HYPERSPACES OF THE DOUBLE ARROW

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Abstract

Let $\mathbb{A}$ and $\mathbb{S}$ denote the double arrow of Alexandroff and the Sorgenfrey line, respectively. We show that for any $n \geq 1$, the space of all unions of at most n closed intervals of $\mathbb{A}$ is not homogeneous. We also prove that the spaces of non-trivial convergent sequences of $\mathbb{A}$ and $\mathbb{S}$ are homogeneous. This partially solves an open question of A. Arhangel'skiǐ [Ar87]. In contrast, we show that the space of closed intervals of $\mathbb{S}$ is homogeneous.

Keywords: Double arrow, Hyperspaces, Homogeneous spaces, Sorgenfrey, Non-trivial convergent sequences.

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1 Introduction.

Given a space X, we denote by $\mathrm{Exp}\,(X)$ the set of all non-empty closed subsets of X. For a non-empty open set V of X, let $[V] = \{F \in \mathrm{Exp}\,(X) : F \subset V\}$ and $\langle V\rangle = \{F \in \mathrm{Exp}\,(X) : F \cap V \neq \emptyset\}$. The collection of all sets $[V]$ and $\langle V\rangle$ is a subbase for a topology on $\mathrm{Exp}\,(X)$ called the *Vietoris topology*. From now on, $\mathrm{Exp}(X)$ will be considered with this topology. Since $\langle \cup_i V_i\rangle = \cup_i\langle V_i\rangle$ for any collection of non-empty open sets V_i; if X is generated by a base, then the Vietoris topology on $\mathrm{Exp}\,(X)$ is generated by the subbase of all sets of the form $[V]$ and $\langle W\rangle$ with V open sets and W basic sets. It is known that if X is compact, then $\mathrm{Exp}(X)$ is also compact. Given a space X, a *hyperspace* of X is any subspace of $\mathrm{Exp}\,(X)$. All subsets of $\mathrm{Exp}\,(X)$ will be considered hyperspaces.

Let X be a Hausdorff space. A set $S \subset X$ will be called a *nontrivial convergent sequence* in X if S is countably infinite and there is $x \in S$ such that $S \setminus V$ is finite for any open neighborhood V of x. The point x is called the limit of S and we will say that S converges to x. The hyperspace of all nontrivial convergent sequences in X will be denoted $\mathcal{S}_c(X)$. The later space was introduced by García-Ferreira and Ortiz-Castillo in [GO15] for metric spaces and studied in a more general setting in [MPP18]. Today has a great interest among topologists. In the usual sense, a convergent sequence in X is a function $f : \omega \to X$ for which there exists $x \in X$ such that for each open

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neighborhood V of x, there is $m \in \omega$ with $f(n) \in U$ for all $n \geq m$. If $f''(\omega)$ is infinite, then $(\{x\} \cup f''(\omega)) \in \mathcal{S}_c(X)$.

A topological space X is *homogeneous* if for every $x, y \in X$ there exists an autohomeomorphism h of X such that $h(x) = y$. Several classic results on homogeneity involve the study of the $\mathrm{Exp}(X)$. In this paper we are motivated by the following general question.

Question 1.1. *When is* $\mathrm{Exp}(X)$ *homogeneous?*

In the 1970's, it was shown by R. Schori and J. West [SW75] that $\mathrm{Exp}([0,1])$ is homeomorphic to the Hilbert cube. In particular, it is possible that the hyperspace $\mathrm{Exp}(X)$ is homogeneous while X is not. On the other hand, if $\kappa > \aleph_1$, then $\mathrm{Exp}\,(2^\kappa)$ is not homogeneous (see [Sce76]). Thus, the question of homogeneity of the hyperspace turns out to be quite subtle.

A. Arhangel'skiĭ in [Ar87] asked the following question (which appears in [AvM13]).

Question 1.2. *Is the hyperspace* $\mathrm{Exp}\,(\mathbb{A})$ *homogeneous?*

In this paper we partially answer Question 1.2, by showing that.

Theorem 1.3. *$\mathcal{C}_m(\mathbb{A})$ is not homogeneous for any $m \geq 1$.*

Where $\mathcal{C}_m(\mathbb{A})$ is the hyperspace of $\mathbb{A}$ consisting of all unions of at most m nonempty closed intervals.

The following result can be seen as a companion of the previous Theorem.

Theorem 1.4. *$\mathcal{S}_c(\mathbb{A})$ is homogeneous.*

In [BM23] the authors show that the symmetric products $\mathcal{F}_m(\mathbb{A})$ are not homogeneous for any $m \geq 2$. Since $\mathcal{F}_m(\mathbb{A}) \subset \mathcal{C}_m(\mathbb{A})$, now we are a little more closer to answer Question 1.2.

The paper is organized as follows. In section 2 we prove that $\mathcal{S}_c(\mathbb{A})$ and $\mathcal{S}_c(\mathbb{S})$ are homogeneous. In section 3 we prove that the space of non-empty closed intervals of $\mathbb{S}$ is homogeneous. In section 4 we give a geometric characterization for spaces of unions of at most m non-empty closed intervals of a compact linearly ordered space and we prove that in the case of the double arrow this spaces are non-homogeneous. Finally, in section 5 we give a metrization theorem that was obtained in our efforts to prove Theorem 1.3 and generalizes a classical result on compact spaces. We will use [En89] as a basic reference on topology and [AvM13] as a reference for homogeneity and hyperspaces.

2 Homogeneity of $\mathcal{S}_c(\mathbb{A})$ and $\mathcal{S}_c(\mathbb{S})$

Let $\mathbb{A}_0 =]0,1] \times \{0\}, \mathbb{A}_1 = [0,1[\times \{1\}$ and $\mathbb{A} = \mathbb{A}_0 \cup \mathbb{A}_1$. Define the lexicographical ordering $\langle a, r\rangle \prec \langle b, s\rangle$ if $a < b$ or $a = b$ and $r < s$. The set $\mathbb{A}$ with the order topology is the *double arrow space*.

Proposition 2.1. *If $S, T \in \mathcal{S}_c(\mathbb{A})$, then there exists a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ such that $h''(S) = T$.*

Proof. Let $S, T \in \mathcal{S}_c(\mathbb{A})$. First, we will prove that if $S = \{x\} \cup \{x_n : n \in \mathbb{Z}^+\}$ and $P = \{\langle 0, 1\rangle\} \cup \{\langle 1/2^n, 1\rangle : n \in \mathbb{Z}^+\}$, then there is a homeomorphism $h_1 : \mathbb{A} \to \mathbb{A}$

such that $h_1''(S) = P$. Since $\mathbb{A}$ is homogeneous, there is a homeomorphism $f : \mathbb{A} \to \mathbb{A}$ with $f(x) = \langle 0, 1\rangle$. We have that the sequence $f(x_n)$ converges to $f(x) = \langle 0, 1\rangle$, so we can define inductively $z_1 = \max\{f(x_n) : n \in \mathbb{Z}^+\}$ and $z_m = \max\{f(x_n) : n \in \mathbb{Z}^+\}\setminus\{z_1, \ldots, z_{m-1}\}$ for $m \geq 2$. By convergence, we can choose a clopen neighborhood V_1 of $\langle 0, 1\rangle$ such that $f(x_n) \in V_1$ for every n with $f(x_n) \neq z_1$ and $z_1 \notin V_1$. Because $\mathbb{A} \setminus V_1$ and $[\langle 1/2, 1\rangle, \langle 1, 0\rangle]$ are homeomorphic to $\mathbb{A}$ and $\mathbb{A}$ is homogeneous, there exists a homeomorphism $g_1 : \mathbb{A} \setminus V_1 \to [\langle 1/2, 1\rangle, \langle 1, 0\rangle]$ such that $g_1(z_1) = \langle 1/2, 1\rangle$. As before, we can choose a clopen neighborhood V_2 of $\langle 0, 1\rangle$ such that $f(x_n) \in V_2$ for every n with $f(x_n) \neq z_1, z_2$ and $z_1, z_2 \notin V_2$. There exists a homeomorphism $g_2 : V_1 \setminus V_2 \to [(\langle 1/2^2, 1\rangle, \langle 1/2, 0\rangle]$ with $g_2(z_2) = \langle 1/2^2, 1\rangle$. Recursively, we can choose a clopen neighborhood V_m of $\langle 0, 1\rangle$ such that $f(x_n) \in V_m$ for every n with $f(x_n) \neq z_1, \ldots, z_m$ and $z_1, \ldots, z_m \notin V_m$. There exists a homeomorphism $g_m : V_{m-1} \setminus V_m \to [\langle 1/2^m, 1\rangle, \langle 1/2^{m-1}, 0\rangle]$ with $g_m(z_m) = \langle 1/2^m, 1\rangle$.

We define the homeomorphism $g = \bigcup g_m :]\langle 0, 1\rangle, \langle 1, 0\rangle] \to]\langle 0, 1\rangle, \langle 1, 0\rangle]$. Hence, we have the homeomorphism $\overline{g} : \mathbb{A} \to \mathbb{A}$ with $\overline{g}(x) = g(x)$ if $x \neq \langle 0, 1\rangle$ and $\overline{g}(\langle 0, 1\rangle) = \langle 0, 1\rangle$. In this way, $h_1 := \overline{g} \circ f$ is the desired homeomorphism.

Finally, by the previous argument there is a homeomorphism $h_2 : \mathbb{A} \to \mathbb{A}$ such that $h_2''(P) = T$. Therefore, the homeomorphism $h := h_2 \circ h_1$ is as required. $\square$

Since the Sorgenfrey line is homeomorphic to $[0, 1[$ with the subspace topology, we will assume that the $\mathbb{S} = [0, 1[$. In a very similar way we can prove the following.

Proposition 2.2. *If $S, T \in \mathcal{S}_c(\mathbb{S})$, then there exists a homeomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $h''(S) = T$.*

Proof of Theorem 1.4:

Proof. Let $S, T \in \mathcal{S}_c(\mathbb{A})$ and h as in the previous proposition. Let us define $\overline{h} : \mathcal{S}_c(\mathbb{A}) \to \mathcal{S}_c(\mathbb{A})$ such that $\overline{h}(X) = h''(X)$. If $X \in \mathcal{S}_c(\mathbb{A})$, then $h^{-1}(X) \in \mathcal{S}_c(\mathbb{A})$, so $\overline{h}(h^{-1}(X)) = X$ and $\overline{h}$ is onto. If $X, Y \in \mathcal{S}_c(\mathbb{A})$ and $\overline{h}(X) = \overline{h}(Y)$, then $h''(X) = h''(Y)$, so $X = Y$ by the injectivity of h. Hence, $\overline{h}$ is bijective and $\overline{h}(S) = T$.

We will prove that $\overline{h}$ is continuous. Let B a basic set of $\mathcal{S}_c(\mathbb{A})$. We have two cases. If $B = \mathcal{S}_c(\mathbb{A}) \cap [V]$ with V an open set of $\mathbb{A}$, then $\overline{h}^{-1}(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}^{-1}([V]) = \mathcal{S}_c(\mathbb{A}) \cap [h^{-1}(V)]$. If $B = \mathcal{S}_c(\mathbb{A}) \cap \langle V\rangle$ with V a basic set of $\mathbb{A}$, then $\overline{h}^{-1}(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}^{-1}(\langle V\rangle) = \mathcal{S}_c(\mathbb{A}) \cap \langle h^{-1}(V)\rangle$. Therefore, $\overline{h}$ is continuous.

To end, we will prove that $\overline{h}$ is an open map. Let B a basic set of $\mathcal{S}_c(\mathbb{A})$. If $B = \mathcal{S}_c(\mathbb{A}) \cap [V]$ with V an open set of $\mathbb{A}$, then $\overline{h}''(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}''([V]) = \mathcal{S}_c(\mathbb{A}) \cap [h''(V)]$. If $B = \mathcal{S}_c(\mathbb{A}) \cap \langle V\rangle$ with V a basic set of $\mathbb{A}$, then $\overline{h}''(B) = \mathcal{S}_c(\mathbb{A}) \cap \overline{h}''(\langle V\rangle) = \mathcal{S}_c(\mathbb{A}) \cap \langle h''(V)\rangle$. $\square$

Analogously, we can prove that

Proposition 2.3. *$\mathcal{S}_c(\mathbb{S})$ is homogeneous.*

3 Homogeneity of $\mathcal{C}_1(\mathbb{S})$

Let $\Delta_2 = \{(x, y) \in \mathbb{S}^2 : x \leq y\}$. Let $\mathcal{C}_n(\mathbb{S}) \subset \mathrm{Exp}(\mathbb{S})$ be the hyperspace of all unions of at most n non-empty closed intervals of $\mathbb{S}$.

Proposition 3.1. *The function $\rho : \Delta_2 \to \mathcal{C}_1(\mathbb{S})$ defined by $\rho(a, b) = [a, b]$ is a homeomorphism.*

Proof. It is easy to see that ρ is a bijection. For the continuity, we will prove that the preimages under ρ of $[V]$ and $\langle W\rangle$, with V an open set and $W = [c, d[$ a basic open set, are open.

Let V an open set of $\mathbb{S}$. There exists basic intervals V_j such that $V = \bigcup_j V_j$. Let $(a, b) \in \rho^{-1}([V]) = \{(x, y) \in \Delta_2 : [x, y] \subset \bigcup_j V_j\}$. We define $B = \bigcup\{V_j : [a, b] \cap V_j \neq \emptyset\}$. We have that B is an interval and open set that contains $[a, b]$. Let $(x, y) \in \Delta_2 \cap B^2$. Since $x, y \in B$, we have that $[x, y] \subset B \subset \bigcup_j V_j = V$. Therefore, $(a, b) \in \Delta_2 \cap B^2 \subset \rho^{-1}([V])$ and $\rho^{-1}([V])$ is open.

Let $W = [c, d[$ a basic interval of $\mathbb{S}$ and $(a, b) \in \rho^{-1}(\langle W\rangle)$. By definition, $[a, b] \cap W \neq \emptyset$. We have two cases.

Case 1. If $c \leq b < d$, let us consider $(x, y) \in \Delta_2 \cap (\mathbb{S} \times W)$. Thus, $[x, y] \cap W \neq \emptyset$. In this way, $(a, b) \in \Delta_2 \cap (\mathbb{S} \times W) \subset \rho^{-1}(\langle W\rangle)$.

Case 2. If $b \geq d$, necessarily $a < d$. Let $(x, y) \in \Delta_2 \cap (] \leftarrow, d[\times [d, \rightarrow [)$. By definition, $[x, y] \cap W \neq \emptyset$. Therefore, $(a, b) \in \Delta_2 \cap (] \leftarrow, d[\times [d, \rightarrow [) \subset \rho^{-1}(\langle W\rangle)$. We conclude that $\rho^{-1}(\langle W\rangle)$ is open.

To show that ρ^{-1} is continuous, we will prove that ρ is an open map. Without loss of generality, let $V = \Delta_2 \cap (C \times \mathbb{S})$ an open set of Δ_2, with C a basic interval of $\mathbb{S}$ and $[a, b] \in \rho''(V)$. Thus, $(a, b) \in V$, that is to say $a \leq b$ and $a \in C$. Let $B = [a, \rightarrow [$ and consider $[x, y] \in \langle C\rangle \cap [B]$. Since $[x, y] \cap C \neq \emptyset$ and $[x, y] \subset B$, we have that $x \in C$, so $(x, y) \in V$. In this way, $[a, b] \in \langle C\rangle \cap [B] \subset \rho''(V)$. Therefore, $\rho''(V)$ is an open set. ☐

Corollary 3.2. *$\mathcal{C}_1(\mathbb{S})$ is homogeneous.*

Proof. By the previous proposition, $\mathcal{C}_1(\mathbb{S})$ is homeomorphic to Δ_2. By [[BM23], Theorem 1.4] the results holds. ☐

Question 3.3. *Is the hyperspace $\mathcal{C}_2(\mathbb{S})$ homogeneous?*

4 Non-homogeneity of $\mathcal{C}_m(\mathbb{A})$

The purpose of this section is to prove Theorem 1.3. It will be convenient to introduce some notation.

We will think of an element of the finite power $x \in {}^mX$ as a function $x : m \to X$. Given a linearly ordered space X, let $\Delta_m(X) = \{x \in {}^mX : \forall i \in m-1(x(i) \leq x(i+1))\}$ and let $\mathcal{F}_m(X)$ be the hyperspace of X consisting of all finite non-empty subsets of cardinality at most m. Let $\rho : \Delta_m(X) \to \mathcal{F}_m(X)$ be the map given by $\rho(x) = \{x(0), \ldots, x(m-1)\}$ and let $\sim$ denote the equivalence relation on $\Delta_m(X)$ defined by $x \sim y$ if and only if $\rho(x) = \rho(y)$. We consider $\Delta_m(X)/\sim$ as a topological space with the quotient topology.

The following classical fact gives us a more geometric representation of $\mathcal{F}_m(X)$.

Proposition 4.1 ([Ga54])**.** *If X is a linearly ordered space, then the map $\tilde{\rho} : \Delta_m(X)/\sim \to \mathcal{F}_m(X)$ given by $\tilde{\rho}([x]) = \rho(x)$ is a homeomorphism.*

Let $(X, <)$ be a linearly ordered space. For $m \geq 1$, we denote $\mathcal{C}_m(X) \subset \mathrm{Exp}(X)$ as the hyperspace of all unions of at most m non-empty closed intervals in X. Let $\varrho : \Delta_{2m}(X) \to \mathcal{C}_m(X)$ be the map defined by $\varrho(x) = \bigcup_{i\in m}[x(2i), x(2i+1)]$ and let $\approx$ the equivalence relation on $\Delta_{2m}(X)$ defined by $x \approx y$ if and only if $\varrho(x) = \varrho(y)$. Let $p : \Delta_{2m}(X) \to \Delta_{2m}(X)/\approx$ be the quotient map. We will sometimes write $[x]$ instead of $p(x)$ to represent the equivalence class. We consider $\Delta_{2m}(X)/\!\approx$ as a topological space with the quotient topology.

Proposition 4.2. *If $(X,<)$ is a linearly ordered space, then ϱ is continuous.*

Proof. We will prove that the preimages under ϱ of $[V]$ and $\langle W\rangle$, with V an open set and W a basic interval, are open.

Let V an open set of X. There exists basic intervals V_j such that $V = \bigcup_{j\in J} V_j$. Let $x \in \varrho^{-1}([V]) = \{y \in \Delta_{2m}(X) : \bigcup_{i\in m}[y(2i), y(2i+1)] \subset \bigcup_{j\in J} V_j\}$. For each $i \in m$ we define $W_i = \bigcup\{V_j : [x(2i), x(2i+1)] \cap V_j \neq \emptyset\}$. We have that W_i is an open interval that contains $[x(2i), x(2i+1)]$. Let $y \in \Delta_{2m}(X) \cap \prod_{i\in m} W_i^2$. For all i, $y(2i)$ and $y(2i+1)$ are in W_i, so $\bigcup_{i\in m}[y(2i), y(2i+1)] \subset \bigcup_{i\in m} W_i \subset \bigcup_{j\in J} V_j = V$. Therefore, $x \in \Delta_{2m}(X) \cap \prod_{i\in m} W_i^2 \subset \varrho^{-1}([V])$ and $\varrho^{-1}([V])$ is open.

Let W be a basic open interval of X and let $x \in \varrho^{-1}(\langle W\rangle)$ be given. By definition, there exists j such that $[x(2j), x(2j+1)] \cap W \neq \emptyset$. If $W = \,]\leftarrow, a[$, then we define $B = \prod_{i\in 2m} B_i$ with $B_i = X$ if $i \neq 2j$ and $B_{2j} = W$. If $y \in \Delta_{2m}(X) \cap B$, then $[y(2j), y(2j+1)] \cap W \neq \emptyset$, that is to say $\bigcup_{i\in m}[y(2i), y(2i+1)] \cap W \neq \emptyset$. In this way, $x \in \Delta_{2m}(X) \cap B \subset \varrho^{-1}(\langle W\rangle)$. The proof for $W = \,]a, \rightarrow[$ is similar. When $W = \,]a, b[$ we have two cases.

Case 1. $a < x(2j+1) < b$. Define $B = \prod_{i\in 2m} B_i$ with $B_i = X$ if $i \neq 2j+1$ and $B_{2j+1} = W$. If $y \in \Delta_{2m}(X) \cap B$, then $[y(2j), y(2j+1)] \cap W \neq \emptyset$. We have that $\bigcup_{i\in m}[y(2i), y(2i+1)] \cap W \neq \emptyset$. In this way, $x \in \Delta_{2m}(X) \cap B \subset \varrho^{-1}(\langle W\rangle)$.

Case 2. $x(2j+1) \geq b$. Necessarily $x(2j) < b$. Define $B = \prod_{i\in 2m} B_i$ with $B_i = X$ if $i \in 2m \setminus \{2j, 2j+1\}$, $B_{2j} = \,]\leftarrow, b[$ and $B_{2j+1} = \,]a, \rightarrow[$. If $y \in \Delta_{2m}(X) \cap B$, then $[y(2j), y(2j+1)] \cap W \neq \emptyset$. Therefore, $x \in \Delta_{2m}(X) \cap B \subset \varrho^{-1}(\langle W\rangle)$.

We conclude that $\varrho^{-1}(\langle W\rangle)$ is open. ☐

Analogously to Proposition 4.1, the following result gives us a more geometric representation of $\mathcal{C}_m(X)$.

Corollary 4.3. *If $(X,<)$ a compact linearly ordered space, then the map $\tilde{\varrho} : \Delta_{2m}(X)/\approx \; \rightarrow \mathcal{C}_m(X)$ given by $\tilde{\varrho}([x]) = \varrho(x)$ is a homeomorphism.*

Proof. Since ϱ is continuous, we have that $\tilde{\varrho}$ is a continuous bijection. Let $x \in {}^{2m}X \setminus \Delta_{2m}(X)$. There are $i, j \in 2m$ such that $i < j$ and $x(i) > x(j)$. Since X is Hausdorff, there exists two disjoint basic intervals V and W with $W < V$ such that $x(i) \in V$ and $x(j) \in W$. Let $A = \prod_{k\in 2m} X_k$ an open neighborhood of x with $X_k = X$ for all $k \in 2m \setminus \{i, j\}$, $X_i = V$ and $X_j = W$. We have that $x \in A \subset {}^{2m}X \setminus \Delta_{2m}(X)$, so $\Delta_{2m}(X)$ is closed in ${}^{2m}X$. Since ${}^{2m}X$ is compact, so is $\Delta_{2m}(X)$. Therefore, $\Delta_{2m}(X)/\approx$ is compact and $\tilde{\varrho}$ is a homeomorphism. ☐

Remark 4.4. *We note that $\Delta_2(X) = \Delta_2(X)/\approx$. By the previous Corollary and Proposition 4.1, we have that $\mathcal{F}_2(X) \cong \mathcal{C}_1(X)$.*

Let $\pi : \mathbb{A} \rightarrow [0,1]$ be the projection onto the first factor $\pi(\langle x, r\rangle) = x$. For any $a \in \mathbb{A}$ we will denote by $\overline{a}$ the constant sequence a of finite length m, where the value of m should be understood by context. Let $\pi_i : {}^m\mathbb{A} \rightarrow \mathbb{A}$ be the projection onto the i-th coordinate, and for any function $h : {}^m\mathbb{A} \rightarrow {}^m\mathbb{A}$, let $h_i = \pi_i \circ h$ denote its i-th coordinate function. Recall that a partial function $f : \mathbb{A} \rightarrow \mathbb{A}$ is monotone if it is either non-decreasing or non-increasing, and f is strictly monotone if it is either strictly increasing or strictly decreasing.

We recall the following results.

Proposition 4.5 ([BM23] Proposition 2.2)**.** *Let $h : \mathbb{A} \to \mathbb{A}$ be a monotone continuous function. Then there is a clopen interval J so that either $h \restriction J$ is constant or $h \restriction J$ is strictly monotone.*

Proposition 4.6 ([BM23] Proposition 3.2)**.** *Every clopen subset of ${}^m\mathbb{A}$ is homeomorphic to ${}^m\mathbb{A}$.*

Lemma 4.7. *If $\mathcal{C}_m(\mathbb{A})$ is homogeneous, then it is homeomorphic to ${}^{2m}\mathbb{A}$.*

Proof. Suppose $\mathcal{C}_m(\mathbb{A})$ is homogeneous, then there is an autohomeomorphism $h : \Delta_{2m}/{\approx} \to \Delta_{2m}/{\approx}$ such that $h([\overline{\langle 0,1\rangle}]) = [x]$, where x is some fixed point such that $\pi(x(0)) < \pi(x(1)) < \cdots < \pi(x(2m{-}1))$. On one hand, notice that, if $J_0 < \cdots < J_{2m-1}$ is a sequence of pairwise disjoint clopen intervals with $x(i) \in J_i$ for $i \in 2m$ and $\max(\pi(J_i)) < \min(\pi(J_{i+1}))$ for $i \in 2m-1$, then $p \restriction \prod\limits_{i\in 2m} J_i : \prod\limits_{i\in 2m} J_i \to \Delta_{2m}/{\approx}$ is an embedding. On the other hand, observe that for any $0 < \epsilon < 1$ the clopen cube ${}^{2m}[\langle 0,1\rangle, \langle \epsilon, 0\rangle]$ is a saturated neighborhood of $\overline{\langle 0,1\rangle}$ such that $p''({}^{2m}[\langle 0,1\rangle,\langle\epsilon,0\rangle])$ is homeomorphic to $\Delta_{2m}/{\approx}$. Since h is continuous, there is an $\epsilon > 0$ such that $h''({}^{2m}[\langle 0,1\rangle,\langle\epsilon,0\rangle]/\approx) \subseteq \prod\limits_{i\in 2m} J_i$. Thus, we have that

$$ {}^{2m}\mathbb{A} \cong \prod_{i\in 2m} J_i \cong h''({}^{2m}[\langle 0,1\rangle,\langle\epsilon,0\rangle]/{\approx}) \cong {}^{2m}[\langle 0,1\rangle,\langle\epsilon,0\rangle]/{\approx} \cong \Delta_{2m}/{\approx} $$

where the second homeomorphism follows from Proposition 4.6. ◻

We are now ready to prove the main result of the section.

Proof of Theorem 1.3:

Proof. We proceed by contradiction. Suppose that there is a homeomorphism $h : \Delta_{2m}/{\approx} \to {}^{2m}\mathbb{A}$, and let $\Gamma = \{[\overline{x}] \in \Delta_{2m}/{\approx} : x \in \mathbb{A}\}$. Recall that the diagonal $\{(x,x) \in {}^2\mathbb{A} : x \in \mathbb{A}\}$ is not a G_δ subspace as $\mathbb{A}$ is a non-metrizable compact space. It follows from this that Γ is not a G_δ in $\Delta_{2m}/{\approx}$ as otherwise this would imply that $\pi''_{\{0,1\}}(p^{-1}(\Gamma)) = \pi''_{\{0,1\}}(\{\overline{x} \in \Delta_{2m} : x \in \mathbb{A}\}) = \{(x,x) \in {}^2\mathbb{A} : x \in \mathbb{A}\}$ would also be one, where $\pi_{\{0,1\}} : {}^{2m}\mathbb{A} \to {}^2\mathbb{A}$ denotes the projection onto the first 2 coordinates. Notice that, since $\mathbb{A} \times {}^{2m-1}\{\langle 0,1\rangle\} = \bigcap_{n\in\omega} \mathbb{A} \times {}^{2m-1}\{[\langle 0,1\rangle, \langle \frac{1}{n}, 0\rangle]\} \subseteq {}^{2m}\mathbb{A}$ and $\mathbb{A}$ is a perfect space, then every closed subset of $\mathbb{A} \times {}^{2m-1}\{\langle 0,1\rangle\}$ is a G_δ set in ${}^{2m}\mathbb{A}$. Analogously, every closed subset of any line parallel to one of coordinates axis, is also a G_δ set in ${}^{2m}\mathbb{A}$. We now consider the embedding $\alpha : \mathbb{A} \to {}^{2m}\mathbb{A}$ given by $\alpha(x) = h([\overline{x}])$. By applying Proposition 4.5 $2m$-times, we can find a clopen interval J such that $\alpha_j := \pi_j \circ \alpha \restriction J$ is monotone for every $j \in 2m$. Since $h''(\Gamma)$ is not a G_δ in ${}^{2m}\mathbb{A}$, it follows, by our previous observations, that there exists $j_0 \neq j_1 \in 2m$ such that α_{j_0} and α_{j_1} are strictly monotone restricted to J. We will assume that both $\alpha_{j_0} \restriction J, \alpha_{j_1} \restriction J$ are strictly increasing, as the other cases are analogous.

The proof of the following result is is analogous to the proof of [[BM23], Claim 3.5].

Claim 4.8. *There is a countable subset $C \subseteq \pi''(J)$ such that*

$$ \pi(\alpha_{j_0}(\langle a,0\rangle)) = \pi(\alpha_{j_0}(\langle a,1\rangle)) $$

and

$$ \pi(\alpha_{j_1}(\langle a,0\rangle)) = \pi(\alpha_{j_1}(\langle a,1\rangle)) $$

for any $a \in \pi''(J) \setminus C$. In other words, $\alpha_{j_k}(\langle a, 1\rangle)$ is the immediate successor of $\alpha_{j_k}(\langle a, 0\rangle)$ for $k \in 2$.

For each $a \in A := \pi''(J) \setminus C$, let $P_a^- = \alpha(\langle a, 0\rangle), Q_a^+ = \alpha(\langle a, 1\rangle)$ and let

$$P_a^+ = \alpha(\langle a, 0\rangle) \restriction_{(2m\setminus\{j_0\})} \cup (j_0, \langle \pi(\alpha_{j_0}(\langle a, 0\rangle)), 1\rangle)$$

and

$$Q_a^- = \alpha(\langle a, 1\rangle) \restriction_{(2m\setminus\{j_1\})} \cup (j_1, \langle \pi(\alpha_{j_1}(\langle a, 0\rangle)), 0\rangle).$$

Pick an element $[x_a]$ belonging to $h^{-1}(\{P_a^+, Q_a^-\}) \setminus \tilde{\varrho}^{-1}([\langle a, 0\rangle, \langle a, 1\rangle])$. Observe that, by our choice of x_a, there is a $\ell_a \in 2m$ so that $\pi(x_a(\ell_a)) \neq a$. Let

$$A^{P,<} = \{a \in A : h([x_a]) = P_a^+, \pi(x_a(\ell_a)) < a\},$$

$$A^{P,>} = \{a \in A : h([x_a]) = P_a^+, \pi(x_a(\ell_a)) > a\},$$

$$A^{Q,<} = \{a \in A : h([x_a]) = Q_a^-, \pi(x_a(\ell_a)) < a\}$$

and

$$A^{Q,>} = \{a \in A : h([x_a]) = Q_a^-, \pi(x_a(\ell_a)) > a\}.$$

We may assume, without loss of generality, that $A^{P,<}$ is uncountable as the other cases are similar. By successively refining $A^{P,<}$, we can find an uncountable subset $B \subseteq A^{P,<}$, a natural number ℓ and a rational number $r \in \mathbb{Q}$ such that $\ell_a = \ell$ and $\pi(x_a(\ell)) < r < a$ for any $a \in B$.

Consider the clopen sets

$$U := \bigcup_{j\in 2m} \pi_j^{-1}([\langle 0, 1\rangle, \langle r, 0\rangle]) \text{ and } V := \bigcap_{j\in 2m} \pi_j^{-1}([\langle r, 1\rangle, \langle 1, 0\rangle]).$$

Claim 4.9. *The sets U and V are saturated.*

Proof. Let $x \in p^{-1}(p''(U))$. There is $y \in U$ such that $\bigcup_{i\in m}[x(2i), x(2i + 1)] = \bigcup_{i\in m}[y(2i), y(2i+1)]$. Since there is $j \in 2m$ and $k \in m$ with $\langle 0, 1\rangle \le y(j) \le \langle r, 0\rangle$ and $y(j) \in [x(2k), x(2k + 1)]$, then $\langle 0, 1\rangle \le x(2k) \le \langle r, 0\rangle$. Thus, $x \in U$.

Let $x \in p^{-1}(p''(V))$. There is $y \in V$ such that $\bigcup_{i\in m}[x(2i), x(2i+1)] = \bigcup_{i\in m}[y(2i), y(2i + 1)]$. Since $y(j) \in [\langle r, 1\rangle, \langle 1, 0\rangle]$ for any $j \in 2m$, we have that $\bigcup_{i\in m}[x(2i), x(2i + 1)] \subset [\langle r, 1\rangle, \langle 1, 0\rangle]$ for any $j \in 2m$. It follows that $x(j) \in [\langle r, 1\rangle, \langle 1, 0\rangle]$ for any $j \in 2m$, that is to say, $x \in V$. □

We have that $\tilde{U} := p''(U)$ and $\tilde{V} := p''(V)$ form a clopen partition of $\Delta_{2m}/{\approx}$. Notice that $X := \{[x_a] : a \in B\} \subset \tilde{U}$ and $Y := \{[\overline{\langle a, 0\rangle}] : a \in B\} \subset \tilde{V}$. Since B is infinite (uncountable) and $\Delta_{2m}/{\approx}$ is compact, then the set of accumulation points X' and Y' of X and Y, respectively, are both non-empty. It follows that $X' \cap Y' = \emptyset$.

Claim 4.10. *The sets $h''(X) = \{P_a^+ : a \in B\}$ and $h''(Y) := \{P_a^- : a \in B\}$ have the same accumulations points.*

Proof. We shall prove that the accumulation points of $h''(X)$ are contained in the accumulation points of $h''(Y)$ as the other case is analogous. Let P be an accumulation point of $h''(X)$ and let $W := \prod_{j\in 2m} J_j$ be a clopen neighborhood of P where each J_j is a clopen interval. Since P is an accumulation point, then there is an infinite subset $B' \subseteq B$ such that $\{P_a^+ : a \in B'\} \subseteq W$. By construction $P_a^-(j) = P_a^+(j)$ for any $j \in 2m \setminus \{j_0\}$ and $a \in B$. In particular, $P_a^-(j) \in J_j$ for any $j \in 2m \setminus \{j_0\}$ and $a \in B'$. Observe that $P_a^+(j_0) \neq P_b^+(j_0)$ for any $a \neq b \in B$ as $\alpha_{j_0} \restriction J$ is strictly monotone. Thus, there is an infinite subset $B'' \subseteq B'$ such that $\pi(P_a^+(j_0)) \notin \{\pi(\min(J_{j_0})), \pi(\max(J_{j_0}))\}$ for all $a \in B''$. It follows that, $\{P_a^- : a \in B''\} \subseteq W$ and hence, P is an accumulation point of $h''(Y)$ as required. $\square$

Since h is a homeomorphism, then X and Y have the same accumulation points which is a contradiction. This finishes the proof of the Theorem. $\square$

It would be interesting to see if the above theorem can be extended to the hyperspace of all finite unions of non-empty closed intervals $\mathcal{C}(\mathbb{A})$.

Question 4.11. *Is the hyperspace $\mathcal{C}(\mathbb{A})$ homogeneous?*

5 A metrization theorem

Proposition 5.1. *Let X be a compact Hausdorff space. If there exists a G_δ-set $C \subset X^2$ homeomorphic to X such that for every $x \in X$ there are $a \in C$ and a unique $b \in C$ with $x = \pi_1(a) = \pi_2(b)$, then X is metrizable.*

Proof. For each $n \in \omega$, let G_n be an open subset of X^2 such that $C = \bigcap_{n\in\omega} G_n$ and $G_{n+1} \subset G_n$. Since X^2 is normal and C is closed, we can define a sequence of open sets U_n as follows. Let $U_0 = G_0$ and for $n > 0$ let U_n such that $C \subset U_n \subset \overline{U_n} \subset U_{n-1} \cap G_n$. It follows that $C = \bigcap_{n\in\omega} \overline{U_n}$. Let $x \in X$ and $(s, x) \in C$. For $n \in \omega$, let $U_n[s] := \{y \in X : (s, y) \in U_n\}$. Hence, $\overline{U_n[s]} \subset U_{n-1}[s]$ for any $n > 0$, since

$$\begin{aligned}\overline{U_n[s]} &= \overline{\pi_2''(U_n \cap (\{s\} \times X))} = \pi_2''(\overline{U_n \cap (\{s\} \times X)}) \subset \pi_2''(\overline{U_n} \cap \overline{\{s\} \times X}) \\ &\subset \pi_2''(U_{n-1} \cap (\{s\} \times X)) = U_{n-1}[s]\end{aligned}$$

where the second equality follows from [[En89], Corollary 3.1.11].

Claim 5.2. *For any open neighborhood V of x, there exists $n \in \omega$ such that*

$$x \in U_n[s] \subset V$$

Proof. We proceed by contradiction. Let V be an open neighborhood of x with $U_n[s] \not\subset V$ for any $n \in \omega$. We choose $x_n \in U_n[s] \setminus V$. It follows that $\bigcap_{n\in\omega} \overline{U_n[s]} = \{x\}$, since if $z \in \bigcap_{n\in\omega} \overline{U_n[s]} \subset \bigcap_{n\in\omega} U_n[s]$, then $(s, z) \in \bigcap_{n\in\omega} U_n \subset C$. Since X is compact, the pseudocharacter and the character of x are equal. Since the sets $U_n[s]$ are open, the pseudocharacter of x is countable. Hence, X is first countable. By [[En89], Theorem 3.10.31] X is sequentially compact. In this way, there exists a convergent subsequence of $(x_n)(n \in \omega)$, let us say with limit L. Since each set $\overline{U_m[s]}$ is closed and contains infinitely many elements of such subsequence, we have that L belongs to each one of them. Thus, $L \in \bigcap_{m\in\omega} \overline{U_m[s]} = \{x\}$. By convergence, there are infinite elements of the subsequence in V, which is a contradiction. $\square$

For each $(s, t) \in C$ and $n \in \omega$ we can choose open neighborhoods $B_{s,n}$ and $B_{t,n}$ such that $B_{s,n} \times B_{t,n} \subset U_n$. Since X is compact, there exists a finite subcover $\mathcal{B}_n$ of $\{B_{t,n} : t \in X\}$. We claim that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ is a countable base for X. Let $x \in X$ and V an open neighborhood of x. By the claim, there is $n \in \omega$ such that $x \in U_n[s] \subset V$. We choose $B_{z,n}$ from $\mathcal{B}_n$ with $x \in B_{z,n}$. Therefore, $x \in B_{z,n} \subset U_n[s] \subset V$.

By the Urysohn's metrization theorem, X is metrizable. $\square$

As a consequence, we obtain the following classical fact.

Corollary 5.3 ([Sn45]). *Let X a compact Hausdorff space. If the diagonal of X is a G_δ-set, then X is metrizable.*

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