CMC FOLIATIONS ON EUCLIDEAN SPACES ARE MINIMAL FOLIATIONS

JOSÉ EDSON SAMPAIO AND EURIPEDES CARVALHO DA SILVA

Abstract. In this article, we give complete answers to some classical problems and conjectures on differential geometry (of foliations). For instance, we give a complete positive answer to the classical conjecture that states that every foliation on $\mathbb{R}^{n+1}$ by (possibly varying) CMC hypersurfaces is a foliation by minimal hypersurfaces. Moreover, if $n \leq 4$ such a CMC foliation must consist of parallel hyperplanes. We prove also that such conjecture holds true in much more general situations, for instance, when the ambient space is a complete Riemannian manifold with non-negative Ricci curvature. We prove also that for a foliation by CMC hypersurfaces on a complete Riemannian manifold M with sectional curvature bounded from below by $-K_0 \leq 0$, then the mean curvature H of the leaves of the foliation satisfies $|H| \leq \sqrt{K_0}$. This gives a complete positive answer to a conjecture due to Meeks III, Pérez and Ros. We give some answers to several other problems.

Contents

2010 *Mathematics Subject Classification.* 53C12, 53C42.

Key words and phrases. Foliations, Minimal hypersurface, Stable hypersurface.

The second-named author was partially supported by CNPq-Brazil grant 310438/2021-7. This work was supported by the Serrapilheira Institute (grant number Serra – R-2110-39576).

1. Introduction

In 1986, Solomon [27] posed the following problem:

Problem 1.1. *Is every foliation of* $\mathbb{R}^{n+1}$*,* $n \leq 7$*, by minimal hypersurfaces a foliation by parallel affine hyperplanes?*

In the case that the leaves are properly embedded, Problem 1.1 has an affirmative answer if $n+1 < 8$ (see [25, Theorems 1 and 3]), and it has a negative answer if $n+1 > 8$, since in those dimensions there are minimal graphs that are not hyperplanes. In the critical dimension $n+1 = 8$, Solomon in [27] stated that it appears quite difficult to settle this question, even in the case where the leaves are properly embedded. When the leaves are not supposed to be properly embedded the problem is much more complicated.

This article deals with the following long-standing and more complicated problems:

Conjecture 1.2. *Every foliation of* $\mathbb{R}^{n+1}$*,* $n \leq 7$*, by (possibly varying) CMC hypersurfaces is a foliation by parallel affine hyperplanes.*

Conjecture 1.3. *Every foliation of* $\mathbb{R}^{n+1}$ *by (possibly varying) CMC hypersurfaces is a foliation by minimal hypersurfaces.*

Although these conjectures were only explicitly stated in 2008 by Meeks III, Pérez and Ros (see [18, Conjecture 5.1]), many mathematicians have been working on them since at least the 1980s. For instance, in 1987, Barbosa, Gomes and Silveira in [3, Theorem 3.12] proved that Conjectures 1.2 and 1.3 have positive answers when $n = 2$ and under the extra hypothesis that all the leaves of the foliation have the same constant mean curvature, and in 1988, Meeks III in [17, Theorem 4.1] presented a beautiful positive answer for Conjectures 1.2 and 1.3 when $n = 2$. Still in [18], Meeks III, Pérez and Ros proved that Conjecture 1.3 has a positive answer if $n \leq 4$. So far the authors of this paper know, these above mentioned results from [17] and [18] are unique cases where these conjectures are known to be true without any additional hypothesis. One can find other partial answers or related results in the references [1], [2], [3], [6], [7], [9], [10], [11] [12], [16], [18], [19], [20], [21], [22], [23], [29] etc.

In this article, we give a complete positive answer for Conjecture 1.3 and prove that Conjecture 1.2 has a positive answer whether $n \leq 4$. More precisely, for a codimension one foliation $\mathfrak{F}$ of $\mathbb{R}^{n+1}$ and denoting by N the vector field on $\mathbb{R}^{n+1}$ that is normal to the leaves of $\mathfrak{F}$, we prove the following:

Theorem 3.7. *Let $\mathfrak{F}$ be a codimension one CMC foliation of $\mathbb{R}^{n+1}$. Then, each leaf of $\mathfrak{F}$ is a minimal and stable hypersurface. Moreover, if $n \leq 4$ or $\|\nabla_N N\| \in L^1(\mathbb{R}^{n+1})$, then $\mathfrak{F}$ consists entirely of parallel hyperplanes.*

Indeed, we prove the following much more general result.

Theorem 3.4. *Let $\mathfrak{F}$ be a codimension one CMC foliation of a complete Riemannian manifold M^{n+1} with non-negative Ricci curvature. Then, each leaf of $\mathfrak{F}$ is a minimal and stable hypersurface.*

We prove even a more general result than Theorem 3.4 (see Theorem 3.1 and Corollary 3.5). It is worth noting that although constant on each leaf L, the mean curvature is allowed to vary with L.

In particular, Theorem 3.4 generalizes [2, Theorem 3.1] and [1, Theorem 1.2], where we remove the hypothesis of compactness of the ambient space in [2, Theorem 3.1] and the hypothesis of the same mean curvature in [1, Theorem 1.2] (see also Corollary 3.5). Note also that Theorem 3.7, in particular, recover the results in [17] and [18] that were mentioned above.

Note that Problem 1.1 and Conjectures 1.2 and 1.3 are closely related to two other long-standing and well-known problems. One proposed by do Carmo (see [7, Question, p. 133]), which is the following:

Problem 1.4. *Is it true that a complete noncompact stable hypersurface $x\colon M \to \mathbb{R}^{n+1}$ with constant mean curvature is minimal?*

The other problem was proposed by Yau (see Problem 102 in [28]), which is the following:

Problem 1.5. *For $n \leq 7$, is an oriented stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ a hyperplane?*

Note that Theorem 3.7 gives a positive answer to Problem 1.4, provided the hypersurface is a leaf of a foliation such that the leaves have constant mean curvature.

It is obvious that if Problem 1.4 has a positive answer in dimension n, then Conjecture 1.3 also has a positive answer in dimension n. Note also that if Problem 1.5 and Conjecture 1.3 have a positive answer in dimension n, then Conjecture 1.2 also has a positive answer in dimension n. Thus, it is a consequence of the proof of Theorem 3.7 that if Problem 1.5 has a positive answer in dimension n, then Conjecture 1.2 has a positive answer in dimension n.

Barbosa, Gomes and Silveira [3] proved that for a foliation $\mathfrak{F}$ of $\tilde{M}^3(c)$ by surfaces L, all with the same constant mean curvature H, then:

i) If $c = 0$, then the leaves of the foliation are planes.

ii) If $c > 0$, no such foliation exists.

iii) If $c < 0$, and $H \geq (-c)^{1/2}$, the leaves of $\mathfrak{F}$ are horospheres with $H = (-c)^{1/2}$.

In the particular case of $\mathbb{R}^3$, as was already mentioned above, Meeks III [17] was able to prove that the above foliation $\mathfrak{F}$ is again given by planes even if the mean curvature, although constant on each leaf L, is allowed to vary with L. In the work [7], do Carmo also stated that he did not know whether this holds for $c \neq 0$ (see the last paragraph of Section 2 in [7]). Thus, the following problem becomes natural:

Problem 1.6 (Generalized do Carmo's problem)**.** *Let $\mathfrak{F}$ be a foliation of $\tilde{M}^{n+1}(c)$ by hypersurfaces L of constant mean curvature H_L, then:*

i) *If $c = 0$, is it true that the leaves of the foliation are minimal?*

ii) *If $c > 0$, is it true that no such foliation exists?*

iii) *If $c < 0$, and $H_L \geq (-c)^{1/2}$, is it true that H_L does not depend on L and $H_L = (-c)^{1/2}$? Moreover, if $n = 2$, is it true that the leaves of $\mathfrak{F}$ are horospheres with $H_L = (-c)^{1/2}$?*

Item ii) of Problem 1.6 was positively answered by Barbosa, Kenmotsu and Oshikiri in [2, Corollary 3.5]. They also gave a partial answer to Item iii) of Problem 1.6, they proved that $\inf H_L = (-c)^{1/2}$ (see [2, Theorem 3.8]). Item iii) of Problem 1.6 was positively answered by Meeks III, Pérez and Ros in [18, Corollary 5.10] when $n = 2$.

In this article, we give complete positive answers to all the items of Problem 1.6 (see Theorem 4.6) and, in particular, we recover the results from [2] and [18] above mentioned.

Another problem approached here, which is closely related to Problem 1.6, is the following conjecture, which was also proposed by Meeks III, Pérez and Ros in [18, Conjecture 5.1.2]:

Conjecture 1.7. *Let $\mathfrak{F}$ be a codimension one foliation of a complete Riemannian manifold M^{n+1}. Assume that M has absolute sectional curvature bounded from above by 1. Suppose that each leaf L of $\mathfrak{F}$ has constant mean curvature H_L. Then $|H_L| \leq 1$.*

Meeks III, Pérez and Ros in [18, Corollary 5.10] proved Conjecture 1.7 has a positive answer in the case that $M = \tilde{M}^3(-1)$. They also proved in

[18, Theorem 5.23] that when M is a homogeneously regular manifold with absolute sectional bounded from above by 1 and $n = 3$ or 4, the absolute mean curvature of any leaf of a codimension one CMC foliation of M is bounded by some constant H_n that only depends on n.

In this paper, we give also a complete positive answer to this conjecture. Indeed, we obtain that Conjecture 1.7 holds true in the more general setting where we ask that the sectional curvature is bounded only from below.

Theorem 4.3. *Let $\mathfrak{F}$ be a codimension one CMC foliation of a complete Riemannian manifold M^{n+1}. Assume that there is $K_0 \geq 0$ such that the sectional curvature of M is bounded from below by $-K_0$. Let $H\colon M \to \mathbb{R}$ be the function that associates to each point the value of the mean curvature of the leaf of $\mathfrak{F}$ that contains that point. Then $|H| \leq \sqrt{K_0}$.*

2. Preliminaries

In this Section, we introduce some basic facts and notations that will appear in the paper.

Here, the Riemannian manifolds are assumed to be connected and without boundary and the foliations are assumed to be C^2 smooth.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold endowed with a **Riemannian metric** $g_M = \sum \omega_A^2$ and $\mathfrak{F}$ is a foliation of codimension one on M.

For a given point $p \in M$ we can choose an orthonormal frame $\{e_1, \cdots, e_n, e_{n+1}\}$ defined around p such that the vectors $e_1, \cdots, e_n$ are tangent to the leaves of $\mathcal{F}$ and e_{n+1} is normal to them. Taking the correspondent dual coframe

$$\{\omega_1, \cdots, \omega_n, \omega_{n+1}\},$$

the **structure equations** on M are given by

$$d\omega_A = \sum_{B=1}^{n+1} \omega_B \wedge \omega_{BA}, \quad \omega_{AB} + \omega_{BA} = 0 \tag{1}$$

$$d\omega_{AB} = \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \tag{2}$$

where

$$\Omega_{AB} = -\frac{1}{2} \sum_{C,D=1}^{n+1} R_{ABCD}\omega_C \wedge \omega_D, \quad R_{ABCD} + R_{ABDC} = 0. \tag{3}$$

The **Ricci curvature** in the direction e_{n+1} is

$$\mathrm{Ric}(e_{n+1}) = \sum_{i=1}^{n} g_M(R(e_{n+1}, e_i)e_{n+1}, e_i). \tag{4}$$

Let ∇ be the **Levi-Civita connection** on M. Then for any tangent field X, we get

$$\nabla_X e_A = \sum_{B=1}^{n+1} \omega_{AB}(X)e_B. \tag{5}$$

Now let θ_A and θ_{AB} denote the restrictions of forms ω_A and ω_{AB} to the tangent vectors of the leaves of $\mathcal{F}$. Then it is obvious that

$$\theta_{n+1} = 0 \;\text{ e }\; \theta_i = \omega_i. \tag{6}$$

Since $\theta_{n+1} = 0$, we obtain from the structure equations

$$0 = d\theta_{n+1} = \sum_{B=1}^{n+1} \theta_B \wedge \theta_{Bn+1} = \sum_{i=1}^{n} \theta_i \wedge \theta_{in+1}.$$

By **Cartan's equation**, we have

$$\theta_{n+1i} = -\sum_{j=1}^{n} h_{ij}\theta_j, \quad h_{ij} = h_{ji}. \tag{7}$$

The **second fundamental form** B of the leaves is then given by

$$\mathcal{B} = \sum_{i=1}^{n} \theta_i \otimes \theta_{in+1} = \sum_{i,j=1}^{n} h_{ij}\theta_i \otimes \theta_j. \tag{8}$$

and its norm is

$$\|\mathcal{B}\|^2 = \sum_{i,j=1}^{n} h_{ij}^2.$$

The **mean curvature vector** is

$$\vec{H} = \frac{1}{n} tr(A) e_{n+1}, \tag{9}$$

where A is the **Weingarten operator** and the **mean curvature function** is $H = \frac{1}{n}tr(A)$.

Observe that the sigh of H depends on the choice of e_{n+1}. The vector field defined locally by $\vec{H}$ is globally defined on each leaf of $\mathcal{F}$. As a consequence, if $H \neq 0$ at each point of the leaf then the leaf is oriented. If N is a unitary vector field normal to the leaves of $\mathcal{F}$ we can choose an adapted frame on an open set in such a way that $N = e_{n+1}$. The mean curvature of the leaf is exactly the mean curvature in the direction of N.

The **divergent of a vector field** V is defined locally over M by

$$\operatorname{div}(V) = \sum_{A=1}^{n+1} g_M(\nabla_{e_A} V, e_A). \tag{10}$$

For a vector field tangent to the leaves of $\mathcal{F}$ the divergent along the leaves can be computed by

$$\operatorname{div}_L(V) = \sum_{i=1}^{n} g_M(\nabla_{e_i} V, e_i). \tag{11}$$

Barbosa et al. in [2] found an equation that relates the foliation with the ambient, more precisely, they obtained the following.

Proposition 2.1. *Let $\mathcal{F}$ be a foliation by hypersurfaces on a Riemannian manifold M and let N be a unit field normal to the leaves of $\mathcal{F}$ on some open set U of M. Then on U, we have*

$$\operatorname{div} N = -nH; \tag{12a}$$

$$\operatorname{div}_L(X) = -nN(H) + \|\mathcal{B}\|^2 + \operatorname{Ric}(N) + \|X\|^2; \tag{12b}$$

$$\operatorname{div} X = \operatorname{div}_L X - \|X\|^2, \tag{12c}$$

where H is the mean curvature in the direction N and $X = \nabla_N N$.

From now on, we will introduce some definitions and basic results about stability for minimal hypersurface. For more details, we recommend the reference [26].

Definition 2.2. *Let M be a Riemannian manifold. We say that $\mathfrak{F}$ is a* **foliation of constant mean curvature** *(or* **CMC foliation***) on M if each leaf $L \in \mathfrak{F}$ is a hypersurface of constant mean curvature (note that the mean curvature possibly varies from leaf to leaf). We say that the foliation $\mathfrak{F}$ is a* **minimal foliation** *if each leaf $L \in \mathfrak{F}$ is a minimal hypersurface.*

The **Riemannian volume** in an n-dimensional Riemannian manifold is the n-dimensional Hausdorff measure determined by its Riemannian metric.

Definition 2.3. *Let M be a connected complete Riemannian manifold and $B(p, r)$ be the geodesic ball centred in $p \in M$ with radius r. The* **volume entropy of** *M is*

$$\mu_M = \limsup_{r \to +\infty} \frac{\ln vol_M\,(B(p, r))}{r},$$

where $vol_M(B)$ *denotes the Riemannian volume in* M *of* B*. The* **lower volume entropy of** M *is*

$$\underline{\mu_M} = \liminf_{r\to+\infty} \frac{\ln vol_M\left(B(p,r)\right)}{r}.$$

- *We say that* M *has* **polynomial volume growth** *if there are a point* $p \in M$*, a non-negative integer* d *and* $a, b > 0$ *such that*

  $$vol_M\left(B(p,r)\right) \le ar^d + b,$$

  for all $r > 0$*.*
- *We say that* M *has* **zero volume entropy** *or* **subexponential volume growth** *if* $\mu_M = 0$*.*
- *We say that* M *has* **zero lower volume entropy** *if* $\underline{\mu_M} = 0$*.*

Note that the choice of p in the above concepts is irrelevant. Note also that "polynomial volume growth" ⇒ "zero volume entropy" ⇒ "zero lower volume entropy".

3. Answer to Conjecture 1.3

In this Section, we prove that Conjecture 1.3 has a positive answer in any dimension.

3.1. Spaces with non-negative Ricci curvature.

In this subsection, we prove that Conjecture 1.3 has a positive answer in any dimension in the following much more general setting:

Theorem 3.1. *Let* $\mathfrak{F}$ *be a transversely oriented codimension one CMC foliation of a complete oriented Riemannian manifold* M^{n+1}*. Assume that* M *has zero lower volume entropy and that* $\mathrm{Ric}(N) + \delta\|\mathcal{B}\|^2 \ge 0$ *for some* $0 < \delta < 1$*, where* N *is a unit vector field on* M *orthogonal to* $\mathfrak{F}$*. Then, each leaf of* $\mathfrak{F}$ *is a minimal and stable hypersurface.*

Proof. We start the proof of Theorem 3.1 by proving the following two lemmas.

Lemma 3.2. *Let* $\mathfrak{F}$ *be a transversely oriented codimension one foliation on a complete oriented Riemannian manifold* M^{n+1}*. Suppose that the mean curvature function* $H\colon M \to \mathbb{R}$*, which associates to each point the value of the mean curvature of the leaf of* $\mathfrak{F}$ *through that point, does not change the sign on* M*. Then,* $\underline{\mu_M} \ge n \inf_{p\in M} |H(p)|$*. Moreover, if additionally,* M^{n+1} *has zero lower volume entropy, then* $\inf_{p\in M} |H(p)| = 0$*.*

Proof of the Lemma 3.2. Since $\mathfrak{F}$ is a transversely oriented foliation on a complete oriented Riemannian manifold M^{n+1}, and the mean curvature function $H\colon M \to \mathbb{R}$, that associates to each point the value of the mean curvature of the leaf of $\mathfrak{F}$ that contains that point, does not change the sign on M, we can choose the normal vector field N or $-N$ in such a way that $H \leq 0$. Since M is a complete Riemannian manifold the flow $\theta_t\colon M \to M$ of the normal vector field N of the foliation is globally defined.

Thus, we define the smooth function $\varphi\colon [0,+\infty) \to (0,+\infty)$ given by

$$\varphi(t) = \mathrm{vol}_M\left(\theta_t(B)\right) = \int_{\theta_t(B)} dM = \int_B \theta_t^* dM,$$

where $B := B(p,r)$ is the geodesic ball centred at p and radius r. Using the compactness of $\overline{B}$ allows us to differentiate under the sign of the integral, we have

$$\begin{aligned}
\varphi'(t_0) &= \left.\frac{d}{dt}\right|_{t=0} \varphi(t+t_0) \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{\theta_{t+t_0}(B)} dM \\
&= \left.\frac{d}{dt}\right|_{t=0} \int_{\theta_{t_0}(B)} \theta_t^* dM \\
&= \int_{\theta_{t_0}(B)} \left.\frac{d}{dt}\right|_{t=0} \theta_t^* dM \\
&= \int_{\theta_{t_0}(B)} \mathrm{div}(N) dM.
\end{aligned}$$

By Equation (12a) of the Proposition 2.1, we have

$$\varphi'(t_0) = \int_{\theta_{t_0}(B)} (-nH) dM. \tag{13}$$

Now, $c_0 \in [0,+\infty)$ such that $\sup_{p\in M} H(p) = -c_0 \leq 0$. Note that $c_0 = \inf_{x\in M} |H(x)|$. Therefore,

$$\begin{aligned}
\varphi'(t) &= \int_{\theta_t(B)} \mathrm{div}(N) dM \\
&= \int_{\theta_t(B)} (-nH) dM \\
&\geq (nc_0) \int_{\theta_t(B)} dM \\
&= (nc_0)\varphi(t),
\end{aligned}$$

for all $t \geq 0$. Note that $\varphi'(t) > 0$, for all $t \geq 0$, consequently, φ is an increasing function. So, $\varphi(t) \geq \varphi(0) = \mathrm{vol}_M(B) > 0$, for all $t \geq 0$. Thus, by using the inequality above and integrating the function $\frac{\varphi'(s)}{\varphi(s)}$ over the interval $[0,t]$, we obtain:

$$\int_0^t \frac{\varphi'(s)}{\varphi(s)} ds \geq \int_0^t (nc_0) ds.$$

Thus,

$$\ln\left(\frac{\varphi(t)}{\mathrm{vol}_M(B)}\right) \geq (nc_0)t.$$

Therefore, $\varphi(t) \geq \mathrm{vol}_M(B)e^{nc_0t}$ for all $t \geq 0$.

Note that $\mathrm{vol}_M(B(p,t+r)) \geq \mathrm{vol}_M(\theta_t(B))$, since $\theta_t(B) \subset B(p,t+r)$, for all $t \geq 0$. Indeed,

$$\text{(14)} \qquad \mathrm{dist}_M(x,\theta_t(x)) \leq \int_0^t \left|\frac{d}{ds}\theta_s(x)\right| ds = \int_0^t \|N(\theta_s(x))\| ds = t,$$

where $\mathrm{dist}_M(x,\theta_t(x))$ the Riemannian distance between x and $\theta_t(x)$. For all $x \in B$ and by triangular inequality, we have

$$\mathrm{dist}_M(p,\theta_t(x)) \leq \mathrm{dist}_M(x,\theta_t(x)) + \mathrm{dist}_M(p,x) < t + r.$$

Then

$$\mathrm{vol}_M(B(p,t+r)) \geq \mathrm{vol}_M(\theta_t(B)) = \varphi(t) = \mathrm{vol}_M(B)e^{nc_0t}, \ \ \forall\, t \geq 0.$$

Therefore,

$$\underline{\mu_M} = \liminf_{t\to+\infty} \frac{\ln \mathrm{vol}_M\,(B(p,t))}{t} \geq \liminf_{t\to+\infty} \frac{\ln(\mathrm{vol}_M(B)e^{nc_0t}))}{t} = nc_0.$$

Finally, if M has zero lower volume entropy, we have that $\underline{\mu_M} = 0$. Thus, $\inf_{x\in M}|H(x)| = 0$. ■

The above lemma was recently proved in [16] in the particular case when each leaf $L \in \mathfrak{F}$ has constant mean curvature $H_L \geq 0$ and when the ambient manifold M has zero volume entropy. Note that the result in [16] follows from [1, Theorem 1.1].

Lemma 3.3. *Let $\mathfrak{F}$ be a transversely oriented codimension one CMC foliation of a complete oriented Riemannian manifold M^{n+1}. Assume that $\mathrm{Ric}(N) + \delta\|\mathcal{B}\|^2 \geq 0$ for some $0 < \delta < 1$, where N is a unit vector field on M orthogonal to $\mathfrak{F}$. Then, $\mathfrak{F}$ is a foliation with the same constant mean curvature.*

Proof of the Lemma 3.3. Let $H\colon M^{n+1} \to \mathbb{R}$ be the function defined as $H = -\frac{1}{n}\,\mathrm{div}(N)$. We have that H is the function that associates to each point the value of the mean curvature of the leaf of $\mathfrak{F}$ that contains that point.

Assume by contradiction that $H\colon M^{n+1} \to \mathbb{R}$ is a non-constant function. Thus, $\nabla H \not\equiv 0$ on M.

We define the set $\mathcal{A} = \{x \in M; \nabla H \neq 0\}$. Thus, $\tilde{N} = \frac{\nabla H}{\|\nabla H\|}$ is well-defined on $\mathcal{A}$. For $x \in \mathcal{A}$, L be a leaf of $\mathfrak{F}$ such that $x \in L$. Then $\tilde{N}$ is a normal vector field to the leaf L. Thus, for each point $x \in \mathcal{A}$, we have that $N(x) = \pm\tilde{N}(x)$, and by continuity, this equality holds in an open neighbourhood $U \subset \mathcal{A}$ of x. Then, $\nabla_N N = \nabla_{\tilde{N}}\tilde{N}$ on $\mathcal{A}$.

Now, we are going to show that $\nabla_N N = 0$ on $\mathcal{A}$. Thus, it is enough to show that $\nabla_{\tilde{N}}\tilde{N}$ on $\mathcal{A}$. Indeed, let $x \in \mathcal{A}$ and L be the leaf of $\mathfrak{F}$ such that $x \in L$. Since $L \subset H^{-1}(s)$ for some $s \in \mathbb{R}$, we have that $\tilde{N}$ is a normal vector field to $L \cap \mathcal{A}$. Since $\tilde{N}$ has unit length, we have $\langle \nabla_{\tilde{N}}\tilde{N}, \tilde{N}\rangle = 0$. Then $\nabla_{\tilde{N}}\tilde{N}$ is tangent to $L \cap \mathcal{A}$.

Let X be an arbitrary smooth vector field on M being tangent to $L \cap \mathcal{A}$. Note that $\tilde{N}(H)$ is constant along $L \cap \mathcal{A}$, so $X(\tilde{N}(H)) = 0$ on $L \cap \mathcal{A}$. Moreover, $X(H) = 0$ and $\tilde{N}(X(H)) = 0$ on $L \cap \mathcal{A}$. Then

$$\begin{aligned}
g_M\left(\nabla_{\tilde{N}}\tilde{N}, X\right) &= g_M\left(\nabla_X \tilde{N}, \tilde{N}\right) - g_M\left(\nabla_{\tilde{N}} X, \tilde{N}\right)\\
&= g_M\left(\tilde{N}, \nabla_X\tilde{N} - \nabla_{\tilde{N}} X\right)\\
&= g_M\left(\tilde{N}, [X, \tilde{N}]\right)\\
&= \frac{1}{\|\nabla H\|}[X, \tilde{N}](H) = 0.
\end{aligned}$$

Therefore $\nabla_{\tilde{N}}\tilde{N} = 0$ on $L \cap \mathcal{A}$. It follows from Equations (12b) and (12c) that

$$nN(H) = \|\mathcal{B}\|^2 + \mathrm{Ric}(N) \geq (1-\delta)\|\mathcal{B}\|^2 \geq 0,$$

on $L \cap \mathcal{A}$. Since x was arbitrarily chosen, we obtain that $N(H) \geq 0$ on $\mathcal{A}$. By continuity, $N(H) \geq 0$ on the closure of $\mathcal{A}$, denoted by $\overline{\mathcal{A}}$.

However, $\nabla H = 0$ on $M \setminus \overline{\mathcal{A}}$ and, in particular, $X(H) = 0$ on $M \setminus \overline{\mathcal{A}}$, for any vector field X on M. Therefore, $N(H) \geq 0$ on M.

Since M is a complete Riemannian manifold the flow $\theta_t\colon M \to M$ of the normal vector field N of the foliation is globally defined.

Since H is a non-constant function, by changing N by $-N$ and H by $-H$, we may assume that there is $p_0 \in \mathcal{A}$ such that $H(p_0) = c_0 > 0$.

Let γ be an integral curve of the unit vector field N. Since M is complete, γ may be extended to $\mathbb{R}$. Since $N(H) \geq 0$ on M, then $H \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ is a non-decreasing function. In particular, if γ is an integral curve of the unit vector field N such that $\gamma(0) = p_0$, then $H \circ \gamma(t) \geq c_0$ for all $t \geq 0$.

Thus, we have two cases to consider.

Case 1. $\gamma((0,+\infty)) \subset \mathcal{A}$.

Now, we define the function $f\colon [0,+\infty) \to \mathbb{R}$ given by $f(s) = H \circ \gamma(s)$.

Since $nN(H) \geq (1-\delta)\|\mathcal{B}\|^2$ on $\mathcal{A}$ and $nH^2 \leq \|\mathcal{B}\|^2$, we obtain that $f'(s) \geq (1-\delta)(f(s))^2$ for all $s \geq 0$. Thus, $\frac{f'(s)}{f(s)^2} \geq 1-\delta$ for all $s \geq 0$.

We define the function function $\phi\colon [0,+\infty) \to \mathbb{R}$ given by $\phi(s) = -\frac{1}{f(s)}$. By the mean value theorem, we have

$$-\frac{1}{f(s)} + \frac{1}{f(0)} = \frac{f'(s^*)}{f(s^*)^2}s,$$

for some $s^* \in (0,s)$. Therefore, $-\frac{1}{f(s)} + \frac{1}{f(0)} \geq (1-\delta)s$, for all $s \geq 0$. Setting $s \to +\infty$, we have that the right-hand side is unbounded, but the left-hand side is bounded, which gives a contradiction.

Case 2. $\gamma((0,+\infty)) \not\subset \mathcal{A}$.

Thus, there is $s_0 > 0$ such that $\gamma(s_0) \in \overline{\mathcal{A}} \setminus \mathcal{A}$ and $\gamma(s) \in \mathcal{A}$ for all $s \in (0,s_0)$. By continuity, $0 = N(H)(\gamma(s_0)) \geq (1-\delta)\|\mathcal{B}(\gamma(s_0))\|^2 \geq 0$. Thus $\mathcal{B}(\gamma(s_0)) = 0$, and this implies that $H(\gamma(s_0)) = 0$, which is a contradiction.

Thus, in any case, we obtain a contradiction. Therefore H must be a constant function. ■

Coming back to the proof of Theorem 3.1, by Lemma 3.3, we obtain that each leaf of the foliation $\mathfrak{F}$ has the same constant mean curvature. In particular, H does not change the sign. By Lemma 3.2, $H \equiv 0$. Therefore $\mathfrak{F}$ is a minimal foliation. Finally, the stability of the leaves follows from Theorem 1 in [20].

■

We obtain the following important consequence.

Theorem 3.4. *Let $\mathfrak{F}$ be a codimension one CMC foliation of a complete Riemannian manifold M^{n+1} with non-negative Ricci curvature. Then, each leaf of $\mathfrak{F}$ is a minimal and stable hypersurface.*

Proof. After possibly lifting to the universal cover of M, we will assume that M is oriented and also that any codimension one CMC foliation of M under consideration is transversely oriented.

It follows from the Bishop–Gromov Inequality (see [14]) that every complete Riemannian manifold with non-negative Ricci curvature has polynomial volume growth. Therefore, M has zero lower volume entropy. By Theorem 3.1, each leaf of $\mathfrak{F}$ is a minimal and stable hypersurface. ■

Theorem 3.1 works in the following setting:

Corollary 3.5. *Let $\mathfrak{F}$ be a transversely oriented codimension one CMC foliation of a complete oriented Riemannian manifold M^{n+1}. Assume that M has zero lower volume entropy and* $\mathrm{Ric}(N) \geq 0$, *where N is a unit vector field on M orthogonal to $\mathfrak{F}$. Then, each leaf of $\mathfrak{F}$ is a minimal and stable hypersurface. Moreover, if a leaf L is such that $\|\nabla_N N\|\big|_L \in L^1(L)$, then L is totally geodesic and* $\mathrm{Ric}(N) = 0$ *on L. If $\|\nabla_N N\| \in L^1(M)$, then $\mathfrak{F}$ is a totally geodesic foliation.*

Proof. The first part follows directly from Therem 3.1.

Now, we are going to prove the second part of Corollary 3.5. Let L be a leaf such that Since $\|\nabla_N N\|\big|_L \in L^1(L)$. By using Equations (12b) and (12c), we obtain the following equality on L

$$\mathrm{div}_L(\nabla_N N) = \|\nabla_N N\|^2 + \|\mathcal{B}\|^2 + \mathrm{Ric}(N).$$

Since $\mathrm{Ric}(N) \geq 0$, we conclude that $\mathrm{div}_L(\nabla_N N)$ does not change the sign on L. By Proposition 1 in [5], we have $\mathrm{div}_L(\nabla_N N) = 0$ on L. Therefore, $\nabla_N N = 0$, $\mathrm{Ric}(N) = 0$ and $\mathcal{B} = 0$ on L, and thus L is a totally geodesic foliation.

Similarly, we prove that if $\|\nabla_N N\| \in L^1(M)$, then $\mathfrak{F}$ is a totally geodesic foliation. ■

Corollary 3.5 generalizes the main results in [2] and [1] (see Theorem 3.1 in [2] and Theorem 1.2 in [1]), where we remove the hypothesis of compactness of the ambient space and the same mean curvature, respectively.

Note that if Σ^n is a Riemannian n-manifold that has zero lower volume entropy, then $\Sigma \times \mathbb{R}$ has also zero lower volume entropy. Thus, as a consequence of Lemma 3.2, we obtain the following version of [12, Corollary 1.2].

Corollary 3.6. *Let Σ^n be a complete oriented Riemannian n-manifold that has zero lower volume entropy. Then, for any graph Γ_f over Σ that has constant mean curvature H, Γ_f is minimal and stable hypersurface in $\Sigma \times \mathbb{R}$.*

3.2. **Euclidean spaces.** The next result says, in particular, that Conjecture 1.3 has a positive answer in any dimension and that Conjecture 1.2 has a positive answer whether $n \leq 4$.

Theorem 3.7. *Let $\mathfrak{F}$ be a codimension one CMC foliation of $\mathbb{R}^{n+1}$. Then, each leaf of $\mathfrak{F}$ is a minimal and stable hypersurface. Moreover, if $n \leq 4$ or $\|\nabla_N N\| \in L^1(\mathbb{R}^{n+1})$, then $\mathfrak{F}$ consists entirely of parallel hyperplanes.*

Proof. By Theorem 3.1, each leaf of $\mathfrak{F}$ is a minimal and stable hypersurface. Moreover, if $\|\nabla_N N\| \in L^1(\mathbb{R}^{n+1})$, then it follows from the second part of Theorem 3.1 that $\mathfrak{F}$ consists entirely of parallel hyperplanes.

Now, assume that $n \leq 4$.

By the first part of the proof of this theorem, the leaves are minimal and stable hypersurfaces. We know that in an orientable space (e.g. in $\mathbb{R}^{n+1}$) a hypersurface is orientable if and only if it is two-sided. Since $\mathfrak{F}$ is transversely oriented, we have that each leaf $L \in \mathfrak{F}$ is oriented. Since $n \leq 4$ and each leaf is a complete hypersurface, we have by main results in [6, 11, 22, 9, 10] that each leaf L of the foliation is a hyperplane. ■

4. Answers to Conjecture 1.7 and Problem 1.6

4.1. **Spaces with sectional curvature bounded from below.** In this Subsection, we give a complete positive answer to Conjecture 1.7.

Theorem 4.1. *Let $\mathfrak{F}$ be a transversely oriented codimension one foliation of a complete oriented Riemannian manifold M^{n+1}. Assume that there is $K_0 \geq 0$ such that $\mathrm{Ric}(N) \geq -nK_0$, where N is a unit vector field on M orthogonal to $\mathfrak{F}$. Suppose that each leaf L of $\mathfrak{F}$ has constant mean curvature H_L such that $|H_L| \geq \sqrt{K_0}$. Then H_L does not depend on L. Moreover, if M^{n+1} has sectional curvature bounded from below by $-K_0$, then $|H| \equiv \sqrt{K_0}$.*

Proof. By Theorem 3.1, we may assume that $K_0 > 0$.

Assume by contradiction that $H\colon M \to \mathbb{R}$ is a non-constant function. In particular, $\nabla H \not\equiv 0$ on M. Changing H by $-H$, if necessary, we may assume that $H \geq \sqrt{K_0}$.

We define the set $\mathcal{A} = \{x \in M; \nabla H \neq 0\}$. In particular, there is $p_0 \in \mathcal{A}$ such that $\delta := H(p_0) > \sqrt{K_0}$.

Let N be a unit vector field on M orthogonal to $\mathfrak{F}$. Similarly, as it was done in the proof of Lemma 3.3, we obtain that $\nabla_N N = 0$ on $\mathcal{A}$. Therefore $nN(H) = \|\mathcal{B}\|^2 + \mathrm{Ric}(N)$ on $\mathcal{A}$. Since $H \geq \sqrt{K_0}$, $\|\mathcal{B}\|^2 \geq nH^2$, and $\mathrm{Ric}(N) \geq -nK_0$, we obtain that $nN(H) \geq nH^2 - nK_0 \geq 0$ on $\mathcal{A}$. Therefore, $N(H) \geq 0$ on M.

Let γ be an integral curve of N. Since M is complete, γ may be extended to $\mathbb{R}$. Since $N(H) \geq 0$ on M, then $H \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ is a non-decreasing function. In particular, if γ is an integral curve of N such that $\gamma(0) = p_0$, then $H \circ \gamma(t) \geq \delta > \sqrt{K_0}$ for all $t \geq 0$.

Thus, we have two cases to consider.

Case 1. $\gamma((0, +\infty)) \subset \mathcal{A}$.

Now, we define the function $f\colon [0,+\infty) \to \mathbb{R}$ given by $f(s) = H \circ \gamma(s)$.

Since $N(H) \geq H^2 - K_0$ on $\mathcal{A}$ and $H \circ \gamma(t) \geq \delta$ for all $t \geq 0$, we obtain that

$$\frac{f'(s)}{f(s)^2} \geq 1 - \frac{K_0}{f(s)} \geq 1 - \frac{K_0}{\delta^2} > 0$$

for all $s \geq 0$.

We define the function $\phi\colon [0,+\infty) \to \mathbb{R}$ given by $\phi(s) = -\frac{1}{f(s)}$. By the mean value theorem, we have

$$-\frac{1}{f(s)} + \frac{1}{f(0)} = \frac{f'(s^*)}{f(s^*)^2} s,$$

for some $s^* \in (0,s)$. Therefore, $-\frac{1}{f(s)} + \frac{1}{f(0)} \geq (1 - \frac{K_0}{\delta^2})s$, for all $s \geq 0$. Setting $s \to +\infty$, we have that the right-hand side is unbounded, but the left-hand side is bounded, which gives a contradiction.

Case 2. $\gamma((0,+\infty)) \not\subset \mathcal{A}$.

Thus, there is $s_0 > 0$ such that $\gamma(s_0) \in \overline{\mathcal{A}} \setminus \mathcal{A}$ and $\gamma(s) \in \mathcal{A}$ for all $s \in (0, s_0)$. By continuity, $0 = N(H)(\gamma(s_0)) \geq \delta^2 - K_0 > 0$, which is a contradiction.

Thus, in any case, we obtain a contradiction.

Therefore, H is a constant function.

For the second part, we assume that M^{n+1} has sectional curvature bounded from below by $-K_0 \leq 0$. Then $\mathrm{Ric} \geq -nK_0$. By the first part of this theorem, H is a constant function and, in particular, it does not change the sign on M. Then, the equality $|H| \equiv \sqrt{K_0}$ is a direct consequence of the following lemma:

Lemma 4.2. *Let $\mathfrak{F}$ be a transversely oriented foliation of a complete oriented Riemannian manifold M^{n+1} with sectional curvature bounded from below by $-K_0 \leq 0$. Suppose that the mean curvature function $H\colon M \to \mathbb{R}$, which associates to each point the value of the mean curvature of the leaf of $\mathfrak{F}$ that contains that point, does not change the sign on M. Then, $\sqrt{K_0} \geq \inf\limits_{p \in M} |H(p)|$.*

Proof. By changing N by $-N$ and H by $-H$, if necessary, we may assume that $H \leq 0$. Let $c_0 \in [0,+\infty)$ such that $\sup_{p \in M} H(p) = -c_0 \leq 0$. Note that $c_0 = \inf_{x \in M} |H(x)|$.

Since M is a complete Riemannian manifold the flow $\theta_t\colon M \to M$ of the normal vector field N of the foliation is globally defined.

Thus, we define the smooth function $\varphi\colon [0,+\infty) \to (0,+\infty)$ given by

$$\varphi(t) = \mathrm{vol}_M\left(\theta_t(B)\right) = \int_{\theta_t(B)} dM = \int_B \theta_t^* dM,$$

where $B := B(p,r)$ is the geodesic ball centred at p and radius r. By proceeding like in the proof of Lemma 3.2, we obtain that

$$\mathrm{vol}_M(B(p,t+r)) \geq \mathrm{vol}_M(\theta_t(B)) = \varphi(t) = \mathrm{vol}_M(B)e^{nc_0t},\ \ \forall\, t \geq 0.$$

By Bishop–Gromov inequality, we obtain

$$\mathrm{vol}_M(B(p,s)) \leq \mathrm{vol}_{\tilde{M}^{n+1}(-K_0)}(B_{\tilde{M}^{n+1}(-K_0)}(\tilde{p},s)),$$

for all $s > 0$, where $\tilde{M}^{n+1}(-K_0)$ is the space form of constant sectional curvature $-K_0$ and $B_{\tilde{M}^{n+1}(-K_0)}(\tilde{p},s)$ is the geodesic ball of $\tilde{M}^{n+1}(-K_0)$ centred at $\tilde{p}$ and of radius s. However, we have that

$$\mathrm{vol}_{\tilde{M}^{n+1}(-K_0)}\left(B_{\tilde{M}^{n+1}(-K_0)}(\tilde{p},s)\right) = c_n\int_0^s \left(\frac{\sinh(\sqrt{K_0}t)}{\sqrt{K_0}}\right)^n dt,$$

where c_n is the n-dimensional volume of the unit sphere in $\mathbb{R}^{n+1}$ (see [2, p. 105] or [8, §III.4.1]). Thus, by using the L'Hospital rule, we obtain

$$n\sqrt{K_0} = \lim_{s\to+\infty}\frac{\ln \mathrm{vol}_{\tilde{M}^{n+1}(-K_0)}\left(B_{\tilde{M}^{n+1}(-K_0)}(\tilde{p},s)\right)}{s}.$$

Since we also have the following equality

$$nc_0 = \lim_{t\to+\infty}\frac{\ln(\mathrm{vol}_M(B)e^{nc_0t}))}{t},$$

then $nc_0 \leq n\sqrt{K_0}$.

Therefore $\inf_{x\in M}|H(x)| \leq \sqrt{K_0}$. ■

■

As a consequence, we obtain that Conjecture 1.7 has a positive answer. Indeed, we obtain that Conjecture 1.7 holds true in the more general setting where we ask that the sectional curvature is bounded only from below.

Theorem 4.3. *Let $\mathfrak{F}$ be a codimension one CMC foliation of a complete Riemannian manifold M^{n+1}. Assume that there is $K_0 \geq 0$ such that the sectional curvature of M is bounded from below by $-K_0$. Let $H\colon M \to \mathbb{R}$ be the function that associates to each point the value of the mean curvature of the leaf of $\mathfrak{F}$ that contains that point. Then $|H| \leq \sqrt{K_0}$.*

Proof. After possibly lifting to the universal cover of M, we will assume that M is oriented and also that any codimension one CMC foliation of M under consideration is transversely oriented.

Assume by contradiction that there exists $p_0 \in M$ such that $|H(p_0)| > \sqrt{K_0}$. Changing H by $-H$, if necessary, we may assume that $H(p_0) > \sqrt{K_0}$.

By Theorem 4.1, there is $q_0 \in M$ such that $H(q_0) < \sqrt{K_0}$. Thus, $\mathcal{A} = \{x \in M; \nabla H \neq 0\} \neq \emptyset$ and we may assume that $p_0 \in \mathcal{A}$. Similarly, as it was done in the proof of Lemma 3.3, we obtain that $\nabla_N N = 0$ on $\mathcal{A}$. Therefore, $nN(H) = \|\mathcal{B}\|^2 + \mathrm{Ric}(N)$ on $\mathcal{A}$.

Since $H(p_0) > \sqrt{K_0}$, $\|\mathcal{B}\|^2 \geq nH^2$, and $\mathrm{Ric}(N) \geq -nK_0$, we obtain that $N(H)(p_0) \geq H(p_0)^2 - K_0 > 0$. Let $\mathcal{D} = \{x \in \mathcal{A}; N(H)(x) > 0\}$.

Let γ be the integral curve of the unit vector field N such that $\gamma(0) = p_0$.

Thus, we have two cases to consider.

Case 1. $\gamma((0, +\infty)) \subset \mathcal{D}$.

Since $N(H) > 0$ on $\mathcal{D}$, then $H \circ \gamma \colon [0, +\infty) \to \mathbb{R}$ is a non-decreasing function. In particular, then $H \circ \gamma(t) \geq H(p_0) > \sqrt{K_0}$ for all $t \geq 0$.

As before, we define the function $f \colon [0, +\infty) \to \mathbb{R}$ given by $f(s) = H \circ \gamma(s)$.

Since $N(H) \geq H^2 - K_0$ on $\mathcal{A}$ and $H \circ \gamma(t) \geq H(p_0)$ for all $t \geq 0$, we obtain that

$$\frac{f'(s)}{f(s)^2} \geq 1 + \frac{c}{f(s)} \geq 1 - \frac{K_0}{H(p_0)^2} > 0$$

for all $s \geq 0$.

We define the function $\phi \colon [0, +\infty) \to \mathbb{R}$ given by $\phi(s) = -\frac{1}{f(s)}$. By the mean value theorem, we have

$$-\frac{1}{f(s)} + \frac{1}{f(0)} = \frac{f'(s^*)}{f(s^*)^2} s,$$

for some $s^* \in (0, s)$. Therefore, $-\frac{1}{f(s)} + \frac{1}{f(0)} \geq (1 - \frac{K_0}{H(p_0)^2})s$, for all $s \geq 0$. Setting $s \to +\infty$, we have that the right-hand side is unbounded, but the left-hand side is bounded, which gives a contradiction.

Case 2. $\gamma((0, +\infty)) \not\subset \mathcal{D}$.

Thus, there is $s_0 > 0$ such that $\gamma(s_0) \in \overline{\mathcal{D}} \setminus \mathcal{D}$ and $\gamma(s) \in \mathcal{D}$ for all $s \in (0, s_0)$. By continuity, $0 = N(H)(\gamma(s_0)) \geq H(p_0)^2 - K_0 > 0$, which is a contradiction.

Thus, in any case, we obtain a contradiction. Therefore $|H| \leq \sqrt{K_0}$. ■

We obtain also the following generalization of Proposition 3.7 in [2].

Corollary 4.4. *Let $\mathfrak{F}$ be a transversely oriented codimension one CMC foliation of a complete oriented Riemannian manifold M^{n+1} with positive Ricci curvature. Assume that M has positive Ricci curvature. Then there is no leaf L of $\mathfrak{F}$ such that $\|\nabla_N N\|\big|_L \in L^1(L)$, where N is a unit vector field on M orthogonal to $\mathfrak{F}$. In particular, $\mathfrak{F}$ has no compact leaf.*

Proof. Assume by contradiction that there is a leaf L of $\mathfrak{F}$ such that $\|\nabla_N N\|\big|_L \in L^1(L)$. By Theorem 3.4, $\mathfrak{F}$ is a minimal foliation. Thus,

$$\operatorname{div}_L \nabla_N N = \|\nabla_N N\|^2 + \|\mathcal{B}\|^2 + \operatorname{Ric}(N) > 0.$$

By [5, Proposition 1], $\operatorname{div}_L \nabla_N N \equiv 0$, which is a contradiction.

Therefore, there is no leaf L of $\mathfrak{F}$ such that $\|\nabla_N N\|\big|_L \in L^1(L)$. ■

Another consequence is the following:

Corollary 4.5. *Let $\mathfrak{F}$ be a codimension one CMC foliation of $M = \tilde{M}^{n+1}(c)$, where $c < 0$. Then, for each leaf L of $\mathfrak{F}$, the mean curvature H_L of L satisfies $|H_L| \leq (-c)^{1/2}$.*

4.2. **Space forms.** In this Subsection, we present a complete positive answer to Problem 1.6.

Theorem 4.6. *Let $\mathfrak{F}$ be a foliation of $\tilde{M}^{n+1}(c)$ by hypersurfaces L of constant mean curvature H_L, then:*

- i) *If $c = 0$, then the leaves of the foliation are minimal. In particular, if $n \leq 4$, then the leaves of the foliation are hyperplanes.*
- ii) *If $c > 0$, no such foliation exists.*
- iii) *If $c < 0$ and $H_L \geq (-c)^{1/2}$, then H_L does not depend on L. Moreover, if $n = 2$, then the leaves of $\mathfrak{F}$ are horospheres with $H_L = (-c)^{1/2}$.*

Proof. Item i) follows from Theorems 3.7 and Item ii) follows from Corollary 4.4

To prove Item iii), if $c < 0$ and $H_L \geq (-c)^{1/2}$, by Theorem 4.1, H_L does not depend on L and $H \equiv (-c)^{1/2}$. Moreover, if $n = 2$, it follows from [3, Theorem 3.12] that the leaves of $\mathfrak{F}$ are horospheres with $H_L = (-c)^{1/2}$. ■

Note that, as it was already said in the introduction, Item ii) above was already proved in [2, Corollary 3.5].

By Corollary 4.4 and Theorem 3.7 and Corollary 4.5, we obtain the following result:

Corollary 4.7. *Let $\mathfrak{F}$ be a codimension one CMC foliation on $\tilde{M}^{n+1}(c)$. Then we have the following:*

i) *If $c = 0$, then $\mathfrak{F}$ is a minimal foliation.*
ii) *If $c > 0$, no such foliation exists.*
iii) *If $c < 0$, then for each leaf L of $\mathfrak{F}$ the mean curvature H_L of L satisfies $|H_L| \leq (-c)^{1/2}$.*

5. Appendix: some examples

In this Section, we analyse the hypotheses of Theorem 3.4.

Note that the hypothesis in Theorem 3.4 the hypothesis that Ricci curvature is non-negative cannot be removed.

Example 5.1. *Let $\mathbb{H}^{n+1} = \left\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}; x_{n+1} > 0\right\}$ be the hyperbolic space with Riemannian metric $g_{\mathbb{H}^{n+1}} = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}$. Let $\mathfrak{F}$ be a foliation on $\mathbb{H}^{n+1}$ given by the family of half-hyperplanes $\mathfrak{F} := \bigcup_{\alpha \in (0,\pi)} L_\alpha$, where*

$$L_\alpha = \{x_{n+1} - \alpha x_n = 0; \alpha \neq 0 \text{ and } x_{n+1} > 0\}.$$

Note that $\mathfrak{F}$ is a transversely orientable codimension one foliation and each leaf L_α has constant mean curvature $H = \cos(\alpha)$.

We can even obtain an example when all the leaves have the same non-zero constant mean curvature.

Example 5.2. *Let $\mathbb{H}^{n+1}$ be the hyperbolic space as in the above example. Fixed $\alpha \in (0, \frac{\pi}{2})$, we define a transversely orientable foliation $\mathfrak{F} := \bigcup_{t \in \mathbb{R}} L_t$ on $\mathbb{H}^{n+1}$, where each leaf L_t is given by*

$$L_t = \{x_{n+1} - \alpha x_n = -\alpha t; x_{n+1} > 0\}.$$

We have that each leaf L_t has constant mean curvature $H = \cos(\alpha)$.

The next example shows that it is not possible to removed the hypothesis of completeness of the Riemannian manifold M in Theorem 3.4.

Example 5.3. *Let $M = \left(\mathbb{R}^3 \setminus \{0\}, can\right)$ be the euclidean space with canonical Riemannian metric. Let $\mathfrak{F}$ be a foliation on M given by the family of spheres $\mathfrak{F} := \bigcup_{\alpha \in (0,+\infty)} L_\alpha$, where*

$$L_\alpha = \left\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = \alpha^2\right\}.$$

Note that $\mathfrak{F}$ is a transversely oriented codimension one foliation and each leaf L_α has mean curvature $H_{L_\alpha} = \frac{1}{\alpha}$.

In the last part of Theorem 3.4, the hypothesis that $\|\nabla_N N\| \in L^1(M)$ cannot be removed.

Example 5.4. *By* [4]*, there is a smooth function $f\colon \mathbb{R}^8 \to \mathbb{R}$ such that Graph(f) is a minimal hypersurface of $\mathbb{R}^9$ that is not a hyperplane. Then, we define the foliation $\mathfrak{F} := \bigcup_{t\in\mathbb{R}} L_t$ on $\mathbb{R}^{n+1}$, where each leaf L_t is given by*

$$L_t = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}; y = f(x) + t\}.$$

We have that each leaf L_t is minimal hypersurface.

It is easy to find examples of foliations on $\mathbb{R}^3$, where the leaves have bounded non-constant mean curvature. In the next example, we present an example of such a foliation given by level sets of a polynomial function.

Example 5.5. *Let $\mathfrak{F}$ be a foliation on $\mathbb{R}^3$ given by the family of algebraic surfaces $\mathfrak{F} := \bigcup_{\alpha\in\mathbb{R}} L_\alpha$, where $L_\alpha = \big\{(x, y, z) \in \mathbb{R}^3; z - \big(x^2 + y^2 + \alpha\big) = 0\big\}$. Note that $\mathfrak{F}$ is a transversely oriented codimension one foliation and each leaf L_α has mean curvature $H_{L_\alpha}(x, y, z) = \frac{2+4(x^2+y^2)}{(1+4x^2+4y^2)^{\frac{3}{2}}}$ and $0 \leq H_{L_\alpha}(x, y, z) \leq 2$.*

Acknowledgements. The authors would like to thank Alexandre Fernandes for their interest in this manuscript, for listening patiently to our first ideas in this research, and for uncountable suggestions.

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José Edson Sampaio: Departamento de Matemática, Universidade Federal do Ceará, Rua Campus do Pici, s/n, Bloco 914, Pici, 60440-900, Fortaleza-CE, Brazil.
E-mail: edsonsampaio@mat.ufc.br

Euripedes Carvalho da Silva: Departamento de Matemática, Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Av. Parque Central, 1315, Distrito Industrial I, 61939-140, Maracanaú-CE, Brazil.
and Departamento de Matemática, Universidade Federal do Ceará, Rua Campus do Pici, s/n, Bloco 914, Pici, 60440-900, Fortaleza-CE, Brazil. E-mail: euripedes.carvalho@ifce.edu.br