpha\beta_r\sin\beta, \tag{2.10}$$

$$r(F_r)_\theta = -\alpha\sin\beta\cdot\beta_\theta, \tag{2.11}$$

$$(F_\theta)_r = \alpha'\sin\beta + \alpha\cos\beta\cdot\beta_r, \tag{2.12}$$

$$F_{\theta\theta} = \alpha\cos\beta\cdot\beta_\theta. \tag{2.13}$$

It follows from (2.11), (2.12) that the equality $(F_r)_\theta = (F_\theta)_r$ turns out to be

$$\frac{\beta_\theta}{r} + \beta_r\frac{\cos\beta}{\sin\beta} = -\frac{\alpha'}{\alpha}. \tag{2.14}$$

Note that the right side of (2.14) is of one variable r, so the left side is as well. Thus

$$\frac{\partial}{\partial\theta}\left(\frac{\beta_\theta}{r} + \beta_r\frac{\cos\beta}{\sin\beta}\right) = 0. \tag{2.15}$$

On the other hand, using (2.10)–(2.13), we can rewrite (2.7) as

$$K = \frac{\alpha(1+\beta_\theta)(r\alpha' - \alpha)}{\varDelta^4}.$$

Here, K must be a non-constant function of one variable r by the assumption of concentric K-contours. It implies that β_θ is a function of one variable r. Therefore, we may set $\beta_\theta = \phi(r)$ and hence

$$\beta = \phi(r)\cdot\theta + \psi(r) \tag{2.16}$$

for some functions $\phi(r)$, $\psi(r)$. It follows from (2.15) with (2.16) that

$$\phi'(r)\{\frac{1}{2}\sin 2\beta - \phi(r)\cdot\theta\} + \psi'(r)\phi(r) = 0.$$

This implies that (i) $\frac{1}{2}\sin 2\beta - \phi(r)\cdot\theta$ is independent of θ or (ii) $\phi'(r) = 0$. In the case (i), by differentiating $\frac{1}{2}\sin 2\beta - \phi(r)\cdot\theta$ by θ, we have $(\cos 2\beta - 1)\phi(r) = 0$, that is,

$$\beta = n\pi \text{ for some integer } n \text{ or } \beta = \psi(r). \tag{2.17}$$

In the case (ii), the function ϕ is constant and $\psi'\phi = 0$. Therefore $(\phi, \psi) = (k, l)$ for some constants k, l or $(\phi, \psi) = (0, \psi(r))$; in other words,

$$\beta = k\theta + l \text{ or } \beta = \psi(r). \tag{2.18}$$

Since the condition (2.18) includes the condition (2.17), we continue to discuss under the condition (2.18).

In the case where $\beta = k\theta + l$, the equation (2.14) reduces to $\alpha' = -k\alpha/r$. Hence we have $\alpha = Cr^{-k}$ for some constant C. It follows from (2.9) that $F = C_1 r^k \cos(k\theta + l) + C_2$ for some constants C_1, C_2. Thus the surface $\boldsymbol{x}$ is congruent to $\boldsymbol{x}_{k,C_1}$.

In the case where $\beta = \psi(r)$, the equation (2.14) reduces to $\psi' \cot \psi = \alpha'/\alpha$. It is solved as $\alpha \sin \psi = C$ for some constant C. The system of equations (2.9) turns out to be

$$rF_r = \alpha(r) \cos(\psi(r)), \quad F_\theta = C.$$

Therefore, we obtain

$$F = C\theta + \int \frac{\alpha(r)}{r} \cos(\psi(r)) dr = C\theta + A(r)$$

for some function $A(r)$. Thus the surface $\boldsymbol{x}$ is helicoidal. $\square$

Proof of Theorem 2.5. Considering a rigid motion in $\mathbb{R}^3$, we may assume that the plane P is the xy-plane and the line l is the z-axis. The surface $\boldsymbol{x}$ is at least locally re-parameterized as $\boldsymbol{x}(r, \theta) = (r\cos\theta, r\sin\theta, F(r, \theta))$. The Gauss map (2.6) is

$$\boldsymbol{n} = \frac{1}{\Delta} \begin{pmatrix} F_\theta \sin\theta - rF_r \cos\theta \\ -rF_r \sin\theta - F_\theta \cos\theta \\ r \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\Delta} \begin{pmatrix} -rF_r \\ -F_\theta \\ r \end{pmatrix}$$

in the column-vector form. Since $\boldsymbol{n}$ is semi-rotational,

(i) the vector-valued function

$$\frac{1}{\Delta} \left(-rF_r, -F_\theta, r\right) \tag{2.19}$$

is of one variable r, or

(ii) there exist $m \in \mathbb{R}$ and $\phi_1 = \phi_1(r), \psi = \psi(r)$ such that

$$\frac{1}{\Delta} \left(-rF_r, -F_\theta, r\right) = \left(\phi_1(r) \cos m\theta, \phi_1(r) \sin m\theta, \psi(r)\right).$$

In the case (i), each component of (2.19) is of one variable r. Hence, Δ, F_r and F_θ are functions of one variable r. This implies that F must be of the form $F = a\theta + \psi(r)$ for a constant a and a function $\psi(r)$. Therefore $\boldsymbol{x}(r, \theta) = (r\cos\theta, r\sin\theta, a\theta + \psi(r))$, that is, $\boldsymbol{x}$ is a helicoidal surface.

In the case (ii), the third component of (2.19) is of one variable r. Hence, Δ is a function of one variable r. Setting $-\phi_1(r) \cdot \Delta = \varphi(r)$, we have

$$\begin{cases} F_r = \frac{\varphi(r)}{r} \cos m\theta \\ F_\theta = \varphi(r) \sin m\theta. \end{cases} \tag{2.20}$$

Thus the equality $(F_r)_\theta = (F_\theta)_r$ turns out to be

$$-m\frac{\varphi(r)}{r} \sin m\theta = \varphi'(r) \sin m\theta. \tag{2.21}$$

In the case where $m = 0$, the system of equations (2.20) turns out to be $F_r = \varphi(r)/r, F_\theta = 0$. Therefore, $F = F(r)$. This implies that $\boldsymbol{x}$ is a rotational surface.

In the case where $m \neq 0$, the equation (2.21) leads to $-m\frac{\varphi(r)}{r} = \varphi'(r)$. Therefore, $\varphi(r) = Cr^{-m}$ (C is a constant.) Then the solution to the system of equations (2.20) is

$$F(r, \theta) = C_1 r^{-m} \cos m\theta + C_2 \quad (C_1, C_2 \text{ are constants}).$$

Thus $\boldsymbol{x}$ is congruent to $\boldsymbol{x}_{-m,C_1}$. $\square$

3. Surfaces with parallel K-contours

We shall discuss here using the same notations and assumption as in Section 2.

Definition 3.1. We say that a surface $\boldsymbol{x} \colon M \to \mathbb{R}^3$ has *parallel K-contours* if there exists a plane in $\mathbb{R}^3$ such that the orthogonal projection $\pi \colon \mathbb{R}^3 \to P$ maps $\mathcal{C}$ to a family of parallel straight lines on P.

3.1. **An example.**

Let k, c be non-zero real numbers. Consider a graph surface of

$$z = ce^{kx} \cos ky,$$

that is,

$$\boldsymbol{p}_{k,c}(x, y) = \left(x, y, ce^{kx} \cos ky\right).$$

See Figure 4. The first and second fundamental forms I, II and a unit normal $\boldsymbol{n}$

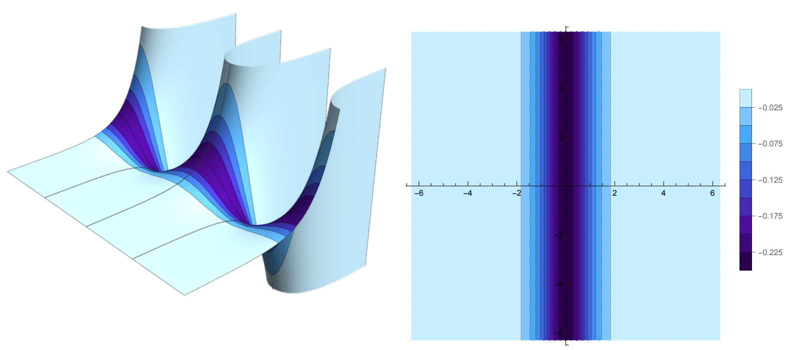


Figure 4. The surface $\boldsymbol{p}_{k,c}$ (left) and its projection to the xy-plane (right) for $c = 1$ and $k = 1$. The surface is gray-scaled by its Gaussian curvature.

are as follows:

$$\begin{aligned} \mathrm{I} &= (1 + c^2k^2e^{2kx} \cos^2 ky)dx^2 - 2c^2k^2e^{2kx} \cos ky \sin ky\, dxdy \\ &\quad + (1 + c^2k^2e^{2kx} \sin^2 ky)dy^2, \\ \boldsymbol{n}(x, y) &= \frac{1}{\sqrt{1 + c^2k^2e^{2kx}}} \left(-cke^{kx} \cos ky, cke^{kx} \sin ky, 1\right), \\ \mathrm{II} &= \frac{ck^2e^{kx}}{\sqrt{1 + c^2k^2e^{2kx}}} \left(\cos ky\, dx^2 - 2 \sin ky\, dxdy - \cos ky\, dy^2\right). \end{aligned}$$

From these, the Gaussian curvature K and the mean curvature H are

$$\begin{aligned} K = K(x) &= -\frac{c^2k^4e^{2kx}}{(1+c^2k^2e^{2kx})^2}, \\ H = H(x,y) &= -\frac{c^3k^4e^{3kx}\cos ky}{2(1+c^2k^2e^{2kx})^{3/2}}. \end{aligned} \tag{3.1}$$

It follows directly from (3.1) that $\boldsymbol{p}_{k,c}$ has parallel K-contours.

3.2. **Characterizations of the surface $\boldsymbol{p}_{k,c}$.** An assertion similar to Theorem 2.4 holds for surfaces with parallel K-contours:

Theorem 3.2. *Let $\boldsymbol{x}\colon M \to \mathbb{R}^3$ be a surface with parallel K-contours. If the area element dA is invariant along each K-contour, then $\boldsymbol{x}$ is locally congruent to a surface $\boldsymbol{p}_{k,c}$ for some k, c.*

We omit the proof because it is quite similar to that of Theorem 2.4 by discussing about a graph surface $(x, y, F(x, y))$.

As well as a surface $\boldsymbol{x}_{m,c}$ in Section 2, the Gauss map $\boldsymbol{n}$ of a surface $\boldsymbol{p}_{k,c}$ satisfies the following property:

$$\boldsymbol{n}(x, y+\alpha) = \mathcal{R}_{-k\alpha} \circ \boldsymbol{n}(x, y),$$

where $\mathcal{R}_{-k\alpha}$ denotes the rotation of angle $-k\alpha$ with respect to the z-axis. Focusing on this property, we give the following definition:

Definition 3.3. An immersed surface $\boldsymbol{x}\colon M \to \mathbb{R}^3$ is said to have *quasi-rotational Gauss map* if there exist a straight line $l \subset \mathbb{R}^3$, a plane $P \subset \mathbb{R}^3$, a vector $\boldsymbol{v}$ parallel to P, and a 1-parameter group $\{\phi_t\}$ of diffeomorphisms of M such that

(1) l is orthogonal to P,
(2) $\pi \circ \boldsymbol{x} \circ \phi_t = T_{t\boldsymbol{v}} \circ \pi \circ \boldsymbol{x}$, and
(3) $\boldsymbol{n} \circ \phi_t = \mathcal{R}_{kt} \circ \boldsymbol{n}$ for some constant k

with a suitable choice of orientations of l and P, where $\pi\colon \mathbb{R}^3 \to P$ is the orthogonal projection, $T_{t\boldsymbol{v}}$ denotes a parallel translation on P of the translation vector $t\boldsymbol{v}$, and $\mathcal{R}_{kt}$ denotes a rotation in $\mathbb{R}^3$ of angle kt with respect to the axis l.

Note that a cylindrical surface has quasi-rotational Gauss map with $k = 0$, however it should be said to have *parallel* Gauss map. So we shall use the term *'strictly quasi-rotational'* in the sense of 'quasi-rotational but not parallel'.

An assertion similar to Corollary 2.6 holds for surfaces with strictly quasi-rotational Gauss map.

Theorem 3.4. *Let $\boldsymbol{x}\colon M \to \mathbb{R}^3$ be a surface with strictly quasi-rotational Gauss map. Then $\boldsymbol{x}$ is locally congruent to a surface $\boldsymbol{p}_{k,c}$ for some k, c.*

We omit the proof because it is quite similar to that of Theorem 2.5 by discussing about a graph surface $(x, y, F(x, y))$.

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