A remark on the B-model Comparison of Hodge structures

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Abstract

In this note we record a comparison theorem on the B-model variation of semi-infinite Hodge structures. This result is considered a folklore theorem by experts in the field. We only take this opportunity to write it down. Our motivation is to apply it in the study of B-model categorical enumerative invariants.

B-model comparison of Hodge structures. Let X be a smooth and projective variety over $\mathbb{C}$. Then its cohomology group carries an integral polarized Hodge structures. Remarkably, a large part of this structure may be reconstructed only from the derived category of X. More precisely, let us denote by $\mathcal{C}_X$ a dg-enhancement of its derived category $D^b\big(\mathsf{Coh}(X)\big)$. By [20], such a dg-enhancement is essentially unique. Following Kontsevich-Soibelman [17] and Kazarkov-Kontsevich-Pantev [12], one obtains a weaker version of Hodge structures from $\mathcal{C}_X$ which was called nc-Hodge structures. The purpose of this note is to compare the nc-Hodge structures of $\mathcal{C}_X$ with the classical Hodge structures of X. By a series of works [19, 26, 27, 13], we recall the following table of comparison results.

Commutative	Non-commutative
Coherent cohomology $\oplus_{p-q=*}H^q(\Omega_X^p)$	Hochschild homology $HH_*(\mathcal{C}_X)$
Coherent cohomology $\oplus_{p+q=*}H^q(\Lambda^p T_X)$	Hochschild cohomology $HH^*(\mathcal{C}_X)$
De Rham cohomology $H^*(X,\Omega_X^\bullet)$, $(*\in\mathbb{Z}/2\mathbb{Z})$	Periodic cyclic homology $HP_*(\mathcal{C}_X)$, $(*\in\mathbb{Z}/2\mathbb{Z})$
Hodge filtration $F^\bullet H^*(X,\Omega_X^\bullet)$	nc-Hodge filtration $F^\bullet HP_*(\mathcal{C}_X)$

Let us denote by $T_{\mathsf{poly}} := \oplus_p \Lambda^p T_X[-p]$ and $A_{\mathsf{poly}} := \oplus_p \Omega_X^p[p]$, both endowed with the zero differential. The comparison map in the above table is given by the Hochschild-Konstant-Rosenberg [11] (HKR) isomorphisms:

$$I^{\mathsf{HKR}} : H^*(X, T_{\mathsf{poly}}) \to HH^*(\mathcal{C}_X),$$
$$I_{\mathsf{HKR}} : HH_*(\mathcal{C}_X) \to H^*(X, A_{\mathsf{poly}}).$$

Motivated from Tsygan formality [25] in the algebraic case, Căldăraru [5] conjectured the following comparison table.

Pairing on $\oplus_{p-q=*}H^q(\Omega_X^p)$	Mukai pairing on $HH_*(\mathcal{C}_X)$
$H^*(X,T_{\mathsf{poly}})$ with exterior product	$HH^*(\mathcal{C}_X)$ with cup product
$H^*(X,A_{\mathsf{poly}})$ as an $H^*(X,T_{\mathsf{poly}})$-module	$HH_*(\mathcal{C}_X)$ as an $HH^*(\mathcal{C}_X)$-module

The comparison of pairing was proved by Ramadoss [21]. The comparison of product structures and module structures were later proved by Calaque-Rossi-Van den Bergh [4], see also Calaque-Rossi [3]. Remarkably the comparison map in Căldăraru's conjecture is no longer the HKR isomorphism, and the proof [4] requires using Kontsevich's formality map [16]. Explicitly, they are given by the following compositions:

$$J^* : H^*(X, T_{\mathsf{poly}}) \xrightarrow{\sqrt{\mathsf{td}(X)}} H^*(X, T_{\mathsf{poly}}) \xrightarrow{I^{\mathsf{HKR}}} HH^*(\mathcal{C}_X),$$

$$J_* : HH_*(\mathcal{C}_X) \xrightarrow{I_{\mathsf{HKR}}} H^*(X, A_{\mathsf{poly}}) \xrightarrow{\sqrt{\mathsf{td}(X)}} H^*(X, A_{\mathsf{poly}}).$$

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Here $\mathsf{td}(X)$ stands for the Todd class of X defined by the series $\frac{z}{1-e^{-z}}$. Following [3, 4], let us also define a third pair of isomorphisms by

$$K^*: H^*(X, T_{\mathsf{poly}}) \xrightarrow{\sqrt{\mathsf{td}'(X)}} H^*(X, T_{\mathsf{poly}}) \xrightarrow{I^{\mathsf{HKR}}} HH^*(\mathcal{C}_X),$$
$$K_*: HH_*(\mathcal{C}_X) \xrightarrow{I_{\mathsf{HKR}}} H^*(X, A_{\mathsf{poly}}) \xrightarrow{\sqrt{\mathsf{td}'(X)}} H^*(X, A_{\mathsf{poly}}).$$

In the above definition, $\mathsf{td}'(X)$ is the modified Todd class defined by the series $\frac{z}{e^{z/2}-e^{-z/2}}$.

The goal of this paper is to further extend the above comparison results. Since the chain-level Hochschild-Konstant-Rosenberg map is compatible intertwines the Connes operator with the de Rham differential, we also obtain an isomorphism denoted by

$$\widetilde{I}_{\mathsf{HKR}}: H^*\big(X, (C_\bullet(X)[[u]], b+uB)\big) \to H^*\big(X, (\Omega_X^\bullet[[u]], ud_{DR})\big).$$

Twisting by the square root of Todd class and the modified Todd class yields

$$\widetilde{J}: H^*\big(X, (C_\bullet(X)[[u]], b+uB)\big) \xrightarrow{\widetilde{I}_{\mathsf{HKR}}} H^*\big(X, (\Omega_X^\bullet[[u]], ud_{DR})\big) \xrightarrow{\wedge\sqrt{\mathsf{td}(X)}} H^*\big(X, (\Omega_X^\bullet[[u]], ud_{DR})\big),$$
$$\widetilde{K}: H^*\big(X, (C_\bullet(X)[[u]], b+uB)\big) \xrightarrow{\widetilde{I}_{\mathsf{HKR}}} H^*\big(X, (\Omega_X^\bullet[[u]], ud_{DR})\big) \xrightarrow{\wedge\sqrt{\mathsf{td}'(X)}} H^*\big(X, (\Omega_X^\bullet[[u]], ud_{DR})\big).$$

Furthermore, let us denote by $\int_X : H^n(X, \omega_X) \to \mathbb{C}$ the trace map from Lipman [18]. Note that by [22], we have

$$\int_X = (-1)^{n(n-1)/2}(\frac{1}{2\pi i})^n \int_{X^{an}}, \tag{1}$$

with $\int_{X^{an}} : H^{2n}(X^{an}, \mathbb{C}) \to \mathbb{C}$ the analytic integration map.

Theorem 0.1. *The following comparison results hold:*

1. *$\widetilde{J}$ intertwines higher residue pairings in the sense that*

$$\langle \alpha, \beta\rangle_{\mathsf{hres}} = (-1)^{n(n+1)/2}\int_X \widetilde{J}(\alpha)\wedge\widetilde{J}(\beta^\vee),$$

   where $\vee$ is the canonical isomorphism denoted by $\vee : HC^-(\mathcal{C}_X) \to HC^-(\mathcal{C}_X^{\mathsf{op}})$. Under the HKR isomorphism, this map corresponds to $(-1)^p$ on the space of p-forms.

2. *Both $\widetilde{J}$ and $\widetilde{K}$ intertwines the rational structures.*

3. *In the family situation, the map $\widetilde{K}$ intertwines the Gauss-Manin connection with the Getzler connection.*

Remark 0.1. *We make a few miscellaneous comments on conventions, signs and factors used in the literature that could be confusing from a first reading. Indeed, Part (1.) is recently proved in [15] which is based on [14]. Note that the formula obtained therein describes the so-called canonical pairing $\langle -, -\rangle_{\mathsf{can}} : HC^-(\mathcal{C}_X)\otimes HC^-(\mathcal{C}_X^{\mathsf{op}}) \to \mathbb{C}[[u]]$ (defined by Shklyarov [24]) by the following formula:*

$$\langle \alpha, \beta\rangle_{\mathsf{can}} = (-1)^{n(n+1)/2}\int_X I_{\mathsf{HKR}}(\alpha)\wedge I_{\mathsf{HKR}}(\beta)\wedge\mathsf{td}(X).$$

It implies (1.) by precomposing with $\mathsf{id}\otimes\vee$. In Ramadoss [21], a different trace map is used, which differs from our $\int_X$ by the sign $(-1)^{n(n+1)/2}$. Finally, throughout the literature the non-commuative Chern character (and hence the definition of Todd class $\mathsf{td}(X)$) differs from the usual Chern character by a factor $(-\frac{1}{2\pi i})^p$ at the (p,p)-component. This is compatible with Equation (1) to deduce classical Hirzebruch-Riemann-Roch formula from (1.) [15, 14].

Remark 0.2. *Part (2.) is proved by Blanc [1]. Indeed, Blanc gave a purely categorical construction of topological K-theory of X, and showed that the HKR isomorphism intertwines the rational structures on $HP_*(\mathcal{C}_X)$ and $H^*(X,\mathbb{C})$. We only need to observe that both $\sqrt{\mathsf{td}(X)}$ and $\sqrt{\mathsf{td}'(X)}$ are rational classes, hence $\widetilde{J}$ and $\widetilde{K}$ still preserve the rational structure.*

Remark 0.3. *Note that the two Todd classes are related by* $\mathsf{td}'(X) = \mathsf{td}(X)\exp(-c_1/2)$*, in the Calabi-Yau case we obtain* $\widetilde{J} = \widetilde{K}$*. In particular, the theorem above verifies a conjecture of Ganatra-Perutz-Sheridan [9] that* $\widetilde{J}$ *is an isomorphism of polarized Variation of Semi-infinite Hodge Structures (VSHS). Furthermore, in the Calabi-Yau case, since* $\mathsf{td}(X)$ *is concentrated at* $(2l, 2l)$ *components, Part (1.) of the theorem above implies the symmetry property of the higher residue pairing:*

$$\langle \alpha, \beta \rangle_{\mathsf{hres}} = (-1)^{n+|\alpha||\beta|} \langle \beta, \alpha \rangle_{\mathsf{hres}}.$$

Calaque-Rossi-Van den Bergh's proof of Căldăraru's conjecture. By the previous remarks, it remains to prove part (3.) of Theorem 0.1. Since the proof is along the same way as that of the Căldăraru's conjecture in [3, 4]. We first briefly sketch the proof in *Loc. Cit.*. The main idea is to use formal geometry [8, 7] and fiberwise apply Kontsevich's and Shoikhet's formality map. Following [3, 4], we introduce some notations:

- Denote by $\mathcal{T}_{\mathsf{poly}}$ the bundle of fiberwise holomorphic poly-vector fields on X.
- Denote by $\mathcal{A}_{\mathsf{poly}}$ the bundle of fiberwise holomorphic differential forms on X.
- Denote by D_{poly} the bundle of holomorphic poly-differential operators on X, and $\mathcal{D}_{\mathsf{poly}}$ the bundle of fiberwise holomorphic poly-differential operators on X.
- Denote by $\widehat{C}_\bullet$ the sheaf of continuous Hochschild chains on X, and $\widehat{\mathcal{C}}_\bullet$ its fiberwise version.

It is well-known that these fiberwise bundles all admit a Fedosov [8] connection D such that the associated de Rham complex of D is a resolution of the corresponding non-fiberwise bundles. For example, in the case of poly-vector fields, there are quasi-isomorphisms

$$\lambda : \big(\Omega^{0,*}(X, T_{\mathsf{poly}}), \overline{\partial}\big) \to \big(\Omega(X, \mathcal{T}_{\mathsf{poly}}), D\big), \quad \pi : \big(\Omega(X, \mathcal{T}_{\mathsf{poly}}), D\big) \to \big(\Omega^{0,*}(X, T_{\mathsf{poly}}), \overline{\partial}\big).$$

The maps λ, π are the "Taylor series" map and "Projection to constant term" map respectively. One may verify both λ and π respects all existing algebraic structures on both sides, see [2, Section 10.2].

Then, the desired comparison map $K^* : H^*(X, T_{\mathsf{poly}}) \to HH^*(X)$, in the chain level, is given by composing along the following diagram.

$$\begin{tikzcd}
\big(\Omega(X, \mathcal{T}_{\mathsf{poly}}), D\big) \arrow[r, "\mathcal{U}_Q"] & \big(\Omega(X, \mathcal{D}_{\mathsf{poly}}), D + d_H\big) \arrow[d, "\pi"] \\
\big(\Omega^{0,*}(X, T_{\mathsf{poly}}), \overline{\partial}\big) \arrow[u, "\lambda"] \arrow[r, "K^*"] & \big(\Omega^{0,*}(X, D_{\mathsf{poly}}), \overline{\partial} + d_H\big)
\end{tikzcd} \tag{2}$$

In the diagram above, we need to explain the construction of the map $\mathcal{U}_Q$. Let us sketch this construction following [2]. Let $U \subset X$ be a holomorphic coordinate chart. Then the Fedosov connection D on both bundles $\mathcal{T}_{\mathsf{poly}}$ is of the form

$$D = \partial + \overline{\partial} + Q_U,$$

for some fiberwise vector fields valued one-form $Q_U \in \Omega^1(U, \mathcal{T})$. Then, on the open subset U, the map $\mathcal{U}_Q$ is defined by

$$\mathcal{U}_{Q_U}(\alpha) := \sum_n \frac{1}{n!} \mathcal{U}_{1+n}(\alpha, Q_U^{\otimes n}),$$

where $\mathcal{U} : \mathcal{T}_{\mathsf{poly}} \to \mathcal{D}_{\mathsf{poly}}$ is the fiberwise Kontsevich formality map [16]. The upshot is that on a different chart V, the difference of the two fiberwise vector fields $Q_U - Q_V$ is a linear fiberwise vector field on $U \cap V$, which implies that we have

$$\begin{aligned}
\mathcal{U}_{Q_U}(\alpha) &= \sum_n \frac{1}{n!} \mathcal{U}_{1+n}(\alpha, Q_U^{\otimes n}) \\
&= \sum_n \frac{1}{n!} \mathcal{U}_{1+n}(\alpha, Q_V^{\otimes n}) \\
&= \mathcal{U}_{Q_V}(\alpha).
\end{aligned}$$

In the second equality, we used the fact that Kontsevich's formality map vanishes whenever one of the input entry is a linear vector field. This calculation shows that we obtain a well-defined global map giving the desired chain map $\mathcal{U}_Q : \big(\Omega(X, \mathcal{T}_{\mathsf{poly}}), D\big) \to \big(\Omega(X, \mathcal{D}_{\mathsf{poly}}), D\big)$. It is then a genius calculation [3] to verify that indeed the induced map in cohomology of the composition $K^* := \pi \mathcal{U}_Q \lambda$ is indeed equal to $I^{\mathsf{HKR}} \circ \sqrt{\mathsf{td}'(X)}$.

Similar construction works in the comparison between $HH_*(X)$ and $H^*(X, A_{\mathsf{poly}})$. In this case, we have the following diagram:

$$\begin{array}{ccc} \big(\Omega(X, \widehat{\mathcal{C}}_\bullet), D + b\big) & \xrightarrow{\mathcal{U}_Q^{\mathsf{Sh}}} & \big(\Omega(X, \mathcal{A}_{\mathsf{poly}}), D\big) \\ \lambda \big\uparrow & & \big\downarrow \pi \\ \big(\Omega^{0,*}(X, \widehat{C}_\bullet), \overline{\partial} + b\big) & \xrightarrow{K_*} & \big(\Omega^{0,*}(X, A_{\mathsf{poly}}), \overline{\partial}\big) \end{array} \tag{3}$$

Note that the comparison map is in the opposite direction $K_* : HH_*(X) \to H^*(X, A_{\mathsf{poly}})$. Let us also briefly sketch the construction of the map $\mathcal{U}_Q^{\mathsf{Sh}}$. Over a local holomorphic chart U, the element Q_U acts on $\widehat{\mathcal{C}}_\bullet)$ and $\mathcal{A}_{\mathsf{poly}}$ by the Lie derivative action. Hence locally on U, we may use Q_U to deform the Tsygan's formality map constructed by Shoihket [25]. This deformed map is given by

$$\mathcal{U}_{Q_U}^{\mathsf{Sh}}(-) := \sum_n \frac{1}{n!} \mathcal{U}_{1+n}^{\mathsf{Sh}}(-, Q_U^{\otimes n}),$$

where $\mathcal{U}^{\mathsf{Sh}} : \widehat{\mathcal{C}}_\bullet \to \mathcal{A}_{\mathsf{poly}}$ is the original fiberwise Tsygan's formality map [25]. Again, one can show these locally defined maps agree on overlaps, which yields a globally defined map $\mathcal{U}_Q^{\mathsf{Sh}} : \big(\Omega(X, \widehat{\mathcal{C}}_\bullet), D\big) \to \big(\Omega(X, \mathcal{A}_{\mathsf{poly}}), D\big)$.

Finishing the proof. We need to set up the context to compare the Gauss-Manin connection with the Getzler connection. Indeed, let us consider a family of complex analytic deformations of X, i.e. a proper and submersive map $p : \mathfrak{X} \to \Delta$ from a complex manifold $\mathfrak{X}$ to a small open ball $\Delta \subset \mathbb{C}^\mu$ centered at the origin. We require that $p^{-1}(0) = X$. We think of $\mathfrak{X}$ as the total space of a deformation of X parametrized by the open ball Δ.

By Ehresmann's fibration theorem, one can always find a *transversely holomorphic trialization*, i.e. a diffeomorphism

$$(\phi, p) : \mathfrak{X} \to X \times \Delta,$$

such that $\phi|_{p^{-1}(0)} = \mathsf{id}$, and fibers of $\phi : \mathfrak{X} \to X$ are holomorphic submanifolds transverse to X.

Thus, we may push-forward the complex structure of $\mathfrak{X}$ to that of $X \times \Delta$ using the diffeomorphism (ϕ, p). Since in the transverse direction the fibers are always biholomorphic to Δ, the complex structure is not changed in the Δ direction. After shrinking Δ if needed, the push-forward complex structure is equivalent to the following data

$$\xi(t_1, \ldots, t_\mu) \in \Omega^{0,1}(X, T_X^{1,0}) \hat{\otimes} \mathcal{O}_\Delta,$$

with $t_1, \ldots, t_\mu$ holomorphic coordinates on Δ and that $\xi(0) = 0$. Furthermore, the integrability of the complex structure is equivalent to the Maurer-Cartan equation

$$\overline{\partial}\xi + \frac{1}{2}[\xi, \xi] = 0.$$

Now, back to the proof of (3.) in Theorem 0.1, we may use the map λ in Diagram (2) to push-forward $\xi(t)$ we obtain Maurer-Cartan elements $\lambda_*\xi$ which, as in the case of Q_U, yields also a fiberwise vector field valued one form. Deforming Diagram (3) using the Maurer-Cartan elements $\xi(t)$ and $\lambda_*\xi(t)$ yields a new diagram:

$$\begin{array}{ccc} \big(\Omega(X, \widehat{\mathcal{C}}_\bullet) \hat{\otimes} \mathcal{O}_\Delta, D + b + L_{\lambda_*\xi}\big) & \xrightarrow{\mathcal{U}_{Q,\lambda_*\xi}^{\mathsf{Sh}}} & \big(\Omega(X, \mathcal{A}_{\mathsf{poly}}) \hat{\otimes} \mathcal{O}_\Delta, D + L_{\lambda_*\xi}\big) \\ \lambda \big\uparrow & & \big\downarrow \pi \\ \big(\Omega^{0,*}(X, \widehat{C}_\bullet) \hat{\otimes} \mathcal{O}_\Delta, \overline{\partial} + b + L_\xi\big) & \xrightarrow{K_*} & \big(\Omega^{0,*}(X, A_{\mathsf{poly}}) \hat{\otimes} \mathcal{O}_\Delta, \overline{\partial} + L_\xi\big) \end{array} \tag{4}$$

Here $\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}$ is the deformed Tsygan's formality map further deformed by the Maurer-Cartan element $\lambda_*\xi$. Explicitly, locally in a holomorphic coordinate chart on $U \subset X$, it is given by

$$\mathcal{U}^{\mathsf{Sh}}_{Q_U,\lambda_*\xi}(-) := \sum_{n,m} \frac{1}{n!m!}\mathcal{U}^{\mathsf{Sh}}_{1+n+m}(-, Q_U^{\otimes n}, (\lambda_*\xi)^{\otimes m}).$$

Since $\lambda_*\xi$ is globally defined, on another such chart V, we still have $\mathcal{U}^{\mathsf{Sh}}_{Q_U,\lambda_*\xi} = \mathcal{U}^{\mathsf{Sh}}_{Q_V,\lambda_*\xi}$ because $Q_U - Q_V$ is a linear fiberwise vector field on $U \cap V$. Thus we obtain a globally defined map $\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}$.

Furthermore, it is proved by Willwatcher [28] that the Shoikhet formality map intertwines the Connes differential with the de Rham differential. Hence the above diagram extends to the following diagram:

$$\begin{array}{ccc} \left(\Omega(X,\widehat{\mathcal{C}}_\bullet)\hat{\otimes}\mathcal{O}_\Delta[[u]], D+b+uB+L_{\lambda_*\xi}\right) & \xrightarrow{\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}} & \left(\Omega(X,\mathcal{A}_{\mathsf{poly}})\hat{\otimes}\mathcal{O}_\Delta[[u]], D+ud_{DR}+L_{\lambda_*\xi}\right) \\ \lambda\Big\uparrow & & \Big\downarrow\pi \\ \left(\Omega^{0,*}(X,\widehat{C}_\bullet)\hat{\otimes}\mathcal{O}_\Delta[[u]], \overline{\partial}+b+uB+L_\xi\right) & \xrightarrow{\widetilde{K}} & \left(\Omega^{0,*}(X,A_{\mathsf{poly}})\hat{\otimes}\mathcal{O}_\Delta[[u]], \overline{\partial}+ud_{DR}+L_\xi\right) \end{array} \tag{5}$$

Note that by definition $\widetilde{K} := \pi \circ \mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi} \circ \lambda$.

There are naturally defined connection operators in the t-variables direction on the complexes in the above diagram, essentially all reflecting the fact that they are all trivialized in the t-direction. For example, let us consider the complex $\left(\Omega^{0,*}(X,A_{\mathsf{poly}})\hat{\otimes}\mathcal{O}_\Delta[[u]], \overline{\partial}+ud_{DR}+L_\xi\right)$ in the lower right corner. Then we may define a connection operator using the Maurer-Cartan element $\xi(t_1,\ldots,t_\mu)$ on it by

$$\nabla^{\mathsf{GM}}_{\frac{\partial}{\partial t_j}} := \frac{\partial}{\partial t_j} - \frac{\iota(\frac{\partial\xi}{\partial t_j})}{u}.$$

Similarly, on its fiberwise bundle we have the connection still denoted by $\nabla^{\mathsf{GM}}_{\frac{\partial}{\partial t_j}} := \frac{\partial}{\partial t_j} - \frac{\iota(\frac{\partial\lambda_*\xi}{\partial t_j})}{u}$. On the non-commutative side (i.e. the left column), following Getlzer [10] we have

$$\nabla^{\mathsf{Get}}_{\frac{\partial}{\partial t_j}} := \frac{\partial}{\partial t_j} - \frac{\iota(\frac{\partial\xi}{\partial t_j})}{u},$$

with $\iota(\frac{\partial\xi}{\partial t_j}) = b^{1|1}(\frac{\partial\xi}{\partial t_j})) + uB^{1|1}(\frac{\partial\xi}{\partial t_j}))$ by the non-commutative Calculus structure. The notations used here borrows that of Sheridan [23]. Again, we use the same notation for its fiberwise version $\nabla^{\mathsf{Get}}_{\frac{\partial}{\partial t_j}} := \frac{\partial}{\partial t_j} - \frac{\iota(\frac{\partial\lambda_*\xi}{\partial t_j})}{u}$.

Lemma 0.1. *The map $\widetilde{K}$ intertwines the Getzler connection with the Gauss-Manin connection after taking cohomology.*

Proof. Since both λ and π respects all existing algebraic structures on both sides, see [2, Section 10.2], it follows that the maps λ and π intertwine the connection operators. Thus, we need to prove that

$$\nabla^{\mathsf{GM}}_{\frac{\partial}{\partial t}} \circ \mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi} - \mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi} \circ \nabla^{\mathsf{Get}}_{\frac{\partial}{\partial t}} = 0,$$

after taking cohomology.

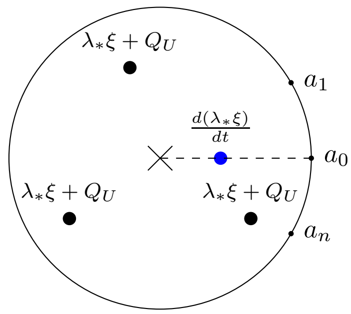
On a local chart U, we can acheive this by a homotopy operator constructed by Cattaneo-Felder-Willwacher [6]. Indeed, consider the configuration space $E_{k+2,n+1}$ of $k+2$ interior marked points $(z_0,\cdots,z_{k+1})$, and $n+1$ boundary marked points $(z_{\overline{0}},\cdots,z_{\overline{n}})$ on the unit disk, such that the first two interior points are such that $z_0 = (0,0)$ and $z_1 = (r,0)$ with $r \in [0,1]$. The notion of admissible graphs are defined in the same way as before. For each admissible graph, set

$$c_\Gamma := \int_{E_{k+2,n+1}} \bigwedge_{e\in E_\Gamma} d\theta_e.$$

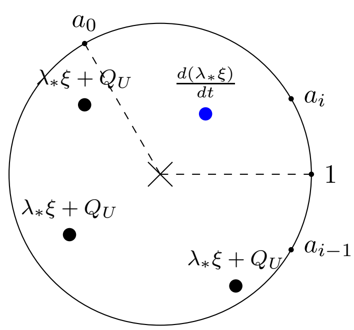
Define an operator $H_{0,U} : \big(\Omega(X,\widehat{\mathcal{C}}_\bullet)[[u]], d+b+uB+L_{Q_U+\lambda_*\xi}\big) \to \big(\Omega(X,\mathcal{A}_{\mathsf{poly}})[[u]], d+ud_{DR}+L_{Q_U+\lambda_*\xi}\big)$ by setting

$$H_{0,U}(a_0|a_1|\cdots|a_n) := \sum_{k\geq 0,\Gamma\in E_{k+2,n+1}} \frac{1}{k!}c_\Gamma\cdot\mathcal{U}_\Gamma^{\mathsf{Sh}}\Big(\frac{d(\lambda_*\xi)}{dt},(\lambda_*\xi+Q_U)^{\otimes k};a_0|a_1|\cdots|a_n\Big).$$

The operator $\mathcal{U}_\Gamma^{\mathsf{Sh}}(\frac{d(\lambda_*\xi)}{dt},(\lambda_*\xi+Q_U)^{\otimes k};a_0|a_1|\cdots|a_n)$ (defined in the same way as in Shoikhet's formula) is illustrated in the following diagram:



Define another operator $H_{1,U}$ in a similar way with the first marked point (corresponding to the point where $\frac{d(\lambda_*\xi)}{dt}$ is inserted) constrainted between the framing of the central point $(0,0)$ and the boundary marked point $\overline{0}$. A typical configuration is illustrated in the following picture.



Then the following identity was proved in [6, Proposition 1.4 and Theorem 4.1]:

$$\begin{aligned}&u\big(\nabla^{\mathsf{GM}}_{\frac{\partial}{\partial t}}\pm\frac{1}{2}d_{DR}\iota(\frac{d(\lambda_*\xi)}{dt})d_{DR}\big)\circ\mathcal{U}^{\mathsf{Sh}}_{Q_U,\lambda_*\xi}-\mathcal{U}^{\mathsf{Sh}}_{Q_U,\lambda_*\xi}\circ(u\nabla^{\mathsf{Get}}_{\frac{\partial}{\partial t}})\\ &=H_U\circ(d+b+uB+L_{Q_U+\lambda_*\xi})+(L_{Q_U+\lambda_*\xi}+d+ud_{DR})\circ H_U,\end{aligned}$$

where $H_U=H_{0,U}+uH_{1,U}$.

Furthermore, since ξ is global, and Q_U-Q_V is a linear fiberwise vector field on $U\cap V$, we have $H_U=H_V$ on the intersection $U\cap V$. Hence the locally defined Homotopy operators glue to form a globally well-defined map H such that

$$\begin{aligned}&u\big(\nabla^{\mathsf{GM}}_{\frac{\partial}{\partial t}}\pm\frac{1}{2}d_{DR}\iota(\frac{d(\lambda_*\xi)}{dt})d_{DR}\big)\circ\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}-\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}\circ(u\nabla^{\mathsf{Get}}_{\frac{\partial}{\partial t}})\\ &=H\circ(D+b+uB+L_{\lambda_*\xi})+(D+L_{\lambda_*\xi}+ud_{DR})\circ H.\end{aligned}$$

This implies, after taking cohomology, we obtain the identity

$$u\big(\nabla^{\mathsf{GM}}_{\frac{\partial}{\partial t}}\pm\frac{1}{2}d_{DR}\iota(\frac{d(\lambda_*\xi)}{dt})d_{DR}\big)\circ\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}=\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}\circ(u\nabla^{\mathsf{Get}}_{\frac{\partial}{\partial t}}).$$

But since X is smooth and proper, it satisfies Hodge-to-de-Rham degeneration, hence in cohomology the operator d_{DR} acts by zero, which then implies our desired identity

$$(u\nabla^{\mathsf{GM}}_{\frac{\partial}{\partial t}})\circ\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}=\mathcal{U}^{\mathsf{Sh}}_{Q,\lambda_*\xi}\circ(u\nabla^{\mathsf{Get}}_{\frac{\partial}{\partial t}}).$$

□

Finally, we observe that the map $\widetilde{K}$, when restricted to $t \in \Delta$, is indeed given by $\sqrt{\mathsf{td}'(X_t)} \circ \widetilde{I}_{\mathsf{HKR}}$. This is because in Diagram 4 the complex structure deformation $\xi(t)$ may be absorbed into the $\overline{\partial}$ operator so that $\overline{\partial}_\xi := \overline{\partial} + L_{\xi(t)}$ is the Dolbeault operator on the deformed space X_t. Thus, we have an isomorphic diagram

$$\begin{array}{ccc} \big(\Omega(X_t, \widehat{\mathcal{C}}_\bullet)\hat{\otimes}\mathcal{O}_\Delta, D_t + b\big) & \xrightarrow{\mathcal{U}^{\mathsf{Sh}}_{Q_t}} & \big(\Omega(X_t, (\mathcal{A}_t)_{\mathsf{poly}})\hat{\otimes}\mathcal{O}_\Delta, D_t\big) \\ \lambda\Big\uparrow & & \Big\downarrow\pi \\ \big(\Omega^{0,*}(X_t, \widehat{C}_\bullet)\hat{\otimes}\mathcal{O}_\Delta, \overline{\partial}_\xi + b\big) & \xrightarrow{K_*} & \big(\Omega^{0,*}(X_t, (A_t)_{\mathsf{poly}})\hat{\otimes}\mathcal{O}_\Delta, \overline{\partial}_\xi\big) \end{array} \tag{6}$$

where locally on $U \subset X_t$, we have a Fedesov operator of the form $D_t = \partial_t + \overline{\partial}_{\xi(t)} + Q_{U,t}$. This shows that we are in the same setup as that of Diagram 3, in which case the computation of [4, 3] shows that in cohomology we have $\widetilde{K} = \sqrt{\mathsf{td}'(X_t)} \circ \widetilde{I}_{\mathsf{HKR}}$. This finishes the proof of part (3.) of Theorem 0.1.

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References

[1] Blanc, A., *Topological K-theory of complex noncommutative spaces,* Compositio Math. 152 (2016) 489–555.

[2] Calaque, D.; Rossi, C., *Lectures on Duflo isomorphisms in Lie algebra and complex geometry,* EMS Series of Lectures in Mathematics 14 (2011), EMS Publishing House.

[3] Calaque, D.; Rossi, C., *Compatibility with cap-products in Tsygan's formality and homological Duflo isomorphism,* Letters in Mathematical Physics, Volume 95, Issue 2, 135–209 (2010).

[4] Calaque, D.; Rossi, C.; Van den Bergh, M., *Căldăraru's conjecture and Tsygan's formality.* Ann. of Math. (2) 176 (2012), no. 2, 865–923.

[5] Căldăraru, A., *The Mukai pairing—II: the Hochschild–Kostant–Rosenberg isomorphism,* Advances in Mathematics, Volume 194, Issue 1, 20 June 2005, Pages 34–66.

[6] Cattaneo, A.; Felder, G.; Willwacher, T., *The character map in deformation quantization,* Advances in Mathematics, Volume 228, Issue 4, 10 November 2011, Pages 1966–1989.

[7] Dolgushev, V., *A formality theorem for Hochschild chains,* Adv. Math. 200 (2006), no. 1, 51–101.

[8] Fedosov, B., *A simple geometrical construction of deformation quantization,* J. Diff. Geom. 40 (1994), 213–238.

[9] Ganatra, S., Perutz T., Sheridan N., *Mirror symmetry: from categories to curve counts,* arXiv:1510.03839.

[10] Getzler, E., *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology,* Israel Math. Conf. Proc. 7 (1993), 1—12.

[11] Hochschild, G.; Kostant, B.; Rosenberg, A., *Differential forms on regular affine algebras,* Trans. AMS 102 (1962), 383–408.

[12] Katzarkov, L.; Kontsevich, M.; Pantev, T., *Hodge theoretic aspects of mirror symmetry.* From Hodge theory to integrability and TQFT tt*-geometry, 87–174, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.

[13] Keller, B., *On the cyclic homology of ringed spaces and schemes,* Documenta Mathematica 3 (1998), 231–259.

[14] Kim, B., *Hirzebruch-Riemann-Roch for global matrix factorizations* , arXiv:2106.00435.

[15] Kim, H.; Kim, T., *Canonical pairing and Hirzebruch-Riemann-Roch formula for matrix factorizations*, arXiv:2304.11675.

[16] Kontsevich, M., *Deformation quantization of Poisson manifolds,* Lett. Math. Phys. 66 (2003), no. 3, 157–216.

[17] Kontsevich, M., Soibelman, Y., *Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I.*, Homological mirror symmetry, 153–219, Lecture Notes in Phys., 757, Springer, Berlin, 2009.

[18] Lipman, J., *Dualizing sheaves, differentials and residues on algebraic varieties.* Asterisque 117, 1984.

[19] Loday, J.-L., *Cyclic homology: a survey, Geometric and algebraic topology,* Banach center Publ., 18, 1986, pp. 281–303.

[20] Lunts, V.; Orlov, D., *Uniqueness of enhancement for triangulated categories,* Journal of American Mathematical Society, Volume 23, Number 3, July 2010, Pages 853–908.

[21] Ramadoss, A., *The Mukai pairing and integral transforms in Hochschild homology.* Mosc. Math. J. 10 (2010), no. 3, 629-645.

[22] Sastry, P.; Tong, YLL., *The Grothendieck Trace and the de Rham Integral.* Canadian Mathematical Bulletin. 2003;46(3):429–440.

[23] Sheridan, N., *Formulae in noncommutative Hodge theory,* J. Homotopy Relat. Struct. 15 (2020), no. 1, 249–299.

[24] Shklyarov, D. *Hirzebruch-Riemann-Roch-type formula for DG algebras.* Proc. Lond. Math. Soc. (3) 106 (2013), no. 1, 1–32.

[25] Shoikhet, B., *A proof of the Tsygan formality conjecture for chains.* Advances in Mathematics, Volume 179, Issue 1, 15 October 2003, Pages 7–37.

[26] Weibel, C., *Weibel, Cyclic homology for schemes,* Proc. Amer. Math. Soc. 124 (1996), no. 6, 1655–1662.

[27] Weibel, C., *The Hodge Filtration and Cyclic Homology,* K-Theory 12 (1997), no. 2, 145–164.

[28] Willwacher, T., *Formality of Cyclic Chains*, International Mathematics Research Notices, Vol. 2011, No. 17, pp. 3939–3956.