# Efficiency analysis for the Perron vector of a reciprocal matrix 

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#### Abstract

In prioritization schemes, based on pairwise comparisons, such as the Analytical Hierarchy Process, it is necessary to extract a cardinal ranking vector from a reciprocal matrix that is unlikely to be consistent. It is natural to choose such a vector only from efficient ones. One of the most used ranking methods employs the (right) Perron eigenvector of the reciprocal matrix as the vector of weights. It is known that the Perron vector may not be efficient. Here, we focus


[^0]on extending arbitrary reciprocal matrices and show, constructively, that two different extensions of any fixed size always exist for which the Perron vector is inefficient and for which it is efficient, with the following exception. If $B$ is consistent, any reciprocal matrix obtained from $B$ by adding one row and one column has efficient Perron vector. As a consequence of our results, we obtain families of reciprocal matrices for which the Perron vector is inefficient. These include known classes of such matrices and many more. We also characterize the 4 -by-4 reciprocal matrices with inefficient Perron vector. Some prior results are generalized or completed.

Keywords: decision analysis, efficient vector, extension, Perron vector, reciprocal matrix.

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## 1 Introduction

The Analytic Hierarchy process was introduced in [19] and is used in decision making. It is based upon "reciprocal matrices" that represent pair-wise ratio comparisons among several alternatives. Such matrices also arise in other multi-criterion decision making models.

An $n$-by- $n$ entry-wise positive matrix $A=\left[a_{i j}\right]$ is called reciprocal if $a_{j i}=\frac{1}{a_{i j}}$, when $1 \leq i, j \leq n$. We denote by $\mathcal{P} \mathcal{C}_{n}$ the set of all such matrices. A matrix $A \in \mathcal{P} \mathcal{C}_{n}$ is further said to be consistent if $a_{i j} a_{j k}=a_{i k}$ for all $i, j, k$ (otherwise it is inconsistent). This is the case if and only if there is a positive vector $w=\left[\begin{array}{lll}w_{1} & \ldots & w_{n}\end{array}\right]^{T}$ such that $a_{i j}=\frac{w_{i}}{w_{j}}$ for all $i, j$. Such a vector $w$ is unique up to a factor of scale and cardinally ranks the alternatives. Any matrix in $\mathcal{P C}_{2}$ is consistent. When $n>2$, consistency of the ratio comparisons is unlikely. However, a cardinal ranking vector, also called a weight vector, should still be obtained from a reciprocal matrix [8, 10, 17].

Many ways of constructing a weight vector from a reciprocal matrix $A$ have been proposed. The classical proposal for such a vector is the (right) Perron vector [19, 20]. Other proposals include the (entry-wise) geometric mean of the columns [5, 12] and any weighted geometric mean of columns of a reciprocal matrix [14].

An important property that a weight vector obtained from a reciprocal matrix should have is efficiency (also called Pareto optimality). Denote by $\mathcal{V}_{n}$ the set of positive $n$-vectors. A vector $w=\left[\begin{array}{lll}w_{1} & \ldots & w_{n}\end{array}\right]^{T} \in \mathcal{V}_{n}$ is
called efficient for $A=\left[a_{i j}\right] \in \mathcal{P C}_{n}$ [5] if, for every other positive vector $v=\left[\begin{array}{lll}v_{1} & \ldots & v_{n}\end{array}\right]^{T} \in \mathcal{V}_{n}$,

$$
\left|a_{i j}-\frac{v_{i}}{v_{j}}\right| \leq\left|a_{i j}-\frac{w_{i}}{w_{j}}\right|, \text { for all } i, j=1, \ldots n,
$$

implies $v$ is a positive multiple of $w$, i.e. no other consistent matrix approximating $A$ is unambiguously better than the one associated with $w$. (Above $|\cdot|$ denotes the absolute value of a real number.) We denote the set of all efficient vectors for $A$ by $\mathcal{E}(A)$.

The efficient vectors for a consistent matrix are the positive multiples of any of its columns (projectively unique). When $A$ is inconsistent, there are infinitely many (non-proportional) efficient vectors for $A$. Clearly, any positive multiple of an efficient vector for $A$ is still efficient.

Vector efficiency has been widely studied for several years. It is known that a vector $w$ is efficient for $A$ if and only if a certain directed graph (digraph) $G(A, w)$, constructed from $A$ and $w$, is strongly connected [5] (see Section (2.3). Any weighted geometric mean of columns of a reciprocal matrix is efficient [5, 14]. The Perron vector of a reciprocal matrix may or may not be efficient. Numerical studies show that the Perron vector is often inefficient in low dimensions but that this quickly becomes rare in higher dimensions (see acknowledgement). A structured class of reciprocal matrices for which inefficiency of the Perron vector occurs was given in [6]. In [1, 2] the authors show that the Perron vector of some perturbed consistent matrices, more precisely, reciprocal matrices obtained from a consistent matrix by changing at most two entries above the main diagonal (and the corresponding reciprocal entries), is efficient. In particular, any matrix in $\mathcal{P} \mathcal{C}_{3}$ has efficient Perron vector, as it is obtained from a consistent matrix by changing a pair of reciprocal entries [15]. When three pairs of reciprocal entries are changed, inefficiency may occur (see [11]). Other developments have been made concerning efficiency of a vector for a reciprocal matrix. In [9, 13], all efficient vectors for the perturbed consistent matrices mentioned above have been described. Recently, a method to generate inductively all efficient vectors for a reciprocal matrix was provided [16]. Several other aspects of efficiency have been studied (see [3, 4, [5, 6, [7]).

Incomplete reciprocal matrices appear in many decision problems. The unknown entries should be estimated in order to obtain a complete reciprocal matrix from which a weight vector is extracted. When using the Perron
vector as the weights, it is important that the Perron vector of the completed matrix is efficient. In this paper we show that any reciprocal matrix $A$ can be extended to one whose Perron vector is efficient, by adding one row (and the reciprocal column). If $A$ is inconsistent, an extension with inefficient Perron vector also exists. However, if $A$ is consistent, we show that any extension of $A$ resulting from adding one row (and the reciprocal column) has efficient Perron vector, though inefficiency can always occur by adding two (or more) rows and columns. We give procedures to construct such extensions. Structured classes of reciprocal matrices with inefficient Perron vector are provided. These include the one in [6]. In addition, we give a characterization of all 4-by-4 reciprocal matrices whose Perron vector is inefficient.

The paper is organized as follows. In Section 2 we introduce some notation and known results that will be helpful throughout. Some new related observations are also made. In particular, a converse of a result in [16] is presented (Theorem (8). In Section 3 we give some technical lemmas that will be used in the proofs of the main results. In Section 4 we show that there is one, and only one, reciprocal matrix $A \in \mathcal{P} \mathcal{C}_{n}$ with prescribed Perron vector and a given principal submatrix $B \in \mathcal{P C}_{n-1}$ (Theorem 16). In Section 5 we show that any matrix $B \in \mathcal{P} \mathcal{C}_{n-1}$ can be extended to a matrix in $\mathcal{P} \mathcal{C}_{n}$ with inefficient Perron vector, unless $B$ is consistent, in which case we prove that such an extension is not possible (Theorem 25), generalizing some recent results. We start by studying the possible extensions to a matrix with constant row sums and inefficient Perron vector (Theorem 19) and then, based on this result, give a procedure to construct reciprocal matrices of arbitrary size with a prescribed principal submatrix and inefficient Perron vector. We also characterize the matrices in $\mathcal{P C}_{4}$ with inefficient Perron vector (Theorem (30). In Section 6 we show constructively that any matrix $B \in \mathcal{P C}_{n-1}$ can be extended to a matrix in $\mathcal{P C}_{n}$ with efficient Perron vector (Theorem (33). Several examples illustrating the theoretical results are provided. We conclude in Section 7 with a summary and some remarks.

## 2 Notation and basic lemmas

### 2.1 Notation

We start by introducing some additional notation used throughout. We denote by $M_{n}$ the set of all $n$-by- $n$ real matrices.

For $A=\left[a_{i j}\right] \in M_{n}$, the principal submatrix of $A$ determined by deleting (by retaining) the rows and columns indexed by a subset $K \subset\{1, \ldots, n\}$ is denoted by $A(K)(A[K])$; we abbreviate $A(\{i\})$ as $A(i)$. Similarly, if $w$ is a vector, we denote by $w(K)$ the vector obtained from $w$ by deleting the entries indexed by $K$ and abbreviate $w(\{i\})$ as $w(i)$. Note that, if $A$ is reciprocal (consistent) then so are $A(K)$ and $A[K]$.

By $\mathbf{J}_{m, n}$ we denote the $m$-by- $n$ matrix with all entries equal to 1 . We write $\mathbf{J}_{n}$ for $\mathbf{J}_{n, n}$ and $\mathbf{e}_{n}$ for the column vector $\mathbf{J}_{n, 1}$. By $I_{n}$ we denote the identity matrix of size $n$.

Given a vector $w$, we denote by $\operatorname{diag}(w)$ the diagonal matrix with diagonal $w$. If $w$ is positive, we say that $\operatorname{diag}(w)$ is a positive diagonal matrix.

### 2.2 Reciprocal matrices and the Perron vector

We recall from Perron-Frobenius theory that the spectral radius of a positive square matrix $A$ is a simple eigenvalue of $A$ and there is a positive associated (right) eigenvector, which is called the Perron vector of $A$ (with a possible normalization as, for example, having the last entry equal to 1) [18]. In fact here, for convenience, we refer to the Perron vector of $A$ as any positive eigenvector of $A$ (all positive eigenvectors of $A$ are positive multiples of one another).

The following lemma, stated in the context of reciprocal matrices, can be easily verified. Observe that, if $A \in \mathcal{P} \mathcal{C}_{n}$ is subjected to either a positive diagonal similarity or a permutation similarity, or both (a monomial similarity), we get a matrix $A^{\prime}$ in $\mathcal{P} \mathcal{C}_{n}$. Moreover, if $A$ is consistent then so is $A^{\prime}$.

Lemma 1 Let $A \in \mathcal{P C}_{n}$ with Perron vector $w$. Let $D \in M_{n}$ be a positive diagonal matrix and let $P \in M_{n-1}$ be a permutation matrix. Let $S$ be either $D$ or $P \oplus[1]$. Then $S w$ is the Perron vector of $S A S^{-1}$. Moreover, $\left(S A S^{-1}\right)(n)=$ $S(n) A(n) S^{-1}(n)$. If $w(n)$ is the Perron vector of $A(n)$ then $(S w)(n)$ is the Perron vector of $\left(S A S^{-1}\right)(n)$.

From the next known lemma, whose proof we include for completeness, it follows the important fact that any reciprocal matrix is diagonally similar to a unique reciprocal matrix with constant row sums.

Lemma 2 For any positive matrix $A \in M_{n}$, there is a unique (up to a positive scalar multiple) positive diagonal matrix $D \in M_{n}$ such that $D A D^{-1}$ has Perron vector $\mathbf{e}_{n}$, that is, has constant row sums.

Proof. Let $w$ be the Perron vector of $A$ and $D^{-1}=\operatorname{diag}(w)$. Since $D w=\mathbf{e}_{n}$ and, by Lemma 1, $D w$ is the Perron vector of $D A D^{-1}$, the existence of $D$ follows. As for the uniqueness, suppose that, for some positive diagonal matrix $D^{\prime}$, the matrix $D^{\prime} A\left(D^{\prime}\right)^{-1}$ has Perron vector $\mathbf{e}_{n}$. Since $D^{\prime} w$ is the Perron vector of $D^{\prime} A\left(D^{\prime}\right)^{-1}$, it follows that $D$ and $D^{\prime}$ are equal (up to a positive multiple).

In what follows we give a sharp lower bound for the sum of all the entries of a reciprocal matrix.

Lemma 3 The sum of the entries of $A \in \mathcal{P C}_{n}$ is at least $n^{2}$, with equality if and only if $A=\mathbf{J}_{n}$.

Proof. Let $A=\left[a_{i j}\right] \in \mathcal{P} \mathcal{C}_{n}$. For any $i, j \in\{1, \ldots, n\}$, with $i>j$, we have

$$
a_{i j}+a_{j i}=a_{i j}+\frac{1}{a_{i j}} \geq 2
$$

with equality if and only if $a_{i j}=a_{j i}=1$. Since there are $\frac{n^{2}-n}{2}$ such pairs $i, j$ and the diagonal entries of $A$ are 1 , the sum of the entries of $A \in \mathcal{P C}_{n}$ is at least $n+2 \frac{n^{2}-n}{2}=n^{2}$, with equality if and only if all entries of $A$ are equal to 1.

Remark 4 From Lemma图, it follows that, if $r_{1} \geq \cdots \geq r_{n}$ are the row sums of $A \in \mathcal{P C}_{n}$, then $r_{1}+\cdots+r_{n} \geq n^{2}$. This implies $r_{1} \geq n$, with equality if and only if $r_{1}=\cdots=r_{n}=n$.

Since, by Lemma2, any $A \in \mathcal{P C}_{n}$ is similar to a reciprocal matrix $A^{\prime}$ with constant row sums (equal to the Perron eigenvalue of $A$ ), the well-known fact that the Perron eigenvalue of a reciprocal matrix $A \in \mathcal{P} \mathcal{C}_{n}$ is greater than or equal to $n$ follows from Remark (4. Moreover, the Perron eigenvalue is $n$ if and only if all row sums of $A^{\prime}$ are $n$, which implies $A^{\prime}=\mathbf{J}_{n}$, by Lemma 3, that is, $A$ is consistent.

Lemma 5 Let $A \in \mathcal{P C}_{n}$ be a consistent matrix. If $r$ is the smallest row sum of $A$ then $r \leq n$, with equality if and only if $A=\mathbf{J}_{n}$.

Proof. Let $A=\left[a_{i j}\right]$ and $r_{i}$ be the sum of the entries in the $i$ th row of $A$. Since $A$ is consistent, there are $w_{1}, \ldots, w_{n}>0$ such that $a_{i j}=\frac{w_{i}}{w_{j}}$ for all $i, j=1, \ldots, n$. Let $l$ be such that $\min _{i=1, \ldots, n} w_{i}=w_{l}$. Then, the smallest row sum of $A$ is

$$
r_{l}=\frac{w_{l}}{w_{1}}+\cdots+\frac{w_{l}}{w_{n}} \leq n
$$

with equality if and only if all $w_{i}$ 's are equal, that is, $A=\mathbf{J}_{n}$.
Note that the claim in Lemma 5 is not true for an arbitrary $A \in \mathcal{P} \mathcal{C}_{n}$.

### 2.3 Results about efficiency

In [5] the authors proved a useful result that gives a characterization of efficiency in terms of a certain digraph. A shorter and matricial proof of this result can be found in [14].

Given $A \in \mathcal{P C}_{n}$ and $w=\left[\begin{array}{lll}w_{1} & \cdots & w_{n}\end{array}\right]^{T} \in \mathcal{V}_{n}$, define $G(A, w)$ as the digraph with vertex set $\{1, \ldots, n\}$ and a directed edge $i \rightarrow j, i \neq j$, if and only if $\frac{w_{i}}{w_{j}} \geq a_{i j}$.

Lemma 6 [5] Let $A \in \mathcal{P C}_{n}$ and $w \in \mathcal{V}_{n}$. The vector $w$ is efficient for $A$ if and only if $G(A, w)$ is a strongly connected digraph, that is, for all pairs of vertices $i, j$, with $i \neq j$, there is a directed path from $i$ to $j$ in $G(A, w)$.

We notice some relevant facts used later. A digraph $G$ is strongly connected if and only if its adjacency matrix is irreducible [18]. Thus, if $G(A, w)$ is not strongly connected, the matrix $\left[\frac{w_{i}}{w_{j}}\right]-A$ is permutationally similar to a matrix of the form

$$
\left[\begin{array}{cc}
Q_{1} & >0 \\
<0 & Q_{2}
\end{array}\right]
$$

for some $Q_{1} \in M_{k}$ and $Q_{2} \in M_{n-k}$, with $1 \leq k<n$. Here $>0($ resp. $<0)$ denotes a block of appropriate size with all entries positive (resp. negative).

For $i \in\{1, \ldots, n\}, G(A(i), w(i))$ is the subgraph of $G(A, w)$ induced by vertices $1, \ldots, i-1, i+1, \ldots, n$. If $w(i)$ is efficient for $A(i)$ and $w$ is inefficient for $A$, then $G(A(i), w(i))$ is strongly connected and $G(A, w)$ is not. Thus, vertex $i$ of $G(A, w)$ is either a sink vertex (that is, a vertex with outdegree 0 ) or a source vertex (that is, a vertex with indegree 0 ).

Next, we recall a result that allows us to simplify our proofs. It concerns how $\mathcal{E}(A)$ changes when $A$ is subjected to a monomial similarity.

Lemma 7 [13] Suppose that $A \in \mathcal{P C}_{n}$ and $w \in \mathcal{E}(A)$. If $D \in M_{n}$ is a positive diagonal matrix ( $P \in M_{n}$ is a permutation matrix), then $D w \in \mathcal{E}\left(D A D^{-1}\right)$ $\left(P w \in \mathcal{E}\left(P A P^{T}\right)\right)$.

We note that, if $D$ is a positive diagonal matrix, then the digraphs $G\left(D A D^{-1}, D w\right)$ and $G(A, w)$ coincide.

In [15] the efficient extensions for $A \in \mathcal{P C}_{n}$ of an efficient vector for a principal submatrix of $A$ in $\mathcal{P C}_{n-1}$ were characterized. That is, for $i \in$ $\{1, \ldots, n\}$, a necessary and sufficient condition on the $i$ th entry of $w \in \mathcal{V}_{n}$ was given for $w$ to be efficient for $A$ when $w(i)$ is efficient for $A(i)$.

In [16] it was shown that, if $w$ is efficient for $A \in \mathcal{P} \mathcal{C}_{n}, n>3$, then there are two $(n-1)$-subvectors of $w$ efficient for the corresponding principal submatrices of $A$. More formally, there are $i, j \in\{1, \ldots, n\}$, with $i \neq j$, such that $w(i)$ is efficient for $A(i)$ and $w(j)$ is efficient for $A(j)$. We note here, for the first time, that a converse of this result also holds. For $A \in \mathcal{P} \mathcal{C}_{n}$, with $n>2$, and $w \in \mathcal{V}_{n}$, if there are two ( $n-1$ )-subvectors of $w$ efficient for the corresponding principal submatrices of $A$, then $w$ is efficient for $A$.

Theorem 8 Let $A \in \mathcal{P C}_{n}$, with $n>2$, and $w \in \mathcal{V}_{n}$. If there are $i, j \in$ $\{1, \ldots, n\}$, with $i \neq j$, such that $w(i)$ is efficient for $A(i)$ and $w(j)$ is efficient for $A(j)$, then $w$ is efficient for $A$.

Proof. The result is a consequence of Lemma 6 and the fact that, if $G(A(i), w(i))$ and $G(A(j), w(j)), i \neq j$, are strongly connected, then so is $G(A, w)$. To see this, let $k, l \in\{1, \ldots, n\}, k \neq l$. If $j \notin\{k, l\}$, then there is a directed path from $k$ to $l$ in $G(A(j), w(j))$. If $j \in\{k, l\}$ and $i \notin\{k, l\}$ then there is a directed path from $k$ to $l$ in $G(A(i), w(i))$. If $k=i$ and $l=j$, since $n>2$, there is a vertex $p \notin\{i, j\}$ in $G(A, w)$. Then, there is a directed path from $k$ to $p$ in $G(A(j), w(j))$ and a directed path from $p$ to $l$ in $G(A(i), w(i))$. In all cases, there is a directed path from $k$ to $l$ in $G(A, w)$.

## 3 Further new lemmas and definitions

We start with an analytical lemma that will be crucial in deriving our main results.

Lemma 9 Let $a_{i} \geq 0$ for $i=1, \ldots, k$. For $x>0$, define

$$
f(x)=\frac{1}{x}+\frac{1}{a_{2}+x}+\cdots+\frac{1}{a_{k}+x}+1-a_{1}-x .
$$

Then, $f$ is a strictly decreasing continuous function with range $\mathbb{R}$. In particular, there is one and only one $x>0$ such that $f(x)=0$. If $a_{i}=0$ for $i=2, \ldots, k, x$ is given by

$$
\begin{equation*}
x=\frac{1-a_{1}+\sqrt{\left(1-a_{1}\right)^{2}+4 k}}{2} . \tag{1}
\end{equation*}
$$

Proof. Clearly, $f$ is continuous and $f^{\prime}(x)<0$ for any $x>0$, implying that $f$ is a strictly decreasing function. Also,

$$
\lim _{x \rightarrow+\infty} f(x)=-\infty \text { and } \lim _{x \rightarrow 0^{+}} f(x)=+\infty
$$

Thus, the first claim follows which, obviously, implies the second one. The last claim follows from a simple calculation.

In the rest of this section we consider a matrix $B \in \mathcal{P C}_{k}$ and denote by $r_{1} \geq \cdots \geq r_{k}$ the row sums of $B$ in nonincreasing order. For $x>0$, let $f$ be the function associated with $B$ defined by

$$
\begin{equation*}
f(x)=\frac{1}{x}+\frac{1}{r_{1}-r_{2}+x}+\cdots+\frac{1}{r_{1}-r_{k}+x}+1-r_{1}-x . \tag{2}
\end{equation*}
$$

Lemma 10 Let $B \in \mathcal{P C}_{k}$. For $f$ as in (2), we have $f(1) \leq 0$, with equality if and only if $B=\mathbf{J}_{k}$. In particular, $f(x)=0$ implies $x \leq 1$.

Proof. Since $r_{1}-r_{i} \geq 0$ for $i=1, \ldots, k$, we have

$$
f(1)=1+\frac{1}{r_{1}-r_{2}+1}+\cdots+\frac{1}{r_{1}-r_{k}+1}-r_{1} \leq k-r_{1} .
$$

From Lemma 3 and Remark (4) we have $k-r_{1} \leq 0$ with equality if and only if $B=\mathbf{J}_{k}$, in which case $f(1)=0$. The second claim follows from the first one and Lemma 9 ,

Definition 11 We say that $B \in \mathcal{P C}_{k}$ is well-behaved of type I if $r_{1}-r_{k} \geq 1$, and is well-behaved of type II if $r_{1}-r_{k}<1$ and
$f\left(1+r_{k}-r_{1}\right)=\frac{1}{1+r_{k}-r_{1}}+\frac{1}{1+r_{k}-r_{2}}+\cdots+\frac{1}{1+r_{k}-r_{k-1}}+1-r_{k} \geq 0$.
We say that $B$ is well-behaved if it is well-behaved of type I or of type II.
Not any matrix in $\mathcal{P C}_{k}$ is well-behaved. A matrix $B \in \mathcal{P} \mathcal{C}_{k}$ is not wellbehaved if $r_{1}-r_{k}<1$ and $f\left(1+r_{k}-r_{1}\right)<0$. Note that the set of matrices $B \in \mathcal{P C}_{k}$ that are not well-behaved is open.

The designation "well-behaved" follows from the fact that a reciprocal matrix in $\mathcal{P C}_{n-1}$ that is not well-behaved cannot be extended to a matrix in $\mathcal{P} \mathcal{C}_{n}$ with efficient Perron vector $\mathbf{e}_{n}$, as will be seen in Section 5.1.

Example 12 The matrices

$$
A_{1}=\left[\begin{array}{ccc}
1 & 2 & \frac{3}{5} \\
\frac{1}{2} & 1 & 3 \\
\frac{5}{3} & \frac{1}{3} & 1
\end{array}\right] \text { and } A_{2}=\left[\begin{array}{ccc}
1 & \frac{6}{5} & 1 \\
\frac{5}{6} & 1 & 1 \\
1 & 1 & 1
\end{array}\right]
$$

are well-behaved of types I and II, respectively. The matrices

$$
A_{3}=\left[\begin{array}{ccc}
1 & 5 & \frac{1}{5} \\
\frac{1}{5} & 1 & 5 \\
5 & \frac{1}{5} & 1
\end{array}\right] \text { and } A_{4}=\left[\begin{array}{ccc}
1 & \frac{1}{5} & \frac{51}{10} \\
5 & 1 & \frac{2}{9} \\
\frac{10}{51} & \frac{9}{2} & 1
\end{array}\right]
$$

(and any sufficiently small reciprocal perturbations of them), are not wellbehaved.

Note that the notion of well-behaved is permutation similarity invariant. Thus, typically, in the discussions to follow, when a row and a column are added by bordering, it could be as well by inserting anywhere in the matrix.

A matrix in $\mathcal{P C}_{k}$ with constant row sums is not well-behaved of type I. Thus, from Lemma 10, we have the following.

Lemma 13 If $B \in \mathcal{P} \mathcal{C}_{k}$ has constant row sums and is different from $\boldsymbol{J}_{k}$ then $B$ is not well-behaved.

The next result will be a consequence of Theorems 19 and 25 given later. However, we give here a direct proof of it.

Proposition 14 If $B \in \mathcal{P C}_{k}$ is consistent then $B$ is well-behaved.
Proof. It is enough to see that, if $r_{1}-r_{k}<1$ then $f\left(1+r_{k}-r_{1}\right) \geq 0$, in which $f$ is as in (2). If $r_{1}-r_{k}<1$, we have

$$
f\left(1+r_{k}-r_{1}\right) \geq \frac{k}{\left(r_{1}-r_{k}\right)+\left(1+r_{k}-r_{1}\right)}-r_{k}=k-r_{k} \geq 0
$$

where the last inequality follows from Lemma 5 .
There are consistent matrices well-behaved of type I and of type II.
Example 15 The consistent matrix $\boldsymbol{J}_{k}$ is well-behaved of type II while the consistent matrix

$$
\left[\begin{array}{cc}
1 & 5 \mathbf{e}_{k-1}^{T} \\
\frac{1}{5} \mathbf{e}_{k-1} & \boldsymbol{J}_{k-1}
\end{array}\right]
$$

is well-behaved of type I.

## 4 Extending a reciprocal matrix to one with a prescribed Perron vector

Theorem 16 For any $B \in \mathcal{P C}_{n-1}$ and $w \in \mathcal{V}_{n}$, there is $A \in \mathcal{P C}_{n}$ with Perron vector $w$ and such that $A(n)=B$. Moreover, $A$ is unique.

Proof. Taking into account Lemmas 1 and 2, we assume $w=\mathbf{e}_{n}$. Moreover, still by Lemma 1, we assume $r_{1} \geq \cdots \geq r_{n-1}$, in which $r_{i}, i=1, \ldots, n-1$, is the sum of the entries in the $i$ th row of $B$.

Since we want $A$ with constant row sums and such that $A(n)=B$, the last column of $A$ should be of the form

$$
\left[\begin{array}{lllll}
x & r_{1}-r_{2}+x & \cdots & r_{1}-r_{n-1}+x & 1 \tag{3}
\end{array}\right]^{T}
$$

for some $x>0$. Since $A$ is reciprocal and the sum of the entries in the last row of $A$ should be $r_{1}+x$, we also have

$$
\begin{equation*}
\frac{1}{x}+\frac{1}{r_{1}-r_{2}+x}+\cdots+\frac{1}{r_{1}-r_{n-1}+x}+1=r_{1}+x \tag{4}
\end{equation*}
$$

By Lemma (9, the existence and uniqueness of $x>0$ satisfying (4) follows. For such $x, A$ is as desired.

By iterating the application of Theorem 16, we get the following.
Corollary 17 Let $B \in \mathcal{P C}_{k}, k<n$, and $w^{(k+i)} \in \mathcal{V}_{k+i}, i=1, \ldots, n-k$. Then, there is $A \in \mathcal{P C}_{n}$ such that $A[\{1, \ldots, k\}]=B$ and $A[\{1, \ldots, k+i\}]$ has Perron vector $w^{(k+i)}, i=1, \ldots, n-k$. Moreover, $A$ is unique.

We also have the following consequence of Theorem 16 .
Corollary 18 Let $A_{1}, A_{2} \in \mathcal{P C}_{n}$ with $A_{1}(n)=A_{2}(n)$. If $w^{(1)}$ and $w^{(2)}$ are the Perron vectors of $A_{1}$ and $A_{2}$, respectively, and $w^{(1)}(n)$ and $w^{(2)}(n)$ are proportional, then $A_{2}=D A_{1} D^{-1}$ for some $D=I_{n-1} \oplus[c]$, with $c>0$.

Proof. Using Lemma 2, we can conclude that there is a (unique) positive diagonal matrix $D$ such that the Perron vector $D w^{(1)}$ of $D A_{1} D^{-1}$ is a multiple of $w^{(2)}$. Moreover, since $w^{(1)}(n)$ and $w^{(2)}(n)$ are multiples, the first $n-1$ entries of $D$ are constant. Thus, $D A_{1} D^{-1}$ satisfies $\left(D A_{1} D^{-1}\right)(n)=A_{2}(n)$ and has Perron vector $w^{(2)}$. By the uniqueness statement in Theorem 16, we have $D A_{1} D^{-1}=A_{2}$, proving the claim.

## 5 Matrices with inefficient Perron vector

In this section we construct matrices $A \in \mathcal{P} \mathcal{C}_{n}$ with inefficient Perron vector and a prescribed principal submatrix. We start by studying the possibility of extending a matrix in $\mathcal{P} \mathcal{C}_{n-1}$ to one in $\mathcal{P} \mathcal{C}_{n}$ with inefficient Perron vector $\mathbf{e}_{n}$. Based on the result obtained, we then extend any reciprocal matrix to one of arbitrary larger size $n$ and with inefficient Perron vector, except if the fixed matrix is consistent of size $n-1$, in which case we show that such an extension does not exist.

### 5.1 Extending a not well-behaved reciprocal matrix to one with inefficient Perron vector $\mathrm{e}_{n}$

Let $A \in \mathcal{P} \mathcal{C}_{n}$ and suppose that $\mathbf{e}_{n}$ is the Perron vector of $A$. We next give sufficient conditions for the Perron vector of $A$ to be inefficient. Moreover, if $\mathbf{e}_{n-1}$ is efficient for $A(n)$, these conditions are also necessary.

Theorem 19 Let $A \in \mathcal{P C}_{n}$ and suppose that $\mathbf{e}_{n}$ is the Perron vector of $A$. If $A(n)$ is not well-behaved, then the Perron vector of $A$ is inefficient for $A$ and vertex $n$ is a sink vertex of $G\left(A, \mathbf{e}_{n}\right)$. As for a converse, if $\mathbf{e}_{n-1}$ is efficient for $A(n)$ and the Perron vector of $A$ is inefficient for $A$, then $A(n)$ is not well-behaved.

Proof. Let $r_{i}$ be the $i$ th row sum of $A(n), i=1, \ldots, n-1$. Taking into account Lemmas 1 and 7, we may assume $r_{1} \geq \cdots \geq r_{n-1}$. Using arguments similar to those in the proof of Theorem 16, the last column of $A$ with Perron vector $\mathbf{e}_{n}$ is as in (3), with $x>0$ satisfying (4).

If $A(n)$ is not well-behaved then

$$
r_{1}-r_{n-1}<1 \text { and } f\left(1+r_{n-1}-r_{1}\right)<0
$$

implying, taking into account Lemma 9,

$$
\begin{equation*}
x<1+r_{n-1}-r_{1} \leq 1 . \tag{5}
\end{equation*}
$$

Since (5) implies $r_{1}-r_{n-1}+x<1$, and, on the other hand,

$$
x \leq r_{1}-r_{2}+x \leq \cdots \leq r_{1}-r_{n-1}+x
$$

we have that all the off-diagonal entries of the last column of $A$ are smaller than 1, implying that $G\left(A, \mathbf{e}_{n}\right)$ is not strongly connected and, thus, by Lemma 6, $\mathbf{e}_{n}$ is inefficient for $A$. Note that vertex $n$ is a sink. This proves the first claim.

As for the second claim, if $\mathbf{e}_{n-1}$ is efficient for $A(n)$, then, by Lemma 6 , $G\left(A(n), \mathbf{e}_{n-1}\right)$ is strongly connected. If $\mathbf{e}_{n}$ is inefficient for $A$, then $G\left(A, \mathbf{e}_{n}\right)$ is not strongly connected. Thus, the off-diagonal entries of the last column of $A$ are all greater than 1 or all less than 1 . Since (4) holds, by Lemma 10 , $x \leq 1$. Hence, the off-diagonal entries of the last column of $A$ are all less than 1. Then, $r_{1}-r_{n-1}+x<1$, that is, $r_{1}-r_{n-1}<1$ and $x<1+r_{n-1}-r_{1}$. By Lemma 9, for $f$ as in (21) with $k=n-1$, we have $f\left(1+r_{n-1}-r_{1}\right)<0$, as $f(x)=0$. This implies that $A(n)$ is not well-behaved, proving the result.

Note that, by Theorem 8, if $\mathbf{e}_{n}$ is inefficient for $A$, then $\mathbf{e}_{n-1}$ is inefficient for $A(i)$, for at least $n-1$ distinct $i$ 's.

By Theorem 16, any matrix $B \in \mathcal{P C}_{n-1}$ can be extended to a matrix in $\mathcal{P C}{ }_{n}$ with Perron vector $\mathbf{e}_{n}$. Following the idea in the proof of that result, in
the next example we construct a matrix $A \in \mathcal{P C}_{4}$ with constant row sums and a prescribed, not well-behaved, principal submatrix in $\mathcal{P C}_{3}$. According to Theorem 19, the Perron vector of $A, \mathbf{e}_{4}$, is inefficient for $A$.
Example 20 We determine $A \in \mathcal{P C}_{4}$ with Perron vector $\mathbf{e}_{4}$ (that is, with constant row sums) and such that

$$
A(4)=\left[\begin{array}{ccc}
1 & \frac{1}{5} & \frac{51}{10}  \tag{6}\\
5 & 1 & \frac{2}{9} \\
\frac{10}{51} & \frac{9}{2} & 1
\end{array}\right]
$$

Note that $A(4)$ is not well-behaved, as observed in Example 12, and has nonincreasing row sums. The matrix $A$ should be of the form

$$
A=\left[\begin{array}{cccc}
1 & \frac{1}{5} & \frac{51}{10} & x \\
5 & 1 & \frac{2}{9} & x+\frac{1}{5}+\frac{51}{10}-5-\frac{2}{9} \\
\frac{10}{51} & \frac{9}{2} & 1 & x+\frac{1}{5}+\frac{51}{10}-\frac{9}{2}-\frac{10}{51} \\
\frac{1}{x} & \frac{1}{x+\frac{1}{5}+\frac{51}{10}-5-\frac{2}{9}} & \frac{1}{x+\frac{1}{5}+\frac{51}{10}-\frac{9}{2}-\frac{10}{51}} & 1
\end{array}\right]
$$

with $x$ satisfying

$$
\frac{1}{x}+\frac{1}{x+\frac{1}{5}+\frac{51}{10}-5-\frac{2}{9}}+\frac{1}{x+\frac{1}{5}+\frac{51}{10}-\frac{9}{2}-\frac{10}{51}}+1=1+\frac{1}{5}+\frac{51}{10}+x
$$

A calculation gives $x=0.39137$, implying that

$$
A=\left[\begin{array}{cccc}
1 & \frac{1}{5} & \frac{51}{10} & 0.39137 \\
5 & 1 & \frac{2}{9} & 0.46915 \\
\frac{10}{51} & \frac{9}{2} & 1 & 0.99529 \\
2.5551 & 2.1315 & 1.0047 & 1
\end{array}\right]
$$

By Theorem 19, A has inefficient Perron vector $\mathbf{e}_{4}$ (4 is a sink vertex of $\left.G\left(A, \mathbf{e}_{4}\right)\right)$.

Theorem 19 leaves the question of characterizing the matrices $A \in \mathcal{P C}_{n}$ with inefficient Perron vector $\mathbf{e}_{n}$ and such that $A(i)$ is well-behaved for all $i \in\{1, \ldots, n\}$. In this case, by the theorem, $\mathbf{e}_{n-1}$ is inefficient for $A(i)$, $i=1, \ldots, n$. Moreover, by Lemma 13, $\mathbf{e}_{n-1}$ is not the Perron vector of $A(i)$. Note that, since $\mathbf{e}_{n-1}$ is inefficient for $A(i)$, the matrices $A(i)$ and $\mathbf{J}_{n-1}$ are distinct. Next we give an example of such a matrix $A$. We show in Section 5.3 that we should have $n>4$.

Example 21 Let

$$
A=\left[\begin{array}{cccccc}
1 & 1.2783 & 0.2364 & 1.0245 & 2.0221 & 4.5197 \\
0.7823 & 1 & 2.4655 & 1.6028 & 2.1091 & 2.1214 \\
4.2304 & 0.4056 & 1 & 1.3002 & 1.7109 & 1.4340 \\
0.9761 & 0.6239 & 0.7691 & 1 & 6.5795 & 0.1324 \\
0.4945 & 0.4741 & 0.5845 & 0.1520 & 1 & 7.3759 \\
0.2213 & 0.4714 & 0.6973 & 7.5555 & 0.1356 & 1
\end{array}\right] \in \mathcal{P C}_{6} .
$$

The matrix $A$ has Perron eigenvector $\mathbf{e}_{6}$, which is inefficient for $A$. It can be seen that, for any $i \in\{1, \ldots, 6\}$, $\mathbf{e}_{5}$ is inefficient for $A(i)$ and $A(i)$ is well-behaved $(A(2)$ is well-behaved of type II and $A(i)$ is well-behaved of type I for $i \neq 2$ ). Observe that the digraph $G(A, w)$ has no sink nor source vertex, as each row of $A$ has entries greater than 1 and entries less than 1.

As a consequence of Theorem [19, we have the following important result that allows us to easily construct reciprocal matrices with inefficient Perron vector.

Corollary 22 Let $A \in \mathcal{P C}_{n}$ and $w \in \mathcal{V}_{n}$ be the Perron vector of $A$. Suppose that $A(n)$ is inconsistent. If $w(n)$ is the Perron vector of $A(n)$ then $w$ is inefficient for $A$ and $n$ is a sink vertex of $G(A, w)$.

Proof. By Lemmas 1, 2 and 7, and since, for a positive diagonal matrix $D \in M_{n}, G\left(D A D^{-1}, D w\right)$ and $G(A, w)$ coincide, we may assume $w=\mathbf{e}_{n}$. Since $A(n)$ is inconsistent with constant row sums, by Lemma 13, it is not well-behaved. Thus, the result follows from Theorem 19 ,

Observe that, if $w=\left[\begin{array}{lll}w_{1} & \cdots & w_{n}\end{array}\right]^{T}$ is the Perron vector of $A$ and $A(n)$ is consistent with Perron vector $w(n)$, then, by the uniqueness statement in Theorem 16, $A=\left[\frac{w_{i}}{w_{j}}\right]$ ( $A$ is consistent) and $w$ is efficient for $A$ (as will follow from Theorem (25). Thus, the assumption in Corollary 22 that $A(n)$ is inconsistent is necessary.

In Corollary 22 we gave sufficient conditions for $G(A, w)$ to have a sink vertex, in which $w$ is the Perron vector of $A$. Another sufficient condition is given next.

Corollary 23 Let $A \in \mathcal{P C}_{n}$ and $w$ be the Perron vector of $A$. If $w$ is inefficient for $A$ and $w(n)$ is efficient for $A(n)$, then $n$ is a sink vertex of $G(A, w)$.

Proof. By Lemmas 1, 2 and 7, we assume that $w=\mathbf{e}_{n}$. If $\mathbf{e}_{n}$ is inefficient for $A$ and $\mathbf{e}_{n-1}$ is efficient for $A(n)$, by Theorem 19, $A(n)$ is not well-behaved, which implies, by the same theorem, that $n$ is a sink vertex of $G(A, w)$.

From Corollary 22, we can easily construct a reciprocal matrix with inefficient Perron vector by adding a column with constant off-diagonal entries (and the corresponding reciprocal row) to an arbitrary inconsistent reciprocal matrix with constant row sums. More precisely, if $T \in \mathcal{P C}_{n-1}$ has constant row sums, say equal to $r$, and is inconsistent, by Corollary 22, any matrix $A$ of the form

$$
A=\left[\begin{array}{cc}
T & a \mathbf{e}_{n-1}  \tag{7}\\
\frac{1}{a} \mathbf{e}_{n-1}^{T} & 1
\end{array}\right]
$$

with $a>0$, has inefficient Perron vector. Note that, for $D^{-1}=I_{n-1} \oplus\left[x_{0}\right]$, in which $x=a x_{0}$ satisfies (1) (with $a_{1}=r$ and $k=n-1$ ), the matrix $D A D^{-1}$ has constant row sums. Thus, the Perron vector of $A$ is $\left[\begin{array}{ll}\mathbf{e}_{n-1}^{T} & x_{0}\end{array}\right]^{T}$.

According to Lemma 7, any matrix monomial similar to a matrix as in (7), with $T$ inconsistent with constant row sums, has inefficient Perron vector.

We now focus on the construction of matrices $T$ as above. First, note that, from Lemma 2, given any $B \in \mathcal{P C}_{k}$ with Perron vector $v$, for $D^{-1}=$ $\operatorname{diag}(v)$, the matrix $T=D B D^{-1}$ has constant row sums. Moreover, if $B$ is inconsistent, then so is $T$.

Next we present structured classes of inconsistent reciprocal matrices $T$ with constant row sums. One such class consists of matrices of the form

$$
T=\left[\begin{array}{ccccccc}
1 & b & 1 & \cdots & \cdots & 1 & \frac{1}{b}  \tag{8}\\
\frac{1}{b} & 1 & b & 1 & \cdots & 1 & 1 \\
1 & \frac{1}{b} & 1 & b & \ddots & & 1 \\
\vdots & 1 & \frac{1}{b} & 1 & \ddots & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \ddots & b & 1 \\
1 & 1 & & \ddots & \ddots & 1 & b \\
b & 1 & 1 & & & \frac{1}{b} & 1
\end{array}\right] \in \mathcal{P C}_{k},
$$

in which $k \geq 3$ and $b$ is an arbitrary positive number different from 1 . The matrices of the form (7) with $T$ as in (8) $(b \neq 1)$ form the class of reciprocal matrices presented in [6] for which inefficiency of the Perron vector was noted. Thus, the class of [6] is part of our much more general construction.

Another class of inconsistent reciprocal matrices with constant row sums consists of the Toeplitz matrices $T \in \mathcal{P C}_{k}$, with $k \geq 3$ odd, whose first row is

$$
\left[\begin{array}{llllllll}
1 & b & \frac{1}{b} & b & \frac{1}{b} & \cdots & b & \frac{1}{b}
\end{array}\right],
$$

with $b \neq 1$. For example, for $k=5$,

$$
T=\left[\begin{array}{ccccc}
1 & b & \frac{1}{b} & b & \frac{1}{b} \\
\frac{1}{b} & 1 & b & \frac{1}{b} & b \\
b & \frac{1}{b} & 1 & b & \frac{1}{b} \\
\frac{1}{b} & b & \frac{1}{b} & 1 & b \\
b & \frac{1}{b} & b & \frac{1}{b} & 1
\end{array}\right] .
$$

Finally, we note that, if $T_{0}, T_{1} \in \mathcal{P C}_{k}$ have constant row sums and least one is different from $\mathbf{J}_{k}$, then

$$
\left[\begin{array}{ll}
T_{0} & T_{1} \\
T_{1} & T_{0}
\end{array}\right] \in \mathcal{P} \mathcal{C}_{2 k}
$$

has constant row sums and is inconsistent. The same holds for

$$
\left[\begin{array}{cc}
T_{0} & x \mathbf{e}_{k} \\
\frac{1}{x} \mathbf{e}_{k}^{T} & 1
\end{array}\right] \in \mathcal{P C}_{k+1}
$$

in which $x$ is given by (1), with $a_{1}$ being the constant row sums of $T_{0}$ (assumed inconsistent). Thus, we can construct inductively infinitely many inconsistent reciprocal matrices of any size $n>3$ with constant row sums.

We conclude this section by showing that, if $A \in \mathcal{P} \mathcal{C}_{n}$ has a consistent ( $n-1$ )-by- $(n-1)$ principal submatrix, then the Perron vector of $A$ is efficient for $A$. This means that inserting a row (and corresponding reciprocal column) to a consistent matrix gives a new reciprocal matrix with efficient Perron vector. This fact generalizes the result of [1] and a special case in [2], and gives a complete picture that includes each.

To give the theorem, we need the following lemma.
Lemma 24 Let $r_{1}, \ldots, r_{k}$ be positive numbers such that $r_{1} \geq k, r_{k} \leq k$ and $r_{1} \geq r_{2} \geq \cdots \geq r_{k}$. If $x_{1}, \ldots, x_{k}$ are positive numbers such that

$$
r_{1}+x_{1}=r_{2}+x_{2}=\cdots=r_{k}+x_{k}=1+\frac{1}{x_{1}}+\cdots+\frac{1}{x_{k}}
$$

then $x_{1} \leq 1$ and $x_{k} \geq 1$.

Proof. Let $r=r_{1}+x_{1}$ and suppose that the hypotheses in the statement hold. Suppose that $x_{1}>1$. Then, $r>k+1$ and $x_{i}>1$ for $i=2, \ldots, k$. The latter implies that

$$
r=1+\frac{1}{x_{1}}+\cdots+\frac{1}{x_{k}}<k+1
$$

a contradiction.
Now suppose that $x_{k}<1$. Then, $x_{i}<1$ for $i=1, \ldots, k-1$. This implies that

$$
r=1+\frac{1}{x_{1}}+\cdots+\frac{1}{x_{k}}>k+1
$$

a contradiction, since $r=r_{k}+x_{k}<k+1$.
Theorem 25 Let $A \in \mathcal{P C}_{n}$. Suppose that $A(n)$ is consistent. Then the Perron vector of $A$ is efficient for $A$.

Proof. Taking into account Lemmas 1, 2 and 7, we may assume that the Perron vector of $A$ is $\mathbf{e}_{n}$. Moreover, with a possible permutation similarity on $A(n)$, we may assume $r_{1} \geq \cdots \geq r_{n-1}$, in which $r_{i}$ is the sum of the entries in the $i$ th row of $A(n)$. Since $A(n)$ is consistent, we have

$$
A(n)=\left[\begin{array}{c}
w_{1} \\
\vdots \\
w_{n-1}
\end{array}\right]\left[\begin{array}{lll}
\frac{1}{w_{1}} & \cdots & \frac{1}{w_{n-1}}
\end{array}\right]
$$

for some positive numbers $w_{i}, i=1, \ldots, n-1$. As $A(n)$ has nonincreasing row sums, we have $w_{1} \geq \cdots \geq w_{n-1}$. Then, the upper diagonal entries of $A(n)$ are greater than or equal to 1 , implying that $G\left(A(n), \mathbf{e}_{n-1}\right)$ contains the path $(n-1) \rightarrow \cdots \rightarrow 2 \rightarrow 1$. Thus, to show that $G\left(A, \mathbf{e}_{n}\right)$ is strongly connected, it is enough to see that $1 \rightarrow n$ and $n \rightarrow(n-1)$ are edges in $G\left(A, \mathbf{e}_{n}\right)$, that is, the entry of $A$ in position $1, n$ is less than or equal to 1 and the one in position $n-1, n$ is greater than or equal to 1 . By Lemma 5 and Remark 4, we have $r_{1} \geq n-1$ and $r_{n-1} \leq n-1$. Since the row sums of $A$ are constant, the claim about the entries of $A$ in positions $1, n$ and $n-1, n$ follows from Lemma 24.

In this section we focused on the extension of a not well-behaved reciprocal matrix (in particular, an inconsistent reciprocal matrix with constant row
sums) to a reciprocal matrix with inefficient Perron vector $\mathbf{e}_{n}$. Using these ideas, in the next section, we summarize how to construct matrices in $\mathcal{P} \mathcal{C}_{n}$ with inefficient Perron vector and a prescribed principal submatrix in $\mathcal{P C}_{k}$, $k<n$.

### 5.2 An algorithm to extend an arbitrary reciprocal matrix to one with an inefficient Perron vector

Here we give an algorithm to construct a matrix $A \in \mathcal{P} \mathcal{C}_{n}$ with prescribed principal submatrix $B \in \mathcal{P C}_{k}, k<n$, and with inefficient Perron vector. We assume that $B$ is inconsistent if $k=n-1$, as otherwise such a construction does not exist by Theorem 25 (however, if $B$ is consistent and $k \leq n-2$, such an $A$ does exist).

Algorithm Let $B \in \mathcal{P C}_{k}$ be given ( $B$ is inconsistent if $k=n-1$ ).
let $S \in \mathcal{P C}_{n-1}$ be an inconsistent matrix such that $S[\{1, \ldots, k\}]=B$ ( $S=B$ if $k=n-1$ )
let $v$ be the Perron vector of $S$
let $R^{-1}=\operatorname{diag}(v)$
let $C=R S R^{-1}$
let $a>0$ be arbitrary
let $A^{\prime}=\left[\begin{array}{cc}C & a \mathbf{e}_{n-1} \\ \frac{1}{a} \mathbf{e}_{n-1}^{T} & 1\end{array}\right]$,
let $D=R \oplus[c]$ for some $c>0$
let $A=D^{-1} A^{\prime} D$
From the discussion in Section 5.1, the Perron vector of the matrix $A^{\prime}$ in the algorithm is inefficient for $A^{\prime}$. Note that the matrix $C$ has constant row sums. By Lemma 7, the Perron vector of the matrix $A$ obtained by the algorithm is inefficient for $A$. Moreover, $A(n)=S$.

If $k<n-1$, the matrix $S$ in the algorithm can be any inconsistent extension of $B \in \mathcal{P C}_{k}$. For example, it can be constructed in $n-k-1$ steps, giving matrices $S_{1} \in \mathcal{P} \mathcal{C}_{k+1}, \ldots, S_{n-k-1} \in \mathcal{P} \mathcal{C}_{n-1}$, with $S_{1}[\{1, \ldots, k\}]=B$ and $S_{i}[\{1, \ldots, k+i-1\}]=S_{i-1}$. As long as $S=S_{n-k-1}$ is inconsistent, each $S_{i}$ can have any desired Perron vector, by applying Theorem [16, or can be constructed to have inefficient Perron vector by applying the algorithm
(if $S_{i-1}$ is not consistent), or to have efficient Perron vector (by applying Theorem (33) in Section (6). Of course, different choices of $S$ produce different matrices $A$.

Example 26 We construct a matrix $A \in \mathcal{P C}_{5}$ with inefficient Perron vector and such that

$$
A([1,2,3])=\left[\begin{array}{ccc}
1 & 2 & \frac{3}{5} \\
\frac{1}{2} & 1 & 3 \\
\frac{5}{3} & \frac{1}{3} & 1
\end{array}\right]
$$

Let

$$
S=\left[\begin{array}{cccc}
1 & 2 & \frac{3}{5} & 2 \\
\frac{1}{2} & 1 & 3 & \frac{1}{2} \\
\frac{5}{3} & \frac{1}{3} & 1 & \frac{3}{2} \\
\frac{1}{2} & 2 & \frac{2}{3} & 1
\end{array}\right]
$$

be an extension in $\mathcal{P C}_{4}$ of the previous matrix in $\mathcal{P C}_{3}$. The Perron vector of $S$ is

$$
v=\left[\begin{array}{llll}
1.3348 & 1.1829 & 1.0946 & 1
\end{array}\right]^{T} .
$$

Let $R^{-1}=\operatorname{diag}(v)$. Then,

$$
C=R S R^{-1}=\left[\begin{array}{cccc}
1 & 1.7724 & 0.4920 & 1.4984 \\
0.5642 & 1 & 2.7761 & 0.4227 \\
2.0324 & 0.3602 & 1 & 1.3704 \\
0.6674 & 2.3658 & 0.7297 & 1
\end{array}\right]
$$

For any $a>0$, the matrix

$$
A^{\prime}=\left[\begin{array}{ccccc}
1 & 1.7724 & 0.4920 & 1.4984 & a \\
0.5642 & 1 & 2.7761 & 0.4227 & a \\
2.0324 & 0.3602 & 1 & 1.3704 & a \\
0.6674 & 2.3658 & 0.7297 & 1 & a \\
\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 1
\end{array}\right]
$$

has inefficient Perron vector. Taking $a=1$ and $D=R \oplus[1]$, we get

$$
A=D^{-1} A^{\prime} D=\left[\begin{array}{ccccc}
1 & 2 & \frac{3}{5} & 2 & 1.3348 \\
\frac{1}{2} & 1 & 3 & \frac{1}{2} & 1.1829 \\
\frac{5}{3} & \frac{1}{3} & 1 & \frac{3}{2} & 1.0946 \\
\frac{1}{2} & 2 & \frac{2}{3} & 1 & 1 \\
0.7492 & 0.8454 & 0.9136 & 1 & 1
\end{array}\right]
$$

Then, $A([1,2,3])=B($ in fact, $A([1,2,3,4])=S)$ and the Perron vector $w$ of $A$ is inefficient for $A$. Moreover, vertex 5 is a sink vertex of $G(A, w)$.

In the example, we could also have started with a 3 -by- 3 consistent matrix and have terminated with a 5 -by- 5 reciprocal matrix whose Perron vector was inefficient.

Finally, we observe that the Perron vector is a continuous function of the entries of a matrix. So, if the Perron vector of $A \in \mathcal{P} \mathcal{C}_{n}$ is inefficient for $A$, then the Perron vector of any sufficiently small reciprocal perturbation of $A$ is also inefficient for $A$ and for the perturbation of $A$ (see [15]).

### 5.3 The 4-by-4 reciprocal matrices with inefficient Perron vector

In this section we give a characterization of all matrices in $\mathcal{P} \mathcal{C}_{4}$ whose Perron vector is inefficient. Recall that the Perron vector of any matrix in $\mathcal{P C}_{n}$, with $n \leq 3$, is efficient. The case $n=2$ is trivial because the matrix is consistent. The case $n=3$ is covered by [1, 9].

When the Perron vector $w$ is inefficient for $A \in \mathcal{P C}_{n}$, it may happen that no ( $n-1$ )-subvector of $w$ is efficient for the corresponding principal submatrix of $A$, as illustrated next for $n=5$ (see also Example 21 for $n=6$ ). However, this does not happen when $n=4$, which is an important fact in obtaining the characterization mentioned above.

Example 27 Consider

$$
A=\left[\begin{array}{ccccc}
1 & 2.032 & 0.53386 & 0.86855 & 0.88385 \\
0.4923 & 1 & 2.1018 & 0.88907 & 0.83513 \\
1.8731 & 0.47578 & 1 & 0.97616 & 0.99334 \\
1.1513 & 1.1248 & 1.0244 & 1 & 1.0176 \\
1.1314 & 1.1974 & 1.0067 & 0.9827 & 1
\end{array}\right] \in \mathcal{P C}_{5}
$$

The vector $\mathbf{e}_{5}$ is the Perron vector of $A$ and is inefficient for $A$. Moreover, for any $i \in\{1,2,3,4,5\}, \mathbf{e}_{4}$ is inefficient for $A(i)$. We note that vertex 4 is a sink vertex of $G\left(A, \mathbf{e}_{5}\right)$.

Theorem 28 Let $A \in \mathcal{P C}_{4}$. If $w$ is the Perron vector of $A$, then there is an $i \in\{1,2,3,4\}$ such that $w(i)$ is efficient for $A(i)$.

Proof. By Lemmas 1, 2 and 7, we may assume that $w=\mathbf{e}_{4}$. Suppose that $\mathbf{e}_{3}$ is not efficient for $A(4)$. Then, by Lemma 6, $G\left(A(4), \mathbf{e}_{3}\right)$ is not strongly connected. Hence, $A(4)$ is permutationally similar to a matrix of one of the following forms

$$
\left[\begin{array}{c|cc}
1 & <1 & <1  \tag{9}\\
\hline>1 & 1 & \geq 1 \\
>1 & \leq 1 & 1
\end{array}\right] \text { or }\left[\begin{array}{cc|c}
1 & \geq 1 & <1 \\
\leq 1 & 1 & <1 \\
\hline>1 & >1 & 1
\end{array}\right]
$$

By $>1,<1, \geq 1$ and $\leq 1$, we denote an entry greater than 1 , less than 1 , greater than or equal to 1 and less than or equal to 1 , respectively. By Lemmas 1 and 7, we may assume that $A(4)$ has one of the forms in (9). Suppose that $A(4)$ is as the first matrix in (9). The proof of the other case is similar. Since $A$ has constant row sums, and these sums are at least 4 (by Remark (4), then the entry in position 1,4 of $A$ is greater than 1 , and thus $A$ has the form

$$
\left[\begin{array}{cccc}
1 & <1 & <1 & >1 \\
>1 & 1 & \geq 1 & \\
>1 & \leq 1 & 1 & \\
<1 & & & 1
\end{array}\right]
$$

Then, using similar arguments, in positions 4,2 or 4,3 the matrix $A$ has an entry greater than 1 . In the first case $A$ has the form

$$
\left[\begin{array}{cccc}
1 & <1 & <1 & >1 \\
>1 & 1 & \geq 1 & <1 \\
>1 & \leq 1 & 1 & \\
<1 & >1 & & 1
\end{array}\right]
$$

implying that $1 \rightarrow 2 \rightarrow 4 \rightarrow 1$ is a cycle in $G\left(A(3), \mathbf{e}_{3}\right)$ and, thus, $\mathbf{e}_{3}$ is efficient for $A(3)$. In the second case, $A$ has the form

$$
\left[\begin{array}{cccc}
1 & <1 & <1 & >1 \\
>1 & 1 & \geq 1 & \\
>1 & \leq 1 & 1 & <1 \\
<1 & & >1 & 1
\end{array}\right]
$$

implying that $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$ is a cycle in $G\left(A(2), \mathbf{e}_{3}\right)$ and, thus, $\mathbf{e}_{3}$ is efficient for $A(2)$.

Note that, if the Perron vector of $A$ is efficient for $A$, Theorem 28 also follows from the results in [16] (see Section 2.3).

Corollary 29 Let $A \in \mathcal{P C}_{4}$ and $w$ be the Perron vector of $A$. Then $w$ is inefficient for $A$ if and only if $G(A, w)$ has a sink vertex.

Proof. The "if" claim follows from Lemma 6. The "only if" claim follows from Theorem 28 and Corollary 23 .

Any reciprocal matrix is similar, via a positive diagonal matrix, to a reciprocal matrix with constant row sums (Lemma (2)). Thus, from Corollary 29 (and Lemma 7), it follows that the matrices in $\mathcal{P C}_{4}$ whose Perron vector is inefficient are those that are diagonal similar, via a positive diagonal matrix, to a reciprocal matrix with constant row sums and with a row in which all the off-diagonal entries are greater than 1 . We formalize this next.

Denote by $\mathcal{S}_{4}$ the set of matrices $A \in \mathcal{P C}_{4}$ with Perron vector $\mathbf{e}_{4}$ (that is, with constant row sums) and such that there is a row in which all the offdiagonal entries are greater than 1 . Denote by $\mathcal{D}_{4}$ the set of positive diagonal matrices of size 4 .

Theorem 30 The Perron vector of $A \in \mathcal{P C}_{4}$ is inefficient for $A$ if and only if $A=D^{-1} B D$ for some $D \in \mathcal{D}_{4}$ and some $B \in \mathcal{S}_{4}$.

We next show that the example presented in [5] of a matrix in $\mathcal{P} \mathcal{C}_{4}$ with inefficient Perron vector follows from Theorem 30.

## Example 31 Let

$$
A=\left[\begin{array}{cccc}
1 & 2 & 6 & 2 \\
\frac{1}{2} & 1 & 4 & 3 \\
\frac{1}{6} & \frac{1}{4} & 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{3} & 2 & 1
\end{array}\right]
$$

The Perron vector of $A$ is $w=\left[\begin{array}{lll}2.9038 & 2.057 & 0.48282 \\ 1\end{array}\right]^{T}$. For $D^{-1}=$ $\operatorname{diag}(w)$, we have

$$
B=D A D^{-1}=\left[\begin{array}{cccc}
1 & 1.4168 & 0.99764 & 0.68876 \\
0.70584 & 1 & 0.93889 & 1.4585 \\
1.0024 & 1.0651 & 1 & 1.0356 \\
1.4519 & 0.68565 & 0.96563 & 1
\end{array}\right]
$$

The matrix $B$ has Perron vector $\mathbf{e}_{4}$ and all the off-diagonal entries in the third row of $B$ are greater than 1 (vertex 3 is a sink vertex of $G(A, w)=G\left(B, \mathbf{e}_{4}\right)$ ). Thus, the Perron vector of $A$ is inefficient for $A$.

## 6 Extending a reciprocal matrix to one with an efficient Perron vector

We next give a constructive proof of the existence of a matrix in $\mathcal{P} \mathcal{C}_{n}$ with efficient Perron vector, extending any given matrix in $\mathcal{P} \mathcal{C}_{n-1}$. We need the following result that is used in the construction of such an extension.

Lemma 32 Let $B \in \mathcal{P C}_{k}$. Then, there is a positive diagonal matrix $D \in M_{k}$ such that $D B D^{-1}$ is well-behaved and $\mathbf{e}_{k}$ is efficient for $D B D^{-1}$.

Proof. Let $D^{-1}=\operatorname{diag}\left(b_{j}\right)$, in which $b_{j}$ is the $j$ th column of $B$. Then, the $j$ th column and the $j$ th row of $B^{\prime}=D B D^{-1}$ are $\mathbf{e}_{k}$ and $\mathbf{e}_{k}^{T}$, respectively. This implies that $\mathbf{e}_{k}$ is efficient for $B^{\prime}$, as any column of $B^{\prime}$ is efficient for $B^{\prime}$ (see [15]).

Let $r_{1} \geq \cdots \geq r_{k}$ be the row sums of $B^{\prime}$. Since the sum of the entries in the $j$ th row of $B^{\prime}$ is $k$, we have $r_{k} \leq k$. Suppose that $r_{1}-r_{k}<1$. Then, for $i=1, \ldots, k$, we have $0 \leq r_{i}-r_{k}<1$, implying

$$
\frac{1}{1+r_{k}-r_{i}} \geq 1
$$

Thus,

$$
\sum_{i=1}^{k} \frac{1}{1+r_{k}-r_{i}} \geq k \geq r_{k}
$$

implying $f\left(1+r_{k}-r_{1}\right) \geq 0$, in which $f$ is as in (2). Thus, $B^{\prime}$ is well-behaved.

Theorem 33 Let $B \in \mathcal{P C}_{n-1}$. Then, there is $A \in \mathcal{P C}_{n}$ with efficient Perron vector and such that $A(n)=B$.

Proof. By Lemma [32, there is a positive diagonal matrix $D$ such that $B^{\prime}=D B D^{-1}$ is well-behaved and $\mathbf{e}_{n-1}$ is efficient for $B$. With a possible permutation similarity on $B^{\prime}$, we assume that $B^{\prime}$ has nonincreasing row sums $r_{1} \geq \cdots \geq r_{n-1}$. Taking into account Lemmas 1 and 7 , we consider $B=B^{\prime}$. Let $A \in \mathcal{P} \mathcal{C}_{n}$ be such that $A(n)=B$ and its last column is as in (3) with $x>0$ satisfying (4). Note that the existence of (a unique) such $x$ is ensured by Lemma 9. Then, $\mathbf{e}_{n}$ is the Perron vector of $A$. Since $B$ is well-behaved and $\mathbf{e}_{n-1}$ is efficient for $B$, by Theorem [19, $\mathbf{e}_{n}$ is efficient for $A$.

Next we construct a matrix in $\mathcal{P C}_{4}$ with efficient Perron vector and a prescribed principal submatrix in $\mathcal{P C}_{3}$. We consider the case in which the well-behaved matrix appearing in the construction is of type I.

Example 34 Let $B$ be the matrix in (6), which is not well-behaved. We construct $A \in \mathcal{P C}_{4}$ with efficient Perron vector and such that $A(4)=B$. Recall that in Example 20 the matrix $B$ was extended to a matrix in $\mathcal{P C}_{4}$ with inefficient Perron vector. Let $D^{-1}=\operatorname{diag}\left(\frac{1}{5}, 1, \frac{9}{2}\right)$. The matrix

$$
B^{\prime}=D B D^{-1}=\left[\begin{array}{ccc}
1 & 1 & 114.75 \\
1 & 1 & 1 \\
0.008715 & 1 & 1
\end{array}\right]
$$

is well-behaved of type $I$ and $\mathbf{e}_{3}$ is efficient for $B^{\prime}$. Let

$$
A^{\prime}=\left[\begin{array}{cccc}
1 & 1 & 114.75 & x \\
1 & 1 & 1 & x+114.75-1 \\
0.008715 & 1 & 1 & x+114.75-0.008715 \\
\frac{1}{x} & \frac{1}{x+114.75-1} & \frac{1}{x+114.75-0.008715} & 1
\end{array}\right]
$$

with

$$
x+2+114.75=1+\frac{1}{x}+\frac{1}{x+114.75-1}+\frac{1}{x+114.75-0.008715} .
$$

A calculation gives $x=0.00864$, implying that

$$
A^{\prime}=\left[\begin{array}{cccc}
1 & 1 & 114.75 & 0.00864 \\
1 & 1 & 1 & 113.76 \\
0.008715 & 1 & 1 & 114.75 \\
115.74 & 0.00879 & 0.008715 & 1
\end{array}\right]
$$

The vector $\mathbf{e}_{4}$ is the Perron vector of $A^{\prime}$ and is efficient for $A^{\prime}$. Thus,

$$
\begin{aligned}
A & =\left(D^{-1} \oplus[1]\right) A^{\prime}(D \oplus[1]) \\
& =\left[\begin{array}{cccc}
1 & \frac{1}{5} & \frac{51}{10} & 0.001728 \\
5 & 1 & \frac{2}{9} & 113.76 \\
\frac{10}{51} & \frac{9}{2} & 1 & 516.38 \\
578.7 & 0.00879 & 0.001937 & 1
\end{array}\right]
\end{aligned}
$$

satisfies $A(4)=B$ and the Perron vector of $A$ is efficient for $A$.

Given $B \in \mathcal{P C}_{n-1}$, we have shown how to construct a matrix $A \in \mathcal{P} \mathcal{C}_{n}$ with efficient Perron vector and such that $A(n)=B$. If $B \in \mathcal{P} \mathcal{C}_{k}$, with $k<n-1$, and we want to construct $A \in \mathcal{P} \mathcal{C}_{n}$ with efficient Perron vector and such that $A[\{1, \ldots, k\}]=B$, we may consider an arbitrary matrix $S \in \mathcal{P C}_{n-1}$ with $S[\{1, \ldots, k\}]=B$, and proceed as above to obtain $A$ with efficient Perron vector and such that $A(n)=S$.

## 7 Conclusions

When prioritizing alternatives, one important property that the weight vector extracted from the reciprocal matrix, of the pair-wise comparisons, should have is efficiency. One of the most used weighting methods employs the right Perron vector of the reciprocal matrix as the vector of weights. It is known that such a vector may not be efficient.

The reciprocal matrix constructed in practice from which a weight vector is obtained may be just partially known. Here we study the existence of an extension of a reciprocal matrix with efficient/inefficient Perron vector. We conclude that any reciprocal matrix can be extended to one with inefficient Perron vector, except if it is consistent of size $n-1$, in which case it is shown that there is no reciprocal extension of size $n$ with inefficient Perron vector. We also show that any reciprocal matrix can be extended to one with efficient Perron vector. Our analysis gives a procedure to construct such extensions. As a consequence, we obtain structured classes of reciprocal matrices with inefficient Perron vector, of which the family presented in [6] is a particular case.

We give sufficient conditions for the digraph $G(A, w)$ (see Section 2.3), associated with an $n$-by- $n$ reciprocal matrix $A$ with inefficient Perron vector $w$, to have a sink vertex, namely, the inefficient Perron vector has a subvector obtained by deleting one entry that is either the Perron vector of the corresponding principal submatrix of $A$ or is efficient for this submatrix. Though this latter condition does not always happen for $n>4$, as illustrated, it holds when $n=4$. This implies that the Perron vector $w$ of a 4-by- 4 reciprocal ma$\operatorname{trix} A$ is inefficient for $A$ if and only if the associated digraph $G(A, w)$ has a sink vertex. Several examples illustrating the theoretical results are provided throughout the paper.

This work leaves some relevant questions for future study, such as determining all extensions of a given reciprocal matrix with inefficient Perron
vector and if there are such extensions for which the associated digraph has a source vertex (we conjecture the answer is negative). Also, studying the existence of reciprocal completions with efficient (inefficient) Perron vector for other patterns of the specified entries is an interesting problem to consider.

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