CONSTANT ENERGY FAMILIES OF HARMONIC MAPS

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Abstract. For a negatively curved manifold M and a continuous map $\psi : \Sigma \to M$ from a closed surface Σ, we study complex submanifolds of Teichmüller space $\mathcal{S} \subset \mathcal{T}(\Sigma)$ such that the harmonic maps $\{h_X : X \to M$ for $X \in \mathcal{S}\}$ in the homotopy class of ψ all have equal energy. When M is real analytic with negative Hermitian sectional curvature, we show that for any such $\mathcal{S}$, there exists a closed Riemann surface Y, such that any h_X for $X \in \mathcal{S}$ factors as a holomorphic map $\phi_X : X \to Y$ followed by a fixed harmonic map $h : Y \to M$. This answers a question posed by both Toledo and Gromov. As a first application, we show a factorization result for harmonic maps from normal projective varieties to M. As a second application, we study homomorphisms from finite index subgroups of mapping class groups to $\pi_1(M)$.

1. Introduction

Let M be a manifold with negative sectional curvatures, and $\psi : \Sigma_g \to M$ be a continuous map from a closed surface of genus g, denoted Σ_g. Then given any marked Riemann surface $f : \Sigma_g \to X$ in Teichmüller space $\mathcal{T}_g$, there exists a unique harmonic map $h : X \to M$ in the homotopy class of $\psi \circ f^{-1}$. The energy of the map h then defines a function $\mathrm{E}_\psi : \mathcal{T}_g \to \mathbb{R}$. It was shown by Toledo [33] that when M has strictly negative Hermitian sectional curvature, E_ψ is plurisubharmonic, and that it is strictly plurisubharmonic when the harmonic map h is an immersion.

In the converse direction, Schiffer variations provide examples of complex submanifolds of $\mathcal{T}_g$ on which E_ψ is constant. Given a fixed Riemann surface Y marked by $\Sigma_{\tilde{g}}$, and a ramified covering map $p : \Sigma_g \to \Sigma_{\tilde{g}}$ branched over some finite set $B \subset \Sigma_{\tilde{g}}$, by moving the branch set B on Y, we obtain a complex submanifold $\mathrm{Schiff}_{Y,p} \subset \mathcal{T}_g$, consisting of marked Riemann surfaces that all admit a holomorphic map to Y in the homotopy class of p. It is then easy to see that $\mathrm{E}_{\psi\circ p}$ is constant on $\mathrm{Schiff}_{Y,p}$, for an arbitrary continuous map $\psi : \Sigma_{\tilde{g}} \to M$.

A natural question, posed by both Toledo [33, Remark 3] and Gromov [15, Remarks (a), §4.6, pp. 53], is do all examples of families of homotopic harmonic maps with constant energy arise in this way? Our main result shows that the answer is affirmative when M is real analytic and convex cocompact, and when the corresponding map is not homotopic into a curve.

Theorem 1. *Let $\mathcal{S} \subset \mathcal{T}_g$ be a connected complex submanifold of Teichmüller space, and let $\psi : \Sigma_g \to M$ be a continuous map into a convex cocompact, real analytic Riemannian manifold M with negative Hermitian sectional curvature, such that ψ is not homotopic into a curve. Then, if E_ψ is constant on $\mathcal{S}$, there exist a closed Riemann surface Y marked by Σ_h, and a continuous map $p : \Sigma_g \to \Sigma_h$, such that any $X \in \mathcal{S}$ admits a holomorphic map to Y homotopic to p, and such that $\psi \simeq \theta \circ p$ for some continuous map $\theta : \Sigma_h \to M$.*

Note that in the equivariant case, an example of constant energy families of harmonic maps was provided by Deroin and Tholozan [10, Theorem 5].

In the setting of Theorem 1, for any $X \in \mathcal{T}_g$, the harmonic map $h : X \to M$ homotopic to ψ factors as $h = h_\theta \circ h_p$, where $h_p : X \to Y$ is the holomorphic map homotopic to p, and $h_\theta : Y \to M$ is the harmonic map homotopic to θ. More precisely, Theorem 1 is essentially equivalent to a factorization result for holomorphic fibrations with (complex) one-dimensonal fibres.

Definition 1.1. We call a holomorphic map $p : E \to B$ between connected complex manifolds an *admissible fibration* if there exists a proper analytic subset $A \subsetneq B$ such that $p : E \setminus p^{-1}(A) \to B \setminus A$ is a holomorphic submersion with connected, closed and (complex) one-dimensional fibres, such that $p : E \setminus p^{-1}(A) \to B \setminus A$ is topologically a product.

Theorem 2. *Let M be a convex cocompact, real analytic Riemannian manifold with negative Hermitian sectional curvature, and $p : E \to B$ be an admissible fibration. Then for any smooth map $F : E \to M$ transverse to p such that the maps $F_b = F|_{p^{-1}(b)} : p^{-1}(b) \to M$, for $b \in B \setminus A$, are all harmonic of equal energy, we have either*

(1) F has image in a closed geodesic in M, or
(2) F factors as $h \circ \phi$, where $\phi : E \to Y$ is a holomorphic map to a closed hyperbolic Riemann surface Y transverse to p, and $h : Y \to M$ is a harmonic map.

1.1. **Factorization of harmonic maps.** As an application of Theorem 2, we show the following.

Theorem 3. *Let X be a normal projective variety equipped with the structure of a Kähler space, and let M be a convex cocompact real analytic Riemannian manifold with negative Hermitian sectional curvature. Then for any non-constant continuous map $f : X \to M$, harmonic on the smooth locus of X, we have either*

(1) f is a pluriharmonic map to a closed geodesic in M, or
(2) f factors as $h \circ \phi$, where $\phi : X \to Y$ is a holomorphic map to a closed hyperbolic Riemann surface Y, and $h : Y \to M$ is harmonic.

When X is a compact Kähler manifold, and when the universal cover of M is a (real) hyperbolic space $\mathbb{H}^d$, this is the result of Carlson–Toledo [8, Theorem 7.2 (a)]. Since in our case, the variety X is allowed to have mild singularities, we work in the setting of Kähler spaces (for precise definitions, the reader can consult [35]).

Since we make use of Theorem 2, we require the variety we start with to have many (complex) one-dimensional submanifolds. In the smooth projective case, this is easily obtained through a modification of Bertini's theorem (see §6.2). Using Hironaka's resolution of singularities [17, 18] this can be extended to arbitrary projective varieties.

1.2. **Mapping class groups.** As a second application of Theorem 2, we consider virtual properties of mapping class groups. We denote by $\mathrm{Mod}_{g,n}$ the pure mapping class group of a surface $\Sigma_{g,n}$ of genus g with n punctures. Recall the Birman exact sequence [12, §4.2]

$$1 \to \pi_1(\Sigma_{g,n}) \xrightarrow{\iota} \mathrm{Mod}_{g,n+1} \xrightarrow{\mathcal{F}} \mathrm{Mod}_{g,n} \to 1.$$

Here $\mathcal{F}$ is the forgetful homomorphism obtained by filling in a puncture, and ι realizes $\pi_1(\Sigma_{g,n})$ as the point-pushing subgroup $\Pi_{g,n} \leq \mathrm{Mod}_{g,n+1}$.

Definition 1.2. Given a subgroup $\Gamma \leq \mathrm{Mod}_{g,n+1}$ and a group G, a homomorphism $\phi : \Gamma \to G$ is called *strongly point-pushing* if $\phi(\Gamma \cap \Pi_{g,n})$ is not trivial or cyclic.

Theorem 4. *Let M be a convex cocompact, real analytic Riemannian manifold with negative Hermitian sectional curvature, and let $\Gamma \leq \mathrm{Mod}_{g,n+1}$ be a finite index subgroup with $g \geq 3, n \geq 0$. Then for any strongly point-pushing homomorphism $\phi : \Gamma \to \pi_1(M)$,*

(1) there exists a closed hyperbolic Riemann surface Y, such that for any marked Riemann surface $X \in \mathcal{T}_{g,n}$, its cover $\hat{X}$ that corresponds to the subgroup $\iota^{-1}(\Pi_{g,n} \cap \Gamma)$ admits a non-constant holomorphic map to Y, and
(2) there exist a finite index subgroup $\Theta \leq \Gamma$ and a homomorphism $\theta : \Theta \to \pi_1(Y)$ such that $\phi|_\Theta$ factors through θ.

Note that when M is a one-holed torus equipped with a complete hyperbolic metric, Theorem 4(1) recovers one direction of a theorem of Marković [24, Theorem 1.2, "if" direction].

1.3. **Outline and organization.** We first show Theorems 1 and 2 in §2–§5. We then show Theorem 3 in §6, and Theorem 4 in §7.

1.3.1. *Main results.* Our point of view in showing Theorems 1 and 2 is Teichmüller theoretic. In §2 we explain how to derive Theorem 2 in general, assuming the case when B is a submanifold of Teichmüller space and E is the restriction of the universal curve to this submanifold. This special case is essentially Theorem 1, and §3–§5 are devoted to its proof.

We now give an outline of the proof of Theorem 1. Let $\mathcal{S} \subset \mathcal{T}_g$ be a submanifold, and $\psi : \Sigma_g \to M$ a continuous map with E_ψ constant on $\mathcal{S}$. Let $\pi : \mathcal{V}_g \to \mathcal{T}_g$ be the universal curve, and write $\mathcal{V}_{\mathcal{S}} = \pi^{-1}(\mathcal{S})$. We also write $F_\psi : \mathcal{V}_g \to M$ for the map that is harmonic and homotopic to ψ on the fibres of π. Our aim is to show that F_ψ factors through a holomorphic map to a closed Riemann surface Y.

The proof relies on three ingredients.

Ingredient 1. Foliation $\mathcal{F}$. Using that E_ψ is constant, we show that F_ψ is pluriharmonic and $\dim_{\mathbb{C}} DF_\psi(T^{1,0}\mathcal{V}_{\mathcal{S}}) \leq 1$. This is a local argument, carried out in Proposition 3.1 in §3. We define a holomorphic foliation $\mathcal{F}$ on $\mathcal{V}_{\mathcal{S}} \setminus \Delta$, where $\dim_{\mathbb{C}} \Delta \leq \dim_{\mathbb{C}} \mathcal{S} - 1$, and where the leaves of $\mathcal{F}$ are tangent to the $\ker \partial F_\psi$. In particular, F_ψ is constant on $\mathcal{F}$. Properties of $\mathcal{F}$ can be found in Corollary 3.3.

Ingredient 2. Schiffer varieties $\Lambda_{Y,p}$. For a Riemann surface Y and a continuous map $p : \Sigma_g \to Y$, define

$$\Lambda_{Y,p} = \{X \in \mathcal{T}_g : \text{there exists a holomorphic map } X \to Y \text{ homotopic to } p\}$$

In §4, we show that $\Lambda_{Y,p}$ is a complex closed submanifold of $\mathcal{T}_g$, and that there is a holomorphic map $\mathcal{V}_{\Lambda_{Y,p}} \to Y$ homotopic to p.

Note that this space is stratified by $\mathrm{Schiff}_{Y,\sigma}$, where σ ranges over branched covers $\Sigma_g \to Y$ homotopic to p. Rather than describing the combinatorics of how the spaces $\mathrm{Schiff}_{Y,\sigma}$ fit together, we choose here to show directly that $\Lambda_{Y,p}$ is a closed complex submanifold using the non-abelian Hodge correspondence and the previous work of the author [34]. Along the way, we also show the analogous result for indiscrete representations $\pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$.

Ingredient 3. Equivalence relations $\sim$. Fix a general point $t \in \mathcal{S}$ and let $X_t = \pi^{-1}(t)$ be the Riemann surface that corresponds to t, and consider the harmonic map $F_\psi|_{X_t} : X_t \to M$. We define a maximal equivalence relation $\sim$ on X_t through which $F_\psi|_{X_t}$ factors holomorphically (Definition 5.3). Specifically, for $x, y \in X_t$ with $(\partial F_\psi|_{X_t})_x, (\partial F_\psi|_{X_t})_y \neq 0$, let $x \sim y$ if there are neighbourhoods U_x (resp. U_y) of x (resp. y), and a bihlomorphism $\phi : U_x \to U_y$ such that $F_\psi \circ \phi = F_\psi|_{U_x}$. We show that the natural quotient map $X_t \to Y_t := X_t/\sim$ is a finite (ramified) cover of closed Riemann surfaces (see §5.2). Moreover, by definition we have $F_\psi|_{X_t} = h_t \circ q_t$, where $h_t : Y_t \to M$ is harmonic.

We claim that Y_t are isomorphic for t ranging over some open subset of $\mathcal{S}$. Fix an arbitrary $t \in \mathcal{S}$. Since F_ψ factors through the local leaf spaces of $\mathcal{F}$, it follows that $F_\psi|_{X_t}$ factors through a holomorphic map $X_t \to Y_s$ followed by $h_s : Y_s \to M$. Thus by definition of $\sim$ on X_t, it follows that Y_t admits a non-constant holomorphic map to Y_s. It easily follows from this that the surfaces $\{Y_t : t \in \mathcal{S}\}$ are all isomorphic over some open subset of $\mathcal{S}$.

By a simple application of the Baire category theorem, it follows that $q_t : X_t \to Y_t$ are all homotopic and Y_t all isomorphic for $t \in \mathcal{U} \subset \mathcal{S}$, over some open subset $\mathcal{U}$. Let $Y = Y_t$ and $q : \Sigma_g \to Y$ be homotopic to q_t, for some fixed $t \in \mathcal{U}$. Thus $\mathcal{U} \subset \Lambda_{Y,q}$. Since $\Lambda_{Y,q}$ is closed, it follows that $\mathcal{S} \subset \Lambda_{Y,q}$, and the proof of Theorem 1 is completed.

1.3.2. *Factorization result.* We show Theorem 3 in §6. The singular version is easily obtained from the case when X is smooth using Hironaka's theorems on resolution of singularities [17, 18] combined with results on singularity removal for harmonic maps [26], so we will assume that X is smooth in this outline.

Suppose that $X \subseteq \mathbb{P}^N$ is d-dimensional. By considering all possible intersections of X with an $(N-d+1)$-dimensional subspace of $\mathbb{P}^N$, we obtain a diagram of holomorphic maps between smooth projective varieties

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "\phi"] & B \\ X & \end{tikzcd}$$

where the general fibre of p is a closed connected Riemann surface, by Bertini's theorem. Here B parameterizes $(N-d+1)$-dimensional subspaces of $\mathbb{P}^N$.

From the Siu–Sampson theorem [28, 30] (the reader can also consult [1, Chapter 6] or [23]) and our local analysis in §3 (see Proposition 3.1), it follows that $(f \circ \phi)|_{p^{-1}(b)}$ has energy independent of $b \in B$.

Note that $p : E \to B$ is not topologically a product, so we are still unable to apply Theorem 2. We therefore take the universal cover $\tilde{B}$ of B, which gives an immersion $\tilde{B} \looparrowright \Lambda_{Y,q}$, for some closed Riemann surface Y and a continuous map $q : \Sigma_g \to Y$. In particular, we get a holomorphic map $\mathcal{V}_{\tilde{B}} \to Y$. An argument using the De Franchis–Severi theorem implies that this map descends to a holomorphic map $\theta : \hat{E} \to Y$, where $\hat{E}$ is the pullback of E via a finite cover $\hat{B} \to B$.

Using the real analyticity of the harmonic map $Y \to M$, we can show that θ is constant on the connected components of the fibres of $\hat{E} \to E \xrightarrow{\phi} X$. Therefore we obtain a map $\hat{\theta} : X \to \mathrm{Sym}^k Y$ for an appropriate integer k, by mapping $x \in X$ to the θ-image of the lifts to $\hat{E}$ of the connected components of $\phi^{-1}(x) \subset E$. We then show that the image of $\hat{\theta}$ has complex dimension one, and that f factors through $\hat{\theta}$. Therefore the map f factors through the holomorphic map from X to a Riemann surface obtained by taking the normalization of the image of $\hat{\theta}$.

1.3.3. *Mapping class groups.* We show Theorem 4 in §7, following the ideas in the joint paper of Marković and the author [25].

Let $\Gamma \leq \mathrm{Mod}_{g,n+1}$ be a finite index subgroup, and $\phi : \Gamma \to \pi_1(M)$ a strongly point-pushing homomorphism. This defines a homomorphism $\phi \circ \iota : \pi_1(\Sigma_{g,n}) \to \pi_1(M)$, with image not contained in a cyclic subgroup of $\pi_1(M)$, that is invariant under Γ.

By a result of Bridson [6], it follows that $\ker \phi$ contains all Dehn multitwists in Γ. In particular, it contains any elements of $\Pi_{g,n}$ that correspond to simple closed curves, including small loops around the punctures. Let $\psi : \pi_1(\Sigma_g) \to \pi_1(M)$ be the extension of $\phi \circ \iota$ over these punctures.

We consider the energy functional $\mathrm{E}_\psi : \mathcal{T}_g \to \mathbb{R}$. This functional is plurisubharmonic, and invariant under the image Γ' of Γ under the forgetful homomoprhism $\mathrm{Mod}_{g,n} \to \mathrm{Mod}_g$. Thus E_ψ descends to a plurisubharmonic map on a finite cover $\mathcal{M}$ of the moduli space of genus g. An analysis based on the same result of Bridson [6] implies that E_ψ is bounded on $\mathcal{M}$. Since $\mathcal{M}$ is quasiprojective, standard results in complex analysis imply that E_ψ is constant. Theorem 4 then follows immediately by Theorem 2.

2. Reduction to submanifolds of Teichmüller space

In this section, we set the stage for the proof of Theorem 1. We first introduce some notation in the setup of Theorem 1 in §2.1, and then we show Theorem 2 assuming Theorem 1 in §2.2.

2.1. **Notation.** Let $\pi : \mathcal{V}_g \to \mathcal{T}_g$ be the universal curve over Teichmüller space, that is a complex fibration such that the fibre of π over $S \in \mathcal{T}_g$ is biholomorphic to S. For a complex submanifold $\mathcal{S}$ of $\mathcal{T}_g$, we denote $\mathcal{V}_{\mathcal{S}} = \pi^{-1}(\mathcal{S})$. Given a homotopy class $\psi : \Sigma_g \to M$ for some convex cocompact manifold M with negative sectional curvatures, we let

$$F_\psi : \mathcal{V}_g \to M$$

be the map that is harmonic and homotopic to ψ on every fibre of $\pi : \mathcal{V}_g \to \mathcal{T}_g$.

Remark 2.1. It follows from the classical work of Eells–Lemaire [11] that the harmonic map $X \to M$ in the homotopy class of ψ depends smoothly on $X \in \mathcal{T}_g$ (the reader can consult the work of Slegers [31] or previous work of the author [34] for the analogous result when the target is a non-compact symmetric space). In particular, the map F_ψ is smooth.

In the entirety of this paper, we will assume the following.

Assumptions 2.2. Let M be a real analytic manifold with a real analytic Riemannian metric, that has negative Hermitian sectional curvature, and is convex cocompact.

Definition 2.3. Given a continuous map $p : \Sigma_h \to \Sigma_g$, we define the *Schiffer variety* $\Lambda_{Y,p}$ associated to $Y \in \mathcal{T}_g$ and p to be the set of marked Riemann surfaces $X \in \mathcal{T}_h$ that admit a holomorphic map to Y in the homotopy class of p.

We will show in Proposition 4.1 that Schiffer varieties are complex submanifolds of Teichmüller space $\mathcal{T}_h$, closed as subsets of $\mathcal{T}_h$. Moreover, we will show that the corresponding map $F_p : \mathcal{V}_{\Lambda_{Y,p}} \to Y$ is holomorphic.

Remark 2.4. Given a covering map $\sigma : \Sigma_{h,m} \to \Sigma_{g,n}$, we get an isometric embedding of Teichmüller spaces $\sigma^* : \mathcal{T}_{g,n} \to \mathcal{T}_{h,m}$, obtained by lifting complex structures from $\Sigma_{g,n}$ to $\Sigma_{h,m}$ via p. Given a Riemann surface $Y \in \mathcal{T}_g$, let $\Delta_Y \subset \mathcal{T}_{g,n}$ be the preimage of Y under the forgetful morphism $\mathcal{T}_{g,n} \to \mathcal{T}_g$. Then, if σ is homotopic to p after filling in the punctures, we see that $\sigma^*(\Delta_Y)$ projects to $\mathrm{Schiff}_{Y,\sigma} \subset \Lambda_{Y,p}$ under the forgetful morphism $\mathcal{T}_{h,m} \to \mathcal{T}_h$. These are the subsets of $\mathcal{T}_h$ that we referred to as Schiffer variations in the introduction. It is easy to see that $\{\mathrm{Schiff}_{Y,\sigma} : \sigma \simeq p\}$ stratify $\Lambda_{Y,p}$.

In §2.2 we show Theorem 2 assuming Theorem 1.

2.2. **Reducing Theorem 2 to Theorem 1.** Let M, p, E, B, F, A be as in the statement of Theorem 2, and assume Theorem 1.

By the universal property of Teichmüller space, there exists a holomorphic map $\iota : B \setminus A \to \mathcal{T}_g$, such that $\iota^*\mathcal{V}_g = E \setminus p^{-1}(A)$. In fact, we abuse notation slightly to denote by ι both the natural map $B \setminus A \to \mathcal{T}_g$ and the natural map that makes the following diagram commute

$$\begin{tikzcd} E \setminus p^{-1}(A) \arrow[r, "\iota"] \arrow[d, "p"] & \mathcal{V}_g \arrow[d, "\pi"] \\ B \setminus A \arrow[r, "\iota"] & \mathcal{T}_g \end{tikzcd}$$

Lemma 2.5. *Let B be a connected complex manifold, $\iota : B \to \mathcal{T}_g$ a holomorphic map, and M be a negatively curved manifold. Then for any continuous map $F : \iota^*\mathcal{V}_g \to M$ transverse to $p := \iota^*\pi : \iota^*\mathcal{V}_g \to B$, such that $F_b = F|_{p^{-1}(b)} : p^{-1}(b) \to M$ is a harmonic map for any $b \in B$, we have*

$$F = F_\psi \circ \iota$$

for some continuous map $\psi : \Sigma_g \to M$.

Proof. Since the fibres of $p : \iota^*\mathcal{V}_g \to B$ are identified with the fibres of $\pi : \mathcal{V}_g \to \mathcal{T}_g$ via ι, they each have a natural marking. Let $\psi : \Sigma_g \to M$ be the continuous map, well-defined up to homotopy, such that $F|_{p^{-1}(b)}$ is in the homotopy class of ψ, for any $b \in B$.

Now let $U \subset B$ be an open subset where F has constant rank. Then by the constant rank theorem, we can locally write ι over a possibly smaller open set $V \subseteq U$ as

$$V \xrightarrow{s} \mathcal{S} \subset \mathcal{T}_g,$$

where s is a submersion and $\mathcal{S}$ is a complex submanifold of $\mathcal{T}_g$. Then $F : s^*\mathcal{V}_{\mathcal{S}} \to M$ is harmonic on the fibres $\pi^{-1}(t)$ for $t \in \mathcal{S}$. Since M is negatively curved, the Jacobi operator of any harmonic map from a closed surface into M is injective. Therefore F descends to a map $\bar{F} : \mathcal{V}_{\mathcal{S}} \to M$. Note that $\bar{F}|_{\pi^{-1}(t)} \simeq \psi$ for any $t \in \mathcal{S}$, so $\bar{F} = F_\psi$. Thus $F = F_\psi \circ \iota$ on V.

Since this equality holds on the open dense subset where ι has locally constant rank, by continuity we have $F = F_\psi \circ \iota$ everywhere on B. □

Applying Lemma 2.5, we get $F = F_\psi \circ \iota$ over $E \setminus p^{-1}(A)$. If ψ is null-homotopic, then F_ψ is constant on the fibres of π. Then F is constant on the fibres of p, which is a contradiction since F is transverse to p. If ψ is homotopic into a closed curve γ, then the image of F_ψ lies in the closed geodesic that corresponds to γ. Then the image of F also lies in this closed geodesic, and Theorem 2 is shown. Assume therefore that ψ is not homotopic into a curve.

Note that the energy of F over $p^{-1}(b)$ is equal to $\mathrm{E}_\psi \circ \iota(b)$. Therefore E_ψ is constant on $\iota(B \setminus A)$. By Theorem 1, the set $\iota(B \setminus A)$ is contained in a Schiffer variety $\Lambda \subset \mathcal{T}_g$, and $F_\psi = h \circ \phi$, where $\phi : \mathcal{V}_\Lambda \to Y$ is a holomorphic map to a closed hyperbolic Riemann surface Y. By the Schwarz lemma, the map $\phi \circ \iota : E \setminus p^{-1}(A) \to Y$ is meromorphic in the sense of Andreotti [2], and hence extends to a map $\bar{\phi} : E \to Y$ by [2, Theorem 4]. Therefore $F = h \circ \bar{\phi}$ on $E \setminus p^{-1}(A)$, and hence by continuity on E, and Theorem 2 is shown.

3. Pluriharmonicity of the combined map

Our main result in this section is the following characterization of constant energy E_ψ complex submanifolds of Teichmüller space in terms of the map F_ψ, that we believe to be of independent interest.

Proposition 3.1. *Let $\psi : \Sigma_g \to M$ be a continuous map into a Riemannian manifold M. Suppose that the pair (M, ψ) satisfies either of the following two conditions*

(1) M has negative Hermitian sectional curvature and is convex cocompact, and ψ is not homotopic into a closed curve in M, or

(2) M is a locally symmetric space of non-positive curvature, and the representation $\psi_* : \pi_1(\Sigma_g) \to \mathrm{Isom}(\tilde{M})$ has trivial centralizer.

Let $\mathcal{S} \subset \mathcal{T}_g$ be a complex submanifold. Then E_ψ is constant on $\mathcal{S}$ if and only if the map F_ψ is pluriharmonic and $R^M(\partial F_\psi|_{\ker \pi_*} \wedge \partial F_\psi) = 0$ on $\mathcal{V}_\mathcal{S}$.

Remark 3.2. Suppose we are in the setting of Proposition 3.1(1). Then observe that the conclusions of the proposition can be restated as

$$\bar{\partial}\partial F_\psi = 0 \text{ and } DF_\psi(T^{1,0}\mathcal{V}_\mathcal{S}) \leq 1,$$

which are the same as the conclusions of the Siu–Sampson rigidity theorem [30, 28]. In particular, the Siu–Sampson theorem can be shown (for targets with strictly negative Hermitian sectional curvature) using Proposition 3.1 using the ideas of Gromov [15, §4.6] for complex manifolds that are closed Riemann surface bundles over closed complex manifolds.

We will mainly use Proposition 3.1 for its complex geometric consequences, layed out in the following corollary shown in §3.5. Given a section ϕ of some holomorphic line bundle L over a complex manifold X, we denote by $\mathbb{V}(\phi)$ the analytic subspace of X defined by the vanishing of ϕ.

Corollary 3.3. *Let (M, ψ) be as in Proposition 3.1(1). If $\mathcal{S} \subset \mathcal{T}_g$ is a connected complex submanifold contained in a level set of E_ψ, then we have the following.*

(1) The bundle $E = F_\psi^* TM \otimes \mathbb{C}$ equipped with the $(0,1)$-part of the Levi–Civita connection on M and the pullback metric from M is a holomorphic Hermitian vector bundle over $\mathcal{V}_\mathcal{S}$.
(2) There exists a holomorphic line subbundle $L \leq E$ that inherits the Hermitian structure from E, such that ∂F_ψ is an L-valued $(1,0)$ form that is closed with respect to the Chern connection.
(3) There is a (complex) codimension two subset $\Delta \subset \mathcal{V}_\mathcal{S}$ and a holomorphic codimension 1 foliation $\mathcal{F}$ on $\mathcal{V}_\mathcal{S} \setminus \Delta$, such that $T\mathcal{F} = \ker \partial F_\psi$ on $\mathcal{V}_\mathcal{S} \setminus \mathbb{V}(\partial F_\psi)$ and such that the connected components of $\mathbb{V}(\partial F_\psi) \setminus \Delta$ are leaves of $\mathcal{F}$.

The harder "only if" direction of Proposition 3.1 will be shown in §3.4. We show the "if" direction in §3.3. The proof of both directions of Proposition 3.1 depends on the computations in [34], that we recall and slightly rephrase in §3.2.

3.1. **Vector fields on the universal curve.** We now rephrase [33, proof of Theorem 3] and [34, Theorem 1.9] in differential geometric language that is more suited to our applications here. Let J be the almost complex structure on $\mathcal{V}_g$ and let $\partial^{\text{vert.}}$ and $\bar{\partial}^{\text{vert.}}$ be the differential operators related to the Riemann surface structure on the fibres of π. Given $t \in \mathcal{T}_g$, let $X_t = \pi^{-1}(t) \subset \mathcal{V}_g$ be the Riemann surface that corresponds to t.

Lemma 3.4. *There is a correspondence between smooth type $(1,0)$ vector fields on $X_t \subset \mathcal{V}_g$ that are lifts of some fixed vector in $T_t^{1,0}\mathcal{T}_g$, and smooth Beltrami forms on X_t, given by*

$$V \longrightarrow \mu = \overline{\bar{\partial}^{\text{vert.}} V}. \tag{$\star$}$$

Moreover, in this case $\omega \circ \mathcal{L}_V J = 2i\mu\omega^{1,0}$ for any 1-form ω on X_t.

Proof. It suffices to construct the inverse to the map $(\star)$. We will use the computations in [25, 34]. We set $X = X_t$ throughout this proof.

The point of view taken in [34] is to consider a fixed topological surface Σ_g, and then consider a family of almost-complex structures J_t on Σ_g, parameterized by points $t \in \mathcal{T}_g$. To use the formulae from [34], we therefore need to identify the fibres of $\mathcal{V}_{\mathcal{S}} \to \mathcal{S}$. Let μ be a smooth Beltrami differential on the marked Riemann surface $X \in \mathcal{T}_g$. As in [25, §2.1], we identify different fibres via solutions to the Beltrami equation $f^{t\mu} : X \to X^{t\mu}$, where $X^{t\mu}$ is a different Riemann surface, t is a small real parameter, and

$$t\mu = \frac{\bar{\partial} f^{t\mu}}{\partial f^{t\mu}}.$$

We therefore define V_μ to be $\frac{\partial f^{t\mu}}{\partial t}$. We also let $V_\mu^{\mathbb{C}} = \frac{1}{2}\left(V_\mu - iV_{i\mu}\right)$. We will show that $\mu \to V_\mu^{\mathbb{C}}$ provides the inverse to $(\star)$.

Rephrasing [25, Claim 3.2] or [34, Claim 3], we get the following claim.

Claim 3.5. For any 1-form ω on X and any smooth Beltrami differential μ on X, we have

$$\omega \circ \mathcal{L}_{V_\mu} J = 2i\left(\mu\omega^{1,0} - \bar{\mu}\omega^{0,1}\right), \text{ and } \omega \circ \mathcal{L}_{V_\mu^{\mathbb{C}}} J = 2i\mu\omega^{1,0}.$$

Proof. The first equality is equivalent to both [34, Claim 3] and [25, Claim 3.2]. The second equality follows from the first by an easy computation,

$$\begin{aligned}
\omega \circ \mathcal{L}_{V_\mu^{\mathbb{C}}} J &= \frac{1}{2}\omega \circ \left(\mathcal{L}_{V_\mu} J - i\mathcal{L}_{V_{i\mu}} J\right) \\
&= i\left(\mu\omega^{1,0} - \bar{\mu}\omega^{0,1} - i\left(i\mu\omega^{1,0} + i\bar{\mu}\omega^{0,1}\right)\right) \\
&= 2i\mu\omega^{1,0}.
\end{aligned}$$

□

Working in a local holomorphic coordinate z on X and setting $\omega = dz$, we see that

$$\left(\mathcal{L}_{V_\mu^{\mathbb{C}}} J\right)(\bar{\partial}_z) = 2i\mu\partial_z.$$

Here we abused notation slightly to denote by μ the coordinate expression for the Beltrami form in z. On the other hand,

$$\begin{aligned}\left(\mathcal{L}_{V_\mu^{\mathbb{C}}}J\right)(\bar\partial_z) &= -i\mathcal{L}_{V_\mu^{\mathbb{C}}}\bar\partial_z - J\mathcal{L}_{V_\mu^{\mathbb{C}}}\bar\partial_z\\ &= -2i[V_\mu^{\mathbb{C}},\bar\partial_z]^{1,0}\\ &= -2i\left(\nabla_{V_\mu^{\mathbb{C}}}\bar\partial_z - \nabla_{\bar z}V_\mu^{\mathbb{C}}\right)^{1,0}\\ &= 2i\bar\partial V_\mu^{\mathbb{C}}.\end{aligned}$$

In particular, $\bar\partial V_\mu^{\mathbb{C}} = \mu$. This shows that $(\star)$ is surjective. Injectivity follows immediately from the non-existence of holomorphic vector fields on X without poles. $\square$

3.2. **Vanishing of the Laplacian of the energy.** Denote by R^M the Riemann curvature tensor on M.

Lemma 3.6. *Fix $t_0 \in \mathcal{T}_g$. Let V be a vector field on $\mathcal{V}_g$ of type $(1,0)$ transverse to the fibres of π, such that V is on $\pi^{-1}(t_0)$ a lift of a vector $\mu \in T_{t_0}\mathcal{T}_g$. Then we have*

$$\bar\partial^{\text{vert.}}d_V F = -\frac{i}{2}\partial^{\text{vert.}}F\circ\mathcal{L}_V J \text{ and } R^M(d_V F, \partial^{\text{vert.}}F) = 0 \tag{3.1}$$

on $\pi^{-1}(t_0)$ if and only if the Laplacian of the energy E_ψ vanishes in the direction μ.

Proof. Set $X = \pi^{-1}(t_0)$. We adopt the notation $V_\mu^{\mathbb{C}}$ from the proof of Lemma 3.4 for the inverse of $(\star)$. By [34, Theorem 1.9], we see that the Laplacian of E_ψ vanishes in the direction defined by a smooth Beltrami form μ on $X \in \mathcal{T}_g$, if and only if

$$\mu\partial^{\text{vert.}}F = \bar\partial^{\text{vert.}}\xi \text{ and } R^M(\xi, \partial^{\text{vert.}}F) = 0, \tag{3.2}$$

for $\xi = \frac{1}{2}\left(\dot F^\mu - i\dot F^{i\mu}\right) = d_{V_\mu^{\mathbb{C}}}F$. Here by $\dot F^\mu$ denotes the derivative of $F\circ f^{t\mu}$ at $t=0$. Combining with Claim 3.5, we get that (3.2) is equivalent to

$$-\frac{i}{2}\partial^{\text{vert.}}F\circ\mathcal{L}_{V_\mu^{\mathbb{C}}}J = \bar\partial^{\text{vert.}}d_{V_\mu^{\mathbb{C}}}F \text{ and } R^M(d_{V_\mu^{\mathbb{C}}}F, \partial^{\text{vert.}}F) = 0.$$

Since any smooth vector field of type $(1,0)$ on $X \subset \mathcal{V}_g$ which is transverse to X and projects under π to some fixed $[\mu] \in T_X\mathcal{T}_g$ can be written in the form $V_\mu^{\mathbb{C}}$ for some smooth Beltrami form μ on X, the result is shown. $\square$

3.3. **"If" direction.** We first claim that if (3.1) holds for some vector field V, then it also holds for $\tilde{V} = fV + W$, where W is any type $(1,0)$ (vertical) vector field on X and $f : X \to \mathbb{C}$ is any smooth function. This is clear for the second equation in (3.1). We show this claim for the first equation in (3.1) in two parts. We first check the first equation in (3.1) for $V = W$,

$$\begin{aligned}\partial^{\text{vert.}} F \circ \mathcal{L}_W J &= \mathcal{L}_W \left(\partial^{\text{vert.}} F \circ J\right) - \left(\mathcal{L}_W \partial^{\text{vert.}} F\right) \circ J \\ &= 2i\mathcal{L}_W \partial^{\text{vert.}} F \circ \frac{\text{id} + iJ}{2} \\ &= 2id^{\text{vert.}} \left(\partial_W F\right) \circ \frac{\text{id} + iJ}{2} = 2i\bar{\partial}^{\text{vert.}} d_W F.\end{aligned}$$

Here we used Cartan's magic formula and the fact that $d^{\text{vert.}}\partial^{\text{vert.}} F = 0$ by harmonicity in going from the second to the third line. We now check how (3.1) transforms for $\tilde{V} = fV$. We have

$$\bar{\partial}^{\text{vert.}} d_{fV} F = f\bar{\partial}^{\text{vert.}} d_V F + d_V F \bar{\partial} f,$$

and

$$\begin{aligned}\partial^{\text{vert.}} F \circ \mathcal{L}_{fV} J &= f\partial^{\text{vert.}} F \circ \mathcal{L}_V J + 2i\partial^{\text{vert.}} F \circ \left(V\bar{\partial} f\right) \\ &= f\partial^{\text{vert.}} F \circ \mathcal{L}_V f + 2id_V F\bar{\partial} f.\end{aligned}$$

where we used the following.

Claim 3.7. If V is a type $(1,0)$ vector field, then

$$\mathcal{L}_{fV} J = f\mathcal{L}_V J + 2iV\bar{\partial} f.$$

Proof. This follows from

$$\begin{aligned}\mathcal{L}_{fV}(J)(W) &= \mathcal{L}_{fV}(JW) - J\mathcal{L}_{fV} W = [fV, JW] - J[fV, W] \\ &= f[V, JW] - Vd_{JW} f - J\left(f[V, W] - Vd_W f\right) \\ &= f\mathcal{L}_V(J)(W) + V\left(-d_{JW} f + id_W f\right) \\ &= f\mathcal{L}_V(J)(W) + 2iV\left(\bar{\partial} f\right)(W).\end{aligned}$$

□

We now let (M, ψ) be as in Proposition 3.1 and suppose that $F_\psi : \mathcal{V}_\mathcal{S} \to M$ is pluriharmonic with $R^M(dF_\psi \wedge dF_\psi) = 0$. By the previous argument, it suffices to check (3.1) locally for any type $(1,0)$ vector field V on X transverse to X. We fix a point in $\mathcal{V}_\mathcal{S}$ and use local coordinates to check (3.1) at that point. Let $(z, t_1, t_2, ..., t_n)$ be holomorphic local coordinates on $\mathcal{V}_S$ and $(t_1, t_2, ..., t_n)$ be local holomorphic coordinates for $\mathcal{S}$, such that π is locally represented by $(z, t_1, t_2, ..., t_n) \to (t_1, t_2, ..., t_n)$.

For $V = \frac{\partial}{\partial t_i}$, the first equation in (3.1) holds, since both sides vanish. Similarly, since $R^M(\partial^{\text{vert.}} F_\psi \wedge \partial F_\psi) = 0$, we see that

$$\sum_{j=1}^{n} R^M \left(\frac{\partial F_\psi}{\partial z}, \frac{\partial F_\psi}{\partial t_j} \right) dz \wedge dt_j = 0,$$

and thus $R^M \left(\partial^{\text{vert.}} F, d_V F \right) = 0$. Therefore by Lemma 3.6 the Laplacian of E_ψ vanishes identically on $\mathcal{S}$. From the computations in [34, §3.3] and [34, Claim 3], it is easy to see that if the Laplacian of E_ψ vanishes in the complex direction defined by μ, then $d_\mu \mathrm{E}_\psi = 0$. Thus E_ψ is constant on $\mathcal{S}$.

3.4. **"Only if" direction.** Note that pluriharmonicity is a local claim, so we may without loss of generality assume that $\mathcal{S}$ is a small holomorphic disk around $t_0 \in \mathcal{T}_g$. Since π is a holomorphic submersion, we may work locally in coordinates (z, t) on $\mathcal{V}_g$ and t on $\mathcal{S}$ such that π is represented locally by the projection $(z, t) \to t$ and such that t_0 corresponds to $t = 0$. Let the Levi form of F be

$$\partial \bar{\partial} F = F_{z\bar{z}} dz \wedge d\bar{z} + F_{t\bar{z}} dt \wedge d\bar{z} + F_{z\bar{t}} dz \wedge d\bar{t} + F_{t\bar{t}} dt \wedge d\bar{t}.$$

We will also use the notation $dF = F_t dt + F_{\bar{t}} d\bar{t} + F_z dz + F_{\bar{z}} d\bar{z}$.

Since $V = \frac{\partial}{\partial t}$ is a holomorphic vector field, its flow preserves the almost complex structure J on $\mathcal{V}_g$. In particular, from the first equation in the conclusion of Lemma 3.6, we see that $F_{t\bar{z}} = 0$. It immediately follows that $F_{z\bar{t}} = 0$. From the harmonicity of F, we see that $F_{z\bar{z}} = 0$. Thus

$$F_{t\bar{t}} dt \wedge d\bar{t} = \partial \bar{\partial} F = \partial \left(F_{\bar{t}} d\bar{t} \right) = \nabla_t F_{\bar{t}} dt \wedge d\bar{t}.$$

It is well-known that the Hessian is symmetric, so we have $\nabla_t F_{\bar{t}} = \nabla_{\bar{t}} F_t$. We therefore have

$$\begin{aligned} \nabla_z F_{t\bar{t}} &= \nabla_z \nabla_t F_{\bar{t}} = \nabla_t \nabla_z F_{\bar{t}} + R^M(F_* \partial_z, F_* \partial_t) F_{\bar{t}} \\ &= \nabla_t F_{z\bar{t}} + R^M(\partial^{\text{vert.}} F, d_V F) d_{\bar{V}} F = 0, \end{aligned}$$

where the first term is the final sum vanishes since $F_{z\bar{t}} = 0$ and the second term vanishes from the second equation in the conclusion of Lemma 3.6. We analogously have

$$\nabla_{\bar{z}} F_{t\bar{t}} = \nabla_{\bar{z}} \nabla_{\bar{t}} F_t = \nabla_{\bar{t}} \nabla_{\bar{z}} F_t + R^M(F_* \partial_{\bar{z}}, F_* \partial_{\bar{t}}) F_t = 0.$$

In particular, $dF_{t\bar{t}} = 0$.

Claim 3.8. We have $R^M(F_{t\bar{t}}, F_z) = 0$.

Proof. If M has strictly negative Hermitian sectional curvature, then $F_t \wedge F_z = 0$, so differentiating we get $F_{t\bar{t}} \wedge F_z + F_t \wedge F_{z\bar{t}} = 0$, so $F_{t\bar{t}} \wedge F_z = 0$ and the result follows.

If M is locally symmetric of non-positive curvature, then

$$0 = \nabla_{\bar{t}} R^M(F_t, F_z) = R^M(F_{t\bar{t}}, F_z) + R^M(F_t, F_{z\bar{t}}) = R^M(F_{t\bar{t}}, F_z).$$

□

Now let W be an arbitrary smooth type $(1,0)$ vector field on $\pi^{-1}(\mathcal{S})$, that is a lift of a nowhere vanishing holomorphic vector field on $\mathcal{S}$. Then in the local cooridnates (z,t), we see that $W = V + \alpha(z,t)\partial_z$, and hence

$$F_{W\bar{W}} = F_{t\bar{t}} + |\alpha|^2 F_{z\bar{z}} + \alpha F_{z\bar{t}} + \bar{\alpha} F_{t\bar{z}} = F_{t\bar{t}},$$

and hence we see that

$$dF_{W\bar{W}} = 0 \text{ and } R^M(F_{W\bar{W}}, \partial^{\text{vert.}} F) = 0. \tag{3.3}$$

If M is locally symmetric of non-positive curvature, then by a result of Sunada [32], the harmonic map realizing $\psi_* : \pi_1(\Sigma_g) \to \mathrm{Isom}(\tilde{M})$ is not unique unless $F_{W\bar{W}} = 0$. However uniqueness follows from the fact that ψ_* has a trivial stabilizer. Therefore $F_{W\bar{W}} = 0$ and the result is shown in this case.

Suppose now that M has negative Hermitian sectional curvature, and that ψ is not homotopic into a graph. We want to show that $F_{W\bar{W}} = 0$, so since $dF_{W\bar{W}} = 0$ it suffices to show that $F_{W\bar{W}}$ vanishes somewhere on every fibre of $\pi : \pi^{-1}(\mathcal{S}) \to \mathcal{S}$. Suppose therefore that $F_{W\bar{W}}$ is nowhere zero on $\pi^{-1}(t_0)$, and set $f = F|_{\pi^{-1}(t_0)}$.

Let L be the rank one subbundle of f^*TM generated by $F_{W\bar{W}}$. Then the second equation in (3.3) implies that ∂f takes values in $L \otimes \mathbb{C}$. In particular, we see that the image of df is contained in L. Therefore we get that

$$df = \omega F_{W\bar{W}},$$

where ω is a harmonic 1-form on $\pi^{-1}(t_0)$. In particular, df has rank at most 1, and hence by a result of Sampson [27], the image of f is contained in a geodesic arc. Since the domain of f is closed, this arc is a closed curve, which contradicts the assumption on ψ.

3.5. **Proof of Corollary 3.3.** We first show (1). Let $\bar{\partial}^E$ be the $(0,1)$-part of the pullback of the Levi–Civita connection on TM to $E := F_\psi^* TM \otimes \mathbb{C}$. We work in local coordinates $(z, t_1, t_2, ..., t_m)$ on $\mathcal{V}_{\mathcal{S}}$ and $(t_1, t_2, ..., t_m)$ on $\mathcal{S}$, such that the projection π locally takes the form $(z, t_1, t_2, ..., t_m) \to (t_1, t_2, ..., t_m)$.

By Proposition 3.1, we see that $R^M(\partial_z F_\psi, \partial_{t_i} F_\psi) = 0$. Since M has negative Hermitian sectional curvature, this implies that $\partial_z F_\psi \wedge$

$\partial_{t_i} F_\psi = 0$. On the set of points where $\partial_z F_\psi \neq 0$, this implies that $d_V F_\psi \wedge d_W F_\psi = 0$, for any two type $(1,0)$ vectors V, W. Note that $\partial_z F_\psi$ has isolated zeros on any $\pi^{-1}(t)$ for $t \in \mathcal{S}$ [29, pp. 10]. By continuity, we therefore have $d_V F_\psi \wedge d_W F_\psi = 0$ for any two $(1,0)$ vectors V, W at any point in $\mathcal{V}_\mathcal{S}$. In particular, $R^M(\partial F_\psi \wedge \partial F_\psi) = 0$.

Conjugating, we see that $R^M(\bar\partial F_\psi \wedge \bar\partial F_\psi) = 0$, and since

$$\bar\partial^E \circ \bar\partial^E = R^M(\bar\partial F_\psi \wedge \bar\partial F_\psi) = 0,$$

finishing the proof of (1) by the Koszul–Malgrange theorem.

We now turn to (2). From Proposition 3.1, and the strict negativity of the Hermitian sectional curvature of M, we see that

$$\partial F_\psi|_{\ker \pi_*} \wedge \partial F_\psi = 0.$$

This implies that over a point where $\partial F_\psi|_{\ker \pi_*} \neq 0$, we have the stronger $\partial F_\psi \wedge \partial F_\psi = 0$. Thus by continuity we see that $\partial F_\psi \wedge \partial F_\psi = 0$ on all of $\mathcal{V}_\mathcal{S}$. We now get a line bundle $L \leq E$ over $\mathcal{V}_\mathcal{S} \setminus \mathbb{V}(\partial F_\psi)$ generated by the image of ∂F_ψ. By [33, Theorem 4 (2)], we see that L extends to $\mathcal{V}_\mathcal{S}$. Finally, since the Chern connection on E is by construction the pullback of the Levi–Civita connection on M, we have

$$d^\nabla \partial F_\psi = \bar\partial \partial F_\psi = 0.$$

The Chern connection on L is the orthogonal projection of the Chern connection on E to L, so $\partial F_\psi \in \Omega^1(L)$ is closed as well.

Finally, we show (3). Let $U \subset \mathcal{V}_\mathcal{S}$ be an open set over which L is trivial, and let $\tau : L|_U \to \mathbb{C}$ be some (holomorphic) trivialization. Let Γ be the local connection 1-form with respect to this trivialization. Since ∂F_ψ is closed, we see that

$$d\tau(\partial F_\psi) + \Gamma \wedge \tau(\partial F_\psi) = 0.$$

In particular, the distribution defined by vanishing of the components of $\tau(\partial F_\psi)$ is integrable by the Frobenius theorem [7, Theorem II.1.1] wherever its rank is locally constant. It follows that there exists a holomorphic codimension 1 foliation $\mathcal{F}$ over $\mathcal{V}_\mathcal{S} \setminus \mathbb{V}(\partial F_\psi)$ with $T\mathcal{F} = \ker \partial F_\psi$.

Suppose that $\dim \mathcal{S} = d$, so that $\dim \mathcal{V}_\mathcal{S} = d + 1$. We write $\mathcal{A} = \mathbb{V}(\partial F_\psi)$ as a disjoint union

$$\mathcal{A} = \mathcal{A}^{\text{sng.}} \cup \mathcal{A}^{\text{reg.}} = \mathcal{A}^{\text{sng.}} \cup \bigcup_{i=0}^{d} \mathcal{A}_i,$$

where

(1) $\mathcal{A}^{\text{sng.}}$ is the singular part of $\mathcal{A}$,
(2) $\mathcal{A}^{\text{reg.}} = \mathcal{A} \setminus \mathcal{A}^{\text{sng.}}$ is the regular (smooth) part of $\mathcal{A}$, and

(3) $\mathcal{A}^{\text{reg.}} = \bigcup_{i=0}^{d} \mathcal{A}_i$ is the decomposition of $\mathcal{A}^{\text{reg.}}$ into the union of its i-dimensional connected components $\mathcal{A}_i$.

We now let $\Delta = \mathcal{A}^{\text{sng.}} \cup \bigcup_{i=0}^{d-1} \mathcal{A}_i \cup \Delta' \cup \Delta''$, where

(1) Δ' is the subset of $\mathcal{A}_d$ where the holomorphic map $\pi|_{\mathcal{A}_d} : \mathcal{A}_d \to \mathcal{S}$ is not a local biholomorphism, and
(2) Δ'' is the subset of $\mathcal{A}_d$ where the order of vanishing of $\partial^{\text{vert.}} F_\psi$ is not locally constant.

It follows from classical complex geometry that $\dim \Delta \leq d-1$. Note that $\mathcal{F}$ is defined on $\mathcal{V}_{\mathcal{S}} \setminus \mathcal{A}$, and we now show how to extend it to $\mathcal{A} \setminus \Delta = \mathcal{A}_d \setminus (\Delta' \cup \Delta'')$.

Fix some point $x \in \mathcal{A}_d \setminus (\Delta' \cup \Delta'')$. Pick a coordinate system $\Phi = (z, t_1, t_2, ..., t_d) : U \to \mathbb{C}^{d+1}$ on an open set $x \in U \subset \mathcal{V}_{\mathcal{S}}$ such that $\Phi(\mathcal{A}_d \cap U) = \mathbb{V}(z)$ and $\Phi(x) = (0, 0, ..., 0)$. By choice of Δ', we may suppose without loss of generality that there is a coordinate system $\Theta = (t_1, t_2, ..., t_d) : \pi(U) \to \mathbb{C}^d$ such that

$$\Theta \circ \pi \circ \Phi^{-1}(z, t_1, t_2, ..., t_d) = (t_1, t_2, ..., t_d).$$

We also choose a trivialization $\tau : L|_U \to \mathbb{C}$. Observe that $\tau(\partial F_\psi)$ vanishes along $\{z = 0\}$, so can be written locally as

$$\Phi_* \tau(\partial F_\psi) = z^n f dz + \sum_{j=1}^{d} g_j dt_j.$$

By choice of Δ'', we may assume that $f(0, 0, ..., 0) \neq 0$. Shrink U if necessary so that $f \neq 0$ on $\Phi(U)$. Let the local connection 1-form for L in the trivialization τ be $\Gamma = \Gamma_0 dz + \sum_{j=1}^{d} \Gamma_j dt_j$. The fact that ∂F_ψ is closed implies that

$$z^n \left(\frac{\partial f}{\partial t_j} + \Gamma_j f \right) = \frac{\partial g_j}{\partial z} + \Gamma_0 g_j \text{ for } j \geq 1. \tag{3.4}$$

Note that, by assumption, ∂F_ψ vanishes along $\mathbb{V}(z)$. In particular, $g_j(0, t_1, t_2, ...) = 0$. Taking derivatives in z of (3.4), we get $\frac{\partial^k g_j}{\partial z^k} = 0$ along $\mathbb{V}(z)$ for $0 \leq k \leq n$. Thus we can write $g_j = z^{n+1} h_j$, and hence

$$\Phi_* \tau(\partial F_\psi) = z^n \left(f dz + z \sum_{j \geq 1} h_j dt_j \right).$$

We now observe that $\Phi_* \mathcal{F}$ coincides with $\ker \omega$ away from $\mathbb{V}(z)$, where

$$\omega = f dz + z \sum_{j \geq 1} h_j dt_j.$$

We will show that the distribution defined by $\ker\omega$ is integrable in $\Phi(U)$, which implies that $\mathcal{F}$ extends over $\mathcal{A}_d \setminus (\Delta' \cup \Delta'')$, and that moreover every connected component of $\mathcal{A}_d \setminus (\Delta' \cup \Delta'')$ is a leaf.

Note that

$$\begin{aligned} d\omega = d(z^{-n}\Phi_*\tau(\partial F_\psi)) &= -\Gamma \wedge \omega - n\frac{dz}{z} \wedge \omega \\ &= -\Gamma \wedge \omega - n\frac{\omega - z\sum_{j\geq 1} h_j dt_j}{zf} \wedge \omega \\ &= \left(\sum_{j\geq 1} \frac{h_j}{f} dt_j - \Gamma\right) \wedge \omega. \end{aligned}$$

By the Frobenius integrability theorem, it follows that $\ker\omega$ is integrable, which concludes the proof of Corollary 3.3 (3).

4. Schiffer varieties

For a representation $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$, let $\Lambda_\rho \subset \mathcal{T}_g$ be the set of marked Riemann surfaces X whose universal cover $\tilde{X}$ admits a ρ-equivariant holomorphic map $\tilde{X} \to \mathbb{H}$. Let $\tilde{\mathcal{V}}_g$ be the universal cover of $\mathcal{V}_g$, and let $\tilde{\mathcal{V}}_{\mathcal{S}}$ be the preimage of $\mathcal{V}_{\mathcal{S}}$ under this universal cover, where $\mathcal{S} \subset \mathcal{T}_g$ is a submanifold. Our main result in this section is the following.

Proposition 4.1. *For any representation $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$ of positive Euler class, the subset $\Lambda_\rho \subset \mathcal{T}_g$ is a locally finite union of connected closed submanifolds of $\mathcal{T}_g$. Moreover, there exists a ρ-equivariant holomorphic map $F : \tilde{\mathcal{V}}_{\Lambda_\rho} \to \mathbb{H}$.*

We use the techniques of [34] to show Proposition 4.1. We now briefly outline the proof of Proposition 4.1. Let $\mathcal{E}(g,k)$ be the moduli space of all harmonic maps $\tilde{X} \to \mathbb{H}$, equivariant with respect to a representation of Euler class k, for a varying marked Riemann surface $X \in \mathcal{T}_g$. This space is naturally a holomorphic fibration over Teichmüller space, that we will describe in §4.1 and show in Appendix A. Let $\mathcal{Z}(g,k) \subset \mathcal{E}(g,k)$ be the subset that consists of holomorphic maps.

Define a distribution $\mathcal{K}$ on $\mathcal{E}(g,k)$ as follows. The non-abelian Hodge correspondence provides a diffeomorphism $\mathrm{NAH} : \mathcal{E}(g,k) \to \mathcal{T}_g \times \chi_{g,k}$. Here $\chi_{g,k}$ denotes the connected component of the representation variety $\pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$ that consists of representations with Euler class k. Now define a distribution $\tilde{\mathcal{K}}$ on $\mathcal{T}_g \times \chi_{g,k}$ as $\tilde{\mathcal{K}}_{(X,\rho)} = \{(\mu, 0) \in T_X\mathcal{T}_g \oplus T_\rho\chi_{g,k} : \Delta_\mu \mathrm{E}_\rho(X) = 0\}$. We then let $\mathcal{K} = \mathrm{NAH}^*\tilde{\mathcal{K}}$. Note that this distribution was previously studied by the author in [34].

We then show that $\mathcal{K}$ restricts to an integrable distribution on $\mathcal{Z}(g,k)$. Let the corresponding foliation of $\mathcal{Z}(g,k)$ be $\mathcal{F}$. Then the connected components of Λ_ρ are the leaves of $\mathrm{NAH}_*\mathcal{F}$ contained in $\mathcal{T}_g \times \{\rho\}$. Local finiteness of this set of leaves then follows easily from the fact that Λ_ρ is a real analytic subset of $\mathcal{T}_g$.

4.1. **Non-abelian Hodge over a moving Riemann surface.** We closely follow [19, §10].

An $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle over a marked Riemann surface $X \in \mathcal{T}_g$ is a triple (L, α, δ), where L is a holomorphic line bundle, $\alpha \in H^0(L^2K_X)$ and $\delta \in H^0(L^{-2}K_X)$. The corresponding Higgs bundle is $E = L \oplus L^{-1}$ with the field

$$\phi = \begin{pmatrix} 0 & \alpha \\ \delta & 0 \end{pmatrix}.$$

Suppose that L is positive. Then this Higgs bundle is stable if and only if $\delta \neq 0$. Note that ϕ is gauge equvialent to $\begin{pmatrix} 0 & c\alpha \\ c^{-1}\delta & 0 \end{pmatrix}$ for any $c \in \mathbb{C} \setminus \{0\}$. Thus the map $(L, \alpha, \delta) \to (\mathrm{div}(\delta), \alpha\delta)$ provides an isomorphism between the moduli space of Higgs bundles over X of Euler class $k = \deg L^2$, and the space of $(D, \Phi) \subset \mathrm{Sym}^{2g-2-k}X \times H^0(K_X^2)$ where $\mathrm{div}(\Phi) \geq D$. This was shown by Hitchin in [19, §10].

This construction carries over to the case of a moving Riemann surface without much difficulty. In this section we state our most general result, and then we prove it for the sake of completeness in Appendix A.

Recall that $\pi : \mathcal{V}_g \to \mathcal{T}_g$ is the universal curve over Teichmüller space $\mathcal{T}_g$, so that the fibre over $X \in \mathcal{T}_g$ is isomorphic to X. Define $\mathrm{Sym}^n\pi : \mathrm{Sym}^n\mathcal{V}_g \to \mathcal{T}_g$ to be the bundle over $\mathcal{T}_g$, whose fibre over X is $\mathrm{Sym}^n X$. We also let $\mathcal{H}_g$ be the vector bundle over $\mathcal{T}_g$, such that the fibre over $X \in \mathcal{T}_g$ naturally parameterizes $H^0(K_X^2)$. This bundle is the π-pushforward of the square of the relative canonical bundle $K_\pi \to \mathcal{V}_g$.

Definition 4.2. The space $\mathcal{E}(g,k)$ is the subhseaf of $(\mathrm{Sym}^n\pi)^*\mathcal{H}_g$ whose fibre over $X \in \mathcal{T}_g$ consists of pairs $(D, \Phi) \in \mathrm{Sym}^{2g-2-k}X \times H^0(K_X^2)$ such that $\mathrm{div}(\Phi) \geq D$.

Using the Riemann–Roch theorem, it is easy to see that the subsheaf $\mathcal{E}(g,k)$ has constant rank, and is thus a vector bundle. Alternatively, in Appendix A we will give a different construction of $\mathcal{E}(g,k)$ in terms of the more classical operations in complex geometry that shows immediately that it is a vector bundle.

The non-abelian Hodge correspondence provides a diffeomorphism between the fibre $\mathcal{E}(g,k)_X$ and $\chi_{g,k}$, that varies smoothly with the underlying Riemann surface $X \in \mathcal{T}_g$. We state this as Theorem 5 below, whose proof we include in Appendix A, for completeness.

Theorem 5. *The non-abelian Hodge correspondence provides a diffeomorphism* $\mathrm{NAH} : \mathcal{E}(g,k) \to \mathcal{T}_g \times \chi_{g,k}$.

4.2. **Foliation.** The following lemma easily implies Proposition 4.1. Recall that $\mathcal{K} = \mathrm{NAH}^*\tilde{\mathcal{K}}$, where $\tilde{\mathcal{K}}$ is the distribution on $\mathcal{T}_g \times \chi_{g,k}$, defined on the slice $\mathcal{T}_g \times \{\rho\}$ as the kernel of the Levi form of E_ρ. The distribution $\mathcal{K}$ is smooth by [34, §4] and Theorem 5.

Lemma 4.3. *The zero section $\mathcal{Z}(g,k)$ in $\mathcal{E}(g,k)$ admits a foliation $\mathcal{F}$ with $T\mathcal{F}|_{\mathcal{Z}(g,k)} = \mathcal{K}|_{\mathcal{Z}(g,k)}$. The leaves of $\mathcal{F}$ take the form $\mathrm{NAH}^{-1}(\Lambda \times \{\rho\})$, where $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$ is a non-elementray representation and $\Lambda \subset \mathcal{T}_g$ is a complex submanifold. For any such leaf, there exists a ρ-equivariant holomorphic map $F : \tilde{\mathcal{V}}_\Lambda \to \mathbb{H}$.*

Note that Lemma 4.3 clearly implies Proposition 4.1, apart from the claim about local finiteness. That follows from the following claim.

Claim 4.4. For any $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$, the set Λ_ρ is a real analytic subset of $\mathcal{T}_g$.

Proof. Fix an $X \in \mathcal{T}_g$, and let $f : \tilde{X} \to \mathbb{H}$ be the ρ-equivariant harmonic map. Denote by $\mathrm{eu}(\rho)$ the Euler class of a representation ρ. Observe that

$$\begin{aligned}\mathrm{E}_\rho(X) - 2\pi\mathrm{eu}(\rho) &= \int_X \left(|\partial f|^2 + \left|\bar{\partial} f\right|^2\right) d\mathrm{area}_X - \int_X \left(|\partial f|^2 - \left|\bar{\partial} f\right|^2\right) d\mathrm{area}_X \\ &= 2\int_X \left|\bar{\partial} f\right|^2 d\mathrm{area}_X.\end{aligned}$$

Hence Λ_ρ is precisely $\mathrm{E}_\rho^{-1}(2\pi\mathrm{eu}(\rho))$, and is thus real analytic as E_ρ is real analytic. □

The rest of this section is devoted to proving Lemma 4.3. We split the proof into three steps, that we show in the following three subsections. We first show that $\mathcal{K}$ is tangent to $\mathcal{Z}(g,k)$ in §4.3. Then we show that $\mathcal{K}$ is integrable over $\mathcal{Z}(g,k)$ in §4.4, and finally we show in §4.5 that the the ρ-equivariant harmonic maps are holomorphic over the parts of $\tilde{\mathcal{V}}_g$ that project to leaves of $\mathcal{F}$.

4.3. **Tangency to the zero section.** The following is a simple consequence of the computations in [34, §3.1].

Lemma 4.5. *Let $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$ be a non-elementary representation. Suppose that $S_0 \in \mathcal{T}_g$ and $f_0 : \tilde{S}_0 \to \mathbb{H}$ is a holomorphic ρ-equivariant map from the universal cover $\tilde{S}_0$ of S_0, and let*

$((S_t, f_t) : t \in \mathbb{D})$ be a complex disk of ρ-equivariant harmonic maps based at S_0 with direction $\mu \in T_{S_0}\mathcal{T}_g$ (see [34, Definition 1.7]*). Assume further that*

$$\mu\partial f = \bar{\partial}\xi \text{ and } \xi \wedge \partial f = 0,$$

for some section ξ of the bundle $f_0^*T\mathbb{H} \otimes \mathbb{C}$. Then if we denote by Φ_t the Hopf differential of f_t, we have $\Phi_t = O\left(|t|^2\right)$ for small t.

Proof. From [34, Theorem 1.9], we see that $\eta = \frac{\partial f_t}{\partial t}\Big|_{t=0} - \xi$ is a parallel section such that $\eta \wedge df_0 = 0$. However, since ρ is non-elementary, df_0 has rank 2 at some point. Therefore η vanishes at this point, and since it is parallel, η is identically zero. Thus $\xi = \frac{\partial f_t}{\partial t}\Big|_{t=0}$.

Now observe that $\Phi_t = \langle \partial f_t \otimes \partial f_t \rangle$, where $\langle \cdot, \cdot \rangle$ is the complex bilinear extension of the pullback metric on $f_t^*T\mathbb{H} \otimes \mathbb{C}$. Therefore we have, setting $t = x + iy$, in the setting of [34, §3.1],

$$\begin{aligned}
\frac{\partial \Phi_t}{\partial x} &= \frac{\partial}{\partial x}\left\langle \left(df_t \circ \frac{\mathrm{id} - iJ_t}{2}\right) \otimes \left(df_t \circ \frac{\mathrm{id} - iJ_t}{2}\right)\right\rangle \\
&= 2\left\langle (2\partial \mathrm{Re}(\xi) + \mu\partial f_0 - \bar{\mu}\bar{\partial} f_0) \otimes \partial f_0 \right\rangle \\
&= 2\left\langle (\partial\xi + \partial\bar{\xi} + \bar{\partial}\xi - \partial\bar{\xi}) \otimes \partial f_0 \right\rangle = 2\langle d\xi \otimes \partial f_0 \rangle \\
&= 2\mu\Phi_0 + 2\langle \partial\xi \otimes \partial f_0 \rangle.
\end{aligned}$$

Note that since f_0 is holomorphic, we have $\Phi_0 = 0$. Moreover, since $\xi \wedge \partial f_0 = 0$ and f_0 is holomorphic, it follows that ξ takes values in $f_0^*T^{1,0}\mathbb{H}$. Since f_0 is holomorphic, it follows that $f_0^*T\mathbb{H} \otimes \mathbb{C} = f_0^*T^{1,0}\mathbb{H} \oplus f_0^*T^{0,1}\mathbb{H}$, and in particular the second fundamental form of $f_0^*T^{1,0}\mathbb{H} \leq f_0^*T\mathbb{H} \otimes \mathbb{C}$ vanishes. Therefore $\partial\xi$ also takes values in $f_0^*T^{1,0}\mathbb{H}$, and hence $\langle \partial\xi \otimes \partial f_0 \rangle = 0$. Thus $\frac{\partial \Phi_t}{\partial x} = 0$, and an identical argument shows that $\frac{\partial \Phi_t}{\partial y} = 0$, concluding the proof. □

4.4. **Integrability.** By [34, Theorem 1.1], the distribution $\mathcal{K}$ is precisely $\mathrm{NAH}^* \ker d\mathcal{R}$, where $\mathcal{R} : \mathcal{T}_g \times \chi_{g,k} \to \chi_{g,k}$ is a certain map defined in [34, pp. 2]. In particular, if we show that $\mathcal{K}$ has constant rank on $\mathcal{Z}(g,k)$, it follows immediately from Frobenius integrability theorem that $\mathcal{K}$ is integrable.

We only show this in the case of even Euler class. Odd Euler class can be handled as in [4, §4] or §A.4. Let (E, Φ) be the stable degree 0 Higgs bundle over $X \in \mathcal{T}_g$ that corresponds to some point $p \in \mathcal{Z}$. From [19, §10], it follows that $E = L \oplus L^{-1}$ for some holomorphic line bundle L over X, such that

$$\Phi = \begin{pmatrix} 0 & \phi \\ 0 & 0 \end{pmatrix},$$

where $\phi \in H^0(L^2K)$. Here L is a negative bundle with degree that depends only on the Euler class. By [34, Theorem 1.6], $\mathcal{K}_p$ consists of directions μ such that

$$\mu\phi = \bar{\partial}\xi,$$

for some ξ that is a section of L^2. The existence of such a ξ is equivalent to $[\mu\phi]$ vanishing in $H^1(L^2)$. By Serre duality, this is equivalent to $\int_S \mu\phi\theta = 0$ for all $\theta \in H^0(KL^{-2})$. Thus

$$\begin{aligned}\dim \mathcal{K}_p &= 3g-3-\dim \phi \otimes H^0(KL^{-2}) = 3g-3-h^0(KL^{-2})\\ &= 3g-3-(h^0(L^2)-\deg L^2+g-1) = 2g-2+\deg L^2\\ &= 2g-2-k,\end{aligned}$$

which is independent of the point in $\mathcal{Z}(g,k)$.

4.5. **Holomorphicity of the combined map.** Let $F : \tilde{\mathcal{V}}_\Lambda \to \mathbb{H}$ be the map which is harmonic and ρ-equivariant on the fibres of $\tilde{\mathcal{V}}_g \to \mathcal{T}_g$. By construction, F is holomorphic on the fibres of $\tilde{\mathcal{V}}_g \to \mathcal{T}_g$. By Proposition 3.1, we see that F is pluriharmonic.

We now work in local coordinates $(z, t_1, t_2, ..., t_n)$ on $\tilde{\mathcal{V}}_\Lambda$ and $(t_1, t_2, ..., t_n)$ on Λ, such that $\tilde{\mathcal{V}}_g \to \mathcal{T}_g$ is locally modelled on $(z, t_1, t_2, ..., t_n) \to (t_1, t_2, ..., t_n)$. It follows from Proposition 3.1 that

$$F_z \wedge F_{t_i} = 0. \tag{4.1}$$

However, since $\langle F_z \otimes F_z\rangle = 0$ by conformality, it follows that $\langle F_{t_i} \otimes F_{t_j}\rangle = 0$ and $\langle F_{t_i} \otimes F_z\rangle = 0$ for any $1 \le i, j \le n$, since by (4.1), F_z and F_{t_i} are colinear. Therefore F is holomorphic when restricted to any complex disk in $\tilde{\mathcal{V}}_\Lambda$, and hence F is holomorphic.

5. Proof of Theorem 1

In this section, we prove Theorem 1. Therefore let $M, \psi, \mathcal{S}$ be as in Theorem 1, and $\Delta, \mathcal{F}$ as in Corollary 3.3.

Recall from the outline that, to show Theorem 1, we define an equivalence relation $\sim$ on X_t, such that the quotient map $q_t : X_t \to Y_t := X_t/\sim$ is a non-constant holomorphic map between Riemann surfaces. We then show that the Riemann surfaces Y_t can be taken to be isomorphic over $\mathcal{S}$.

The organization of this section is as follows. In §5.1, we show some preliminary facts on real analytic maps, that will be used later to show properties of $\sim$. We then define $\sim$ in §5.2 and use the reuslts of §5.1 to show that the quotient $X_t/\sim$ carries a natural structure of a closed Riemann surface. In §5.3, we show the semicontinuity of the family $\{Y_t : t \in \mathcal{S}\}$. More precisely, we show in Lemma 5.5 that for any $t \in \mathcal{S}$, there exists a neighbourhood $t \in \mathcal{U} \subset \mathcal{S}$ such that, for any $s \in \mathcal{U}$, there exists a non-constant holomorphic map $Y_t \to Y_s$. Using this observation, we finish the proof of Theorem 1 in §5.4.

5.1. **Real analytic maps.** Recall that the definition of $\sim$ over X_t is, roughly, two points $x, y \in X_t$ have $x \sim y$, if and only if there exists a biholomorphism $\phi : U_x \to U_y$ between neighbourhoods $U_x \ni x, U_y \ni y$ in X_t, such that $F_\psi \circ \phi = F_\psi|_{U_x}$. The main result in this subsection is Lemma 5.1 below, that allows us to show that accumulation points of $\sim \subset X_t \times X_t$ also lie in $\sim$, under suitable conditions on the limit point. The proof of this lemma also shows a uniqueness result in Corollary 5.2, that we will make use of in the sequel.

Lemma 5.1. *Let $F, G : \mathbb{D} \to \mathbb{R}^N$ be real analytic maps with $\frac{\partial G}{\partial z}(0) \neq 0$. Suppose that $x_1, x_2, \ldots \in \mathbb{D}$ is a sequence of points with connected neighbourhoods $U_n \ni x_n$ and holomorphic maps $\phi_n : U_n \to \mathbb{D}$, such that*

(1) $x_n \to 0$ and $\phi_n(x_n) \to 0$, and
(2) $G \circ \phi_n = F|_{U_n}$.

Then there exists a connected neighbourhood $U \ni 0$, and a holomorphic map $\phi : U \to \mathbb{D}$ with $\phi(0) = 0$, such that $G \circ \phi = F|_U$. If $\frac{\partial F}{\partial z}(0) \neq 0$, then we can take ϕ to be injective. Finally, ϕ and ϕ_n agree on $U \cap U_n$.

Proof. We will show that there exists a unique power series ϕ at 0, with a positive radius of convergence, such that $F \circ \phi = G$. We first briefly observe that $G(x_n) = F \circ \phi_n(x_n)$, so after taking limits we see that $F(0) = G(0)$.

We differentiate $F = G \circ \phi_n$ and apply Faà di Bruno's formula to get

$$\frac{\partial^{i+j} F}{\partial z^i \partial \bar{z}^j} = \sum_{\pi_1 \in \Pi(i), \pi_2 \in \Pi(j)} \frac{\partial^{|\pi_1|+|\pi_2|} G}{\partial z^{|\pi_1|} \partial \bar{z}^{|\pi_2|}} \prod_{b \in \pi_1} \frac{\partial^b \phi_n}{\partial z^b} \overline{\prod_{b \in \pi_2} \frac{\partial^b \phi_n}{\partial \bar{z}^b}}. \tag{5.1}$$

Here $\Pi(n)$ denotes the set of all partitions of the non-negative integer n. Partitions here are represented as multisets of integers (e.g. the partition $6 = 1+1+2+2$ is represented as $\{1, 1, 2, 2\}$). Set $y_n = \phi_n(x_n)$.

We now evaluate (5.1) for $j = 0$ at x_n, to obtain

$$\begin{aligned}\frac{\partial^i F}{\partial z^i}(x_n) &= \sum_{\pi\in\Pi(i)} \frac{\partial^{|\pi|}G}{\partial z^{|\pi|}}(y_n)\prod_{b\in\pi}\frac{\partial^b\phi_n}{\partial z^b}(x_n)\\ &= \frac{\partial G}{\partial z}(y_n)\frac{\partial^i\phi_n}{\partial z^i}(x_n) + \text{lower-order derivatives of } \phi_n.\end{aligned}$$

Since $y_n \to 0$ and $\frac{\partial G}{\partial z}(0) \neq 0$ by assumption, it follows by induction on $i \geq 0$ that $\frac{\partial^i\phi_n}{\partial z^i}(x_n) \to \tilde{\phi}_i$. We remark here that an identical argument with Faà di Bruno's formula applied to $G\circ\phi = F$ shows the uniqueness of ϕ (provided it exists). Taking the limit $n \to \infty$ in (5.1) evaluated at x_n, we see that

$$\frac{\partial^{i+j}F}{\partial z^i\partial\bar{z}^j}(0) = \sum_{\pi_1\in\Pi(i),\pi_2\in\Pi(j)} \frac{\partial^{|\pi_1|+|\pi_2|}G}{\partial z^{|\pi_1|}\partial\bar{z}^{|\pi_2|}}(0)\prod_{b\in\pi_1}\tilde{\phi}_b\overline{\prod_{b\in\pi_2}\tilde{\phi}_b}.$$

It follows that, if we let $\phi(z) = \sum_{i\geq 0}\frac{\tilde{\phi}_i}{i!}z^i$ be a formal power series, we have the formal equality $F = G\circ\phi$. Observe that $\tilde{\phi}_1 = \frac{F_z(0)}{G_z(0)}$ vanishes if and only if $F_z := \frac{\partial F}{\partial z}(0)$ vanishes, so the claim on injectivity of ϕ follows automatically, after possibly shrinking U. We now show that ϕ has a positive radius of convergence.

Since $\frac{\partial G}{\partial z}(0) \neq 0$, choose a component $k \in \{1, 2, ..., N\}$ such that $\frac{\partial G_k}{\partial z}(0) \neq 0$. Since $G_k : \mathbb{D} \to \mathbb{R}$ is a real analytic function, in some neighbourhood U of the origin, we can write $G_k(z) = A(z) + \bar{z}B(z)$, where $A : U \to \mathbb{C}$ is a holomorphic function, and $B : U \to \mathbb{C}$ is real analytic. We similarly write $F_k(z) = C(z) + \bar{z}D(z)$. By comparing power series, from $F_k = G_k\circ\phi$, we have another formal equality

$$C = A\circ\phi.$$

Now observe that, since $\frac{\partial G_k}{\partial z}(0) \neq 0$, we have $A'(0) \neq 0$. Moreover, since $F(0) = G(0)$, we have $C(0) = A(0)$. Therefore, possibly after shrinking U, the map A is a biholomorphism from some neighbourhood $V \ni 0$ to $C(U)$. Thus we can define a surjective holomorphic map $\hat{\phi} : U \to V$ by $\hat{\phi} = A^{-1}\circ C$. Note that ϕ is the power series expansion of $\hat{\phi}$, and hence has a positive radius of convergence.

Since $G\circ\phi_n = F|_{U_n}$, we have by an identical argument $A\circ\phi_n = C|_{U_n\cap U} = A\circ\phi$. Therefore ϕ_n and ϕ agree on $U_n\cap U$. □

Note that the proof of Lemma 5.1 immediately implies the following corollary.

Corollary 5.2. *Let $F : \mathbb{D} \to \mathbb{R}^N$ be a real analytic map with $\frac{\partial F}{\partial z}(0) \neq 0$. Suppose that $\phi_1 : U \to \mathbb{D}$ and $\phi_2 : V \to \mathbb{D}$ are holomorphic maps with*

$\phi_i(0) = 0$ *defined on neighbourhoods* U, V *of the origin* $0 \in \mathbb{C}$*, such that* $F \circ \phi_1 = F \circ \phi_2$*. Then* $\phi_1 = \phi_2$*.*

Proof. Apply Faà di Bruno's formula as in the proof of Lemma 5.1 to conclude that ϕ_1, ϕ_2 agree to infinite order at the origin. □

5.2. **Holomorphic equivalence relation.** We now construct a holomorphic map $X_t \to Y_t$ for $t \in \mathcal{S}^\circ$, where Y_t is a family of closed Riemann surfaces.

Fix an arbitrary $t \in \mathcal{S}^\circ = \mathcal{S} \setminus \pi(\Delta)$. We set $X_t^\circ = X_t \setminus \mathbb{V}(\partial^{\text{vert.}} F_\psi)$, and

(5.2)

$$X_t^\bullet = X_t^\circ \cup \left\{ z \in \mathbb{V}(\partial^{\text{vert.}} F_\psi) : \begin{array}{c} \text{there exist a neighbourhood } V \ni z \\ \text{and a holomorphic map } \phi : V \to X_t^\circ \\ \text{with } F_\psi \circ \phi = F_\psi|_V \end{array} \right\}.$$

Definition 5.3. We define an equivalence relation $\sim$ on $X_t^\bullet$ as follows. For $z, w \in X_t^\circ$, we let $z \sim w$ if there exist neighbourhoods U_z of z and U_w of w in X_t°, and a biholomorphism $\phi : U_z \to U_w$ such that $\phi(z) = w$ and $F_\psi \circ \phi = F_\psi$ on U_z. For $z \in X_t^\bullet \setminus X_t$ and $w \in X_t^\circ$, we let $z \sim w$ if there exists a map ϕ as in (5.2) such that $\phi(z) = w$.

We define $Y_t^\bullet := X_t^\bullet / \sim$ as a topological space, and let $q_t : X_t^\bullet \to Y_t^\bullet$ be the natural projection. We show that $Y_t^\bullet$ is a Riemann surface wtih $q_t : X_t^\bullet \to Y_t^\bullet$ holomorphic in two steps.

Step 1. The equivalence classes are finite. Suppose to the contrary that $x_1, x_2, ... \in X_t^\bullet$ is a sequence of distinct points such that $x_1 \sim x_n$ for all $n \geq 2$. Since $X_t^\bullet \setminus X_t^\circ$ is finite, suppose without loss of generality that $x_n \in X_t^\circ$. We pass to a subsequence so that $x_n \to x \in X_t$. Let U_n be a neighbourhood of x_n, and let $\phi_n : U_n \to X_t$ be injective holomorphic maps with $\phi_n(x_n) = x_1$, such that $F_\psi \circ \phi_n = F_\psi|_{U_n}$. By Lemma 5.1, there exists a neighbourhood $U \ni x$ and a holomorphic map $\phi : U \to X_t$ with $\phi(x) = x_1$, such that $F_\psi \circ \phi = F_\psi|_U$ and $\phi_n|_{U_n \cap U} = \phi|_{U_n \cap U}$. But then $\phi(x_n) = \phi_n(x_n) = x_1$. Thus since x_n are distinct points that accumulate at x, it follows that ϕ is constant. Thus F_ψ is constant on U, and hence by real analyticity on all of X_t, which is a contradiction.

Step 2. q_t is a ramified covering map. Let $\{x_1, x_2, ..., x_k, y_1, y_2, ..., y_\ell\}$ be a $\sim$-equivalence class, such that $y_i \in X_t^\bullet \setminus X_t^\circ$ and $x_i \in X_t^\circ$. Here we allow $\ell = 0$, but note by definition of $X_t^\bullet$ that $k \geq 1$. Let $U_i \ni x_i$ and $V_i \ni y_i$ be neighbourhoods such that $\phi_i : U_i \to U_1$ are biholomorphisms with $\phi_i(x_i) = x_1$, and $\theta_i : V_i \to U_1$ are holomorphic maps with

$\theta_i(y_i) = x_1$, such that $F_\psi|_{U_i} = F_\psi \circ \phi_i$ and $F_\psi|_{V_i} = F_\psi \circ \theta_i$. Note that $\phi_1 = \mathrm{id}_{U_1}$.

Claim 5.4. After possibly shrinking U_i, V_i, for any $x \in U_1$, the equivalence class of x consists of $\{\phi_i^{-1}(x) : i = 1, 2, ..., k\} \cup \bigcup_{i=1}^{\ell} \theta_i^{-1}(x)$.

This shows in particular that the map

$$Q = \bigcup_{i=1}^{k} \phi_i \cup \bigcup_{i=1}^{\ell} \theta_i : \bigcup_{i=1}^{k} U_i \cup \bigcup_{i=1}^{\ell} V_i \to U_1$$

is a local model for the map q_t, that is $q_t = \xi \circ Q$, where $\xi : U_1 \to Y_t^\bullet$ is a homeomorphism to a neighbourhood of $q_t(x_1)$ in $Y_t^\bullet$. Then q_t is a ramified covering map to the topological surface $Y_t^\bullet$. Moreover, since Q is holomorphic, it follows that $Y_t^\bullet$ admits the structure of a Riemann surface that makes $q_t : X_t^\bullet \to Y_t^\bullet$ into a holomorphic map. In this case, since $X_t^\bullet$ is a finite type boundaryless Riemann surface, so is $Y_t^\bullet$. Denote by Y_t the closed Riemann surface resulting from filling in the punctures of $Y_t^\bullet$. The map q_t extends to a holomorphic map $q_t : X_t \to Y_t$.

Note that by Definition 5.3, there exists a map $h_t : Y_t \to M$ such that $F_\psi|_{X_t} = h_t \circ q_t$. Since $q_t, F_\psi|_{X_t}$ are holomorphic and harmonic, respectively, it follows that h_t is harmonic.

Proof of Claim 5.4. Suppose that there is a sequence $z_1, z_2, ...$ converging to x_1, such that $z_n \sim w_n$ for some $w_n \in X_t^\bullet \setminus \{\phi_i^{-1}(z_n) : i = 1, 2, ..., k\} \cup \bigcup_{i=1}^{\ell} \theta_i^{-1}(z_n)$. After passing to a subsequence, we can assume that $w_n \to w \in X_t$. Let W_n be a neighbourhood of w_n and $\eta_n : W_n \to U_1$ be holomorphic maps with $\eta_n(w_n) = z_n$, such that $F_\psi \circ \eta_n = F_\psi|_{W_n}$. Since $\partial F_\psi(x_1) \neq 0$, by Lemma 5.1, there exists a neighbourhood $W \ni w$ and a holomorphic map $\eta : W \to X_t^\circ$ with $\eta(w) = x_1$ and $F_\psi|_W = F_\psi \circ \eta$. But then by definition, we get $w \in X_t^\bullet$ and $w \sim x_1$. Therefore $w \in \{x_1, x_2, ..., x_k, y_1, y_2, ..., y_\ell\}$, and Corollary 5.2 η is equal to one of ϕ_i or θ_i. Since $\eta(z_n) = \eta_n(z_n) = w_n$, it follows that $w_n \in \{\phi_i^{-1}(z_n) : i = 1, 2, ..., k\} \cup \bigcup_{i=1}^{\ell} \theta_i^{-1}(z_n)$ for all n large enough, which is a contradiction. $\square$

5.3. **Semicontinuity of Y_t.** Our main result here is the following.

Lemma 5.5. *For any $t \in \mathcal{S}^\circ$, there exists a neighbourhood $\mathcal{U} \ni t$ in $\mathcal{S}^\circ$, such that for any $s \in \mathcal{U}$, there is a non-constant holomorphic map $\tau_{t,s} : Y_t \to Y_s$ with $h_t = h_s \circ \tau_{t,s}$.*

Proof. Fix a $t \in \mathcal{S}^\circ$, and let $U_1, U_2, ..., U_N$ be open subsets of $\mathcal{V}_\mathcal{S}$ that cover X_t, such that there exist holomorphic maps $\phi_i : U_i \to \Lambda_i$, where

Λ_i is a Riemann surface homeomorphic to a disk, and the fibres of ϕ_i are connected open subsets of the leaves of $\mathcal{F}$. Note that F_ψ is constant on the leaves of $\mathcal{F}$, so we can write $F_\psi|_{U_i} = \tilde{h}_i \circ \phi_i$.

Let $\mathcal{U} \subset \mathcal{S}^\circ$ be a neighbourhood of t, such that $\mathcal{V}_\mathcal{U} \subseteq \bigcup_{i=1}^N U_i$ and $\phi_i(U_i \cap X_s) = \Lambda_i$ for all $s \in \mathcal{U}$. For $s \in \mathcal{U}$, let $D_s \subset X_s$ be the finite set that consists of the points $z \in X_s$ where ϕ_i is not an immersion at z, for some $i = 1, 2, ..., N$.

Claim 5.6. For any $s \in \mathcal{U}$ and $i \in \{1, 2, ..., N\}$, there exists a holomorphic map $\xi_i : \Lambda_i \to Y_s$ such that $q_s = \xi_i \circ \phi_i$ on $X_s \cap U_i$ and $h_s \circ \xi_i = \tilde{h}_i$ on Λ_i.

Proof. Suppose $a, b \in U_i \cap X_s \setminus D_s$ are such that $\phi_i(a) = \phi_i(b)$. Then since ϕ_i is a local biholomorphism near a, b, there exist neigbhourhoods $A \ni a, B \ni b$ in $X_s \cap U_i$, and a biholomorphism $\eta : A \to B$ such that $\phi \circ \eta = \phi|_A$.

Since F_ψ is constant on the leaves of $\mathcal{F}$, there exists a harmonic map $h_i : \Lambda_i \to M$ such that $F_\psi|_{U_i} = h_i \circ \phi_i$. Then $F_\psi \circ \eta = F_\psi|_A$, and hence $a \sim b$. Therefore $q_s(a) = q_s(b)$.

It follows that there exists a holomorphic map $\xi_i : \phi_i(U_i \cap X_s \setminus D_s) \to Y_s$ such that $q_s|_{U_i \cap X_s \setminus D_s} = \xi_i \circ \phi_i|_{U_i \cap X_s \setminus D_s}$. Since $\phi_i(D_s \cap U_i)$ is finite and $\phi_i(U_i \cap X_s) = \Lambda_i$, the map ξ_i extends to Λ_i and the factorization $q_s = \xi_i \circ \phi_i$ extends over D_s by continuity.

Observe that $h_s \circ \xi_i \circ \phi_i = h_s \circ q_s = F_\psi = \tilde{h}_i \circ \phi_i$. Since ϕ_i is non-constant, it follows that $h_s \circ \xi_i = \tilde{h}_i$ on some open set, so by real analyticity $h_s \circ \xi = \tilde{h}_i$ on all of Λ_i. $\square$

From Claim 5.6, it follows that we have the following commutative diagram for any $s \in \mathcal{U}$,

$$\begin{tikzcd} U_i \cap X_s \arrow[r, "\phi_i"] \arrow[d, "q_s"] & \Lambda_i \arrow[d, "\tilde{h}_i"] \arrow[dl, "\xi_i"] \\ Y_s \arrow[r, "h_s"] & M \end{tikzcd}$$

Denote $X_t^- = X_t^\circ \setminus D_t$, and let $s \in \mathcal{U}$ be arbitrary. Let ξ_i be as in Claim 5.6. Define $f : X_t^- \to Y_s$ by $f|_{U_i \cap X_t^-} = \xi_i \circ \phi_i|_{U_i \cap X_t^-}$.

Claim 5.7. The map f is well-defined and constant on the fibres of q_t.

Proof. We prove both claims simultaneously. Suppose $a \in X_t^- \cap U_i, b \in X_t^- \cap U_j$ have neighbourhoods $A \ni a, B \ni b$ in X_t^-, and a biholomorphism $\eta : A \to B$ such that $F_\psi \circ \eta = F_\psi|_A$. We will show that $\xi_i(\phi_i(a)) = \xi_j(\phi_j(b))$. Note that

(1) when $a = b$ and $\eta = \mathrm{id}$, this shows that f is well-defined, and

(2) when $a \sim b$, this shows that f is constant on the fibres of q_t.

Since $a, b \notin D_t$, the corresponding maps $\phi_i : A \to \Lambda_i$ and $\phi_j : B \to \Lambda_j$ are local biholomorphisms, where we shrink A, B without mention as necessary. Thus there exists a holomorphic map $\bar{\eta} : \phi_i(A) \to \phi_j(B)$ such that $\bar{\eta} \circ \phi_i = \phi_j \circ \eta$. In particular, $\bar{\eta}(\phi_i(a)) = \phi_j(b)$.

We have

$$\tilde{h}_j \circ \bar{\eta} \circ \phi_i = F_\psi \circ \eta = F_\psi = \tilde{h}_i \circ \phi_i$$

on A. Thus, since ϕ_i is a local biholomorphism, we have $\tilde{h}_j \circ \bar{\eta} = \tilde{h}_i|_{\phi_i(A)}$. Therefore $h_s \circ \xi_j \circ \bar{\eta} = h_s \circ \xi_i$ on $\phi_i(A) \subseteq \Lambda_i$. By Corollary 5.2, it follows that $\xi_j \circ \bar{\eta} = \xi_i$ on some open neighbourhood of $\phi_i(a)$. In particular, $\xi_j(\phi_j(b)) = \xi_i(\phi_i(a))$, and the result is shown. □

We now finish the proof of Lemma 5.5. Note that f extends to a holomorphic map $f : X_t \to Y_s$ that is also constant on the fibres of q_t. Therefore $f = \tau_{t,s} \circ q_t$, where $\tau_{t,s} : Y_t \to Y_s$ is the desired holomorphic map. Note that $h_s \circ \tau_{t,s} \circ q_t = h_s \circ \xi_i \circ \phi_i = \tilde{h}_i \circ \phi_i = F_\psi = h_t \circ q_t$, so since $q_t : X_t \to Y_t$ is surjective, we see that $h_s \circ \tau_{t,s} = h_t$ as claimed. □

5.4. **Finishing the proof.** Denote by $g(t)$ the genus of Y_t for $t \in \mathcal{S}^\circ$. Since $\tau_{t,s}$ is a non-constant holomorphic map $Y_t \to Y_s$ for any s in some neighbourhood $\mathcal{U}$ of t, it follows that $g(t) \geq g(s)$. Therefore

$$g(t) \geq \limsup_{s \to t} g(s).$$

Hence $g : \mathcal{S}^\circ \to \mathbb{Z}_{\geq 0}$ is an upper semicontinuous map. Let g_0 be the minimum of g. Thus there is a connected open subset $\mathcal{U} \subset \mathcal{S}^\circ$ such that $g(t) = g_0$ for $t \in \mathcal{U}$. Now fix $t \in \mathcal{U}$. After shrinking $\mathcal{U}$, we can assume that for all $s \in \mathcal{U}$, the map $\tau_{t,s} : Y_t \to Y_s$ is a holomorphic map between closed Riemann surfaces of the same genus, so by Riemann–Hurwitz it is an isomorphism.

Note now that for any $s \in \mathcal{U}$, the Riemann surface X_s admits a non-constant holomorphic map $\tau_{t,s}^{-1} \circ q_s : X_s \to Y := Y_t$. Hence $\mathcal{U}$ is covered by $\bigcup_{[p] \in [\Sigma_g, Y]} \Lambda_{Y,p}$. Here $[\Sigma_g, Y]$ denotes the set of free homotopy classes of maps $\Sigma_g \to Y$. Thus, by the Baire category theorem [20, Theorem 34, pp. 200], for some homotopy class $p : \Sigma_g \to Y$, the set $\mathcal{U} \subset \Lambda_{Y,p}$, after possibly shrinking $\mathcal{U}$ to a smaller open set. Since $\mathcal{S}$ is connected and $\Lambda_{Y,p}$ is closed, it follows that $\mathcal{S} \subseteq \Lambda_{Y,p}$.

Finally, fix an arbitrary $t \in \mathcal{U}$, and observe that since $q_t \simeq p$, it follows that $\psi \simeq h_t \circ p$.

6. Factorization results

This section is devoted to the proof of Theorem 3, and is organized as follows. In §6.1, we pass from X to its resolution of sinuglarities X' using Hironaka's results [17, 18], and apply the Siu–Sampson theorem [30, 28] to the lift of f to X'. In §6.2 we construct a diagram of holomorphic maps

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \Big\downarrow{\scriptstyle \phi} & & \\ X' & & \end{array}$$

where the fibres of p are closed connected Riemann surfaces, and ϕ is a proper dominant holomorphic map. Then by the conclusion of the Siu–Sampson theorem, and Proposition 3.1 show that the energy of the lift of f over $p^{-1}(b)$ is constant as a function of $b \in B$. In §6.3, we apply Theorem 2 to finish the proof of Theorem 3. In §6.4 and §6.5 we show two technical claims made in §6.3.

6.1. **Siu–Sampson argument.** Let $f : X \to M$ be as in Theorem 3. Let $\rho : X' \to X$ be a bimeromorphic map, where X' is a smooth projective variety. Such a map exists by Hironaka's results on resolution of singularities [17, 18] (the reader can also consult [21, Corollary 3.22]). Since ϕ is projective, it is also a Kähler morphism [3], and hence X' admits the structure of a compact Kähler manifold as in [35, Proposition 1.3.1 (vi)]. Moreover, for this Kähler structure, the map $f \circ \rho$ is harmonic on $X' \setminus \rho^{-1}(X^{\text{sing.}})$. Here $X^{\text{sing.}}$ is the singular locus of X.

Note that the real codimension of $\rho^{-1}(X^{\text{sing.}})$ in X' is at least two, so the relative 2-capacity vanishes:

$$\operatorname{cap}_2\left(\rho^{-1}(X^{\text{sing.}})\right) = 0.$$

By a result of Meier [26, Theorem 1], it follows that $f \circ \rho$ extends to a harmonic map $f' : X' \to M$. Applying Sampson's theorem [27] (see also [23]), we see that f' is pluriharmonic and

$$\operatorname{rank}_{\mathbb{C}}(Df'(T^{1,0}X')) \leq 1.$$

6.2. **Constructing a cover by curves.** Suppose that $X' \subseteq \mathbb{P}^N$ is a complex submanifold of N-dimensional projective space $\mathbb{P}^N$. Set $d = \dim_{\mathbb{C}} X'$.

We denote by $\operatorname{Gr}(k, n)$ the Grassmanian variety parameterizing k-dimensonal subspaces of $\mathbb{C}^n$. Recall that the tautological bundle

$$\mathcal{O}_{\mathbb{P}^N}(-1) \to \mathbb{P}^N$$

is a subbundle of the trivial $\mathbb{C}^{N+1}$-bundle over $\mathbb{P}^N$, such that the fibre of $\mathcal{O}_{\mathbb{P}^N}(-1)$ over a line $L \in \mathbb{P}^N$ is precisely $L \leq \mathbb{C}^{N+1}$. We denote by $\mathcal{O}_{X'}(-1)$ the restriction of $\mathcal{O}_{\mathbb{P}^N}(-1)$ to X'.

Define

$$E = \{(x, V) \in X' \times \mathrm{Gr}(N-d+2, N+1) : x \in V\}.$$

Set $B = \mathrm{Gr}(N-d+2, N+1)$, and let $\phi : E \to X'$ be the projection to the first factor, and $p : E \to B$ be the projection to the second factor.

Claim 6.1. The diagram

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle \phi} & & \\ X' & & \end{array}$$

has the following properties

(1) E is a smooth projective variety and ϕ is a proper, dominant morphism,
(2) a general fibre of p is a smooth, connected, closed Riemann surface, and
(3) the closed equivalence relation on X' generated by

$$\{(x_1, x_2) \in X' : \text{there exists } b \in B \text{ with } \phi^{-1}(x_i) \cap p^{-1}(b) \neq \emptyset \text{ for } i = 1, 2\}$$

has a single equivalence class.

Proof. We show the different claims in turn.

(1) Note that $\phi : E \to X$ is the Grassman $(N-d+1)$-plane bundle of the vector bundle $\mathbb{C}^{N+1}/\mathcal{O}_X(-1)$, and in particular E is a smooth projective variety and $\phi : E \to X$ is proper and dominant.
(2) By [16, Theorem 1.22, pp. 108], the generic fibre of p is smooth. Moreover, since $\phi : E \to X$ is the Grassman $(N-d+1)$-plane bundle of the vector bundle $\mathbb{C}^{N+1}/\mathcal{O}_X(-1)$, we have

$$\dim E = d + \dim \mathrm{Gr}(N-d+1, N) = d + (N-d+1)(d-1),$$

and hence the general fibre of $p : E \to B$ has dimension $\dim E - \dim B = \dim E - (N-d+2)(d-1) = 1$.

By [13, Theorem 1.1], the map $p : E \to B$ has a non-empty connected fibre over a general point.
(3) Immediate from the fact that any two points in X' lie on some $(N-d+2)$-dimensional subspace.

$\square$

6.3. **Applying Theorem 2.** Let $A \subset B$ be the exceptional subset to Claim 6.1(2). Thus for any $b \in B \setminus A$, the fibre $p^{-1}(b)$ is a smooth connected closed Riemann surface.

Let $q : \tilde{B} \to B \setminus A$ be the universal cover of $B \setminus A$, and suppose that

$$\begin{array}{ccc} q^*E & \xrightarrow{q^*p} & \tilde{B} \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q} \\ E \setminus p^{-1}(A) & \xrightarrow{p} & B \setminus A \end{array}$$

is a pullback diagram of the bundle $p : E \setminus p^{-1}(A) \to B \setminus A$. Note that q^*p is topologically a product, since $\tilde{B}$ is simply connected. By Proposition 3.1 applied to the map $F \circ q : q^*E \to M$, where $F = f' \circ \phi$, we see that the energy of $(F \circ q)_b$ does not depend on $b \in \tilde{B}$. To apply Theorem 2, it remains to show that $F \circ q : q^*E \to M$ is transverse to the fibres of q^*p. Assume to the contrary, then $F \circ q$ is constant on each fibre of q^*E. Thus $F = f' \circ \phi$ is constant on the fibres of $p : E \setminus p^{-1}(A) \to B \setminus A$. By Claim 6.1(3), it follows that f' is constant. Hence f is constant, and there is nothing to prove.

Thus by Theorem 2, $F \circ q$ either has image contained in a closed geodesic, or factors as $\tilde{h} \circ \tilde{\phi}$, where $\tilde{\phi} : q^*E \to Y$ is a holomorphic map to a closed hyperbolic Riemann surface Y and $\tilde{h} : Y \to M$ is harmonic. If the image of $F \circ q$ is contained in a geodesic, so is the image of f, and there is nothing to prove.

Assume therefore that we are in the latter case, $F \circ q = \tilde{h} \circ \tilde{\phi}$. Recall that $\phi : E \to X'$ is the holomorphic map from Claim 6.1. Thus we have a commuting diagram

$$\begin{array}{ccccc} q^*E & \xrightarrow{q} & E \setminus p^{-1}(A) & \xrightarrow{\phi} & X' \\ \Big\downarrow{\scriptstyle \tilde{\phi}} & & & & \Big\downarrow{\scriptstyle f'} \\ Y & & \xrightarrow{\quad\tilde{h}\quad} & & M \end{array}$$

Recall that $\pi_1(B \setminus A)$ acts on $\tilde{B}$ by covering transformations. This induces an action of $\pi_1(B \setminus A)$ on q^*E.

Claim 6.2. The diagram above has the following properties:

(1) There exists a finite index subgroup $\Gamma \leq \pi_1(B \setminus A)$ such that $\tilde{\phi}$ is invariant under γ, and
(2) $\tilde{\phi}$ is constant on the connected components of any general fibre of $\phi \circ q$.

We defer the proof of Claim 6.2 to the next subsection.

Let $m = [\pi_1(B \setminus A) : \Gamma]$ be the index of Γ. We define tne a map $\hat{\phi} : E \setminus p^{-1}(A) \to \mathrm{Sym}^m Y$ by

$$\hat{\phi}(x) = \{\tilde{\phi}(\gamma\tilde{x}) : \gamma\Gamma \in \mathrm{Deck}(q)/\Gamma\},$$

where $\tilde{x} \in q^*E$ is some arbitrary lift of $x \in E \setminus p^{-1}(A)$. The right-hand side of this equation denotes a multiset, and $\mathrm{Sym}^d Y$ denotes the configuration space of d (unlabelled) points on the Riemann surface Y. It is easy to see that $\hat{\phi}$ is holomorphic.

Claim 6.3. We have $\mathrm{rank}_{\mathbb{C}} D\hat{\phi} = 1$ on an open dense subset of E.

We defer the proof of Claim 6.3 until §6.5.

Hence the image of $\hat{\phi}$ is a (complex) one-dimensional constructible subset $D \subset \mathrm{Sym}^m Y$, by Chevalley's theorem. Let $A' \subset D$ be such that $D \setminus A$ is a qusiprojective variety of complex dimension one.

Since $E \setminus (p^{-1}(A) \cup \hat{\phi}^{-1}(A'))$ is a complex manifold, by standard properties of normalization [14, Proposition, pp. 180], $\hat{\phi}$ factors as

$$E \setminus (p^{-1}(A) \cup \hat{\phi}^{-1}(A')) \xrightarrow{\phi} \hat{D} \xrightarrow{\theta} D,$$

where $\theta : \hat{D} \to D$ is the normalization of D. Normal one-dimensional complex analytic spaces are smooth, so $\hat{D}$ is in fact a finite type Riemann surface. We fill in the punctures of $\hat{D}$, and denote the resulting closed Riemann surface $\hat{D}$ as well, by a slight abuse of notation. Note that by the Schwarz lemma, the map $\phi : E \setminus (p^{-1}(A) \cup \hat{\phi}^{-1}(A')) \to \hat{D}$ is meromorphic in the sense of Andreotti [2], and can hence be extended to a holomorphic map $\theta : E \to \hat{D}$ by [2, Theorem 4]. Moreover, it is clear that F factors through θ.

By Claim 6.2(2), it follows that θ is constant on the connected components of the fibres of ϕ. We let $\phi \circ \rho = \alpha \circ \beta$ be the Stein factorization of $\phi \circ \rho : E \to X$, such that $\beta : E \to S$ has connected fibres, and such that $\alpha : S \to X$ is finite. Let $K \subset X$ be the proper analytic subset containing the singular locus of X, such that $\alpha^{-1}(x)$ has exactly $\deg \alpha$ distinct points, for $x \in X \setminus K$. Thus we define $\hat{\theta} : X \setminus K \to \mathrm{Sym}^{\deg \alpha} \hat{D}$ by

$$\hat{\theta}(x) = \{\theta(y) : y \in \alpha^{-1}(x)\}.$$

From an argument that is identical to the proof of Claim 6.3 and the paragraph following its statement, it follows that there is a holomorphic map $\xi : X \to C$ for a closed Riemann surface C, such that f factors through ξ.

6.4. **Proof of Claim 6.2.**

(1) Fix $b \in B \setminus A$ and one of its q-preimages $\tilde{b} \in \tilde{B}$. Since Y is a closed hyperbolic Riemann surface, there are only finitely many holomorphic maps $p^{-1}(b) \to Y$. Thus there is a finite index subgroup $\Gamma \leq \mathrm{Deck}(q)$ such that $\tilde{\phi} \circ \gamma = \tilde{\phi}$ on $(q^*p)^{-1}(\tilde{b})$. By uniqueness of harmonic maps in a given homotopy class, if $\tilde{\phi} \circ \gamma = \tilde{\phi}$ on one fibre of q^*E, then $\tilde{\phi} \circ \gamma = \tilde{\phi}$ on all of q^*E. Thus $\tilde{\phi} \circ \gamma = \tilde{\phi}$ for all $\gamma \in \Gamma$.

(2) Let $t \in X'$ be arbitrary such that $\phi^{-1}(t)$ is a closed complex submanifold (which is a Zariski open condition). Thus $q^{-1}(\phi^{-1}(t) \setminus p^{-1}(A))$ is a complex submanifold, on which $\tilde{h} \circ \tilde{\phi}$ is constant by definition. However, note that by real analyticity, the fibres of $\tilde{h}$ are at most one-dimensional (as analytic subsets of Y). Thus we have a holomorphic map $\tilde{\phi}$ from a complex manifold $q^{-1}(\phi^{-1}(t) \setminus p^{-1}(A))$ to a closed Riemann surface Y, with image of real dimension one. Thus $\tilde{\phi}$ is constant on $q^{-1}(\phi^{-1}(t) \setminus p^{-1}(A))$.

6.5. **Proof of Claim 6.3.** Denote by $\mathrm{Deck}(q) \cong \pi_1(B \setminus A)$ the deck group of q.

Let $\tilde{x} \in q^*E$ be a point such that $\tilde{\phi}$ has surjective derivative in a nieghbourhood U of $\tilde{x}$ and such that $\tilde{h}$ is locally an immersion near $\tilde{\phi}(\gamma\tilde{x})$ for any $\gamma \in \mathrm{Deck}(q)$. Such points lie in comeagre subset of q^*E.

Let $U \ni \tilde{x}$ be a small neighbourhood of $\tilde{x}$ such that

(1) $\tilde{\phi}$ is a holomorphic submersion over $\bigcup_{\gamma \in \mathrm{Deck}(q)} \gamma U$, and
(2) $\tilde{h}$ is a local immersion on $\bigcup_{\gamma\Gamma \in \mathrm{Deck}(q)/\Gamma} \tilde{\phi}(\gamma U)$,

which exists since $[\mathrm{Deck}(q) : \Gamma] < \infty$ and $\tilde{\phi}$ is Γ-invariant.

Let $\mathcal{F}$ be the foliation on $\mathrm{Deck}(q)U$ whose leaves are connected components of the fibres of $\tilde{\phi}$. Since $\tilde{h}$ is locally an immersion, the leaves of $\mathcal{F}$ are precisely the connected components of the fibres of $F \circ q$. In particular, $\mathcal{F}$ is $\mathrm{Deck}(q)$-invariant. Let Λ be the Riemann surface parameterizing the leaves of $\mathcal{F}$.

Then $\tilde{\phi} = \alpha \circ \beta$, where $\beta : \mathrm{Deck}(q)U \to \Lambda$ is the natural projection, and $\alpha : \Lambda \to Y$ is some holomorphic map. In particular, since β is $\mathrm{Deck}(q)$-equivariant, it follows that

$$\hat{\phi} = \alpha \left(\gamma \cdot \beta(x) : \gamma\Gamma \in \mathrm{Deck}(q)/\Gamma \right).$$

It follows that the image of $\hat{\phi}$ lies within the α image of the map $z \to \{\gamma z : \gamma\Gamma \in \mathrm{Deck}(q)/\Gamma\}$ defined on Λ. In particular, since $\dim_{\mathbb{C}} \Lambda = 1$, it follows that $\mathrm{rank}_{\mathbb{C}} D\hat{\phi} \leq 1$.

Since $\tilde{\phi}$ is not constant, neither is $\hat{\phi}$, and hence $\operatorname{rank}_{\mathbb{C}} D\hat{\phi} \geq 1$ at a general point, concluding the proof of the claim.

7. Mapping class groups

In this section, we show Theorem 4, following the argument of [25]. In §7.1, we recall the result of Bridson [6] that is used in the proof of Theorem 4, that we then show in §7.2.

7.1. **A result of Bridson.** We first state some preliminary results due to Bridson [6, Theorem B, Remark 1] that will be crucial in our proof of Theorem 4.

Proposition 7.1. *Let $\Gamma \leq \mathrm{Mod}_{g,n}$ be a finite index subgroup of the mapping class group of a surface of genus $g \geq 3$ with $n \geq 0$ punctures. Let X be a $\mathrm{CAT}(0)$ space and $\phi : \Gamma \to \mathrm{Isom}(X)$ be a homomorphism whose image consists of hyperbolic isometries. Then any power of a Dehn multitwist that lies in Γ also lies in $\ker \phi$.*

We will apply this with X being the universal cover of M. Since M is convex cocompact, the deck group of this cover $X \to M$ consists entirely of hyperbolic isometries. Using the fact that any point-pushing mapping class that corresponds to a simple closed curve is a mutlitwist, we get the following corollary.

Corollary 7.2. *Let M be as in Assumptions 2.2, and let $\Gamma \leq \mathrm{Mod}_{g,n+1}$ be a finite index subgroup, with $g \geq 3, n \geq 0$. Then for any homomorphism $\phi : \Gamma \to \pi_1(M)$ and for any simple closed curve γ on $\Sigma_{g,n}$, we have $\phi(\iota(\gamma)^k) = 0$, whenever $\iota(\gamma)^k \in \Gamma$.*

Recall that ι is the map that embeds $\pi_1(\Sigma_{g,n})$ as the point-pushing subgroup $\Pi_{g,n} \leq \mathrm{Mod}_{g,n+1}$.

7.2. **Proof of Theorem 4.** Let M be a manifold satisfying Assumptions 2.2, $\Gamma \leq \mathrm{Mod}_{g,n+1}$ be a finite index subgroup and $\phi : \Gamma \to \pi_1(M)$ be a strongly point-pushing homomorphism. Recall the Birman exact sequence

$$1 \to \pi_1(\Sigma_{g,n}) \xrightarrow{\iota} \mathrm{Mod}_{g,n+1} \xrightarrow{\mathcal{F}} \mathrm{Mod}_{g,n} \to 1,$$

and the point-pushing subgroup $\Pi_{g,n} = \mathrm{im}(\iota) \leq \mathrm{Mod}_{g,n+1}$.

Let $K = \iota^{-1}(\Pi_{g,n} \cap \Gamma) \leq \pi_1(\Sigma_{g,n})$. Then, as $[\mathrm{Mod}_{g,n+1} : \Gamma] < \infty$, we have $[\pi_1(\Sigma_{g,n}) : K] < \infty$. Thus $K \leq \pi_1(\Sigma_{g,n})$ corresponds to a finite covering map $p : \Sigma_{h,m} \to \Sigma_{g,n}$ via $\mathrm{im}(p_*) = K$. Note that by Corollary 7.2, the map $\phi \circ \iota \circ p_*$ annihilates any simple closed curve on $\Sigma_{h,m}$. In particular, it descends to a map $\pi_1(\Sigma_h) \to \pi_1(M)$. Let $\psi : \Sigma_h \to M$

be a continuous map representing $\phi \circ \iota \circ p_*$, which exists since both Σ_h and M are Eilenberg–MacLane spaces for their respective fundamental groups.

We denote by σ_p the holomorphic map $\mathcal{T}_{g,n} \to \mathcal{T}_h$ obtained by lifting a complex structure $X \in \mathcal{T}_{g,n}$ via p to $\Sigma_{h,m}$, and then filling in the punctures of the resulting Riemann surface. Note that Theorem 4(1) follows immediately from Theorem 1 once we show that E_ψ is constant on the image of σ_p. We now show how to conclude Theorem 4(2) from the constancy of $\mathrm{E}_\psi \circ \sigma_p$, and then we show that $\mathrm{E}_\psi \circ \sigma_p$ is constant in the next subsection.

Assume that $\mathrm{E}_\psi \circ \sigma_p$ is constant. By Theorem 1 (and Proposition 4.1), the map F_ψ factors over $\sigma_p(\mathcal{T}_{g,n})$ through a holomorphic map $\xi : \mathcal{T}_{g,n+1}/\Gamma \cap \Pi_{g,n} \to Y$ to a closed hyperbolic Riemann surface Y. By [24, Propositions 2.3 and 2.4], the map ξ is invariant under a finite index subgroup $\Gamma' \leq \mathrm{Mod}_{g,n+1}$. This shows Theorem 4(2) with $\Theta = \Gamma \cap \Gamma'$.

7.2.1. *The energy* E_ψ *is constant on the image of* σ_p. We first observe that $\mathrm{E}_\psi \circ \sigma_p$ is invariant under the action of the finite index subgroup $\mathcal{F}(\Gamma) \leq \mathrm{Mod}_{g,n}$. This follows from uniqueness of harmonic maps, and the fact that the free homotopy class of ψ is invariant under the action of Γ.

Claim 7.3. The energy $\mathrm{E}_\psi \circ \sigma_p$ is bounded.

We first show how to prove that $\mathrm{E}_\psi \circ \sigma_p$ is constant assuming Claim 7.3. By the result of Boggi–Pikaart [5, Corollary 2.10], there exists a finite index subgroup $\bar{\Gamma} \leq \mathcal{F}(\Gamma)$, such that $\mathcal{T}_{g,n}/\bar{\Gamma}$ has a compactification $\mathcal{M}$ which is a smooth projective variety. Then by standard theory in complex analysis [9, Theorem (5.24)], the function $\mathrm{E}_\psi \circ \sigma_p : \mathcal{T}_{g,n}/\bar{\Gamma} \to \mathbb{R}$ extends to a bounded plurisubharmonic function $\mathcal{M} \to \mathbb{R}$. Then by the strong maximum principle, $\mathrm{E}_\psi \circ \sigma_p$ is constant.

Proof of Claim 7.3. The proof is nearly identical to that of [25, Proposition 6.4], so we only give a sketch.

Suppose that $X_1, X_2, ... \in \mathcal{T}_{g,n}$ has $\mathrm{E}_\psi(\sigma_p(X_n)) \to \infty$. Since $\mathcal{F}(\Gamma) \leq \mathrm{Mod}_{g,n}$ is finite index, there exist mapping classes $T_n \in \mathcal{F}(\Gamma)$ such that $T_n X_n \to Y$, where Y is a marked noded Riemann surface.

Let Z be the marked noded Riemann surface obtained by lifting the complex structure of Y via p. Let $\gamma_1, \gamma_2, ..., \gamma_k \in \pi_1(\Sigma_h)$ be the (disjoint) simple closed curves that correspond to nodes of Z. Then by Corollary 7.2, we have $\psi_*(\gamma_i) = 1$ for $1 \leq i \leq k$.

After applying a suitable homotopy, we may assume without loss of generality that $\psi : Z \to M$ is smooth on Z and constant in a neighbourhood of each node. But then

$$\mathrm{E}_\psi \circ \sigma_p(X_n) = \mathrm{E}_\psi \circ \sigma_p(T_n X_n) \to \mathrm{E}_\psi(Z) \le \int_Z |D\psi|^2 \, d\mathrm{vol}_Z < \infty,$$

leading to the desired contradiction. □

Appendix A. Non-abelian Hodge correspondence over a moving Riemann surface

The goal of this section is to prove Theorem 5, i.e. the non-abelian Hodge correspondence for $\mathrm{PSL}(2,\mathbb{R})$ over a moving Riemann surface. If $\chi_{g,k}$ denotes the representation variety $\pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$ consisting of representations of Euler class k, then Theorem 5 identifies $\mathcal{T}_g \times \chi_{g,k}$ via the non-abelian Hodge correspondence to the total space of a holomorphic fibration $\mathcal{E}(g,k) \to \mathcal{T}_g$. For $k > 0$, the fibres parameterize the divisor of the holomorphic energy, and the Hopf differential of the associated equivariant harmonic map to the hyperbolic plane $\mathbb{H}$.

An analogous statement was shown by Hitchin over a fixed Riemann surface [19, Theorem (10.8)]. Over a moving Riemann surface, the moduli spaces of solutions to the Hitchin equations for $\mathrm{GL}(2,\mathbb{C})$ were constructed by the author in the previous paper [34, §4]. Since only representations of even Euler class can be lifted to $\mathrm{SL}(2,\mathbb{R})$, here we show Theorem 5 by combining this construction from [34] with the arguments from [4, §4].

We first recall some notation and define the Euler class in §A.1. Then we construct $\mathcal{E}(g,k)$ in §A.2. We then show Theorem 5 for k even in §A.3, and for k odd in §A.4.

A.1. **Notation and preliminary bundless.** Recall that $\pi : \mathcal{V}_g \to \mathcal{T}_g$ is the universal curve, and that $\mathrm{Sym}^n\pi : \mathrm{Sym}^n\mathcal{V}_g \to \mathcal{T}_g$ is the holomorphic fibration over $\mathcal{T}_g$ whose fibre over $S \in \mathcal{T}_g$ is $\mathrm{Sym}^n S$.

Let χ_g be the space of representations $\pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$. For any representation $\rho \in \chi_g$, define its *Euler class* $\mathrm{eu}(\rho)$ to be the Euler number of the flat $\mathbb{RP}^1$ bundle associated to ρ (via the natural action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{RP}^1$) [19, §10, pp. 117]. Then the Euler class completely classifies the connected components of $\chi_{g,k}$.

We will make use of the fibred product

$$\begin{tikzcd}
\mathrm{Sym}^n\mathcal{V}_g \times_{\mathcal{T}_g} \mathcal{V}_g \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{V}_g \arrow[d, "\pi"] \\
\mathrm{Sym}^n\mathcal{V}_g \arrow[r, "\mathrm{Sym}^n\pi"] & \mathcal{T}_g
\end{tikzcd}$$

Recall that the fibre of $\mathrm{Sym}^n\mathcal{V}_g \times_{\mathcal{T}_g} \mathcal{V}_g$ over a point $S \in \mathcal{T}_g$ is the product $\mathrm{Sym}^n S \times S$. Let $\mathcal{U}_n$ be the tautological line bundle over $\mathrm{Sym}^n\mathcal{V}_g \times_{\mathcal{T}_g} \mathcal{V}_g$ that restricts to $\mathcal{O}(D)$ over the section of $\mathrm{Sym}^n S \times S$ that corresponds $D \in \mathrm{Sym}^n S$, i.e. to the effective degree n divisor D on S. Let K_π be the relative canonical bundle of π, i.e. the bundle over $\mathcal{V}_g$ that restricts to K_S over the fibre of π that corresponds to $S \in \mathcal{T}_g$.

A.2. **Constructing $\mathcal{E}(g,k)$.** We construct $\mathcal{E}(g,k)$ as a holomorphic vector bundle over $\mathrm{Sym}^{2g-2-k}\mathcal{V}_g$. Consider the bundle $\mathcal{U}_{2g-2-k}^{-1} \otimes \mathrm{pr}_2^* K_\pi^2 \to \mathrm{Sym}^n\mathcal{V}_g \times_{\mathcal{T}_g} \mathcal{V}_g$. This is a holomorphic vector bundle that resticts over $\{D\} \times S \subset \mathrm{Sym}^{2g-2-k} S \times S = (\pi \circ \mathrm{pr}_2)^{-1}(S)$ to the line bundle $K_S^2(-D)$. Let $\mathcal{E}(g,k)$ be the pushforward of this bundle $\mathcal{U}_{2g-2-k}^{-1} \otimes \mathrm{pr}_2^* K_\pi^2$ via the map $\mathrm{pr}_1 : \mathrm{Sym}^{2g-2-k}\mathcal{V}_g \times_{\mathcal{T}_g} \mathcal{V}_g \to \mathrm{Sym}^{2g-2-k}\mathcal{V}_g$.

Thus the fibre of $\mathcal{E}(g,k)$ over $D \in \mathrm{Sym}^{2g-2-k} S$ for $S \in \mathcal{T}_g$, is naturally isomorphic to the sections of $K_S^2(-D)$. Here in the notation of §4.1, D represents the divisor of δ, and the fibre $\mathcal{E}(g,k)_S = H^0(K_S^2(-D))$ represents the space of holomorphic quadratic differentials that vanish along D.

A.3. **Even Euler class.** Here we show Theorem 5 for even k. We will define a map $\mathrm{T} : \mathcal{E}(g,k) \to \mathcal{T}_g \times \chi_{g,k}$ in two steps: we first define T locally as a smooth diffeomorphism, then we show that the different local definitions agree.

Step 0. Local choices. Therefore we fix a point (X_0, D_0, Φ_0) in (the total space of) $\mathcal{E}(g,k)$, that consists of

(1) a marked Riemann surface $X_0 \in \mathcal{T}_g$,
(2) an effective degree $2g-2-k$ divisor D_0 on X_0, and
(3) a holomorphic section $\Phi_0 \in H^0(K_{X_0}^2(-D_0))$.

Let U be a small neighbourhood of this point in $\mathcal{E}(g,k)$. We will freely shrink U without mention throughout the proof if necessary.

Let $\mathcal{U}_{2g-2-k}^{1/2}$ be a square root of $\mathcal{U}_{2g-2-k}$ over

$$\begin{aligned} V &= \mathrm{pr}_1^{-1}(\{(D,X) : (X,D,\Phi) \in U \text{ for some } \Phi\}) \\ &= \{(D,x) : (X,D,\Phi) \in U \text{ for some } \Phi, \text{ and } x \in X\} \subset \mathrm{Sym}^{2g-2-k}\mathcal{V}_g \times_{\mathcal{T}_g} \mathcal{V}_g, \end{aligned}$$

which exists since k is even. We similarly pick a consistent square root $K_X^{1/2}$ of K_X over V.

Step 1. Definition of T. Given $(X,D,\Phi) \in U$, consider the Higgs bundle $E = L \oplus L^{-1}$ over X, where $L = \mathcal{U}_{2g-2-k}^{-1/2}\Big|_{\{D\}\times X} \otimes K_X^{1/2}$

with Higgs field

$$\phi = \begin{pmatrix} & \Phi\sigma^{-1} \\ \sigma & \end{pmatrix},$$

where σ is an arbitrary section of $K_X L^{-2} \cong \mathcal{O}(D)$ with divisor D. Note that there is ambiguity here in the scaling of σ, but all such Higgs bundles are gauge equivalent by [19, Proposition (10.2)]. By [19, Proposition (10.2)], (E, ϕ) is stable, the harmonic metric on (E, ϕ) is diagonal with respect to the splitting $E = L \oplus L^{-1}$, and the associated flat bundle has holonomy $\rho : \pi_1(\Sigma_g) \to \mathrm{SL}(2, \mathbb{R})$ which projects to $\chi_{g,k}$. We set $\mathrm{T}(X, D, \Phi) = (X, \rho)$. Note that once we show that T is a diffeomorphism, the description of the zero section follows immediately since $\operatorname{tr}\left(\phi^2\right) = 2\Phi$ is the Hopf differential of the ρ-equivariant harmonic map (see [22, §5.2])

Step 2. T **is well-defined.** We need to show that T does not depend on the local choices of square roots of $\mathcal{U}_{2g-2-k}$ and K. Fix a triple $(X, D, \Phi) \in U$, and let L be as above. Let $\tilde{L}$ be a line bundle obtained as above with a different choice of square roots of $\mathcal{U}_{2g-2-k}, K_X$, and let $\rho, \tilde{\rho} : \pi_1(\Sigma_g) \to \mathrm{SL}(2, \mathbb{R})$ be the respective representations. Then $\tilde{L} = L\xi$, where ξ is a line bundle such that $\xi^{\otimes 2} \cong \mathcal{O}_X$. In particular the Higgs bundle with this choice $\tilde{L}$ is

$$\tilde{E} = \tilde{L} \oplus \tilde{L}^{-1} = \xi \otimes \left(L \oplus L^{-1}\right) = \xi \otimes E.$$

In particular, the projective bundles $\mathbb{P}(E)$ and $\mathbb{P}(\tilde{E})$ coincide. Thus (E, ϕ) and $(\tilde{E}, \tilde{\phi})$ agree as holomorphic $\mathrm{PSL}(2, \mathbb{C})$-Higgs bundles, so $q \circ \rho = q \circ \tilde{\rho}$, where $q : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$ is the natural quotient map.

Step 3. T **is smooth.** It suffices to show that the harmonic metric on L depends smoothly on (X, D, Φ). Note that L is a line bundle over V. We can also choose σ to be a holomorphic section of KL^{-2} over V. Let ℓ_0 be an arbitrary background Hermitian metric on L, and let $h \in C^\infty(T^*\mathcal{V}_g^{\otimes 2})$ that restricts to the hyperbolic metric on the fibres of $\pi : \mathcal{V}_g \to \mathcal{T}_g$.

The Hitchin equation for the diagonal metric on $E = L \oplus L^{-1}$ for (E, ϕ) over $X \in \mathcal{T}_g$ is

$$i\Lambda_\omega \Theta_\ell + \left\|\Phi\sigma^{-1}\right\|_{\ell,h}^2 - \|\sigma\|_{\ell,h}^2 = 0,$$

where ω is the volume form, and ℓ is the metric on L [4, §2.3, equation (2.5)]. If we write $\ell = e^\phi \ell_0$, then the equation becomes

$$i\Lambda_\omega \Theta_{\ell_0} + \Delta_g \phi - e^{2\phi}\left\|\Phi\sigma^{-1}\right\|_{\ell_0,h}^2 + e^{-2\phi}\|\sigma\|_{\ell_0,h}^2 = 0. \tag{A.1}$$

This equation is analogous to [4, §2.3, equation (2.6)]. We know by the non-abelian Hodge theorem that (A.1) admits a unique solution for any $(X, D, \Phi) \in U$. The following claim shows that this solution depends smoothly on the data $\left\|\Phi\sigma^{-1}\right\|_{\ell_0}^2$ and $\|\sigma\|_{\ell_0}^2$, that in turn depend smoothly on $(X, D, \Phi) \in U$. Thus T is smooth.

Claim A.1. Let Σ be a closed surface. Suppose that g_0 is a Riemannian metric on Σ, and that $\lambda_0, \mu_0, \phi_0 : \Sigma \to \mathbb{R}$ are smooth functions, such that at least one of λ_0, μ_0 does not vanish identically. Then there exists a neighbourhood U of $(g_0, \lambda_0, \mu_0) \in C^\infty(\mathrm{Sym}^2 T^*\Sigma) \times C^\infty(\Sigma) \times C^\infty(\Sigma)$ such that, for any $(g, \lambda, \mu) \in U$, there exists a smooth function ϕ solving the equation

$$\Delta_g \phi + \mu^2 e^{-2\phi} - \lambda^2 e^{2\phi} = \Delta_{g_0}\phi_0 + \mu_0^2 e^{-2\phi_0} - \lambda_0^2 e^{2\phi_0}, \tag{A.2}$$

where Δ_g is the Laplacian with respect to the Riemannian metric g. Moreover, this solution depends smoothly on (g, λ, μ).

Proof. Consider the map $U \times C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$(g, \lambda, \mu, \phi) \longrightarrow \Delta_g \phi + \mu^2 e^{-2\phi} - \lambda^2 e^{2\phi}.$$

This is a smooth map with derivative in the ϕ-direction given by

$$F[\dot\phi] = \Delta_g \dot\phi - \left(\lambda^2 e^{2\phi} + \mu^2 e^{-2\phi}\right)\dot\phi,$$

which is an elliptic self-adjoint operator. Note that F is injective by the strong maximum principle, and thus also surjective. Hence F is an isomorphism, and the result follows by the implicit function theorem. ∎

Step 4. **T is a diffeomorphism.** The fact that T is smooth implies that T is a diffeomorphism from the implicit function theorem, in the version stated below.

Claim A.2. Let $\mathrm{T} : P_1 \to P_2$ be a smooth map between smooth fibrations $p_i : P_i \to X$ over a manifold X, i.e. a smooth map making the following diagram commute

$$\begin{tikzcd} P_1 \arrow[rr, "\mathrm{T}"] \arrow[dr, "p_1"'] & & P_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Then if $\mathrm{T} : p_1^{-1}(x) \to p_2^{-1}(x)$ is a diffeomorphism for all $x \in X$, then T is a diffeomorphism between total spaces $P_1 \to P_2$.

Proof. It suffices to show that T is a local diffeomorphism since it is clearly a bijection. We pick local coordinates (x, y) of P_1, (x, z) of P_2, and x on X, such that p_1 takes the form $(x, y) \to x$ and p_2 takes the form $(x, z) \to x$. Here $x \in \mathbb{R}^{\dim X}, y, z \in \mathbb{R}^{\dim P_i - \dim X}$. We represent T locally as a map $(x, y) \to (x, F(x, y))$. Then $F(x, -)$ is a local diffeomorphism, so in particular $\frac{\partial F}{\partial y}$ is an isomorphism. Therefore by the implicit function theorem, there is a smooth map $G(x, z)$ such that $F(x, G(x, z)) = z$. Then T has a local smooth inverse $\mathrm{id}_X \times G$. $\square$

A.4. **Odd Euler class.** We now show Theorem 5 for odd k. Let $p : \Sigma_{2g-1} \to \Sigma_g$ be an arbitrary unramified double cover. Given $(X, D, \Phi) \in \mathcal{E}(g, k)$, by lifting everything via p, we get an element $(\hat{X}, \hat{D}, \hat{\Phi}) \in \mathcal{E}(2g-1, 2k)$. The even Euler class case from §A.3 then defines a map

$$\begin{aligned} \hat{\mathrm{T}} : \mathcal{E}(g, k) &\longrightarrow \mathcal{T}_{2g-1} \times \chi_{g,2k} \\ (X, D, \Phi) &\longrightarrow \mathrm{T}(\hat{X}, \hat{D}, \hat{\Phi}) = (\hat{X}, \hat{\rho}). \end{aligned}$$

Let (E, ϕ) be as in §A.3. Then by [4, §4], the $\mathrm{PSL}(2, \mathbb{C})$-Higgs bundle $(\mathbb{P}(E), \phi)$ over $\hat{X}$ descends to a $\mathrm{PSL}(2, \mathbb{C})$-Higgs bundle on X. From the uniqueness of the solutions to the Hitchin equations, we see that the harmonic metric over $\hat{X}$ must be a lift of the harmonic metric on X. Therefore $\hat{\rho} : \pi_1(\Sigma_{2g-1}) \to \mathrm{PSL}(2, \mathbb{R})$ can be uniquely extended to a representation $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{R})$ such that $\rho \circ p_* = \tilde{\rho}$. It follows that $\hat{\mathrm{T}}$ descends to a smooth map $\mathcal{E}(g, k) \to \mathcal{T}_g \times \chi_{g,k}$. The fact that this map is a diffeomorphism follows as in Step 4 in §A.3.

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