SUMS OF THE TRIPLE DIVISOR FUNCTION OVER VALUES OF SOME QUADRATIC FORMS

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Abstract. Let $\tau_3(n)$ be the triple divisor function. It is proved that

$$\sum_{1\leq n_1,n_2,n_3\leq\sqrt{x}}\tau_3(n_1^2+n_2^2+n_3^2)=c_1x^{\frac{3}{2}}(\log x)^2+c_2x^{\frac{3}{2}}\log x+c_3x^{\frac{3}{2}}+O_\varepsilon(x^{\frac{13}{10}+\varepsilon})$$

for some constants c_1, c_2 and c_3, updating a result of the second author and Zhang. Moreover, we show that

$$\sum_{1\leq n_1,n_2,n_3,n_4\leq\sqrt{x}}\tau_3(n_1^2+n_2^2+n_3^2+n_4^2)=c_4x^2(\log x)^2+c_5x^2\log x+c_6x^2+O_\varepsilon\left(x^{\frac{5}{3}+\varepsilon}\right)$$

for some constants c_4, c_5 and c_6, which improves the results in [6].

1. Introduction

Let $\tau_k(n)$ be the k-th divisor function which is the number of the solutions of the equation $d_1d_2\cdots d_k=n$ in natural numbers. Sums of $\tau_k(n)$ over values of quadratic forms have been studied by many authors for $k=2$, i.e., the usual divisor function $\tau_2(n)=d(n)$ (see Hooley [5], Yu [12], Calderón and de Velasco [2], Guo and Zhai [4], Zhao [13] and so on). Sums of the triple divisor function over special values are also very interesting. For example, Friedlander and Iwaniec [3] showed that, for $x\geq 3$,

$$\sum_{\substack{n_1^2+n_2^6\leq x\\(n_1,n_2)=1}}\tau_3(n_1^2+n_2^6)=cx^{\frac{2}{3}}(\log x)^2+O\left(x^{\frac{2}{3}}(\log x)^{\frac{7}{4}}(\log\log x)^{\frac{1}{2}}\right),$$

where c is a constant. For interesting results involving k-th divisor function over special values, see Blomer [1], Lapkova and Zhou [8] and the references therein.

In [10], the second author and Zhang considered sums of triple divisor function over values of a ternary quadratic form, and proved that

$$\sum_{1\leq n_1,n_2,n_3\leq\sqrt{x}}\tau_3(n_1^2+n_2^2+n_3^2)=c_1x^{\frac{3}{2}}(\log x)^2+c_2x^{\frac{3}{2}}\log x+c_3x^{\frac{3}{2}}+O_\varepsilon(x^{\frac{11}{8}+\varepsilon}),\qquad(1.1)$$

where, throughout the paper, c_j, $j=1,2,3\cdots$, denote absolute constants. By adapting the method of [10], Hu and Yang [6] studied the case of average values of $\tau_3(n)$ over a

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quaternary quadratic form and proved that

$$\begin{aligned}
&\sum_{1\le n_1,n_2,n_3,n_4\le\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2+n_4^2) \\
&= c_4x^2(\log x)^2+c_5x^2\log x+c_6x^2+O_\varepsilon\left(x^{\frac{7}{4}+\varepsilon}\right). \qquad (1.2)
\end{aligned}$$

The main purpose of this paper is to improve the O-terms in (1.1) and (1.2). To state our results, we introduce some notations. Let $\overline{a}$ denote the multiplicative inverse of a mod q, $S(a,b;c)$ denotes the classical Kloosterman sum,

$$G(a,b;q)=\sum_{d \bmod q} e\Big(\frac{ad^2+bd}{q}\Big)$$

is the Gauss sum and

$$P_0(n,q) = 1, \qquad P_1(n,q)=\frac{5}{3}\log n-3\log q+3\gamma-\frac{1}{3\tau(n)}\sum_{d|n}\log d, \qquad (1.3)$$

$$\begin{aligned}
P_2(n,q) &= (\log n)^2-5\log q\log n+\frac{9}{2}(\log q)^2+3\gamma^2-3\gamma_1+7\gamma\log n-9\gamma\log q \\
&\quad +\tau(n)^{-1}\Big((\log n+\log q-5\gamma)\sum_{d|n}\log d-\frac{3}{2}\sum_{d|n}(\log d)^2\Big) \qquad (1.4)
\end{aligned}$$

with γ and γ_1 being Euler's constant and Stieltjes' constant, respectively. We have the following results.

Theorem 1.1. *For any $x\ge x_1$ (x_1 is a large constant) and any $\varepsilon>0$, we have*

$$\begin{aligned}
\sum_{1\le n_1,n_2,n_3\le\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2) &= \frac{\mathcal{C}_0\mathcal{J}_0}{4}x^{\frac{3}{2}}(\log x)^2+\frac{1}{2}\left(\mathcal{C}_1\mathcal{J}_0+\mathcal{C}_0\mathcal{J}_1\right)x^{\frac{3}{2}}\log x \\
&\quad +\frac{1}{2}\Big(\mathcal{C}_2\mathcal{J}_0+\mathcal{C}_1\mathcal{J}_1+\frac{1}{2}\mathcal{C}_0\mathcal{J}_2\Big)x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{13}{10}+\varepsilon}\right),
\end{aligned}$$

where for $\ell=0,1,2$,

$$\mathcal{J}_\ell=\int_{-\infty}^{\infty}\Big(\int_0^3(\log u)^\ell e(-\beta u)\mathrm{d}u\Big)\Big(\int_0^1 e(\beta v^2)\mathrm{d}v\Big)^3\mathrm{d}\beta$$

and

$$\mathcal{C}_\ell=\sum_{q=1}^{\infty}q^{-5}\sum_{n|q}n\tau(n)P_\ell(n,q)\sum_{\substack{a=1\\(a,q)=1}}^{q}G(a,0;q)^3S(-\overline{a},0;q).$$

Here $P_j(n,q)$ ($j=0,1,2$) are given in (1.3)-(1.4).

Theorem 1.2. *For any $x \geq x_2$ (x_2 is a large constant) and any $\varepsilon > 0$, we have*

$$\begin{aligned}\sum_{1\leq n_1,n_2,n_3,n_4\leq\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2+n_4^2) &= \frac{\mathfrak{C}_0\mathfrak{J}_0}{4}x^2(\log x)^2 + \frac{1}{2}\left(\mathfrak{C}_1\mathfrak{J}_0+\mathfrak{C}_0\mathfrak{J}_1\right)x^2\log x \\ &\quad + \frac{1}{2}\Big(\mathfrak{C}_2\mathfrak{J}_0+\mathfrak{C}_1\mathfrak{J}_1+\frac{1}{2}\mathfrak{C}_0\mathfrak{J}_2\Big)x^2 + O_\varepsilon\left(x^{\frac{5}{3}+\varepsilon}\right),\end{aligned}$$

where for $\ell = 0, 1, 2$,

$$\mathfrak{J}_\ell = \int_{-\infty}^{\infty}\left(\int_0^4 (\log u)^\ell e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta v^2)\mathrm{d}v\right)^4 \mathrm{d}\beta$$

and

$$\mathfrak{C}_\ell = \sum_{q=1}^{\infty} q^{-6}\sum_{n|q} n\tau(n)P_\ell(n,q)\sum_{\substack{a=1\\(a,q)=1}}^{q} G(a,0;q)^4 S(-\overline{a},0;q).$$

Here $P_j(n,q)$ ($j = 0, 1, 2$) are given in (1.3)-(1.4).

Obviously the asymptotic formulas in Theorem 1.1-1.2 improve (1.1) and (1.2), respectively. Similar asymptotic formulas can be derived for the modified and in some sense simpler sums

$$\sum_{1\leq n_1^2+n_2^2+\cdots+n_\ell^2\leq x} \tau_3(n_1^2+n_2^2+\cdots+n_\ell^2) = \sum_{1\leq n\leq x}\tau_3(n)r_\ell(n)$$

where for $\ell = 3, 4$,

$$r_\ell(n) = \#\left\{(n_1,n_2,\ldots,n_\ell)\in\mathbb{Z}^3 : n_1^2+n_2^2+\cdots+n_\ell^2 = n\right\}.$$

We have the following results.

Theorem 1.3. *For any $x \geq x_1$ (with x_1 as in Theorem 1.1) and any $\varepsilon > 0$, we have*

$$\begin{aligned}\sum_{\substack{1\leq n_1^2+n_2^2+n_3^2\leq x\\(n_1,n_2,n_3)\in\mathbb{Z}^3}} \tau_3(n_1^2+n_2^2+n_3^2) &= 2\mathcal{C}_0\mathcal{K}_0x^{\frac{3}{2}}(\log x)^2 + 4\big(\mathcal{C}_1\mathcal{K}_0+\mathcal{C}_0\mathcal{K}_1\big)x^{\frac{3}{2}}\log x \\ &\quad + 4\Big(\mathcal{C}_2\mathcal{K}_0+\mathcal{C}_1\mathcal{K}_1+\frac{1}{2}\mathcal{C}_0\mathcal{K}_2\Big)x^{\frac{3}{2}} + O_\varepsilon\Big(x^{\frac{13}{10}+\varepsilon}\Big),\end{aligned}$$

where C_ℓ's are as in Theorem 1.1 and

$$\mathcal{K}_\ell = \int_{-\infty}^{\infty}\Big(\int_0^1 (\log u)^\ell e(-\beta u)\mathrm{d}u\Big)\Big(\int_0^1 e(\beta v^2)\mathrm{d}v\Big)^3\mathrm{d}\beta.$$

Theorem 1.4. *For any $x \geq x_2$ (with x_2 as in Theorem 1.2) and any $\varepsilon > 0$, we have*

$$\begin{aligned}\sum_{\substack{1\leq n_1^2+n_2^2+n_3^2++n_4^2\leq x\\(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4}} \tau_3(n_1^2+n_2^2+n_3^2+n_4^2) &= 2\mathfrak{C}_0\mathfrak{K}_0x^2(\log x)^2 + 4\left(\mathfrak{C}_1\mathcal{K}_0+\mathfrak{C}_0\mathfrak{K}_1\right)x^2\log x \\ &\quad + 4\Big(\mathfrak{C}_2\mathcal{K}_0+\mathfrak{C}_1\mathfrak{K}_1+\frac{1}{2}\mathfrak{C}_0\mathfrak{K}_2\Big)x^2 + O_\varepsilon\left(x^{\frac{5}{3}+\varepsilon}\right),\end{aligned}$$

where $\mathfrak{C}_\ell$'s are as in Theorem 1.2 and

$$\mathfrak{K}_\ell = \int_{-\infty}^{\infty} \Big(\int_0^1 (\log u)^\ell e(-\beta u) \mathrm{d}u \Big) \Big(\int_0^1 e(\beta v^2) \mathrm{d}v \Big)^4 \mathrm{d}\beta.$$

The asymptotic formulas in Theorem 1.3-1.4 improve Theorem 1.2 in [10] and Theorem 1.2 in [6], respectively.

We end the introduction by an outline of the proof of Theorem 1.1. As in [10] we first reduce the long sum into (smooth) sums over dyadic intervals in Section 2. To deal with the smooth sums, instead of the Hardy-Littlewood-Kloosterman circle method used in [10], we apply the classical circle method which in fact lowers the conductor of the circle method by treating the contribution from the minor arcs separately by Weyl's inequality and Hua's lemma (see Lemmas 2.4 and 2.5 in [11]). For the contribution from the major arcs (now with a conductor less than square-root of the length of the sum) we proceed as in [10] by applying the Voronoi formula for $\tau_3(n)$ and an asymptotic formula for the exponential sum $\sum_{n\leq\sqrt{x}} e(\alpha n^2)$ to express the aimed sum as 16 sums, which can be estimated similarly as in [10].

2. Proof of Theorem 1.1

Let $\mathcal{V}$ denote the set $[1, \sqrt{x}] \cap \mathbb{Z}$ and $r_3^*(n) = \sum_{\substack{n_1^2+n_2^2+n_3^2=n \\ (n_1,n_2,n_3)\in\mathcal{V}^3}} 1$. Following [10], by dyadic subdivision, we decompose the aimed sum into partial sums

$$\sum_{1\leq n_1,n_2,n_3\leq\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2) = \sum_{j\geq1} \sum_{3x/2^j<n\leq 3x/2^{j-1}} \tau_3(n) r_3^*(n), \tag{2.1}$$

where the inner sum in (2.1) vanishes for $j > \log 6x/\log 2$. Thus we are treating $O(\log x)$ sums of the form

$$\sum_{X_j/2<n\leq X_j} \tau_3(n) r_3^*(n),$$

where $X_j = 3x/2^{j-1}$. Let $\phi(y)$ be a smooth function supported on $[1/2, 1]$, identically equal 1 on $[1/2 + M^{-1}, 1 - M^{-1}]$ with $M > 4$, and satisfy $\phi^{(\ell)}(y) \ll_\ell M^\ell$ for any integer $\ell \geq 0$. Using the bound $r_3^*(n) \leq n^{\frac{1}{2}+\varepsilon}$, we have

$$\sum_{X_j/2<n\leq X_j} \tau_3(n) r_3^*(n) = \sum_{n\geq1} \tau_3(n) r_3^*(n) \phi(n/X_j) + O_\varepsilon(X_j^{\frac{3}{2}+\varepsilon} M^{-1}). \tag{2.2}$$

So it remains to study the smoothed sum

$$\mathscr{S}(X) = \sum_{n\geq1} \tau_3(n) r_3^*(n) \phi(n/X).$$

We have the following result.

Proposition 2.1. *For any $1 < X \le 3x$ and any $\varepsilon > 0$, we have*

$$\mathscr{S}(X) \;=\; \frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0 x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}M^{\frac{1}{4}} + x^{\frac{3}{2}+\varepsilon}M^{-1}\right),$$

where for $\ell = 0, 1, 2$,

$$\mathcal{C}_\ell = \sum_{q=1}^{\infty} q^{-5} \sum_{n|q} n\tau(n)P_\ell(n,q) \sum_{a=1}^{q}{}^{*}\, G(a,0;q)^3 S\left(-\overline{a},0;qn^{-1}\right),$$

and

$$\mathcal{I}_\ell(X) = \int_{-\infty}^{\infty}\left(\int_{X/2}^{X} e(-\beta u)(\log u)^\ell \mathrm{d}u\right)\left(\int_0^1 e(\beta x v^2)\mathrm{d}v\right)^3 \mathrm{d}\beta.$$

We set the proof of Proposition 2.1 aside and continue the derivation of Theorem 1.1. Applying Proposition 2.1 to the sum on the right side of (2.2) and taking $M = x^{\frac{1}{5}}$ we get

$$\sum_{X_j/2<n\le X_j} \tau_3(n)r_3^*(n) = \frac{1}{2}\mathcal{I}_0(X_j)\mathcal{C}_2 x^{\frac{3}{2}} + \frac{1}{2}\mathcal{I}_1(X_j)\mathcal{C}_1 x^{\frac{3}{2}} + \frac{1}{4}\mathcal{I}_2(X_j)\mathcal{C}_0 x^{\frac{3}{2}} + O_\varepsilon\left(x^{\frac{13}{10}+\varepsilon}\right). \tag{2.3}$$

Then by the same arguments as in Section 2 of [10] (comparing (2.3) with (2.4) in [10]), we obtain Theorem 1.1.

2.1. **Proof of Proposition 2.1.** We will prove Porposition 2.1 by the classical circle method. Recall

$$\mathscr{S}(X) = \sum_{n\ge1} \tau_3(n) r_3^*(n)\phi\left(\frac{n}{X}\right),$$

where $r_3^*(n) = \sum_{\substack{n_1^2+n_2^2+n_3^2=n \\ (n_1,n_2,n_3)\in\mathcal{V}^3}} 1$, $\mathcal{V} = [1, \sqrt{x}] \cap \mathbb{Z}$. For any $\alpha \in \mathbb{R}$, define

$$\mathscr{F}(\alpha) = \sum_{n\in\mathcal{V}} e(\alpha n^2), \qquad \mathscr{G}(\alpha) = \sum_{n\ge1} \tau_3(n)e(-\alpha n)\phi\left(\frac{n}{X}\right). \tag{2.4}$$

By the orthogonality relation

$$\textstyle\int_0^1 e(n\alpha)d\alpha = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{Z}\backslash\{0\}, \end{cases}$$

we have

$$\mathscr{S}(X) = \int_0^1 \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)d\alpha.$$

Let Q be a parameter to be chosen later. Note that $\mathscr{F}^3(\alpha)\mathscr{G}(\alpha)$ is a periodic function of period 1, one further has

$$\mathscr{S}(X) = \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)d\alpha.$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1+1/Q]$ can be written in the form

$$\alpha = \frac{a}{q} + \beta, \qquad |\beta| \le \frac{1}{qQ},$$

where a, q are some integers satisfying $1 \le a \le q \le Q$, $(a,q)=1$. Let $P = x/Q$, $P < Q$. We define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{1\le q\le P} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \mathfrak{M}(a,q), \qquad \mathfrak{M}(a,q) = \Big\{\alpha : \Big|\alpha - \frac{a}{q}\Big| \le \frac{1}{qQ}\Big\},$$

$$\mathfrak{m} = \Big[\frac{1}{Q}, 1+\frac{1}{Q}\Big]\backslash\mathfrak{M}.$$

Then we have

$$\mathscr{S}(X) = \int_{\mathfrak{M}} \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)d\alpha + \int_{\mathfrak{m}} \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)d\alpha. \tag{2.5}$$

We first consider the contribution from the minor arcs. For $\mathscr{F}(\alpha)$ in (2.4) and $\alpha \in \mathfrak{m}$, by Weyl's inequality (see Lemma 2.4 in [11]) one has

$$\mathscr{F}(\alpha) \ll x^{\frac12+\varepsilon}\Big(q^{-1} + x^{-\frac12} + qx^{-1}\Big)^{\frac12} \ll x^{\frac12+\varepsilon}P^{-\frac12} + x^{\frac14+\varepsilon}, \tag{2.6}$$

recalling that $PQ = x$. Moreover, by Hua's Lemma (see Lemma 2.5 in [11]),

$$\int_0^1 |\mathscr{F}(\alpha)|^4 d\alpha \ll x^{1+\varepsilon}. \tag{2.7}$$

Thus by applying Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_{\mathfrak{m}} \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)d\alpha &\ll \sup_{\alpha\in\mathfrak{m}} |\mathscr{F}(\alpha)| \Big(\int_0^1 |\mathscr{F}(\alpha)|^4 d\alpha\Big)^{\frac12} \Big(\int_0^1 |\mathscr{G}(\alpha)|^2 d\alpha\Big)^{\frac12} \\ &\ll x^{\frac12+\varepsilon}\Big(x^{\frac12+\varepsilon}P^{-\frac12} + x^{\frac14+\varepsilon}\Big)\Big(\sum_{n\le 1} \tau_3^2(n)\phi^2(n/X)\Big)^{\frac12} \\ &\ll x^{\frac32+\varepsilon}P^{-\frac12} + x^{\frac54+\varepsilon}. \end{aligned} \tag{2.8}$$

Next we consider the contribution from the major arcs. We have

$$\int_{\mathfrak{M}} \mathscr{F}^3(\alpha)\mathscr{G}(\alpha)d\alpha = \sum_{q\le P} \int_{|\beta|\le\frac{1}{qQ}} \sideset{}{^*}\sum_{a=1}^{q} \mathscr{F}^3\Big(\frac{a}{q}+\beta\Big)\mathscr{G}\Big(\frac{a}{q}+\beta\Big)d\beta, \tag{2.9}$$

where, throughout the paper the $*$ denotes the condition $(a,q)=1$. For an asymptotic formula of $\mathscr{F}(a/q+\beta)$, we quote the following result (see Theorem 4.1 in [11] or Lemma 4.1 in [13]).

Lemma 2.1. *Suppose that $(a,q)=1$, $q\le Q$ and $|\beta|\le 1/(qQ)$. We have*

$$\mathscr{F}\Big(\frac{a}{q}+\beta\Big)=\frac{G(a,0;q)}{q}\Psi_0(\beta)+\sum_{-\frac{3q}{2}<b\le\frac{3q}{2}}G(a,b;q)\Psi(b,q,\beta), \tag{2.10}$$

where $G(a,b;q)$ is the Gauss sum

$$G(a,b;q)=\sum_{d \bmod q}e\Big(\frac{ad^2+bd}{q}\Big),$$

$\Psi_0(\beta)$ is the integral

$$\Psi_0(\beta)=\int_0^{\sqrt{x}}e(\beta u^2)\mathrm{d}u, \tag{2.11}$$

and $\Psi(b,q,\beta)$ satisfies

$$\sum_{-\frac{3q}{2}<b\le\frac{3q}{2}}|\Psi(b,q,\beta)|\ll\log(q+2). \tag{2.12}$$

To transform $\mathscr{G}(a/q+\beta)$, we use the Vornoi formula for $\tau_3(n)$, which was first proved by Ivić [7]. Here we will use a more explicit formula derived by Li in [9]. For $\phi(y)\in C_c(0,\infty)$, $k=0,1$ and $\sigma>-1-2k$, set

$$\Phi_k(y)=\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma}\left(\pi^3y\right)^{-s}\frac{\Gamma\left((1+s+2k)/2\right)^3}{\Gamma\left(-s/2\right)^3}\widetilde{\phi}(-s-k)\mathrm{d}s,$$

where $\widetilde{\phi}(s)=\int_0^\infty\phi(u)u^{s-1}\mathrm{d}u$ is the Mellin transform of ϕ, and

$$\Phi^{\pm}(y)=\Phi_0(y)\pm\frac{1}{i\pi^3y}\Phi_1(y).$$

Lemma 2.2. *For $\phi(y)\in C_c^\infty(0,\infty)$, $a,\overline{a},q\in\mathbb{Z}^+$ with $a\overline{a}\equiv 1(\mathrm{mod} q)$, we have*

$$\begin{aligned}
&\sum_{n\ge1}\tau_3(n)e\left(\frac{an}{q}\right)\phi(n)\\
=\;&\frac{q}{2\pi^{\frac32}}\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)S\left(\pm m,\overline{a};qn^{-1}\right)\Phi^{\pm}\left(\frac{mn^2}{q^3}\right)\\
&+\frac{1}{2q^2}\widetilde{\phi}(1)\sum_{n|q}n\tau(n)P_2(n,q)S\left(0,\overline{a};qn^{-1}\right)\\
&+\frac{1}{2q^2}\widetilde{\phi}'(1)\sum_{n|q}n\tau(n)P_1(n,q)S\left(0,\overline{a};qn^{-1}\right)\\
&+\frac{1}{4q^2}\widetilde{\phi}''(1)\sum_{n|q}n\tau(n)S\left(0,\overline{a};qn^{-1}\right),
\end{aligned}$$

where $P_j(n,q)$ $(j=0,1,2)$ *are given in* (1.3)-(1.4) *and*

$$\sigma_{0,0}(k,l)=\sum_{\substack{d_1|l\\ d_1>0}}\ \sum_{\substack{d_2|d_1^{-1}l\\ d_2>0,(d_2,k)=1}}1,$$

Applying (2.10) and Lemma 2.2 with $\phi_\beta(y)=\phi\left(y/X\right)e(-\beta y)$, one has

$$\sum_{a=1}^{q}{}^{*}\mathscr{F}^3\left(\frac{a}{q}+\beta\right)\mathscr{G}\left(\frac{a}{q}+\beta\right)=\sum_{j=1}^{16}\mathscr{B}_j(q,\beta), \tag{2.13}$$

where $\mathscr{B}_j(q,\beta)=:\mathscr{B}_j$, $j=1,\ldots,16$, are defined by

$$\begin{aligned}
\mathscr{B}_1 &= \frac{q}{2\pi^{\frac{3}{2}}}\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\\
&\quad\times\sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ 1\le j\le3}}\mathscr{C}(b_1,b_2,b_3,n,\pm m;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), &(2.14)\\
\mathscr{B}_2 &= \frac{3}{2\pi^{\frac{3}{2}}}\Psi_0(\beta)\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\\
&\quad\times\sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ j=1,2}}\mathscr{C}(0,b_1,b_2,n,\pm m;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), &(2.15)\\
\mathscr{B}_3 &= \frac{3}{2\pi^{\frac{3}{2}}}\frac{\Psi_0(\beta)^2}{q}\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\\
&\quad\times\sum_{-\frac{3q}{2}<b\le\frac{3q}{2}}\mathscr{C}(0,0,b,n,\pm m;q)\Psi(b,q,\beta), &(2.16)\\
\mathscr{B}_4 &= \frac{1}{2\pi^{\frac{3}{2}}}\frac{\Psi_0(\beta)^3}{q^2}\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\\
&\quad\times\mathscr{C}(0,0,0,n,\pm m;q), &(2.17)\\
\mathscr{B}_5 &= \frac{1}{2}\frac{\widetilde{\phi}_\beta(1)}{q^2}\sum_{n|q}n\tau(n)P_2(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ 1\le j\le3}}\mathscr{C}(b_1,b_2,b_3,n,0;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), &(2.18)
\end{aligned}$$

$$\mathscr{B}_6 = \frac{1}{2}\frac{\widetilde{\phi}'_\beta(1)}{q^2}\sum_{n|q} n\tau(n)P_1(n,q) \sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ 1\le j\le 3}} \mathscr{C}(b_1,b_2,b_3,n,0;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), \tag{2.19}$$

$$\mathscr{B}_7 = \frac{1}{4}\frac{\widetilde{\phi}''_\beta(1)}{q^2}\sum_{n|q} n\tau(n) \sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ 1\le j\le 3}} \mathscr{C}(b_1,b_2,b_3,n,0;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), \tag{2.20}$$

$$\mathscr{B}_8 = \frac{3}{2}\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q} n\tau(n)P_2(n,q) \sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ j=1,2}} \mathscr{C}(0,b_1,b_2,n,0;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), \tag{2.21}$$

$$\mathscr{B}_9 = \frac{3}{2}\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q} n\tau(n)P_1(n,q) \sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ j=1,2}} \mathscr{C}(0,b_1,b_2,n,0;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), \tag{2.22}$$

$$\mathscr{B}_{10} = \frac{3}{4}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q} n\tau(n) \sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\ j=1,2}} \mathscr{C}(0,b_1,b_2,n,0;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), \tag{2.23}$$

$$\mathscr{B}_{11} = \frac{3}{2}\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n)P_2(n,q) \sum_{-\frac{3q}{2}<b\le\frac{3q}{2}} \mathscr{C}(0,0,b,n,0;q)\Psi(b,q,\beta), \tag{2.24}$$

$$\mathscr{B}_{12} = \frac{3}{2}\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n)P_1(n,q) \sum_{-\frac{3q}{2}<b\le\frac{3q}{2}} \mathscr{C}(0,0,b,n,0;q)\Psi(b,q,\beta), \tag{2.25}$$

$$\mathscr{B}_{13} = \frac{3}{4}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n) \sum_{-\frac{3q}{2}<b\le\frac{3q}{2}} \mathscr{C}(0,0,b,n,0;q)\Psi(b,q,\beta), \tag{2.26}$$

$$\mathscr{B}_{14} = \frac{1}{2}\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0;q), \tag{2.27}$$

$$\mathscr{B}_{15} = \frac{1}{2}\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)P_1(n,q)\mathscr{C}(0,0,0,n,0;q), \tag{2.28}$$

$$\mathscr{B}_{16} = \frac{1}{4}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)\mathscr{C}(0,0,0,n,0;q), \tag{2.29}$$

where $P_\ell(n,q)$ ($\ell=1,2$) are defined in (1.3) and (1.4),

$$\Phi_\beta^{\pm}(y)=\Phi_0(y,\beta)\pm\frac{1}{i\pi^3 y}\Phi_1(y,\beta) \tag{2.30}$$

with

$$\Phi_k(y,\beta)=\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma}\left(\pi^3 y\right)^{-s}\frac{\Gamma\left((1+s+2k)/2\right)^3}{\Gamma\left(-s/2\right)^3}\widetilde{\phi}_\beta(-s-k)\mathrm{d}s \tag{2.31}$$

and

$$\mathscr{C}(b_1,b_2,b_3,n,m;q)=\sum_{a=1}^{q}{}^{*}G(a,b_1;q)G(a,b_2;q)G(a,b_3;q)S\left(-\overline{a},m;q/n\right). \tag{2.32}$$

We will show that $\mathscr{B}_j$, $1\le j\le 13$, contribute the remainder terms, and $\mathscr{B}_j$, $14\le j\le 16$, contribute the main terms. We need the following results (see Propositions 5.1-5.2 in [10]).

Lemma 2.3. *Let $q=q_1q_0q_3'$, $q_1|n$, $4q_0$ square-full and q_3' square-free. For any $\varepsilon>0$, we have*

$$\mathscr{C}(b_1,b_2,b_3,n,m;q)\ll_\varepsilon\frac{(q_1q_0)^{3+\varepsilon}q_3'^{\frac{5}{2}+\varepsilon}}{\sqrt{n}}. \tag{2.33}$$

Lemma 2.4. *For any $\varepsilon>0$, we have*

$$\sum_{\pm}\sum_{m\ge1}\frac{1}{m}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\left|\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\right|\ll_\varepsilon X^\varepsilon n^\varepsilon(M+|\beta|^2X^2). \tag{2.34}$$

2.2. **Contribution of $\mathscr{B}_j$, $1\le j\le 4$.** By the second derivative test, $\Psi_0(\beta)$ in (2.11) is bounded by

$$\Psi_0(\beta)\ll\left(\frac{x}{1+|\beta|x}\right)^{1/2}. \tag{2.35}$$

Let q be as in Lemma 2.3. By (2.12), (2.14) and Lemmas 2.3-2.4, we have

$$\begin{aligned}\mathscr{B}_1 \quad&\ll\quad q\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\left|\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\right|\\&\qquad\times\sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\1\le j\le3}}|\Psi(b_1,q,\beta)||\Psi(b_2,q,\beta)||\Psi(b_3,q,\beta)||\mathscr{C}(b_1,b_2,b_3,n,m;q)|\\&\ll_\varepsilon\quad x^\varepsilon(M+|\beta|^2x^2)\sum_{n\le P}n^{-\frac{3}{2}+\varepsilon}q_1^4q_0^4q_3'^{\frac{7}{2}}.\end{aligned}$$

Recall that $PQ=x$. Thus

$$
\begin{aligned}
&\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_1(q,\beta)\mathrm{d}\beta\\
\ll_\varepsilon\ & x^\varepsilon\sum_{n\le P}n^{-\frac32+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le P/q_1\\ q_3' \text{ square-free}}}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}}q_1^4q_0^4q_3'^{\frac72}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(M+|\beta|^2x^2)\mathrm{d}\beta\\
\ll_\varepsilon\ & x^\varepsilon\sum_{n\le P}n^{-\frac32+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le P/q_1}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}}q_1^4q_0^4q_3'^{\frac72}\left(\frac{M}{q_1q_0q_3'Q}+\frac{x^2}{(q_1q_0q_3'Q)^3}\right)\\
\ll_\varepsilon\ & \frac{x^\varepsilon M}{Q}\sum_{n\le P}n^{-\frac32+\varepsilon}\sum_{q_1|n}q_1^3\sum_{q_3'\le P/q_1}q_3'^{\frac52}\left(\frac{P}{q_1q_3'}\right)^{\frac72}\\
&+\frac{x^{2+\varepsilon}}{Q^3}\sum_{n\le P}n^{-\frac32+\varepsilon}\sum_{q_1|n}q_1\sum_{q_3'\le P/q_1}q_3'^{\frac12}\left(\frac{P}{q_1q_3'}\right)^{\frac32}\\
\ll_\varepsilon\ & \frac{x^\varepsilon MP^{\frac72}}{Q}+\frac{x^{2+\varepsilon}P^{\frac32}}{Q^3}\\
\ll_\varepsilon\ & x^{-1+\varepsilon}MP^{\frac92}.
\end{aligned}\tag{2.36}
$$

By (2.12), (2.15), (2.35) and Lemmas 2.3-2.4, we have

$$
\begin{aligned}
\mathscr{B}_2 &\ll_\varepsilon x^\varepsilon\left(\frac{x}{1+|\beta|x}\right)^{\frac12}\sum_{n\le P}n^{-\frac32}q_1^3q_0^3q_3'^{\frac52}(M+|\beta|^2x^2)\\
&\ll_\varepsilon x^{\frac12+\varepsilon}\left((1+|\beta|x)^{\frac32}+\frac{M}{\sqrt{1+|\beta|x}}\right)\sum_{n\le P}n^{-\frac32+\varepsilon}q_1^3q_0^3q_3'^{\frac52}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_2(q,\beta)\mathrm{d}\beta\\
\ll_\varepsilon\ & x^{\frac12+\varepsilon}\sum_{n\le P}n^{-\frac32+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le P/q_1\\ q_3' \text{ square-free}}}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}}q_1^3q_0^3q_3'^{\frac52}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(1+|\beta|x)^{\frac32}\mathrm{d}\beta\\
&+x^\varepsilon M\sum_{n\le P}n^{-\frac32+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le P/q_1\\ q_3' \text{ square-free}}}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}}q_1^3q_0^3q_3'^{\frac52}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}\frac{1}{\sqrt{x^{-1}+|\beta|}}\mathrm{d}\beta
\end{aligned}
$$

$$
\begin{aligned}
&\ll_\varepsilon \quad x^{\frac{1}{2}+\varepsilon}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le P/q_1}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}} q_1^3q_0^3q_3'^{\frac{5}{2}}\left(\frac{1}{q_1q_0q_3'Q}+\frac{x^{\frac{3}{2}}}{(q_1q_0q_3'Q)^{\frac{5}{2}}}\right)\\
&\qquad +x^\varepsilon M\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le P/q_1}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}} q_1^3q_0^3q_3'^{\frac{5}{2}}(q_1q_0q_3'Q)^{-\frac{1}{2}}\\
&\ll_\varepsilon \quad \frac{x^{\frac{1}{2}+\varepsilon}}{Q}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n} q_1^2\sum_{q_3'\le P/q_1} q_3'^{\frac{3}{2}}\left(\frac{P}{q_1q_3'}\right)^{\frac{5}{2}}\\
&\qquad +\frac{x^{2+\varepsilon}}{Q^{\frac{5}{2}}}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n} q_1^{\frac{1}{2}}\sum_{q_3'\le P/q_1} \frac{P}{q_1q_3'}\\
&\qquad +\frac{x^\varepsilon M}{Q^{\frac{1}{2}}}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n} q_1^{\frac{5}{2}}\sum_{q_3'\le P/q_1} q_3'^2\left(\frac{P}{q_1q_3'}\right)^{3}\\
&\ll_\varepsilon \quad \frac{x^{\frac{1}{2}+\varepsilon}P^{\frac{5}{2}}}{Q}+\frac{x^{2+\varepsilon}P}{Q^{\frac{5}{2}}}+\frac{x^\varepsilon MP^3}{Q^{\frac{1}{2}}}\\
&\ll \quad x^{-\frac{1}{2}+\varepsilon}MP^{\frac{7}{2}}. \qquad (2.37)
\end{aligned}
$$

Similarly, by (2.12), (2.16), (2.35) and Lemmas 2.3-2.4, we have

$$
\begin{aligned}
\mathscr{B}_3 &\ll_\varepsilon \frac{x^{1+\varepsilon}}{1+|\beta|x}\sum_{n\le P} n^{-\frac{3}{2}}q_1^2q_0^2q_3'^{\frac{3}{2}}\,X^\varepsilon n^\varepsilon(M+|\beta|^2x^2)\\
&\ll_\varepsilon x^{1+\varepsilon}\left(1+|\beta|x+\frac{M}{1+|\beta|x}\right)\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}q_1^2q_0^2q_3'^{\frac{3}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_3(q,\beta)\mathrm{d}\beta\\
&\ll_\varepsilon x^{1+\varepsilon}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le P/q_1\\ q_3' \text{ square-free}}}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}} q_1^2q_0^2q_3'^{\frac{3}{2}}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(1+|\beta|x)\mathrm{d}\beta\\
&\quad +x^\varepsilon M\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le P/q_1\\ q_3' \text{ square-free}}}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0 \text{ square-full}}} q_1^2q_0^2q_3'^{\frac{3}{2}}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(x^{-1}+|\beta|)^{-1}\mathrm{d}\beta
\end{aligned}
$$

$$
\begin{aligned}
&\ll_\varepsilon \quad x^{1+\varepsilon}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le P/q_1}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0\ \mathrm{square-full}}} q_1^2q_0^2{q_3'}^{\frac{3}{2}}\left(\frac{1}{q_1q_0q_3'Q}+\frac{x}{(q_1q_0q_3'Q)^2}\right)\\
&\qquad +x^\varepsilon M\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le P/q_1}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0\ \mathrm{square-full}}} q_1^2q_0^2{q_3'}^{\frac{3}{2}}\\
&\ll_\varepsilon \quad \frac{x^{1+\varepsilon}P^{\frac{3}{2}}}{Q}+\frac{x^{2+\varepsilon}P^{\frac{1}{2}}}{Q^2}+x^\varepsilon MP^{\frac{5}{2}}\\
&\ll_\varepsilon \quad x^\varepsilon MP^{\frac{5}{2}}. \qquad (2.38)
\end{aligned}
$$

Moreover, by (2.12), (2.17), (2.35) and Lemmas 2.3-2.4, we have

$$
\mathscr{B}_4 \ll_\varepsilon \left(x^{\frac{3}{2}+\varepsilon}\,(1+|\beta|x)^{\frac{1}{2}}+\frac{x^\varepsilon M}{(x^{-1}+|\beta|)^{\frac{3}{2}}}\right)\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}q_1q_0{q_3'}^{\frac{1}{2}},
$$

and

$$
\begin{aligned}
&\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_4(q,\beta)\mathrm{d}\beta\\
&\ll_\varepsilon \quad x^{\frac{3}{2}+\varepsilon}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le P/q_1\\ q_3'\ \mathrm{square-free}}}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0\ \mathrm{square-full}}} q_1q_0{q_3'}^{\frac{1}{2}}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(1+|\beta|x)^{\frac{1}{2}}\mathrm{d}\beta\\
&\qquad +x^\varepsilon M\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{\substack{q_3'\le P/q_1\\ q_3'\ \mathrm{square-free}}}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0\ \mathrm{square-full}}} q_1q_0{q_3'}^{\frac{1}{2}}\int_{|\beta|\le\frac{1}{q_1q_0q_3'Q}}(x^{-1}+|\beta|)^{-\frac{3}{2}}\mathrm{d}\beta\\
&\ll_\varepsilon \quad x^{\frac{3}{2}+\varepsilon}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le P/q_1}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0\ \mathrm{square-full}}} q_1q_0{q_3'}^{\frac{1}{2}}\left(\frac{1}{q_1q_0q_3'Q}+\frac{x^{\frac{1}{2}}}{(q_1q_0q_3'Q)^{\frac{3}{2}}}\right)\\
&\qquad +Mx^{\frac{1}{2}+\varepsilon}\sum_{n\le P} n^{-\frac{3}{2}+\varepsilon}\sum_{q_1|n}\sum_{q_3'\le P/q_1}\sum_{\substack{q_0\le P/(q_1q_3')\\ 4q_0\ \mathrm{square-full}}} q_1q_0{q_3'}^{\frac{1}{2}}\\
&\ll_\varepsilon \quad \frac{x^{\frac{3}{2}+\varepsilon}P^{\frac{1}{2}}}{Q}+\frac{x^{2+\varepsilon}}{Q^{\frac{3}{2}}}+x^{\frac{1}{2}+\varepsilon}MP^{\frac{3}{2}}\\
&\ll \quad x^{\frac{1}{2}+\varepsilon}MP^{\frac{3}{2}}. \qquad (2.39)
\end{aligned}
$$

Since $P<x^{1/2}$, by (2.36)-(2.39) the contribution from $\mathscr{B}_j$, $j=1,2,3,4$, is at most

$$
x^{-1+\varepsilon}MP^{\frac{9}{2}}+x^{-\frac{1}{2}+\varepsilon}MP^{\frac{7}{2}}+x^\varepsilon MP^{\frac{5}{2}}+x^{\frac{1}{2}+\varepsilon}MP^{\frac{3}{2}}\ll x^{\frac{1}{2}+\varepsilon}MP^{\frac{3}{2}}. \qquad (2.40)
$$

2.3. **Contribution of $\mathscr{B}_j$, $5 \le j \le 13$.** Note that (recall (1.3) and (1.4))

$$P_\ell(n,q) \ll (\log(n+2)(q+2))^\ell, \qquad \ell = 1,2, \tag{2.41}$$

and

$$\widetilde{\phi_\beta}^{(\ell)}(1) = \int_0^\infty \phi\left(\frac{u}{X}\right) e(-\beta u)(\log u)^\ell \mathrm{d}u \ll \frac{x(\log x)^\ell}{1+|\beta|x}, \qquad \ell = 0,1,2. \tag{2.42}$$

Bounding the character sum $\mathscr{C}(b_1,b_2,b_3,n,m;q)$ in (2.32) by Weil's bound for Kloosterman sums, one has

$$\mathscr{C}(b_1,b_2,b_3,n,m;q) \ll q^{\frac{5}{2}}\left(qn^{-1}\right)^{\frac{1}{2}}\tau\left(qn^{-1}\right) \ll_\varepsilon q^{3+\varepsilon}n^{-\frac{1}{2}}. \tag{2.43}$$

By (2.12), (2.18)-(2.20) and (2.41)-(2.43), we have, for $j = 5,6,7$,

$$\mathscr{B}_j \ll_\varepsilon \frac{1}{q^2}\frac{x(\log x)^2}{1+|\beta|x}\sum_{n|q} n\tau(n)q^{3+\varepsilon}n^{-\frac{1}{2}}(\log(q+2))^5 \ll_\varepsilon \frac{x^\varepsilon q^{\frac{3}{2}+\varepsilon}}{x^{-1}+|\beta|}$$

and

$$\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}} \mathscr{B}_j(q,\beta)\mathrm{d}\beta \ll_\varepsilon x^\varepsilon \sum_{q\le P} q^{\frac{3}{2}+\varepsilon}\int_{|\beta|\le\frac{1}{qQ}} \frac{1}{x^{-1}+|\beta|}\mathrm{d}\beta \ll_\varepsilon x^\varepsilon P^{\frac{5}{2}}. \tag{2.44}$$

Similarly, by (2.12), (2.21)-(2.23), (2.35) and (2.41)-(2.43), we have, for $j = 8,9,10$,

$$\begin{aligned}\mathscr{B}_j &\ll_\varepsilon \frac{1}{q^3}\frac{x(\log x)^2}{1+|\beta|x}\left(\frac{x}{1+|\beta|x}\right)^{\frac{1}{2}}\sum_{n|q} n\tau(n)q^{3+\varepsilon}n^{-\frac{1}{2}}(\log(q+2))^4 \\ &\ll_\varepsilon x^\varepsilon q^{\frac{1}{2}+\varepsilon}\left(\frac{1}{x^{-1}+|\beta|}\right)^{\frac{3}{2}}\end{aligned}$$

and

$$\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}} \mathscr{B}_j(q,\beta)\mathrm{d}\beta \ll_\varepsilon x^\varepsilon \sum_{q\le P} q^{\frac{1}{2}+\varepsilon}\int_{|\beta|\le\frac{1}{qQ}} \left(\frac{1}{x^{-1}+|\beta|}\right)^{\frac{3}{2}}\mathrm{d}\beta \ll_\varepsilon x^{\frac{1}{2}+\varepsilon} P^{\frac{3}{2}}. \tag{2.45}$$

Further, by (2.12), (2.24)-(2.26), (2.35) and (2.41)-(2.43), we have, for $j = 11,12,13$,

$$\begin{aligned}\mathscr{B}_j &\ll_\varepsilon \frac{1}{q^4}\frac{x(\log x)^2}{1+|\beta|x}\frac{x}{1+|\beta|x}\sum_{n|q} n\tau(n)q^{3+\varepsilon}n^{-\frac{1}{2}}(\log(q+2))^3 \\ &\ll_\varepsilon x^\varepsilon q^{-\frac{1}{2}+\varepsilon}\left(\frac{1}{x^{-1}+|\beta|}\right)^2\end{aligned}$$

and

$$\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}} \mathscr{B}_j(q,\beta)\mathrm{d}\beta \ll_\varepsilon x^\varepsilon \sum_{q\le P} q^{-\frac{1}{2}+\varepsilon}\int_{|\beta|\le\frac{1}{qQ}} \left(\frac{1}{x^{-1}+|\beta|}\right)^2\mathrm{d}\beta \ll_\varepsilon x^{1+\varepsilon} P^{\frac{1}{2}}. \tag{2.46}$$

By (2.44)-(2.46), for the contribution from $\mathscr{B}_j$, $5 \le j \le 13$, is at most

$$x^{\varepsilon} P^{\frac{5}{2}} + x^{\frac{1}{2}+\varepsilon} P^{\frac{3}{2}} + x^{1+\varepsilon} P^{\frac{1}{2}} \ll x^{1+\varepsilon} P^{\frac{1}{2}}. \tag{2.47}$$

2.4. **Computation of the main terms.** The three sums $\mathscr{B}_{14}$, $\mathscr{B}_{15}$ and $\mathscr{B}_{16}$ in (2.27)-(2.29) contribute the main terms. Following [10], we replace $\widetilde{\phi_\beta}^{(\ell)}(1)$ by

$$\vartheta^{\flat,\ell}(\beta) = \int_{X/2}^{X} e(-\beta u)(\log u)^{\ell} \mathrm{d}u, \qquad \ell = 0, 1, 2, \tag{2.48}$$

and estimate the remainder terms from

$$\vartheta^{\sharp,\ell}(\beta) = \widetilde{\phi_\beta}^{(\ell)}(1) - \vartheta^{\flat,\ell}(\beta), \qquad \ell = 0, 1, 2,$$

Write correspondingly

$$\mathscr{B}_j^{\sharp}(q,\beta) = \mathscr{B}_j - \mathscr{B}_j^{\flat}(q,\beta), \qquad j = 14, 15, 16.$$

Firstly, we evaluate the remainder terms from $\mathscr{B}_j^{\sharp}(q,\beta)$, $j = 14, 15, 16$. Notice that

$$\vartheta^{\sharp,\ell}(\beta) = \int_{X/2}^{X} \left(\phi\left(\frac{u}{X}\right) - 1\right) e(-\beta u)(\log u)^{\ell} \mathrm{d}u \ll XM^{-1}(\log X)^{\ell}.$$

Hence

$$\begin{aligned}
& \sum_{q \le P} \int_{|\beta| \le \frac{1}{qQ}} \mathscr{B}_{14}^{\sharp}(q,\beta) \mathrm{d}\beta \\
= \quad & \frac{1}{2} \sum_{q \le P} q^{-5} \int_{|\beta| \le \frac{1}{qQ}} \vartheta^{\sharp,0}(\beta) \Psi_0(\beta)^3 \sum_{n \mid q} n\tau(n) P_2(n,q) \mathscr{C}(0,0,0,n,0;q) \mathrm{d}\beta \\
\ll_\varepsilon \quad & X^{1+\varepsilon} M^{-1} \sum_{q \le P} q^{-2+\varepsilon} \sum_{n \mid q} n^{\frac{1}{2}} \tau(n) |P_2(n,q)| \int_{|\beta| \le \frac{1}{qQ}} \left(\frac{1}{x^{-1} + |\beta|}\right)^{\frac{3}{2}} \mathrm{d}\beta \\
\ll_\varepsilon \quad & x^{\frac{3}{2}+\varepsilon} M^{-1}.
\end{aligned} \tag{2.49}$$

Here we have used (2.35), (2.41) and (2.43). Similarly,

$$\sum_{q \le P} \int_{|\beta| \le \frac{1}{qQ}} \mathscr{B}_{15}^{\sharp}(q,\beta) \mathrm{d}\beta \ll_\varepsilon \quad x^{\frac{3}{2}+\varepsilon} M^{-1}$$

and

$$\sum_{q \le P} \int_{|\beta| \le \frac{1}{qQ}} \mathscr{B}_{16}^{\sharp}(q,\beta) \mathrm{d}\beta \ll_\varepsilon x^{\frac{3}{2}+\varepsilon} M^{-1}.$$

Next, we compute the contribution from $\mathscr{B}_j^{\flat}(q,\beta)$ which constitute the main terms. We write

$$\sum_{q \le P} \int_{|\beta| \le \frac{1}{qQ}} \mathscr{B}_{14}^{\flat}(q,\beta) \mathrm{d}\beta = \sum_{q \le P} \int_{-\infty}^{\infty} \mathscr{B}_{14}^{\flat}(q,\beta) \mathrm{d}\beta - \sum_{q \le P} \int_{|\beta| > \frac{1}{qQ}} \mathscr{B}_{14}^{\flat}(q,\beta) \mathrm{d}\beta, \tag{2.50}$$

where the second term on the right side of (2.50) is

$$\begin{aligned}
&\frac{1}{2}\sum_{q\le P} q^{-5}\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0;q)\int_{|\beta|>\frac{1}{qQ}}\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta\\
\ll_\varepsilon\ &\sum_{q\le P} q^{-2+\varepsilon}\sum_{n|q} n^{\frac{1}{2}}\tau(n)|P_2(n,q)|\int\limits_{|\beta|>\frac{1}{qQ}}|\beta|^{-\frac{5}{2}}\mathrm{d}\beta\\
\ll_\varepsilon\ & x^\varepsilon PQ^{\frac{3}{2}} \ll x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}}.
\end{aligned}$$

Here we have used (2.35), (2.41), (2.43) and the estimate $\vartheta^{\flat,\ell}(\beta)\ll_\ell x(\log x)^\ell/(1+|\beta|x)$. Thus

$$\begin{aligned}
\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}^\flat_{14}(q,\beta)\mathrm{d}\beta \ &=\ \frac{1}{2}\sum_{q\le P} q^{-5}\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0;q)\\
&\quad\times\int\limits_{-\infty}^{\infty}\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta + O_\varepsilon\left(x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}}\right). \qquad (2.51)
\end{aligned}$$

Moreover, by (2.11),

$$\Psi_0(\beta)=\int_0^{\sqrt{x}} e(\beta v^2)\mathrm{d}v = x^{\frac{1}{2}}\int_0^1 e(\beta x v^2)\mathrm{d}v. \qquad (2.52)$$

It follows that

$$\int\limits_{-\infty}^{\infty}\vartheta^{\flat,0}(\beta)\Psi_0(\beta)^3\mathrm{d}\beta = x^{\frac{3}{2}}\int_{-\infty}^{\infty}\left(\int_{X/2}^{X} e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta x v^2)\mathrm{d}v\right)^3\mathrm{d}\beta := x^{\frac{3}{2}}\mathcal{I}_0(X),$$

where

$$\mathcal{I}_0(X)=\int_{-\infty}^{\infty}\left(\int_{X/2}^{X} e(-\beta u)\mathrm{d}u\right)\left(\int_0^1 e(\beta x v^2)\mathrm{d}v\right)^3\mathrm{d}\beta.$$

Substituting in (2.51) and by (2.49), we have

$$\begin{aligned}
&\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_{14}(q,\beta)\mathrm{d}\beta\\
=\ &\frac{1}{2}\mathcal{I}_0(X)x^{\frac{3}{2}}\sum_{q\le P} q^{-5}\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0,0;q) + O_\varepsilon\left(x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}} + x^{\frac{3}{2}+\varepsilon}M^{-1}\right).
\end{aligned}$$

Further, by (2.41) and (2.43), we have

$$\sum_{q>P} q^{-5}\sum_{n|q} n\tau(n)P_2(n,q)\mathscr{C}(0,0,0,n,0,0;q)\ll_\varepsilon P^{-\frac{1}{2}+\varepsilon}.$$

Therefore,

$$\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_{14}(q,\beta)\mathrm{d}\beta=\frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right),\qquad(2.53)$$

where

$$\mathcal{C}_2=\sum_{q=1}^{\infty}q^{-5}\sum_{n|q}n\tau(n)P_2(n,q)\sum_{a=1}^{q}{}^{*}\,G(a,0;q)^3S\left(-\overline{a},0;qn^{-1}\right).$$

Similarly, we have

$$\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_{15}(q,\beta)\mathrm{d}\beta=\frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right)\qquad(2.54)$$

and

$$\sum_{q\le P}\int_{|\beta|\le\frac{1}{qQ}}\mathscr{B}_{16}(q,\beta)\mathrm{d}\beta=\frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right),\qquad(2.55)$$

where

$$\mathcal{I}_\ell(X)=\int_{-\infty}^{\infty}\left(\int_{X/2}^{X}e(-\beta u)(\log u)^\ell\mathrm{d}u\right)\left(\int_0^1e(\beta xv^2)\mathrm{d}v\right)^3\mathrm{d}\beta,\quad \ell=0,1,2,\qquad(2.56)$$

and

$$\mathcal{C}_\ell=\sum_{q=1}^{\infty}q^{-5}\sum_{n|q}n\tau(n)P_\ell(n,q)\sum_{a=1}^{q}{}^{*}\,G(a,0;q)^3S\left(-\overline{a},0;qn^{-1}\right),\quad \ell=0,1,2.\qquad(2.57)$$

By (2.53)-(2.55), the contribution from $\mathscr{B}_j$, $14\le j\le 16$, is

$$\frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2x^{\frac{3}{2}}+\frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1x^{\frac{3}{2}}+\frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right),\qquad(2.58)$$

where $\mathcal{I}_\ell$, $\mathcal{C}_\ell$, $\ell=0,1,2$, are defined in (2.56) and (2.57), respectively.

2.5. **Conclusion.** Recall that $P<x^{1/2}$. By (2.9), (2.13), (2.40), (2.47) and (2.58), we have

$$\begin{aligned}\int_{\mathfrak{M}}\mathscr{F}^3(\alpha)\mathscr{G}(\alpha)d\alpha &= \frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2x^{\frac{3}{2}}+\frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1x^{\frac{3}{2}}+\frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0X^{\frac{3}{2}}\\ &\quad+O_\varepsilon\left(x^{\frac{1}{2}+\varepsilon}MP^{\frac{3}{2}}+x^{\frac{3}{2}+\varepsilon}P^{-\frac{1}{2}}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right).\end{aligned}$$

Take $P=x^{\frac{1}{2}}M^{-\frac{1}{2}}$. Then the O-term above is

$$O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}M^{\frac{1}{4}}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right).$$

These combined with (2.5) and the estimate from the minor arcs in (2.8) give

$$\mathscr{S}(X)=\frac{1}{2}\mathcal{I}_0(X)\mathcal{C}_2x^{\frac{3}{2}}+\frac{1}{2}\mathcal{I}_1(X)\mathcal{C}_1x^{\frac{3}{2}}+\frac{1}{4}\mathcal{I}_2(X)\mathcal{C}_0x^{\frac{3}{2}}+O_\varepsilon\left(x^{\frac{5}{4}+\varepsilon}M^{\frac{1}{4}}+x^{\frac{3}{2}+\varepsilon}M^{-1}\right).$$

This proves Propositon 2.1.

3. Proof of Theorem 1.2

Let $r_4^*(n) = \sum_{\substack{n_1^2+n_2^2+n_3^2+n_4^2=n \\ (n_1,n_2,n_3,n_4)\in\mathcal{V}^4}} 1$. As in the proof of Theorem 1.1, we have

$$\sum_{1\le n_1,n_2,n_3,n_4\le\sqrt{x}} \tau_3(n_1^2+n_2^2+n_3^2+n_4^2) = \sum_{j\ge 1}\ \sum_{4x/2^j<n\le 4x/2^{j-1}} \tau_3(n) r_4^*(n), \tag{3.1}$$

where the right side of (3.1) vanishes for $j > \log 8x/\log 2$. To study the inner sum over n in (3.1), we write

$$\sum_{X_j/2<n\le X_j} \tau_3(n) r_4^*(n) = \sum_{n\ge 1} \tau_3(n) r_4^*(n)\phi\big(n/X_j\big) + O_\varepsilon(X_j^{2+\varepsilon}M^{-1}), \tag{3.2}$$

where $X_j = 4x/2^{j-1}$ and $\phi(y)$ is the same as in Section 2. Here we have used the estimate $r_4^*(n) \ll n^{1+\varepsilon}$. To study the first term on the right side of (3.2), we set

$$\mathscr{T}(X) = \sum_{n\ge 1} \tau_3(n) r_4^*(n)\phi\left(\frac{n}{X}\right) \tag{3.3}$$

and prove the following asymptotic formula.

Proposition 3.1. *For any $1 < X \le 4x$ and any $\varepsilon > 0$, we have*

$$\mathscr{T}(X) \ = \ \frac{1}{2}\mathfrak{I}_0(X)\mathfrak{C}_2 x^2 + \frac{1}{2}\mathfrak{I}_1(X)\mathfrak{C}_1 x^2 + \frac{1}{4}\mathfrak{I}_2(X)\mathfrak{C}_0 x^2 + O_\varepsilon\left(x^{\frac{3}{2}+\varepsilon}M^{\frac{1}{2}} + x^{2+\varepsilon}M^{-1}\right),$$

where $\mathfrak{I}_\ell(X)$ and $\mathfrak{C}_\ell$, $\ell = 0,1,2$, are absolute constants defined in (3.30) and (3.31), respestively.

Applying Proposition 3.1 to the sum on the right side of (3.2) and taking $M = x^{\frac{1}{3}}$ we get

$$\sum_{X_j/2<n\le X_j} \tau_3(n) r_4^*(n) = \frac{1}{2}\mathfrak{I}_0(X_j)\mathfrak{C}_2 x^2 + \frac{1}{2}\mathfrak{I}_1(X_j)\mathfrak{C}_1 x^2 + \frac{1}{4}\mathfrak{I}_2(X_j)\mathfrak{C}_0 x^2 + O_\varepsilon\left(x^{\frac{5}{3}+\varepsilon}\right). \tag{3.4}$$

Then Theorem 1.2 follows from (3.4) by the same arguments as in Section 2 of [10].

Proof of Proposition 3.1. Let $\mathscr{F}(\alpha)$ and $\mathscr{G}(\alpha)$ be as in (2.4). As in the proof of Theorem 1.1, $\mathscr{T}(X)$ in (3.3) can be written as

$$\mathscr{T}(X) = \int_0^1 \mathscr{F}^4(\alpha)\mathscr{G}(\alpha)d\alpha = \int_{1/Q^*}^{1+1/Q^*} \mathscr{F}^4(\alpha)\mathscr{G}(\alpha)d\alpha,$$

where Q^* is a parameter to be chosen later. By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q^*, 1+1/Q^*]$ can be written in the form

$$\alpha = \frac{a}{q} + \beta, \qquad |\beta| \le \frac{1}{qQ^*},$$

where a, q are some integers satisfying $1 \le a \le q \le Q^*$, $(a,q)=1$. Let $P^* = x/Q^*$, $P^* < Q^*$. We define the major arcs $\mathfrak{M}^*$ and the minor arcs $\mathfrak{m}^*$ as follows:

$$\mathfrak{M}^* = \bigcup_{1\le q\le P^*} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \mathfrak{M}^*(a,q), \qquad \mathfrak{M}^*(a,q) = \Big\{\alpha : \Big|\alpha - \frac{a}{q}\Big| \le \frac{1}{qQ^*}\Big\},$$

$$\mathfrak{m}^* = [1/Q^*, 1+1/Q^*]\backslash\mathfrak{M}^*.$$

Then we have

$$\mathscr{T}(X) = \int_{\mathfrak{M}^*} \mathscr{F}^4(\alpha)\mathscr{G}(\alpha)d\alpha + \int_{\mathfrak{m}^*} \mathscr{F}^4(\alpha)\mathscr{G}(\alpha)d\alpha. \tag{3.5}$$

We first consider the contribution from the minor arcs. By (2.6), (2.7) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{\mathfrak{m}^*} \mathscr{F}^4(\alpha)\mathscr{G}(\alpha)d\alpha &\ll \sup_{\alpha\in\mathfrak{m}} |\mathscr{F}(\alpha)|^2\Big(\int_0^1 |\mathscr{F}(\alpha)|^4 d\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1 |\mathscr{G}(\alpha)|^2 d\alpha\Big)^{\frac{1}{2}}\\ &\ll x^{1+\varepsilon}\left(x^{\frac{1}{2}+\varepsilon}P^{*-\frac{1}{2}} + x^{\frac{1}{4}+\varepsilon}\right)^2\\ &\ll x^{2+\varepsilon}P^{*-1} + x^{\frac{3}{2}+\varepsilon}. \end{aligned}\tag{3.6}$$

Next we consider the contribution from the major arcs. Applying (2.10) and Lemma 2.2 with $\phi_\beta(y) = \phi\,(y/X)\,e(-\beta y)$, one has

$$\begin{aligned}\int_{\mathfrak{M}^*} \mathscr{F}^4(\alpha)\mathscr{G}(\alpha)d\alpha &= \sum_{q\le P^*}\int_{|\beta|\le\frac{1}{qQ^*}} \sideset{}{^*}\sum_{a=1}^{q} \mathscr{F}^4\left(\frac{a}{q}+\beta\right)\mathscr{G}\left(\frac{a}{q}+\beta\right)d\beta\\ &= \sum_{j=1}^{20}\sum_{q\le P^*}\int_{|\beta|\le\frac{1}{qQ^*}} \widetilde{\mathscr{B}_j}(q,\beta)d\beta, \end{aligned}\tag{3.7}$$

where $\widetilde{\mathscr{B}_j}(q,\beta) =: \widetilde{\mathscr{B}_j}$, $j=1,\ldots,20$, are defined by

$$\begin{aligned}\widetilde{\mathscr{B}_1} &= \frac{q}{2\pi^{\frac{3}{2}}}\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\\ &\times \sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\1\le j\le4}}\widetilde{\mathscr{C}}(b_1,b_2,b_3,b_4,n,\pm m;q)\prod_{k=1}^{4}\Psi(b_k,q,\beta), \end{aligned}\tag{3.8}$$

$$\begin{aligned}\widetilde{\mathscr{B}_2} &= \frac{2}{\pi^{\frac{3}{2}}}\Psi_0(\beta)\sum_{\pm}\sum_{n|q}\sum_{m\ge1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)\\ &\times \sum_{\substack{-\frac{3q}{2}<b_j\le\frac{3q}{2}\\1\le j\le3}}\widetilde{\mathscr{C}}(0,b_1,b_2,b_3,n,\pm m;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), \end{aligned}\tag{3.9}$$

$$\widetilde{\mathscr{B}_3} = \frac{3}{\pi^{\frac{3}{2}}}\frac{\Psi_0(\beta)^2}{q}\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)$$

$$\times \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}}\widetilde{\mathscr{C}}(0,0,b_1,b_2,n,\pm m;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), \tag{3.10}$$

$$\widetilde{\mathscr{B}_4} = \frac{2}{\pi^{\frac{3}{2}}}\frac{\Psi_0(\beta)^3}{q^2}\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)$$

$$\times \sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}}\widetilde{\mathscr{C}}(0,0,0,b,n,\pm m;q)\Psi(b,q,\beta), \tag{3.11}$$

$$\widetilde{\mathscr{B}_5} = \frac{1}{2\pi^{\frac{3}{2}}}\frac{\Psi_0(\beta)^4}{q^3}\sum_{\pm}\sum_{n|q}\sum_{m\geq 1}\frac{1}{nm}\sum_{n_1|n}\sum_{n_2|\frac{n}{n_1}}\sigma_{0,0}\left(\frac{n}{n_1n_2},m\right)\Phi_\beta^{\pm}\left(\frac{mn^2}{q^3}\right)$$

$$\times \widetilde{\mathscr{C}}(0,0,0,0,n,\pm m;q), \tag{3.12}$$

$$\widetilde{\mathscr{B}_6} = \frac{1}{2}\frac{\widetilde{\phi}_\beta(1)}{q^2}\sum_{n|q}n\tau(n)P_2(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 4}}\widetilde{\mathscr{C}}(b_1,b_2,b_3,b_4,n,0;q)\prod_{k=1}^{4}\Psi(b_k,q,\beta), \tag{3.13}$$

$$\widetilde{\mathscr{B}_7} = \frac{1}{2}\frac{\widetilde{\phi}'_\beta(1)}{q^2}\sum_{n|q}n\tau(n)P_1(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 4}}\widetilde{\mathscr{C}}(b_1,b_2,b_3,b_4,n,0;q)\prod_{k=1}^{4}\Psi(b_k,q,\beta),$$

$$\widetilde{\mathscr{B}_8} = \frac{1}{4}\frac{\widetilde{\phi}''_\beta(1)}{q^2}\sum_{n|q}n\tau(n)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 4}}\widetilde{\mathscr{C}}(b_1,b_2,b_3,b_4,n,0;q)\prod_{k=1}^{4}\Psi(b_k,q,\beta), \tag{3.14}$$

$$\widetilde{\mathscr{B}_9} = 2\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q}n\tau(n)P_2(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 3}}\widetilde{\mathscr{C}}(0,b_1,b_2,b_3,n,0;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), \tag{3.15}$$

$$\widetilde{\mathscr{B}_{10}} = 2\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q}n\tau(n)P_1(n,q)\sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 3}}\widetilde{\mathscr{C}}(0,b_1,b_2,b_3,n,0;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), \tag{3.16}$$

$$\widetilde{\mathscr{B}_{11}} = \frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)}{q^3}\sum_{n|q} n\tau(n) \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ 1\leq j\leq 3}} \widetilde{\mathscr{C}}(0,b_1,b_2,b_3,n,0;q)\prod_{k=1}^{3}\Psi(b_k,q,\beta), \tag{3.17}$$

$$\widetilde{\mathscr{B}_{12}} = 3\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n)P_2(n,q) \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}} \widetilde{\mathscr{C}}(0,0,b_1,b_2,n,0;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), \tag{3.18}$$

$$\widetilde{\mathscr{B}_{13}} = 3\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n)P_1(n,q) \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}} \widetilde{\mathscr{C}}(0,0,b_1,b_2,n,0;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), \tag{3.19}$$

$$\widetilde{\mathscr{B}_{14}} = \frac{3}{2}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)^2}{q^4}\sum_{n|q} n\tau(n) \sum_{\substack{-\frac{3q}{2}<b_j\leq\frac{3q}{2}\\ j=1,2}} \widetilde{\mathscr{C}}(0,0,b_1,b_2,n,0;q)\prod_{k=1}^{2}\Psi(b_k,q,\beta), \tag{3.20}$$

$$\widetilde{\mathscr{B}_{15}} = 2\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)P_1(n,q) \sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}} \widetilde{\mathscr{C}}(0,0,0,b,n,0;q)\Psi(b,q,\beta), \tag{3.21}$$

$$\widetilde{\mathscr{B}_{16}} = 2\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n)P_2(n,q) \sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}} \widetilde{\mathscr{C}}(0,0,0,b,n,0;q)\Psi(b,q,\beta), \tag{3.22}$$

$$\widetilde{\mathscr{B}_{17}} = \frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)^3}{q^5}\sum_{n|q} n\tau(n) \sum_{-\frac{3q}{2}<b\leq\frac{3q}{2}} \widetilde{\mathscr{C}}(0,0,0,b,n,0;q)\Psi(b,q,\beta), \tag{3.23}$$

$$\widetilde{\mathscr{B}_{18}} = \frac{1}{2}\frac{\widetilde{\phi}_\beta(1)\Psi_0(\beta)^4}{q^6}\sum_{n|q} n\tau(n)P_2(n,q)\widetilde{\mathscr{C}}(0,0,0,0,n,0;q), \tag{3.24}$$

$$\widetilde{\mathscr{B}_{19}} = \frac{1}{2}\frac{\widetilde{\phi}'_\beta(1)\Psi_0(\beta)^4}{q^6}\sum_{n|q} n\tau(n)P_1(n,q)\widetilde{\mathscr{C}}(0,0,0,0,n,0;q), \tag{3.25}$$

$$\widetilde{\mathscr{B}_{20}} = \frac{1}{4}\frac{\widetilde{\phi}''_\beta(1)\Psi_0(\beta)^4}{q^6}\sum_{n|q} n\tau(n)\widetilde{\mathscr{C}}(0,0,0,0,n,0;q), \tag{3.26}$$

where $P_\ell(n,q)$ $(\ell=1,2)$ are defined in (1.3) and (1.4), respectively, $\Phi^{\pm}_\beta(y)$ is defined in (2.30), and

$$\widetilde{\mathscr{C}}(b_1,b_2,b_3,b_4,n,m;q) = \sum_{a=1}^{q}{}^{*}\, S\left(-\overline{a},m;q/n\right)\prod_{j=1}^{4}G(a,b_j;q).$$

Repeating the arguments in Sections 2.3-2.4 and using Lemma 3.3 in [6], we have

$$\begin{aligned}
&\sum_{j=1}^{5}\sum_{q\le P^*}\int_{|\beta|\le\frac{1}{qQ^*}}\widetilde{\mathscr{B}}_j(q,\beta)\mathrm{d}\beta\\
\ll\ & x^{-1+\varepsilon}MP^{*5}+x^{-\frac12+\varepsilon}MP^{*4}+x^{\varepsilon}MP^{*3}+x^{\frac12+\varepsilon}MP^{*2}+x^{1+\varepsilon}MP^{*}\\
\ll\ & x^{1+\varepsilon}MP^{*} \qquad (3.27)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=6}^{17}\sum_{q\le P^*}\int_{|\beta|\le\frac{1}{qQ^*}}\widetilde{\mathscr{B}}_j(q,\beta)\mathrm{d}\beta\\
\ll_\varepsilon\ & x^{\varepsilon}P^{*3}+x^{\frac12+\varepsilon}P^{*2}+x^{1+\varepsilon}P^{*}+x^{\frac32+\varepsilon}\ll x^{\frac32+\varepsilon}. \qquad (3.28)
\end{aligned}$$

Moreover, repeating the arguments in Section 2.5 and using the trivial estimate

$$\widetilde{\mathscr{C}}(b_1,b_2,b_3,b_4,n,m;q)\ll_\varepsilon q^{\frac72+\varepsilon}n^{-\frac12},$$

one has

$$\begin{aligned}
&\sum_{j=18}^{20}\sum_{q\le P^*}\int_{|\beta|\le\frac{1}{qQ^*}}\widetilde{\mathscr{B}}_j(q,\beta)\mathrm{d}\beta\\
=\ & \frac12\mathfrak{I}_0(X)\mathfrak{C}_2x^2+\frac12\mathfrak{I}_1(X)\mathfrak{C}_1x^2+\frac14\mathfrak{I}_2(X)\mathfrak{C}_0x^2+O_\varepsilon\left(x^{2+\varepsilon}P^{*-1}+x^{2+\varepsilon}M^{-1}\right), \qquad (3.29)
\end{aligned}$$

where for $\ell=0,1,2$,

$$\mathfrak{I}_\ell(X)=\int_{-\infty}^{\infty}\left(\int_{X/2}^{X}e(-\beta u)(\log u)^\ell\mathrm{d}u\right)\left(\int_0^1 e(\beta xv^2)\mathrm{d}v\right)^4\mathrm{d}\beta \qquad (3.30)$$

and

$$\mathfrak{C}_\ell=\sum_{q=1}^{\infty}q^{-6}\sum_{n|q}n\tau(n)P_\ell(n,q)\sideset{}{^*}\sum_{a=1}^{q}G(a,0;q)^4S\left(-\overline{a},0;qn^{-1}\right). \qquad (3.31)$$

As in the proof of Theorem 1.1, we take $P^*=x^{\frac12}M^{-\frac12}$. Then Propositon 3.1 follows from (3.5)-(3.7) and (3.27)-(3.31).

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