A new family of counterexamples to the Zariski Cancellation Problem in positive characteristic

Parnashree Ghosh* and Ananya Pal**

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute,

203 B.T.Road, Kolkata-700108, India

*e-mail : *parnashree_r@isical.ac.in, ghoshparnashree@gmail.com*

*e-mail : **palananya1995@gmail.com*

Abstract

In this paper, over a field of positive characteristic we exhibit an infinite family of counter examples to the Zariski Cancellation Problem in higher dimensions ($\geqslant 3$) which are pairwise non-isomorphic and also non-isomorphic to the existing family of counter examples, demonstrated by Gupta in [10].

Keywords. Polynomial ring, Zariski Cancellation Problem, Derksen invariant, Makar-Limanov invariant.

2020 MSC. Primary: 14R10; Secondary: 13B25, 13A50, 13A02

1 Introduction

Throughout this paper k will denote a field. For an integral domain R, $R^{[n]}$ denotes a polynomial ring in n variables over R. We begin with one of the fundamental problems in the area of Affine Algebraic Geometry, namely the Zariski Cancellation Problem (ZCP).

Question 1: For an integer $n \geqslant 1$, is $k^{[n]}$ cancellative? That is if B be an affine domain such that $B^{[1]} = k^{[n+1]}$, then does this imply that $B = k^{[n]}$?

The answer to the above question is affirmative for $n = 1$ (cf. [1]). For $n = 2$, when k is a field of characteristic zero, the cancellative property of $k^{[2]}$ is proved by Miyanishi, Sugie and Fujita ([13], [5]). This result is extended over perfect fields by Russell in [15]. However, in [11], Bhatwadekar and Gupta have dropped the condition of perfect fields and extended the result over arbitrary fields. Thus proving the cancellative properly of $k^{[2]}$, for any field.

We now discuss the case for $n \geqslant 3$. When k is a field of characteristic zero, the problem is still open. However, potential candidate for counterexample to this problem in characteristic zero is given by the Russell-Koras threefold whose coordinate ring is given by $k[X, Y, Z, T]/(X^2Y - X - Z^2 - T^3)$. However, over fields of positive characteristic, the work of Gupta in [9] and [10] produced counterexamples ZCP for all $n \geqslant 3$. She has considered varieties whose coordinate rings are given by

$$B = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))}, \tag{1}$$

where $k[Z,T]/(f) = k^{[1]}$, i.e., f is a line in $k[Z,T]$. For $m = 1$, such rings were introduced by Asanuma as an illustration of non-trivial $\mathbb{A}^2$-fibration over a DVR not containing $\mathbb{Q}$. These varieties are contained in a larger class of varieties called "Generalised Asanuma varieties" which are well studied in [6] and [7]. In [10], Gupta has shown that $B^{[1]} = k^{[m+3]}$ but $B \neq k^{[m+2]}$ whenever f is a *non-trivial line*, i.e., $k[Z,T]/(f) = k^{[1]}$ but $k[Z,T] \neq k[f]^{[1]}$. Since non-trivial lines exist only over fields of positive characteristic (cf. [2], [17], [16], [14]), the rings of the form B produced counterexamples to ZCP over fields of positive characteristic. Such rings were further studied in [6] and an infinite family of pairwise non-isomorphic varieties having coordinate rings of type (1) are obtained, which are counterexamples to the ZCP in positive characteristic.

In this paper we study varieties whose coordinate rings are given by the following equation.

$$A := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T))}, \tag{2}$$

such that $\alpha \notin k$ and either $\deg_Z F \geqslant 1$ or $\deg_T F \geqslant 1$. Let $x_1, \ldots, x_m, y, z, t$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in A. When $\alpha(X_1, \ldots, X_m) = a_1(X_1) \cdots a_m(X_m)$ and $F = f(Z,T)$, we denote the corresponding rings as $A_{(a_1, \ldots, a_m)}$, that is

$$A_{(a_1, \ldots, a_m)} := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(a_1(X_1) \cdots a_m(X_m)Y - f(Z,T))}.$$

We show that when k is a field of positive characteristic and f is a non-trivial line in $k[Z,T]$, the family

$$\left\{ A_{(a_1, \ldots, a_m)} \mid a_1, \ldots, a_m \in k^{[1]} \right\}$$

contains an infinite subfamily of pairwise non-isomorphic rings which are counterexamples to the ZCP but none of them is isomorphic to the varieties of type (1). Thus we produce a new family of counterexamples to the ZCP over fields of positive characteristic (Corollary 3.11).

In the next section we will record some basic facts on exponential maps on k-algebras and recall some known results which will be used later in this paper.

2 Preliminaries

We begin this section with the basic concept of exponential maps on k-algebras.

Definition. Let B be a k-algebra and $\phi_U : B \to B[U]$ be a k-algebra homomorphism. We say that $\phi = \phi_U$ is an *exponential map* on B, if ϕ satisfies the following two properties:

(i) $\varepsilon_0 \phi_U = id_B$, where $\varepsilon_0 : B[U] \to B$ is the evaluation map at $U = 0$.

(ii) $\phi_V \phi_U = \phi_{V+U}$; where $\phi_V : B \to B[V]$ is extended to a k-algebra homomorphism $\phi_V : B[U] \to B[U,V]$ by defining $\phi_V(U) = U$.

Given an exponential map ϕ on a k-algebra B, the ring of invariants of ϕ is a subring of B given by

$$B^{\phi} = \{a \in B \mid \phi(a) = a\}.$$

An exponential map ϕ on B is said to be non-trivial if $B^{\phi} \neq B$. We denote the set of all exponential maps on B by $\mathrm{EXP}(B)$. The *Derksen invariant* of B is a subring of B defined by

$$\mathrm{DK}(B) = k[B^{\phi} \mid \phi \text{ is a non-trivial exponential map on } B]$$

and the *Makar-Limanov invariant* of B is a subring of B defined by

$$\mathrm{ML}(B) = \bigcap_{\phi \in \mathrm{EXP}(B)} B^{\phi}.$$

We list below some useful properties of exponential maps (cf. [12, Chapter I], [3], [9]) on a k-domain.

Lemma 2.1. *Let ϕ be a non-trivial exponential map on a k-domain B. Then the following holds:*

(i) *B^{ϕ} is factorially closed in B i.e., for any $a, b \in B \setminus \{0\}$, if $ab \in B^{\phi}$ then $a, b \in B^{\phi}$. Hence, B^{ϕ} is algebraically closed in B.*

(ii) $\mathrm{tr.deg}_k(B^{\phi}) = \mathrm{tr.deg}_k(B) - 1$.

(iii) *Let S be a multiplicative closed subset of $B^{\phi} \setminus \{0\}$. Then ϕ extends to a non-trivial exponential map $S^{-1}\phi$ on $S^{-1}B$ defined by $S^{-1}\phi(a/s) = \phi(a)/s$, for all $a \in B$ and $s \in S$. Moreover, the ring of invariants of $S^{-1}\phi$ is $S^{-1}(B^{\phi})$.*

(iv) *ϕ extends to a non-trivial exponential map $\phi \otimes id$ on $B \otimes_k \overline{k}$ such that $(B \otimes_k \overline{k})^{\phi \otimes id} = B^{\phi} \otimes_k \overline{k}$.*

We define below a *proper* and *admissible* $\mathbb{Z}$*-filtration* on a k-domain B.

Definition. A collection $\{B_n \mid n \in \mathbb{Z}\}$ of k-linear subspaces of B is said to be a *proper* $\mathbb{Z}$*-filtration* if the following properties hold:

(i) $B_n \subseteq B_{n+1}$ for every $n \in \mathbb{Z}$.

(ii) $B = \cup_{n \in \mathbb{Z}} B_n$.

(iii) $\cap_{n \in \mathbb{Z}} B_n = \{0\}$.

(iv) $(B_n \setminus B_{n-1})(B_m \setminus B_{m-1}) \subseteq B_{m+n} \setminus B_{m+n-1}$ for all $m, n \in \mathbb{Z}$.

Any proper $\mathbb{Z}$-filtration $\{B_n\}_{n \in \mathbb{Z}}$ on B determines an *associated* $\mathbb{Z}$*-graded integral domain*

$$\mathrm{gr}(B) := \bigoplus_{n \in \mathbb{Z}} \frac{B_n}{B_{n-1}}$$

and there exists a natural map $\rho : B \to \mathrm{gr}(B)$ defined by $\rho(a) = a + B_{n-1}$, if $a \in B_n \setminus B_{n-1}$.

Definition. A proper $\mathbb{Z}$-filtration $\{B_n\}_{n \in \mathbb{Z}}$ on an affine k-domain B is said to be *admissible* if there exists a finite generating set Γ of B such that, for any $n \in \mathbb{Z}$ and $a \in B_n$, a can be written as a finite sum of monomials in $k[\Gamma] \cap B_n$.

We now record a theorem on homogenization of exponential map by H. Derksen, O. Hadas and L. Makar-Limanov [4]. The following version can be found in [3, Theorem 2.6].

Theorem 2.2. *Let B be an affine k-domain with an admissible proper $\mathbb{Z}$-filtration and $gr(B)$ be the induced associated $\mathbb{Z}$-graded domain. Let ϕ be a non-trivial exponential map on B, then ϕ induces a non-trivial, homogeneous exponential map $\overline{\phi}$ on $gr(B)$ such that $\rho(B^{\phi}) \subseteq gr(B)^{\overline{\phi}}$, where $\rho : B \to \mathrm{gr}(B)$ denotes the natural map.*

Next we quote some important results from [8] which will be used in this paper.

Theorem 2.3. *Let R be an affine UFD over a field k and*

$$A_R := \frac{R[X,Y,Z,T]}{(\alpha(X)X^dY - F(X,Z,T))} \text{ with } \alpha(0) \neq 0 \text{ and } F(0,Z,T) \neq 0.$$

Let x,y,z,t denote the images of X,Y,Z,T in A_R respectively. Let w_R be a degree function on A_R such that $w_R(r) = 0$ for every $r \in R$, $w_R(x) = -1$, $w_R(y) = d$ and $w_R(z) = w_R(t) = 0$. Suppose $\gcd_{R[Z,T]}(\alpha(0), F(0,Z,T)) = 1$, then for any generating set $\{c_1, \dots, c_n\}$ of the affine k-algebra R, w_R induces an admissible $\mathbb{Z}$-filtration on A_R with respect to the generating set $\{c_1, \dots, c_n, x, y, z, t\}$ such that

$$gr(A_R) \cong \frac{R[X,Y,Z,T]}{(\alpha(0)X^dY - F(0,Z,T))}.$$

We now quote special cases of two results from [8] which are going to be used in the subsequent section of this paper. We first record the following result ([8, Proposition 4.20]).

Proposition 2.4. *Let*

$$A = \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} \alpha_1(X_1) \cdots \alpha_m(X_m)Y - f(Z,T) - X_1 h(X_1, \dots, X_m, Z, T))}$$

be an affine domain such that $r_i > 1$, $\alpha_i(0) \neq 0$ for every i, $1 \leqslant i \leqslant m$ and for some $h \in k^{[m+2]}$. Let w_1 be the degree function on A given by $w_1(x_1) = -1, w_1(y) = r_1, w_1(x_i) = w_1(z) = w_1(t) = 0$, $2 \leqslant i \leqslant m$. If $\mathrm{DK}(A)$ contains an element with positive w_1-value (in particular, if $\mathrm{DK}(A) = A$), then the following statements hold.

(i) *When $m = 1$ or k is an infinite field then there exist a system of coordinates Z_1, T_1 of $k[Z,T]$ and $a_0, a_1 \in k^{[1]}$, such that $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$.*

(ii) *When f is a line in $k[Z,T]$, then $k[Z,T] = k[f]^{[1]}$.*

The next theorem follows from [8, Theorem 4.22].

Theorem 2.5. *Let A be an affine domain defined as follows*

$$A = \frac{k[X_1, \dots, X_m, Y, Z, T]}{(a_1(X_1) \cdots a_m(X_m)Y - f(Z,T))},$$

such that each $a_i(X_i)$ has at least one separable multiple root in $\overline{k}$. Then the following are equivalent.

(i) *$A = k^{[m+2]}$.*

(ii) *$k[Z,T] = k[f]^{[1]}$.*

3 Main Theorems

We begin with the following lemma.

Lemma 3.1. *Let A be the affine domain as in (2) i.e.,*

$$A = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T))}.$$

Then the following statements hold:

(i) $k[x_1, \ldots, x_m, z, t] \subseteq \mathrm{DK}(A)$.

(ii) *Suppose for every prime divisor p of α in $k[X_1, \ldots, X_m]$, $p \notin A^*$ and $F \notin k[X_1, \ldots, X_m]$. Then* $\mathrm{ML}(A) \subseteq k[x_1, \ldots, x_m]$.

Proof. (i) At first we define two exponential maps ϕ_1, ϕ_2 on A.
$\phi_1 : A \to A[U]$ is defined as follows:

$$\begin{array}{lcl}
\phi_1(x_i) & = & x_i,\ 1 \leqslant i \leqslant m \\
\phi_1(z) & = & z + \alpha(x_1, \ldots, x_m)U \\
\phi_1(t) & = & t \\
\phi_1(y) & = & \frac{F(x_1, \ldots, x_m, z+\alpha U, t)}{\alpha} = y + Uv(x_1, \ldots, x_m, z, t, U), \text{ for some } v \in k[x_1, \ldots, x_m, z, t, U]
\end{array}$$

Next $\phi_2 : A \to A[U]$ is defined below:

$$\begin{array}{lcl}
\phi_2(x_i) & = & x_i,\ 1 \leqslant i \leqslant m \\
\phi_2(z) & = & z \\
\phi_2(t) & = & t + \alpha(x_1, \ldots, x_m)U \\
\phi_2(y) & = & \frac{F(x_1, \ldots, x_m, z, t+\alpha U)}{\alpha} = y + Uw(x_1, \ldots, x_m, z, t, U), \text{ for some } w \in k[x_1, \ldots, x_m, z, t, U]
\end{array}$$

Clearly

$$k[x_1, \ldots, x_m, t] \subseteq A^{\phi_1} \subseteq k[x_1, \ldots, x_m, \alpha^{-1}, t] \tag{3}$$

and

$$k[x_1, \ldots, x_m, z] \subseteq A^{\phi_2} \subseteq k[x_1, \ldots, x_m, \alpha^{-1}, z] \tag{4}$$

Therefore, $k[x_1, \ldots, x_m, z, t] \subseteq \mathrm{DK}(A)$.
(ii) We now assume that $p \notin A^*$ for every prime divisor p of α and $F \notin k[X_1, \ldots, X_m]$. Then it follows that $\mathrm{ML}(A) \subseteq A^{\phi_1} \cap A^{\phi_2} \cap A = k[x_1, \ldots, x_m]$. ☐

The following result describes the isomorphisms between two affine domains of the form A as in (2) upto certain condition on their Makar-Limanov and Derksen invariants.

Theorem 3.2. *Let*

$$A_1 := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha_1(X_1, \ldots, X_m)Y - F_1(X_1, \ldots, X_m, Z, T))}$$

and

$$A_2 := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(\alpha_2(X_1, \ldots, X_m)Y - F_2(X_1, \ldots, X_m, Z, T))}.$$

Let $x_1, \ldots, x_m, y, z, t$ and $x'_1, \ldots, x'_m, y', z', t'$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in A_1 and A_2 respectively. Suppose that $B_1 := \mathrm{DK}(A_1) = k[x_1, \ldots, x_m, z, t]$, $E_1 := \mathrm{ML}(A_1) = k[x_1, \ldots, x_m]$, $B_2 := \mathrm{DK}(A_2) = k[x'_1, \ldots, x'_m, z', t']$ and $E_2 := \mathrm{ML}(A_2) = k[x'_1, \ldots, x'_m]$. Let $\alpha_1 = \prod_{i=1}^{n} p_i^{r_i}$ and $\alpha_2 = \prod_{j=1}^{n'} q_j^{s_j}$ be the prime factorisations of α_1 and α_2 in $k[X_1, \ldots, X_m]$ respectively. Suppose $\phi : A_1 \to A_2$ is an isomorphism, then the following hold:

(i) *ϕ restricts to an isomorphism from B_1 to B_2 and from E_1 to E_2.*

(ii) *For every j, $1 \leqslant j \leqslant n'$, there exists i, $1 \leqslant i \leqslant n$ such that $\phi(p_i) = \lambda_{ij} q_j$ for some $\lambda_{ij} \in k^*$ and $n = n'$.*

(iii) *If $\phi(p_i) = \lambda_{ij} q_j$, then $r_i = s_j$.*

(iv) *$\phi(\alpha_1) = \gamma\alpha_2$ for some $\gamma \in k^*$.*

(v) *$\phi((\alpha_1, F_1)B_1) = (\alpha_2, F_2)B_2$.*

Proof. (i) Since $B_i = \mathrm{DK}(A_i)$ and $E_i = \mathrm{ML}(A_i)$ for $i = 1, 2$, the assertion follows.

(ii) Since $\phi : A_1 \to A_2$ is an isomorphism and (i) holds, identifying $\phi(A_1)$ to A_1 we can assume that $A_1 = A_2 = A$, $B_1 = B_2 = B$ and $E_1 = E_2 = E$. Now note the following

$$B \hookrightarrow A \hookrightarrow B\left[\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right] \tag{5}$$

and

$$B \hookrightarrow A \hookrightarrow B\left[\frac{1}{q_1}, \ldots, \frac{1}{q_{n'}}\right]. \tag{6}$$

Since $y' = \dfrac{F_2(x'_1, \ldots, x'_m, z', t')}{q_1^{s_1} \cdots q_{n'}^{s_{n'}}} \in A \setminus B$ and (5) holds, there exists $l > 0$ such that

$$(p_1 \cdots p_n)^l y' = \frac{(p_1 \cdots p_n)^l F_2(x'_1, \ldots, x'_m, z', t')}{q_1^{s_1} \cdots q_{n'}^{s_{n'}}} \in B.$$

Since A is an integral domain, for every j, $q_j \nmid F_2(x'_1, \ldots, x'_m, z', t')$ in B, and hence there exists i, $1 \leqslant i \leqslant n$ and $\lambda_{ij} \in k^*$ such that

$$p_i = \lambda_{ij} q_j. \tag{7}$$

Thus $n' \leqslant n$. Further using the fact that $y \in A \setminus B$, by (6) we have $n \leqslant n'$. Therefore the assertion follows.

(iii) By the given condition we can assume that (7) holds. We now show that $r_i = s_j$. If possible suppose $r_i < s_j$. Consider the ideal $I = q_j^{s_j} A \cap B = (q_j^{s_j}, F_2(x'_1, \ldots, x'_m, z', t'))B$. Further using (7), we have $I = p_i^{s_j} A \cap B = (p_i^{s_j}, p_i^{s_j - r_i} F_1(x_1, \ldots, x_m, z, t))B \subseteq p_i B = q_j B$. But it contradicts the fact that $q_j \nmid F_2(x'_1, \ldots, x'_m, z', t')$ in B. Thus $r_i \geqslant s_j$.

Next if $r_i > s_j$, considering the ideal $J = p_i^{r_i} A \cap B$ and by similar arguments as above, we get that $F_1(x_1, \ldots, x_m, z, t) \in p_i B$, which is a contradiction. Thus $r_i = s_j$.

(iv) By (7) and (iii) we have $p_i^{r_i} = \mu_{ij} q_j^{s_j}$ for some $\mu_{ij} \in k^*$ and $r_i = s_j$. Now using the factorisations of $\alpha_1(x_1, \ldots, x_m)$ and $\alpha_2(x'_1, \ldots, x'_m)$ in B we have

$$\alpha_1(x_1, \ldots, x_m) = \gamma\alpha_2(x'_1, \ldots, x'_m), \tag{8}$$

for some $\gamma \in k^*$. Thus the assertion follows.

(v) By (8), we have $\alpha_1(x_1, \ldots, x_m)A \cap B = \alpha_2(x'_1, \ldots, x'_m)A \cap B$. Thus

$$(\alpha_1(x_1, \ldots, x_m), F_1(x_1, \ldots, x_m, z, t))B = (\alpha_2(x'_1, \ldots, x'_m), F_2(x'_1, \ldots, x'_m, z', t'))B \quad (9)$$

and hence the result follows. $\square$

From now on wards for the rest of this section, we assume that A be as in (2), with $\alpha(X_1, \ldots, X_m) = a_1(X_1) \cdots a_m(X_m)$ and $F = f(Z,T) + (\prod_{j=1}^{n} p_j)h(X_1, \ldots, X_m, Z, T)$ where p_j's are all prime factors of $\alpha(X_1, \ldots, X_m)$ in $k[X_1, \ldots, X_m]$. That means

$$A = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{\Big(a_1(X_1) \cdots a_m(X_m)Y - f(Z,T) - (\prod_{j=1}^{n} p_j)h(X_1, \ldots, X_m, Z, T)\Big)}. \quad (10)$$

Remark 3.3. (i) Let λ_i be a root of $a_i(X_i)$ in $\overline{k}$ with multiplicity $r_i \geqslant 1$, for some $i, 1 \leqslant i \leqslant m$. Now note that

$$A \hookrightarrow \overline{A} := A \otimes_k \overline{k} := \frac{\overline{k}[X_1, \ldots, X_m, Y, Z, T]}{((X_i - \lambda_i)^{r_i}\alpha'(X_1, \ldots, X_m)Y - F(X_1, \ldots, X_m, Z, T))},$$

where $\alpha'(X_1, \ldots, \lambda_i, \ldots, X_m) \neq 0$. Consider the degree function ω_{λ_i} on $\overline{A}$ given by,

$$\omega_{\lambda_i}(x_i - \lambda_i) = -1,\ \omega_{\lambda_i}(y) = r_i,\ \omega_{\lambda_i}(x_j) = 0 \text{ for } 1 \leqslant j \leqslant m,\ j \neq i,\ \omega_{\lambda_i}(z) = 0,\ \omega_{\lambda_i}(t) = 0.$$

By Theorem 2.3, ω_{λ_i} induces an admissible $\mathbb{Z}$-filtration on $\overline{A}$ with respect to the generating set $\{x_1, \ldots, x_i - \lambda_i, \ldots, x_m, y, z, t\}$.

(ii) Since the filtration induced by ω_{λ_i} is admissible with respect to $\{x_1, \ldots, x_i - \lambda_i, \ldots, x_m, y, z, t\}$, if $b \in A$ be such that $\omega_{\lambda_i}(b) \leqslant 0$, then $b \in \overline{k}[x_1, \ldots, x_m, (x_i - \lambda_i)^{r_i}y, z, t]$.

(iii) Let $b \in A$ be such that $\omega_{\lambda_i}(b) > 0$, then the highest degree homogeneous summand of b with respect to ω_{λ_i} is divisible by y in $\overline{A}$.

The following result describes $\mathrm{ML}(A)$ when $\mathrm{DK}(A) = k[x_1, \ldots, x_m, z, t]$.

Proposition 3.4. *Let ϕ be a non-trivial exponential map on A such that $A^{\phi} \subseteq k[x_1, \ldots, x_m, z, t]$. Then $k[x_1, \ldots, x_m] \subseteq A^{\phi}$. In particular, if $\mathrm{DK}(A) = k[x_1, \ldots, x_m, z, t]$, then $\mathrm{ML}(A) = k[x_1, \ldots, x_m]$.*

Proof. If possible suppose $x_i \notin A^{\phi}$, for some $i, 1 \leqslant i \leqslant m$. Consider the ring $\overline{A} := A \otimes_k \overline{k}$. From (10), it is clear that

$$\overline{A} = \frac{\overline{k}[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m}\alpha_1(X_1) \cdots \alpha_m(X_m)Y - f(Z,T) - \overline{h}(X_1, \ldots, X_m, Z, T))}, \quad (11)$$

for some $r_j \geqslant 1$, $\overline{h} \in \overline{k}^{[m+2]}$, $\alpha_j \in \overline{k}^{[1]}$ with $\alpha_j(0) \neq 0$ for every j, $1 \leqslant l \leqslant m$ and $X_i \mid \overline{h}$. Let $\overline{x_1}, \ldots, \overline{x_m}, \overline{y}, \overline{z}, \overline{t}$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $\overline{A}$. Consider the non-trivial exponential map $\overline{\phi}$ on $\overline{A}$ induced by ϕ (cf. Lemma 2.1(iv)). Since $x_i \notin A^{\phi}$, it follows that $\overline{x_i} \notin \overline{A}^{\overline{\phi}}$. Further, since $A^{\phi} \subseteq k[x_1, \ldots, x_m, z, t]$, we have $\overline{A}^{\overline{\phi}} \subseteq$

$\overline{k}[\overline{x_1},\ldots,\overline{x_m},\overline{z},\overline{t}]$. By Lemma 2.1(ii), $\operatorname{tr.deg}_k(\overline{A}^{\overline{\phi}}) = m+1$. Let $\{f_1,\ldots,f_{m+1}\}$ be an algebraically independent set of elements in $\overline{A}^{\overline{\phi}}$. Suppose

$$f_j = g_j(\overline{x_1},\ldots,\overline{x_{i-1}},\overline{x_{i+1}},\ldots,\overline{x_m},\overline{z},\overline{t}) + \overline{x_i}h_j(\overline{x_1},\ldots,\overline{x_m},\overline{z},\overline{t}),$$

for every j, $1 \leqslant j \leqslant m+1$. If there exists a polynomial P such that $P(g_1,\ldots,g_{m+1}) = 0$ then $\overline{x_i} \mid P(f_1,\ldots,f_{m+1})$. Now this would contradict the fact that $\overline{x_i} \notin \overline{A}^{\overline{\phi}}$ as $P(f_1,\ldots,f_{m+1}) \in \overline{A}^{\overline{\phi}}$ and $\overline{A}^{\overline{\phi}}$ is factorially closed (cf. Lemma 2.1(i)). Therefore, $\{g_1,\ldots,g_{m+1}\}$ is an algebraically independent set of elements in $\overline{k}[\overline{x_1},\ldots,\overline{x_{i-1}},\overline{x_{i+1}},\ldots,\overline{x_m},\overline{z},\overline{t}]$. We now consider the degree function ω_i on $\overline{A}$, given by $\omega_i(\overline{x_i}) = -1$, $\omega_i(\overline{y}) = r_i$, $\omega_i(\overline{x_j}) = 0$, for $1 \leqslant j \leqslant m$, $j \neq i$, $\omega_i(\overline{z}) = \omega_i(\overline{t}) = 0$. Then by Theorem 2.3, ω_i induces an admissible $\mathbb{Z}$-filtration on $\overline{A}$ and the associated graded ring is

$$\tilde{A} = \frac{\overline{k}[X_1,\ldots,X_m,Y,Z,T]}{(X_i^{r_i}\tilde{\alpha}(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_m)Y - f(Z,T))}, \tag{12}$$

for some $\tilde{\alpha} \in \overline{k}[X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_m]$. For any element $b \in \overline{A}$, let $\tilde{b}$ denotes its image in $\tilde{A}$. Now by Theorem 2.2, $\overline{\phi}$ induces a non-trivial exponential map $\tilde{\phi}$ on $\tilde{A}$ such that $\tilde{f_j} \in \tilde{A}^{\tilde{\phi}}$ for every j, $1 \leqslant j \leqslant m+1$. Note that

$$\tilde{f_j} = g_j(\widetilde{x_1},\ldots,\widetilde{x_{i-1}},\widetilde{x_{i+1}},\ldots,\widetilde{x_m},\tilde{z},\tilde{t}), \text{ for every } j, 1 \leqslant j \leqslant m.$$

Since $g_1,\ldots,g_{m+1}$ are algebraically independent it follows that $\overline{k}[\widetilde{x_1},\ldots,\widetilde{x_{i-1}},\widetilde{x_{i+1}},\ldots,\widetilde{x_m},\tilde{z},\tilde{t}]$ is algebraic over $\overline{k}[g_1,\ldots,g_{m+1}]$. Hence $\overline{k}[\widetilde{x_1},\ldots,\widetilde{x_{i-1}},\widetilde{x_{i+1}},\ldots,\widetilde{x_m},\tilde{z},\tilde{t}] \subseteq \tilde{A}^{\tilde{\phi}}$. But by (12), this contradicts that $\tilde{\phi}$ is non-trivial. Thus we have $x_i \in A^{\phi}$ and since i is arbitrary, $k[x_1,\ldots,x_m] \subseteq A^{\phi}$.

If $\mathrm{DK}(A) = k[x_1,\ldots,x_m,z,t]$, then for every prime divisor p of a, $p \notin A^*$ and $f \notin k$. Therefore, by Lemma 3.1, we have $\mathrm{ML}(A) = k[x_1,\ldots,x_m]$. □

Now we record an easy lemma which will be useful in proving the next two results which will describe $\mathrm{DK}(A)$, upto a certain condition on the roots of $a_i(X_i)$ for every i, $1 \leqslant i \leqslant m$.

Lemma 3.5. *Let $f(Z,T) \in k[Z,T]$ be such that $k[Z,T]/(f) = k^{[1]}$ and there exists a system of coordinates $\{Z_1,T_1\}$ in $k[Z,T]$ such that $f(Z,T) = b_0(Z_1) + b_1(Z_1)T_1$ for some $b_0, b_1 \in k^{[1]}$. Then $k[Z,T] = k[f]^{[1]}$. In particular, when k is a field of positive characteristic and $g(Z,T)$ is a non-trivial line, then $g(Z,T)$ is not linear with respect to any system of coordinate in $k[Z,T]$.*

Lemma 3.6. *Suppose for every i, $1 \leqslant i \leqslant m$ there exists a multiple root λ_i of $a_i(X_i)$ in $\overline{k}$, such that $f(Z,T)$ is not linear with respect to any system of coordinate in $\overline{k}[Z,T]$. Then the following hold.*

(i) *Let w_{λ_i} denotes the degree function on $\overline{A} := A \otimes_k \overline{k}$ as in Remark* 3.3(i), then for every element $a \in \mathrm{DK}(A)$, $w_{\lambda_i}(a) \leqslant 0$. In particular, $\mathrm{DK}(A) \subsetneq A$.

(ii) If μ denotes the multiplicity of the root λ_i in $a_i(X_i)$, then $\mathrm{DK}(A) \subseteq \overline{k}[x_1,\ldots,x_m,(x_i-\lambda_i)^{\mu}y,z,t]$.

Proof. (i) Follows from Proposition 2.4.

(ii) Follows from (i) and Remark 3.3(ii). □

Corollary 3.7. *For every i, let $a_i(X_i) = \prod_{j=1}^{n_i}(X_i - \lambda_{ij})^{\mu_{ij}}$ in $\overline{k}[X_i]$, with $\mu_{ij} > 1$. Suppose that $f(Z,T)$ is not linear with respect to any coordinate system in $\overline{k}[Z,T]$. Then* $\mathrm{DK}(A) = k[x_1, \dots, x_m, z, t]$.

Proof. By Lemma 3.6(ii), $\mathrm{DK}(A) \subseteq \bigcap_{i=1}^{m}(\bigcap_{j=1}^{n_i} \overline{k}[x_1, \dots, x_m, (x_i - \lambda_{ij})^{\mu_{ij}} y, z, t]) = \overline{k}[x_1, \dots, x_m, z, t]$. Therefore, $\mathrm{DK}(A) \subseteq k[x_, \dots, x_m, z, t]$. Now the assertion follows by Lemma 3.1(i). $\square$

The following result describes isomorphisms between two rings of the form A as in (10), when every root of $a_i(X_i)$ is a multiple root.

Theorem 3.8. *Let*

$$A_1 := \frac{k[X_1, \dots, X_m, Y, Z, T]}{(a_1(X_1) \dots a_m(X_m)Y - f(Z,T) - h_1(X_1, \dots, X_m, Z, T))}$$

and

$$A_2 := \frac{k[X_1, \dots, X_m, Y, Z, T]}{(b_1(X_1) \dots b_m(X_m)Y - g(Z,T) - h_2(X_1, \dots, X_m, Z, T))}$$

be such that $a_i(X_i) := \prod_{1 \leqslant j \leqslant n}(X_i - \lambda_{ij})^{d_{ij}}$ and $b_i(X_i) := \prod_{1 \leqslant j \leqslant n'}(X_i - \mu_{ij})^{e_{ij}}$ in $\overline{k}[X_i]$ with $d_{ij}, e_{ij} > 1$ for all i, j and $\prod_{1 \leqslant j \leqslant n}(X_i - \lambda_{ij}) \mid h_1$, $\prod_{1 \leqslant j \leqslant n}(X_i - \mu_{ij}) \mid h_2$ in $\overline{k}[X_1, \dots, X_m]$. Let $x_1, \dots, x_m, y, z, t$ and $x_1', \dots, x_m', y', z', t'$ denote the images of $X_1, \dots, X_m, Y, Z, T$ in A_1 and A_2 respectively. Let $E_1 := k[x_1, \dots, x_m]$, $E_2 := k[x_1', \dots, x_m']$, $B_1 := k[x_1, \dots, x_m, z, t]$ and $B_2 := k[x_1', \dots, x_m', z', t']$. Suppose $f(Z,T)$ and $g(Z,T)$ are not linear with respect to any system of coordinate in $\overline{k}[Z,T]$ and $\phi : A_1 \to A_2$ is an isomorphism. Then the following hold.

(i) *ϕ restricts to an isomorphism from B_1 to B_2 and from E_1 to E_2.*

(ii) *For every l, $1 \leqslant l \leqslant m$, there exists i, $1 \leqslant i \leqslant m$ such that $\phi(x_i) = \nu x_l' + \mu$, for some $\nu \in k^*$ and $\mu \in k$.*

(iii) *If $\phi(x_i) = \nu x_l' + \mu$, then $a_i(X_i)$ and $b_l(X_l)$ have equal number of roots in $\overline{k}$ with equal multiplicities. Further $\phi(a_i(x_i)) = \gamma b_l(x_l')$ for some $\gamma \in k^*$.*

Proof. (i) Since $f(Z,T)$ and $g(Z,T)$ are both not linear with respect to any system of coordinate in $\overline{k}[Z,T]$, by Corollary 3.7, $\mathrm{DK}(A_i) = B_i$ and hence by Proposition 3.4, $\mathrm{ML}(A_i) = E_i$ for $i = 1, 2$. Thus the result follows.

(ii) Since $\phi : A_1 \to A_2$ is an isomorphism, it induces an isomorphism $\overline{\phi} : \overline{A_1} := A_1 \otimes_k \overline{k} \to \overline{A_2} := A_2 \otimes_k \overline{k}$. Further from Corollary 3.7 it follows that $\mathrm{DK}(\overline{A_i}) = \overline{B_i} := B_i \otimes_k \overline{k}$ and $\mathrm{ML}(\overline{A_i}) = \overline{E_i} := E_i \otimes_k \overline{k}$. Now since $\overline{\phi}$ is an isomorphism, from Theorem 3.2(ii) we know that for every l, $1 \leqslant l \leqslant m$ and a prime divisor $(x_l' - \mu_{lj_l})$ of $b_l(x_l')$ in $\overline{B_2}$, there exist some $i, 1 \leqslant i \leqslant m$ and a prime divisor $(x_i - \lambda_{ij_i})$ of $a_i(x_i)$ in $\overline{B_1}$ such that $\overline{\phi}(x_i - \lambda_{ij_i}) = \nu(x_l' - \mu_{lj_l})$ for some $\nu \in \overline{k}^*$. Now since $\overline{\phi}\,|_{B_1} = \phi$ and $\phi(B_1) = B_2$, we have the desired result.

(iii) If $\phi(x_i) = \nu x_l' + \mu$, considering $\overline{\phi} : \overline{A_1} \to \overline{A_2}$ we get that a prime divisor $(x_i - \lambda_i)$ of $a_i(x_i)$ is mapped to a prime divisor $(x_l' - \mu_l)$ of $b_l(x_l')$. Now from Theorem 3.2(ii), it follows that there is an one to one correspondence between the roots of $a_i(X_i)$ and $b_l(X_l)$ in $\overline{k}$. Now for a root λ_{ij_i} of $a_i(X_i)$ if $\overline{\phi}(x_i - \lambda_{ij_i}) = (x_l' - \mu_{lj_l})$ for some root μ_{lj_l} of $b_l(X_l)$, then by Theorem 3.2(iii), λ_{ij_i} and μ_{lj_l} have the same multiplicity. Therefore, it follows that $\phi(a_i(x_i)) = \gamma b_l(x_l')$ for some $\gamma \in k^*$. $\square$

The next result characterizes the automorphisms of A as in (10) when every root of $a_i(X_i)$ is a multiple root for every i, $1 \leqslant i \leqslant m$.

Theorem 3.9. *Let A be the affine domain as in* (10)*, with $a_i(X_i) = \prod_{1\leqslant j\leqslant n_i}(X_i - \lambda_{ij})^{d_{ij}}$ in $\overline{k}[X_i]$ with $d_{ij} > 1$, $1 \leqslant i \leqslant m$. Let $x_1, \dots, x_m, y, z, t$ be the images of $X_1, \dots, X_m, Y, Z, T$ in A. Suppose that $f(Z,T)$ is not linear with respect to any co-ordinate system in $\overline{k}[Z,T]$. If $\phi \in \mathrm{Aut}_k(A)$, then the following hold:*

(i) *ϕ restricts to an automorphism of $E := k[x_1, \dots, x_m]$ and $B := k[x_1, \dots, x_m, z, t]$.*

(ii) *For every i, $1 \leqslant i \leqslant m$, there exists j, $1 \leqslant j \leqslant m$ such that $\phi(a_i(x_i)) = \gamma a_j(x_j)$ for some $\gamma \in k^*$.*

(iii) *$\phi(I) = I$, where $I = (\alpha(x_1, \dots, x_m), F(x_1, \dots, x_m, z, t))\, B$.*

Conversely, if $\phi \in \mathrm{End}_k(A)$ satisfies conditions (i) and (iii), then $\phi \in \mathrm{Aut}_k(A)$.

Proof. (i), (ii) and (iii) follows from Theorem 3.8(i), Theorem 3.8(iii) and Theorem 3.2(v) respectively.

We now show the converse part. From (i) and (iii) it follows that $\phi(B) = B$, $\phi(E) = E$ and $\phi(I) = I$. Hence $\phi(I \cap E) = I \cap E$. Since $I \cap E = (\alpha(x_1, \dots, x_m))\, E$,

$$\phi(\alpha(x_1, \dots, x_m)) = \gamma\alpha(x_1, \dots, x_m), \tag{13}$$

for some $\gamma \in k^*$. Now since ϕ is an automorphism of B and $A \subseteq B[\alpha(x_1, \dots, x_m)^{-1}]$, using (13) it follows that ϕ is an injective endomorphism of A. Therefore, it is enough to show that ϕ is surjective. For this, it is enough to find a preimage of y in A. Since $\phi(I) = I$, we have

$$F = \alpha(x_1, \dots, x_m)u(x_1, \dots, x_m, z, t) + \phi(F)v(x_1, \dots, x_m, z, t), \tag{14}$$

for some $u, v \in B$. Since $y = \frac{F(x_1,\dots,x_m,z,t)}{\alpha(x_1,\dots,x_m)}$, using (13) and (14) we have

$$y = u(x_1, \dots, x_m, z, t) + \frac{\phi(F)v(x_1, \dots, x_m, z, t)}{\gamma^{-1}\phi(\alpha(x_1, \dots, x_m))}. \tag{15}$$

Since $\phi(B) = B$, there exist $\widetilde{u}, \widetilde{v} \in B$ such that $\phi(\widetilde{u}) = u$ and $\phi(\widetilde{v}) = v$. And hence from (15), we get that $y = \phi(\widetilde{u} + \gamma y\widetilde{v})$, where $\widetilde{u} + \gamma y\widetilde{v} \in A$. ☐

We now recall the following result from [10, Proposition 3.6].

Theorem 3.10. *Let R be an integral domain, $\pi_1, \dots, \pi_n \in R$ and $\pi = \pi_1 \cdots \pi_n$. Let $G(Z,T) \in R[Z,T]$ be such that $R[Z,T]/(\pi, G(Z,T)) \cong_R (R/\pi)^{[1]}$. For positive integers $r_1, \dots, r_n$, set*

$$D := \frac{R[Y,Z,T]}{(\pi_1^{r_1} \cdots \pi_n^{r_n} Y - G(Z,T))}.$$

Then $D^{[1]} = R^{[3]}$.

Let

$$A(r_1, \dots, r_n, f) := \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))},$$

where $r_i > 1$ for i, $1 \leqslant i \leqslant m$ and f is a non-trivial line in $k[Z,T]$. Let

$$\Omega_1 := \{A(r_1, \ldots, r_m, f) \mid r_i > 1 \text{ and } f \text{ is a non-trivial line in } k[Z,T]\}.$$

In [10] Gupta has shown the rings $A(r_1, \ldots, r_m, f)$ are counter example to the Zariski Cancellation Problem in positive characteristic. Whereas in [6, corollary 4.4], it has been shown that Ω_1 contains an infinite family of pairwise non-isomorphic rings which are counter example to the Zariski Cancellation Problem.

Corollary 3.11. *Let k be a field of positive characteristic. Then there exists an infinite family of pairwise non-isomorphic varieties which are counterexamples to the Zariski Cancellation Problem in higher dimensions($\geqslant 3$), and are also non-isomorphic to the members of Ω_1.*

Proof. Let $f(Z,T)$ be a non-trivial line in $k[Z,T]$. Consider the following family of varieties:

$$\Omega_2 := \{A_{(a_1,\ldots,a_m)} := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(a_1(X_1) \cdots a_m(X_m)Y - f(Z,T))} \mid \text{ each } a_i \text{ has only multiple roots in } \overline{k}$$
$$\text{and some } a_i \text{ has at least two distinct roots } \}.$$

By Theorem 3.10, $A^{[1]}_{(a_1,\ldots,a_m)} = k^{[m+3]}$, for every $A_{(a_1,\ldots,a_m)} \in \Omega$. Now by Theorem 2.5, $A_{(a_1,\ldots,a_m)} \otimes_k \overline{k} \neq \overline{k}^{[m+2]}$. Hence $A_{(a_1,\ldots,a_m)}$ is counterexample to the Zariski Cancellation Problem.

Now since $f(Z,T)$ be a non-trivial line in $k[Z,T]$, by Lemma 3.5, $f(Z,T)$ is not linear with respect to any coordinate system in $\overline{k}[Z,T]$. Hence by Theorem 3.8(ii) and (iii), $A_{(a_1,\ldots,a_m)} \in \Omega_2$ is not isomorphic to the rings in Ω_1. Now we consider two sets of polynomials $S_1 = \{a_1(X_1), \ldots, a_m(X_m)\}$ and $S_2 = \{b_1(X_1), \ldots, b_m(X_m)\}$ such that each $a_i(X_i)$, $b_i(X_i)$ has only multiple roots and total number of roots of $a_1 \ldots a_m$ is different from total number of roots of $b_1 \ldots b_m$. Then $A_{(a_1,\ldots,a_m)} \ncong A_{(b_1,\ldots,b_m)}$, by Theorem 3.8(iii). Therefore, Ω_2 contains infinitely many pairwise non-isomorphic varieties which are counterexamples to the ZCP in dimension $m+2$ for $m \geqslant 1$. □

References

[1] S.S. Abhyankar, P. Eakin and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra **23** (1972), 310–342.

[2] S. S. Abhyankar and T. T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148–166.

[3] A. J. Crachiola, *The hypersurface $x + x^2y + z^2 + t^3 = 0$ over a field of arbitrary characteristic*, Proc. Amer. Math. Soc. **134** (2005), 1289–1298.

[4] H. Derksen, O. Hadas and L. Makar-Limanov, *Newton polytopes of invariants of additive group actions*, J. Pure Appl. Algebra **156** (2001), 187–197.

[5] T. Fujita, *On Zariski problem*, Proc. Japan Acad. **55A** (1979) 106–110.

[6] P. Ghosh and N. Gupta, *On the triviality of a family of linear hyperplanes*, Adv. Math. **428** (2023), 109166.

[7] P. Ghosh and N. Gupta, *On Generalised Danielewski and Asanuma varieties*, J. Algebra **632** (2023), 226–250.

[8] P. Ghosh, N. Gupta and A. Pal, *On the Epimorphism Problem for a family of linear hypersurfaces in* $\mathbb{A}_k^n$, preprint.

[9] N. Gupta, *On the cancellation problem for the affine space* $\mathbb{A}^3$ *in characteristic* p, Invent. Math. **195** (2014), 279–288.

[10] N. Gupta, *On Zariski's Cancellation Problem in positive characteristic*, Adv. Math. **264** (2014), 296–307.

[11] S.M. Bhatwadekar and Neena Gupta, *A note on the cancellation property of* $k[X, Y]$, Journal of Algebra and its Applications (special issue in honour of Prof. Shreeram S. Abhyankar), **14(9)** (2015), 15400071–5.

[12] M. Miyanishi, *Lectures on Curves on rational and unirational surfaces*, Narosa Publishing House, New Delhi (1978).

[13] M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ. **20** (1980) 11–42.

[14] M. Nagata, *On automorphism group of* $k[X, Y]$, Kyoto Univ. Lec. Math. **5**, Kinokuniya, Tokyo (1972).

[15] P. Russell, *On affine-ruled rational surfaces*, Math. Ann. **255** (1981) 287–302.

[16] B. Segre, *Corrispondenze di Möbius e trasformazioni cremoniane intere*, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956/1957), 3–19.

[17] M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace* $\mathbb{C}^2$, J. Math. Soc. Japan **26** (1974), 241–257.