

Second-Order Identification Capacity of AWGN Channels

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Abstract—In this paper, we establish the second-order randomized identification capacity (RID capacity) of the Additive White Gaussian Noise Channel (AWGNC). On the one hand, we obtain a refined version of Hayashi’s theorem to prove the achievability part. On the other, we investigate the relationship between identification and channel resolvability, then we propose a finer quantization method to prove the converse part. Consequently, the second-order RID capacity of the AWGNC has the same form as the second-order transmission capacity. The only difference is that the maximum number of messages in RID scales double exponentially in the blocklength.

Index Terms—randomized identification, AWGN channels, channel resolvability, quantization.

I. INTRODUCTION

WITH the emergence of IoT applications [1], [2], modern communications in the next-generation wireless network framework (XG) require robust and ultra-reliable low latency information exchange between a large pool of potential smart devices. Numerous XG applications are event-triggered communication systems, such as vehicle-X communication [3]–[5], tactile internet [6], [7], industry 4.0, Online sales, etc. The emergence of novel communication tasks, including control systems [8], the automotive domain [9], watermarking [10]–[12], recommendation systems [13], and other scenarios requiring quick or small checks, presents new challenges. For many of these problems, signaling a user, rather than transmitting a bulk of data, becomes the main task. As such, the identification approach suggested by Ahlswede and Dueck [14] is more suitable than the transmission scheme as studied by Shannon [15]. Some other possible applications of identification codes have been pointed out in [16], [17].

In identification, receiver- i only cares whether message- i is transmitted. Once receiver- i believes message- i is not sent, it does not attempt to decode that message. In randomized identification, message- i is sent by randomly transmitting a codeword from receiver- i ’s codebook, and it was proved that the optimal code size scales double exponentially in the block length with vanishing type-I and type-II error rates. Han and Verdú [18] provided a new perspective named channel resolvability to further discuss the identification capacity. Subsequently, Steinberg [19] proposed a much more tighter converse bound for the randomized identification capacity. Hayashi [20] extended the previous results to wiretap channels

and established the error exponents. Han obtained the identification capacity of the continuous input channels in [21]. The second-order RID capacity of the discrete memoryless channels (DMCs) was demonstrated in [22] which relies on the finiteness assumption of the input alphabet, hence can not apply to continuous input channels directly. Based on constant-weight codes (CWCs) that result from concatenating a CWC initialization with outer linear block codes, Verdú and Wei [23] suggested the first explicit ID code construction.

In the deterministic setup, given a discrete memoryless channel (DMC), the number of messages grows exponentially with the blocklength. R. Ahlswede and Ning Cai determined the deterministic identification capacity (DID capacity) for DMCs [24]. In particular, Jájá [25] showed that the deterministic identification (DI) capacity of a binary symmetric channel is 1 bit per channel use, as one can exhaust the entire input space and assign (almost) all binary n -tuples as codewords. Recently, Salariseddigh et al. established the deterministic identification capacity for AWGN channels and AWGN channels with slow and fast fading coefficients [26], [27]. Li et al. [28] proposed the deterministic identification capacity for block memoryless fading channels without CSI.

Figure 1 provides a brief overview of the connections between our work and previous research. We extend the results presented by Watanabe to the continuous input alphabet, specifically, by determining the second-order RID capacity for the AWGNCs.

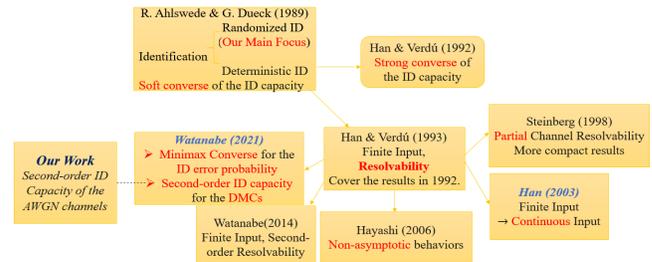


Fig. 1: Research status of the Randomized Identification Problems.

In this paper, we derive the second-order RID capacity of the AWGNC. Our contributions are summarized as follows:

- 1) We propose a refined version of Hayashi's theorem [20], which results in the achievability part.
- 2) We propose a finer quantization method than that in [21] to obtain the converse part.

The remainder of this paper is structured as follows. Section II provides the background of the identification and the resolvability. Section III introduces the main results of the paper: the second-order RID capacity of the AWGNC. Finally, we draw conclusions in Section IV. The main differences between our work and the previous studies are provided in Table I.

Notation Conventions: In this paper, the lowercase letters x, y, z, \dots represent the value of random variables, and uppercase letters X, Y, Z, \dots represent random variables. The probability distribution of random variable X is specified by a cumulative distribution function (CDF) $F_X(x) = \Pr(X \leq x)$, or alternatively, by a probability density function (PDF) $P_X(x)$. $X^n = (X_1, \dots, X_n)$ and $x^n = (x_1, x_2, \dots, x_n)$ are random vectors of length- n and its realization, respectively. \mathcal{X}^n and \mathcal{Y}^n represent the n th Cartesian product of input alphabet and output alphabet. The ℓ^2 -norm of x^n is denoted by $\|x^n\|$. A probability distribution P_{X^n} is called an M -type, if for any integer $M > 0$ and every $x^n \in \mathcal{X}^n$,

$$P_{X^n}(x^n) \in \{0, \frac{1}{M}, \frac{2}{M}, \dots, 1\}.$$

The variational distance between two probability distributions P_{X_1} and P_{X_2} defined on the same measurable space (Ω, \mathcal{F}) is $d(P_{X_1}, P_{X_2}) = 2 \sup_E |P_{X_1}(E) - P_{X_2}(E)|$. Denote $\mathcal{P}(\mathcal{X}^n)$ the set of all distributions supported on \mathcal{X}^n . $\log x$ is the base-2 logarithm of x and \mathbf{I}_n is the $n \times n$ identity matrix.

II. BACKGROUND

In this section, we introduce the identification and resolvability in AWGNC, refer to [14], [18]–[22] for the results in DMCs.

A. Identification via Channels

In this section, we review some basic results of randomized identification. In AWGNC, the received sequence $Y^n = X^n + Z^n$, where $Z^n \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_n)$. Denote the signal-to-noise ratio by $\text{SNR} = \frac{P}{\sigma^2}$, and without loss of generality, we assume $\sigma^2 = 1$. The channel W^n is $W^n(y^n | x^n) = (2\pi)^{-\frac{n}{2}} e^{-\|y^n - x^n\|^2/2}$. The receiver shall identify if message $i \in \{1, \dots, N\}$ was transmitted or not. A randomized identification code is defined by a family $\{(Q_i, \mathcal{D}_i)_{i=1}^N\}$, where $Q_i \in \mathcal{P}(\mathcal{X}^n(P))$ ($i = 1, \dots, N$) are randomized encoders with $\mathcal{X}^n(P) = \{x^n \in \mathcal{X}^n \mid \|x^n\|^2 \leq nP\}$ and P is the power constraint, $\mathcal{D}_i \subset \mathbb{R}^n$ ($i = 1, \dots, N$) are decoding regions.

For a given ID code $(Q_i, \mathcal{D}_i)_{i=1}^N$, we define the type-I error (missed detection error) and type-II error (false activation rate) as

$$P_{\text{I}} \triangleq \max_{1 \leq i \leq N} Q_i W^n(\mathcal{D}_i^c), \quad (1)$$

$$P_{\text{II}} \triangleq \max_{1 \leq j \neq i \leq N} Q_i W^n(\mathcal{D}_j), \quad (2)$$

where $Q_i W^n$ is the output distribution induced by the input distribution Q_i , i.e.,

$$Q_i W^n(y^n) = \sum_{x^n \in \mathcal{X}^n(P)} Q_i(x^n) W^n(y^n | x^n). \quad (3)$$

Note that the codewords $c_i \sim Q_i$ ($i = 1, \dots, N$).

For all $0 < \varepsilon, \delta < 1$ with $0 < \varepsilon + \delta < 1$, an ID code $(Q_i, \mathcal{D}_i)_{i=1}^N$ is an $(n, N, \varepsilon, \delta)$ -ID code if $P_{\text{I}} \leq \varepsilon$ and $P_{\text{II}} \leq \delta$.

Denote $N^*(\varepsilon, \delta | W^n)$ the optimal code size, i.e.,

$$N^*(\varepsilon, \delta | W^n) \triangleq \sup \{N \mid (n, N, \varepsilon, \delta)\text{-ID code exists}\}. \quad (4)$$

Since the decoding regions can be overlapped in randomized identification while must be disjoint in transmission, the RID allows much more messages. Furthermore, it is necessary to emphasize that random coding is one of the main factors responsible for the performance gain. Shannon [15] proved that the optimal message number scales exponentially with n , and provided the transmission capacity in the *exponential scale*, i.e., the channel capacity. While in RID, the capacity in the *exponential scale* is infinite, Ahlswede and Dueck [14] further pointed out that the optimal message number scales double exponentially with n for DMCs, they denoted

$$C_{\text{ID}}(\varepsilon, \delta | W^n) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \log N^*(\varepsilon, \delta | W^n)$$

the RID capacity in the *double-exponential scale*. Surprisingly, $C_{\text{ID}}(\varepsilon, \delta | W^n)$ equals the Shannon capacity in DMCs. The result was extended to AWGNC by Han [21], i.e.,

$$C_{\text{ID}}(\varepsilon, \delta | W^n) = C(P) = \frac{1}{2} \log(1 + P), \quad (5)$$

as long as $0 < \varepsilon + \delta < 1$. Notice that the RID capacity and the channel capacity are only numerically equal, while the meanings of the two are different. In fact, the number of messages supported by the RID far exceeds that in transmission!

Additionally, in [17], the identification code for AWGN is analyzed without randomized encoding. The performance remains superior in terms of transmission, yet with $N^*(\varepsilon, \delta | W^n) = 2^{Rn \log n}$ where R is the optimal rate (*super-exponential* but not *double-exponential*!).

However, the above first-order approximation (5) is only valid when n is sufficiently large, thus we still need to propose more accurate expressions to estimate the performance of identification in the finite length regime. It is worth noting that there exists a constructive proof of finite-length identification codes [29] with a code construction which was analyzed in [30] and implemented in [13].

In the next section, we provide our main results, i.e., the second-order RID capacity of the AWGNC. In order to prove it, we need to introduce the channel resolvability [18] which plays a major role in the proof of the converse bound.

B. Channel Resolvability

Given a target distribution P_{Y^n} , the channel resolvability minimizes the size M of the codebook $c^M = (c_1, \dots, c_M)$

TABLE I: Comparisons between this work and the previous studies.

Properties		Results
R. Ahlswede & G. Dueck (1989) [14]	Finite Input Alphabets, exponentially decay error probability	Soft Converse
T. S. Han & Verdú (1993) [18]	Finite Input & Strong Converse Property	Resolvability & Strong Converse
Y. Steinberg (1998) [19]	Finite Input & Partial Channel Resolvability	Tighter Identification converse bound
T. S. Han (2003) [21]	Finite Input Alphabet \rightarrow Continuous Input Alphabet	$\log \log N^*(\varepsilon, \delta W^n) = nC(P)$
M. Hayashi (2006) [20]	Finite Input & Wiretap Channels	ID Converse & Error Exponents
S. Watatnabe (2022) [22]	Minimax Converse for Identification via arbitrary finite input channels & Second-Order Identification Capacity of the DMCs, $\delta \rightarrow 0$	$\log \log N^*(\varepsilon, \delta W^n) = nC(W) - \sqrt{nV(\varepsilon W)}Q^{-1}(\varepsilon) + o(\sqrt{n})$
This work	Second-order Identification Capacity of the AWGN Channels, $\delta \rightarrow 0$	Theorem 1, $\log \log N^*(\varepsilon, \delta W^n) = nC(P) - \sqrt{nV(P)}Q^{-1}(\varepsilon) + O(\log n)$

such that when the codewords are equiprobably selected, the output distribution

$$P_{Y^n[c^M]} = \frac{1}{M} \sum_{i=1}^M W(Y^n|c_i) \quad (6)$$

approximates the target distribution P_{Y^n} well, apparently, the output distribution $P_{Y^n[c^M]}$ is an M -type.

Definition 1. A resolvability code for input distribution P_{X^n} is called an (n, M, ξ) -resolvability code if

$$\limsup_{n \rightarrow \infty} d(P_{Y^n}, P_{Y^n[c^M]}) \leq \xi, \quad (a)$$

where P_{Y^n} denotes the channel output induced by P_{X^n} .

The optimal code size of resolvability is defined by

$$M^*(\xi|W^n) \triangleq \inf\{M \mid (n, M, \xi)\text{-resolvability code exists for all input distributions } P_{X^n} \in \mathcal{P}(\mathcal{X}^n(P))\}. \quad (b)$$

The channel resolvability capacity of the AWGNC is defined as $C_{\text{RE}}(\xi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log M^*(\xi|W^n)$.

It is worth mentioning that Han [21] pointed out that the channel resolvability capacity equals to the identification capacity of the AWGNC as long as $0 < \varepsilon + \delta + \xi < 1$. A substitute approach for Han and Verdú's resolvability was subsequently proposed by Ahlswede in a more combinatorial manner. This concept is elaborated upon in [31].

C. Quantization

The minimax converse of identification was given by Watanabe [22] for DMCs. The proof relies on the finiteness assumption of the input alphabet. Nevertheless, it becomes impossible to obtain an upper bound on the number of distinct types for continuous input alphabet, as in AWGNC. Therefore, we use quantization method to overcome this difficulty.

In [21], Han proposed the quantization method to derive the first-order identification capacity (5) in AWGNC. Fig 2 shows the basic ideas of Han's quantization method. An alternative to Han's quantization method is suggested in [16], and this approach is employed in [29] to determine the capacity of secure identification codes.

Specifically, the input alphabet is $\mathcal{X}^n(P) = \{x^n \mid \|x^n\|^2 \leq nP\}$. Find the minimal hypercube $V_n(P)$ with edge length $l_n = 2\sqrt{nP}$ covering $\mathcal{X}^n(P)$ and partition $V_n(P)$ into small hypercubes $\Lambda_n^{(i)}$ with edge length Δ_n ($i = 1, \dots, k_n(P)$). The number of small hypercubes is $k_n(P) = (\frac{l_n}{\Delta_n})^n$. We choose a representative point u_i^n in each $\Lambda_n^{(i)}$ and set

$$\mathcal{R}_n(P) = \{u_1^n, \dots, u_{k_n(P)}^n\}.$$

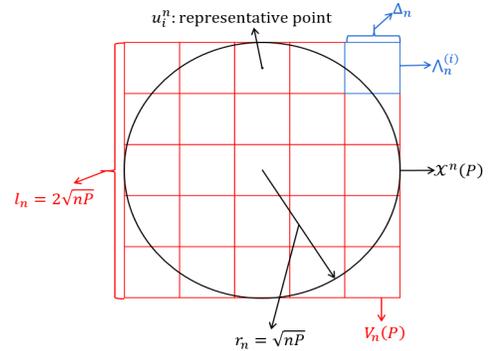


Fig. 2: Intuition behind the quantization method.

For a distribution Q supported on $\mathcal{X}^n(P)$, we denote \bar{Q} supported on $\mathcal{R}^n(P)$ by

$$\bar{Q}(u_i^n) = Q(\Lambda_n^{(i)}) (i = 1, 2, \dots, k_n(P)). \quad (7)$$

When the edge length Δ_n is small enough, the output distribution induced by \bar{Q} approximates the real output distribution induced by Q (see (3)) well. Therefore, the problem boils down to the finite input alphabet $\mathcal{R}_n(P)$, which can be solved as that in DMCs.

III. SECOND-ORDER RID CAPACITY OF THE AWGNC

In this section, we obtain the second-order RID capacity of the AWGNC (Theorem 1). In the achievability part, we choose the uniform distribution on the power shell as the input distribution $P_{X^n}(x^n)$, so the components of the output distribution $P_{Y^n}(y^n)$ induced by $P_{X^n}(x^n)$ are not *i.i.d.*, thus

is complex to analysis. In order to overcome this difficulty, we extend Hayashi's Theorem [20] by utilizing capacity-achieving output distribution $Q_{Y^n}^*(y^n) = \mathcal{N}(0, (1+P)\mathbf{I}_n)$ as the auxiliary output distribution instead of P_{Y^n} . In the converse part, the key idea is to quantize the input alphabet and then convert the problem to a DMC scenario, which is solved by Watanabe [22]. This procedure plays an important role in the derivation of the first-order RID capacity of the AWGNC. In order to get the second-order RID capacity, we directly partition the power shell into small sectors. Compared to the partitions on hypercubes in [21], our quantization method is more accurate.

Let $Q^{-1}(\cdot)$ be the inverse of the complementary CDF of the standard normal distribution. In [22], the second-order RID capacity of the DMCs is proved to have the same form as the transmission capacity. It was shown by Polyanskiy et.al. [32] that the second-order transmission capacity of the AWGNC is

$$\log M_{\text{T}}^*(\epsilon|W^n) = nC(P) - \sqrt{nV(P)}Q^{-1}(\epsilon) + O(\log n),$$

where $M_{\text{T}}^*(\epsilon|W^n)$ is the optimal code size of the transmission code with the maximal block error rate less than ϵ and $V(P) = \log^2 e \frac{P(P+2)}{2(P+1)^2}$ is the channel dispersion.

We would naturally ask: whether the second-order RID capacity and transmission capacity still have the same form in AWGNC? Our answer is "yes".

Theorem 1. Given an AWGNC W^n , when the type-II error δ vanishes, $\forall 0 < \epsilon < 1$, the optimal code size of RID (defined in (4)) is

$$\log \log N^*(\epsilon, \delta|W^n) = nC(P) - \sqrt{nV(P)}Q^{-1}(\epsilon) + O(\log n), \quad \delta \rightarrow 0. \quad (8)$$

Compared with the first-order approximation (5) given by Han [21], we provide a more accurate approximation which gives the exact second-order term in the asymptotic expansion. We divide the proof into two parts: the achievability part and the converse part.

Proof of achievability. The key point is the extension of Hayashi's Theorem 1 [20] by utilizing the auxiliary output distribution. The achievability part is a result of Lemma 1.

Lemma 1. Given an arbitrary channel W^n and input distribution P_{X^n} , denote by P_{Y^n} the output distribution induced by P_{X^n} . Assume that real numbers $c, c', d, d', \tau, \zeta > 0$ satisfy

$$\begin{aligned} \zeta \log\left(\frac{1}{\tau} - 1\right) &> \log 2 + 1, \quad 0 < \tau < \frac{1}{3}, \quad 0 < \zeta < 1, \\ 1 &> \frac{1}{c} + \frac{1}{c'}, \quad f := 1 - \frac{1}{d} - \frac{1}{d'} > 0. \end{aligned}$$

Then, for any integer $M > 0$ and real number $K > 0$, there exists an (N, ϵ, δ) -ID code such that

$$\epsilon \leq cd\Pr(\tilde{i}(X; Y) \leq \log K), \quad (9)$$

$$\delta \leq \zeta + c'd' \frac{1}{K} \lceil \frac{M}{f} \rceil, \quad (10)$$

$$N = \lfloor \frac{e^{\tau M}}{Me} \rfloor \quad (11)$$

provided that

$$cd\Pr(\tilde{i}(X; Y) \leq \log K) + c'd' \frac{1}{K} \lceil \frac{M}{f} \rceil < 1, \quad (12)$$

where $\tilde{i}(X; Y) \triangleq \log \frac{W(Y|X)}{Q_Y(Y)}$ and $Q_Y(Y)$ is an auxiliary distribution.

In the following, we choose appropriate parameters in Lemma 1 to prove the achievability part.

Let W^n be the length- n AWGNC. Choose $Q_{Y^n}^*$ as the auxiliary output distribution, namely,

$$Q_{Y^n}^*(y^n) = \mathcal{N}(0, (1+P)\mathbf{I}_n). \quad (13)$$

Applying the Central Limit Theorem [33], we have

$$\begin{aligned} \frac{\tilde{i}(x^n; Y^n) - nC(P)}{\sqrt{nV(P)}} &= \frac{\log e}{2\sqrt{nV(P)}} [\frac{\|Y^n\|^2}{1+P} - \|Z^n\|^2] \\ &\Rightarrow \mathcal{N}(0, 1), \end{aligned} \quad (14)$$

where $\tilde{i}(x^n; Y^n) = \log \frac{W^n(Y^n|x^n)}{Q_{Y^n}^*(Y^n)}$ and " \Rightarrow " means weak convergence.

Set $c = d = 1 + \frac{2}{n}$, $c' = d' = n + 2$, $\tau = \frac{1}{n+2}$, $\zeta = \frac{1+\log 2}{\log n}$. For $K > 0$, we apply Lemma 1 by setting $M = \lceil \frac{\log K}{(n+2)^4} \rceil$, then, there exist a constant $F > 0$ and a sequence of $(N^*(\epsilon, \delta|W^n), \epsilon_n, \delta_n)$ -ID codes such that

$$\log \log N^*(\epsilon, \delta|W^n) \geq \log K - F \log n,$$

$$\epsilon_n \leq (1 + \frac{2}{n})^2 \Pr(\tilde{i}(x^n; Y^n) \leq \log K),$$

$$\delta_n \leq \frac{1 + \log 2}{\log n} + \frac{2}{n+2},$$

if

$$(1 + \frac{2}{n})^2 \Pr(\tilde{i}(x^n; Y^n) \leq \log K) + \frac{2}{n+2} \leq 1.$$

Lemma 2. (Berry-Esseen Theorem for Functions of i.i.d. Random Vectors [33], [34]). Assume that X_1^k, \dots, X_n^k are \mathbb{R}^k -valued, zero-mean, i.i.d. random vectors with positive definite covariance $\text{Cov}(X_1^k)$ and finite third absolute moment $\xi := \mathbb{E}[\|X_1^k\|^3]$. Let $\mathbf{f}(\mathbf{x})$ be a vector-valued function from \mathbb{R}^k to \mathbb{R}^l that is also twice continuously differentiable in a neighborhood of $\mathbf{x} = \mathbf{0}$. Let $\mathbf{J} \in \mathbb{R}^{l \times k}$ be the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{0}$, i.e., its elements are

$$J_{ij} = \left. \frac{\partial f_i(\mathbf{x})}{\partial x_j} \right|_{\mathbf{x}=\mathbf{0}},$$

where $i = 1, \dots, l$ and $j = 1, \dots, k$. Then, for every $n \in \mathbb{N}$, we have

$$\sup_{\mathcal{C} \in \mathcal{C}_l} \left| \Pr\left(\mathbf{f}\left(\frac{1}{n} \sum_{i=1}^n X_i^k\right) \in \mathcal{C}\right) - \Pr(Z^l \in \mathcal{C}) \right| \leq \frac{c}{\sqrt{n}}$$

where $c > 0$ is a finite constant, and Z^l is a Gaussian random vector in \mathbb{R}^l with mean vector and covariance matrix respectively given as

$$\mathbb{E}[Z^l] = \mathbf{f}(\mathbf{0}), \quad \text{and} \quad \text{Cov}(Z^l) = \frac{\mathbf{J} \text{Cov}(X_1^k) \mathbf{J}'}{n}.$$

Denote for arbitrary κ

$$\log K = nC(P) - \kappa\sqrt{nV(P)}.$$

It is easily to check that for every $x^n \in \mathbb{R}^n$ such that $\|x^n\|_2^2 = nP$, the moments of $\tilde{i}(x^n; Y^n)$ satisfy that

$$\begin{aligned} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \log \frac{W(Y_i | x_i)}{Q_Y^*(Y_i)} \right] &= C(P), \\ \text{Var} \left[\frac{1}{n} \sum_{i=1}^n \log \frac{W(Y_i | x_i)}{Q_Y^*(Y_i)} \right] &= \frac{V(P)}{n}, \\ \mathbb{E} \left| \frac{1}{n} \sum_{i=1}^n \log \frac{W(Y_i | x_i)}{Q_Y^*(Y_i)} \right|^3 &< +\infty. \end{aligned}$$

According to Lemma 2, we have

$$\left| \Pr(\tilde{i}(x^n; Y^n) \leq \log K) - Q(\kappa) \right| \leq \frac{B}{\sqrt{n}}, \quad (15)$$

where $B > 0$ is a finite constant.

For sufficiently large n , let

$$\kappa = Q^{-1}\left(\left(1 + \frac{2}{n}\right)^{-2}\varepsilon - \frac{B}{\sqrt{n}}\right).$$

Then, from (15) we obtain

$$\varepsilon_n \leq \left(1 + \frac{2}{n}\right)^2 \Pr(\tilde{i}(x^n; Y^n) \leq \log K) \leq \varepsilon. \quad (16)$$

Hence,

$$\begin{aligned} \log \log N^*(\varepsilon, \delta | W^n) &\geq nC(P) - \sqrt{nV(P)}Q^{-1}(\varepsilon) \\ &\quad + O(\log n), \quad \delta \rightarrow 0. \end{aligned} \quad (17)$$

Proof of converse. In this part, we develop a finer quantization method depicted in Fig 3.

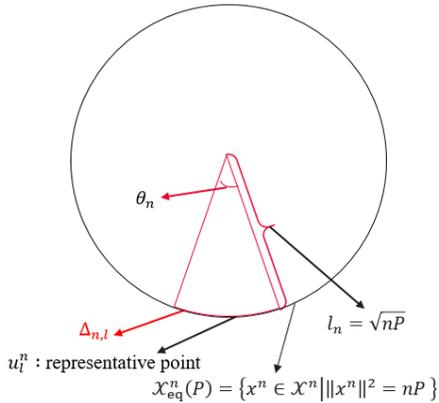


Fig. 3: A finer quantization method.

Let us provide some intuition for the quantization method above. First, we partition the power shell $\mathcal{X}_{\text{eq}}^n(P)$ into small sectors $\Delta_{n,l} (l = 1, \dots, m_n(P))$ with angle θ_n . Then, let

$$\mathcal{V}_n(P) = \{u_1^n, \dots, u_{m_n(P)}^n\}$$

and denote \bar{Q} supported on $\mathcal{V}_n(P)$ by

$$\bar{Q}(u_l^n) = Q(\Delta_{n,l}) (l = 1, 2, \dots, m_n(P)), \quad (c)$$

in other words, \bar{Q} is obtained by concentrating the mass of Q in $\Delta_{n,l}$ to the representative point u_l^n .

When the angle θ_n is small, the output distribution induced by \bar{Q} approximates the real output induced by Q (see (3) by replacing $\mathcal{X}^n(P)$ by $\mathcal{X}_{\text{eq}}^n(P)$) well. Fig 3 proposes a finer quantization which leads to the second-order RID capacity.

Remark 1. Define an n -dimensional spherical coordinate system determined by angles $\phi_1, \dots, \phi_{n-1}$ and radius l_n ,

$$\begin{cases} x_1 = l_n \cos \phi_1 \\ x_2 = l_n \sin \phi_1 \cos \phi_2 \\ \dots \\ x_{n-2} = l_n \sin \phi_1 \dots \sin \phi_{n-2} \cos \phi_{n-1} \\ x_{n-1} = l_n \sin \phi_1 \dots \sin \phi_{n-2} \sin \phi_{n-1} \end{cases}$$

where $0 \leq \phi_m \leq \pi (1 \leq m \leq n-2)$, $0 \leq \phi_{n-1} \leq 2\pi$.

Then, the number of the sectors is

$$m_n(P) = \left(\frac{\pi}{\theta_n}\right)^{n-2} \frac{2\pi}{\theta_n}. \quad (18)$$

Step 1. Let $l_n = \sqrt{nP}$ be the radius of the power shell $\mathcal{X}_{\text{eq}}^n(P)$ and substitute $\theta_n = \exp(-n)$ into (18), we have

$$\frac{\log \log m_n(P)}{\log n} \rightarrow C, \quad (19)$$

where $C > 0$ is a constant.

Here, we give a brief proof of (19).

Substituting θ_n into (18), so that $m_n(P)$ can be calculated as

$$m_n(P) = \frac{2\pi^{n-1}}{(\exp(-n))^{n-1}}. \quad (20)$$

The result shown in (19) immediately follows by (20).

Step 2. Using the quantization distribution $\bar{Q}(\cdot)$ given by (c) and the Pinsker's inequality, we can show that

$$d(QW^n, \bar{Q}W^n) \leq \sqrt{P}n^{3/2}e^{-n}. \quad (21)$$

Now, we give the proof of (21).

For any $x^n, u_l^n \in \Delta_{n,l}$, from the definition of the KL divergence between two multivariate Gaussian distributions, we obtain

$$\begin{aligned} D(W^n(\cdot | x^n) || W^n(\cdot | u_l^n)) &= \frac{1}{2}n\|x^n - u_l^n\|^2 \\ &\leq \frac{n(n-1)}{2}(\sqrt{nP}e^{-n})^2. \end{aligned} \quad (22)$$

By the Pinsker's inequality [35] and the triangle inequality, we have

$$\begin{aligned}
d(QW^n, \bar{Q}W^n) &= 2 \sup_{\mathcal{B} \subset \mathbb{R}^n} |QW^n(\mathcal{B}) - \bar{Q}W^n(\mathcal{B})| \\
&\leq 2 \sum_{l=1}^{m_n(P)} \sum_{x^n \in \Delta_{n,l}} |W^n(\mathcal{B}|x^n) - W^n(\mathcal{B}|u_l^n)| Q(x^n) \\
&\leq 2 \sum_{l=1}^{m_n(P)} \frac{1}{2} \sqrt{2D(W^n(\cdot|x^n) \| W^n(\cdot|u_l^n))} \bar{Q}(u_l^n) \\
&\leq \sqrt{P} n^{3/2} e^{-n}. \tag{23}
\end{aligned}$$

Step 3. Next, we investigate the relation between ID and resolvability [14], [18]–[22]. We have the following lemma.

Lemma 3. For arbitrary $0 < \varepsilon, \delta, \xi < 1$ with $0 < \varepsilon + \delta + \xi + \sqrt{P} n^{3/2} e^{-n} < 1$, we have

$$\log \log N^*(\varepsilon, \delta | W^n) \leq \log M^*(\xi | W^n) + O(\log n), \tag{24}$$

where $N^*(\varepsilon, \delta | W^n)$ and $M^*(\xi | W^n)$ are the optimal code size of the (ε, δ) -ID code and the ξ -resolvability code, respectively.

Proof. Let $N_n = N^*(\varepsilon, \delta | W^n)$, $M_n = M^*(\xi | W^n)$, by definition, there exist an $(n, N_n, \varepsilon_n, \delta_n)$ -ID code and M_n -type probability distributions $\tilde{Q}_j (j = 1, \dots, N_n)$ supported on $\mathcal{X}_{\text{eq}}^n(P)$ satisfying

$$\varepsilon_n \leq \varepsilon, \quad \delta_n \leq \delta, \tag{25}$$

$$d(Q_j W^n, \tilde{Q}_j W^n) \leq \xi, \tag{26}$$

for all $j = 1, \dots, N_n$, where Q_j is the random encoder.

Let $\tilde{\tilde{Q}}_j (j = 1, \dots, N_n)$ be the quantization distribution of the M_n -type distribution \tilde{Q}_j . Therefore, $\tilde{\tilde{Q}}_j$ s are also M_n -types. By (21), we have

$$d(\tilde{Q}_j W^n, \tilde{\tilde{Q}}_j W^n) \leq \sqrt{P} n^{3/2} e^{-n}. \tag{27}$$

Combine (26) and (27), we obtain

$$d(Q_j W^n, \tilde{\tilde{Q}}_j W^n) \leq \xi + \sqrt{P} n^{3/2} e^{-n}. \tag{28}$$

Suppose that there exists $j \neq k$, such that $\tilde{\tilde{Q}}_j = \tilde{\tilde{Q}}_k$, then, by the triangular inequality and the definition of the TV distance,

$$2(1 - \varepsilon - \delta) \leq d(Q_j W^n, Q_k W^n) \leq 2\xi + 2\sqrt{P} n^{3/2} e^{-n},$$

i.e., $\varepsilon + \delta + \xi \pm \sqrt{P} n^{3/2} e^{-n} \geq 1$, which is a contradiction!

Therefore, $\tilde{\tilde{Q}}_j (j = 1, \dots, N_n)$ must be distinct M_n -type distributions, since the number of distinct M_n -types is upper bounded by $m_n(P)^{M_n}$. Hence, it must hold that $N_n \leq m_n(P)^{M_n}$, which implies

$$\log \log N_n \leq \log M_n + \log \log m_n(P). \tag{29}$$

By (19)

$$\log \log N_n \leq \log M_n + O(\log n). \tag{30}$$

This completes the proof of (24). \square

A channel $W = \{(\mathcal{X}, \mathcal{F}), (\mathcal{Y}, \mathcal{G}), K\}$ is defined by an input alphabet \mathcal{X} with σ -algebra \mathcal{F} , an output alphabet \mathcal{Y} with σ -algebra \mathcal{G} , and a stochastic kernel K that specifies the stochastic transition between the input and output alphabets. Subsequently, the lemma presented below provides a second-order achievability bound for the channel resolvability capacity, paving the way for establishing the second-order RID capacity for AWGNCs.

Lemma 4. (Frey's Theorem 3 [36]) Given a channel $W = \{(\mathcal{X}, \mathcal{F}), (\mathcal{Y}, \mathcal{G}), K\}$ and an input distribution Q_X such that the information density $i(X; Y)$ has finite central second moment V and finite absolute third moment ρ , $\xi > 0$ and $c > 1$, suppose the rate R depends on n in the following way:

$$R = I(X; Y) + \sqrt{\frac{V}{n}} Q^{-1}(\xi) + c \frac{\log n}{n}.$$

Then, for any $d \in (0, c - 1)$ and n that satisfy $n^{(c-d)/2} \geq 6$, we have

$$\begin{aligned}
P_e(\|P_{Y^n|cM} - QW^n\|_2 \geq \mu(1 + \frac{1}{\sqrt{n}}) + \frac{1}{\sqrt{n}}) \\
\leq \exp(-\frac{1}{3} n \mu \exp(nR)) \\
+ (\frac{7}{6} + \sqrt{3\pi/2} \exp(\frac{3}{4})) \exp(-n^{\frac{1}{2}(c-d-1)}),
\end{aligned}$$

where

$$\mu := Q(Q^{-1}(\xi) + d \frac{\log n}{\sqrt{nV}}) + \frac{\rho}{V^{\frac{3}{2}} \sqrt{n}}$$

tends to ξ for $n \rightarrow \infty$ and $\|x\|_2$ denote the ℓ^2 -norm of x .

Remark 2. Given that the aforementioned lemma applies to any channel, when we consider the channel as an AWGNC, we can derive the converse bound for the second-order RID capacity of the AWGNCs.

Step 4. Let

$$\xi = 1 - \varepsilon - \delta - \sqrt{P} n^{3/2} e^{-n}. \tag{31}$$

Let $\delta \rightarrow 0$, i.e., the type-II error vanishes, from (24) and Lemma 4, we obtain

$$\begin{aligned}
\log \log N^*(\varepsilon, \delta | W^n) &\leq \log M^*(\xi | W^n) + O(\log n) \\
&\leq nC(P) + \sqrt{nV(P)} Q^{-1}(1 - \varepsilon - \sqrt{P} n^{3/2} e^{-n}) + O(\log n) \\
&= nC(P) - \sqrt{nV(P)} Q^{-1}(\varepsilon + \sqrt{P} n^{3/2} e^{-n}) + O(\log n) \\
&\leq nC(P) - \sqrt{nV(P)} Q^{-1}(\varepsilon) + O(\log n), \tag{32}
\end{aligned}$$

where the last inequality is due to the Taylor expansion

$$\begin{aligned}
&Q^{-1}(\varepsilon + \sqrt{P} n^{3/2} e^{-n}) \\
&= Q^{-1}(\varepsilon) - \underbrace{\frac{dQ^{-1}(x)}{dx}}_{O(1)} \Big|_{x=\varepsilon} \sqrt{P} n^{3/2} e^{-n} + o(n^{3/2} e^{-n}) \\
&\geq Q^{-1}(\varepsilon) + o(\frac{1}{\sqrt{n}}).
\end{aligned}$$

Thus, we prove the converse part.

The desired results follow from (17) and (32). \square

IV. CONCLUSION

The second-order RID capacity of the AWGNC has been investigated in this paper. It has been proved that the second-order RID capacity have the same form as the transmission capacity in AWGNC. An extension of Hayashi's Theorem to the auxiliary output distribution leads to the achievability part and a finer quantization method has been proposed in the proof of the converse part which gives the $O(\log n)$ term. As a future direction, it is tempting to apply the quantization approach to other identification problems. Furthermore, it remains an open problem to determine the second-order (ε, δ) -ID capacity of the AWGNC for non-vanishing type-II error δ , i.e., $\delta \geq \varepsilon_0$ with $\varepsilon_0 > 0$ a positive constant. It is also an interesting work to derive the third-order (ε, δ) -ID capacity of the AWGNC.

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