THE ANDONI-NAOR-NEIMAN INEQUALITIES AND ISOMETRIC EMBEDDABILITY INTO A CAT(0) SPACE

TETSU TOYODA

arXiv:2404.13871v2 [math.MG] 24 Apr 2024

Abstract. Andoni, Naor and Neiman (2018) established a family of quadratic metric inequalities that hold true in every CAT(0) space. As stated in their paper, this family seems to include all previously used quadratic metric inequalities that hold true in every CAT(0) space. We prove that there exists a metric space that satisfies all inequalities in this family but does not admit an isometric embedding into any CAT(0) space. More precisely, we prove that the 6-point metric spaces constructed by Nina Lebedeva, which are known to admit no isometric embedding into any CAT(0) space, satisfy all inequalities in this family.

1. Introduction

A condition for points $x_1, \dots, x_n$ in a metric space (X, d_X) is called a *quadratic metric inequality on n points* if it can be written in the form

$$0 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} d_X(x_i, x_j)^2 \tag{1}$$

by some real $n \times n$ matrix (a_{ij}). If (1) holds true for all $x_1, \dots, x_n \in X$, then we say that the metric space X *satisfies the quadratic metric inequality* (1). We call a quadratic metric inequality a CAT(0) *quadratic metric inequality* if every CAT(0) space satisfies it. Andoni, Naor and Neiman [2] proved the following theorem.

Theorem 1.1 (Andoni, Naor and Neiman [2])**.** *An n-point metric space X admits an isometric embedding into a* CAT(0) *space if and only if X satisfies all* CAT(0) *quadratic metric inequalities on n points.*

For the original statement of Theorem 1.1 in full generality, see [2, Proposition 3]. To find a characterization of those metric spaces that admit an isometric embedding into a CAT(0) space is a longstanding open problem stated by Gromov in [4, §15] and [3, Section 1.19+] (see also [1, Section 7], [2, Section 1.4] and [6, Section 1]). Theorem 1.1 tells us that we can answer this problem if we can characterize CAT(0) quadratic inequalities. To see some recent developments in this direction, we recall the following family of CAT(0) quadratic metric inequalities on 4 points.

Definition 1.2. We say that a metric space (X, d_X) satisfies the $\boxtimes$*-inequalities* if we have

$$0 \leq (1-s)(1-t)d_X(x,y)^2 + s(1-t)d_X(y,z)^2 + std_X(z,w)^2 + (1-s)td_X(w,x)^2 - s(1-s)d_X(x,z)^2 - t(1-t)d_X(y,w)^2.$$

for any $s, t \in [0,1]$ and any $x, y, z, w \in X$.

Gromov [4] and Sturm [5] proved independently that every CAT(0) space satisfies the $\boxtimes$-inequalities. The name "$\boxtimes$-inequalities" is based on a notation used by Gromov [4]. Sturm [5] called these inequalities the *weighted quadruple inequalities*. The present author [6] proved the following theorem.

This work was supported in part by JSPS KAKENHI Grant Number JP21K03254.

Theorem 1.3 ([6])**.** *If a metric space X satisfies the $\boxtimes$-inequalities, then X satisfies all* CAT(0) *quadratic metric inequalities on* 5 *points.*

It follows from Theorem 1.1 and Theorem 1.3 that a metric space X with $|X| \leq 5$ admits an isometric embedding into a CAT(0) space if and only if X satisfies the $\boxtimes$-inequalities. On the other hand, Nina Lebedeva constructed 6-point metric spaces that satisfy the $\boxtimes$-inequalities but do not admit an isometric embedding into any CAT(0) space (see [1, §7.2]). Therefore, there exist CAT(0) quadratic metric inequalities on 6 points that do not follow from the $\boxtimes$-inequalities. However to the best of my knowledge, no such CAT(0) quadratic metric inequality is known explicitly at this moment.

Toward a characterization of CAT(0) quadratic metric inequalities, Andoni, Naor and Neiman proposed a general procedure to obtain CAT(0) quadratic metric inequalities, and presented the following large family of CAT(0) quadratic metric inequalities (see Lemma 27 and Section 5.1 in [2]). Throughout this paper, we denote by $[n]$ the set $\{1, 2, \dots, n\}$ for each positive integer n.

Theorem 1.4 (Andoni, Naor and Neiman [2])**.** *Fix positive integers m and n. Let $c_1, \dots, c_m$ be positive real numbers. Suppose for every $k \in [m]$, $p_1^k, \dots, p_n^k, q_1^k, \dots, q_n^k$ are positive real numbers that satisfy $\sum_{i=1}^n p_i^k = \sum_{j=1}^n q_j^k = 1$. Suppose for every $k \in [m]$, $A_k = (a_{ij}^k)$ and $B_k = (b_{ij}^k)$ are $n \times n$ matrices with nonnegative real entries that satisfy*

$$\sum_{s=1}^n a_{is}^k + \sum_{s=1}^n b_{sj}^k = p_i^k + q_j^k$$

for any $i, j \in [n]$. Then we have

$$\sum_{k=1}^m \sum_{i,j \in [n] \text{ s.t. } a_{ij}^k + b_{ij}^k > 0} \frac{c_k a_{ij}^k b_{ij}^k}{a_{ij}^k + b_{ij}^k} d_X(x_i, x_j)^2 \leq \sum_{k=1}^m \sum_{i=1}^n \sum_{j=1}^n c_k p_i^k q_j^k d_X(x_i, x_j)^2 \tag{2}$$

for any CAT(0) *space (X, d_X) and any points $x_1, \dots, x_n \in X$.*

It is easily seen that many known CAT(0) quadratic metric inequalities, including the $\boxtimes$-inequalities, are of the form (2). Moreover, as stated in [2, Section 5.1], it seems that all of the previously used CAT(0) quadratic metric inequalities are of the form (2). So it is natural to ask whether the inequalities of the form (2) capture the totality of CAT(0) quadratic metric inequalities. In this paper, we answer this question negatively by proving the following theorem, which is our main result.

Theorem 1.5. *There exists a* 6*-point metric space (X, d_X) that satisfies the following conditions:*

(i) *For every positive integer n, any points $x_1, \dots, x_n \in X$ satisfy the inequality*

$$\frac{1}{2} \sum_{i,j \in [n] \text{ s.t. } \pi_{ij} + \pi_{ji} > 0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij} + \pi_{ji}} d_X(x_i, x_j)^2 \leq \sum_{i=1}^n \sum_{j=1}^n p_i q_j d_X(x_i, x_j)^2$$

for any nonnegative real numbers $p_1, \dots, p_n, q_1, \dots, q_n$ and any $n \times n$ matrix (π_{ij}) with nonnegative real entries that satisfy

$$\sum_{k=1}^n p_k = \sum_{k=1}^n q_k = 1, \quad \sum_{k=1}^n \pi_{ik} = p_i + q_i$$

for every $i \in [n]$.

(ii) *X does not admit an isometric embedding into any* CAT(0) *space.*

More precisely, we prove that Lebedeva's 6-point metric spaces that we have mentioned above satisfy the condition (i) in Theorem 1.5. As we will show in Section 2, Theorem 1.5 implies the following corollary.

Corollary 1.6. *There exists a* 6*-point metric space* (X, d_X) *that satisfies the following conditions:*

(i) *For every positive integer* n*, any points* $x_1, \dots, x_n \in X$ *satisfy the inequality* (2) *for any* m*,* $c_1, \dots, c_m$*,* $\left(p_i^1\right)_{i\in[n]}, \dots, \left(p_i^m\right)_{i\in[n]}$*,* $\left(q_j^1\right)_{j\in[n]}, \dots, \left(q_j^m\right)_{j\in[n]}$*,* $A_1, \dots, A_m$*,* $B_1, \dots, B_m$ *as in the statement of Theorem 1.4.*

(ii) X *does not admit an isometric embedding into any* CAT(0) *space.*

1.1. **Organization of the paper.** This paper is organized as follows. In Section 2, we slightly reformulate and generalize the Andoni-Naor-Neiman inequalities (2), and prove that Theorem 1.5 implies Corollary 1.6. In Section 3, we recall the definition of Lebedeva's 6-point metric spaces. In Section 4, we prove that Lebedeva's 6-point metric spaces satisfy the condition (i) in Theorem 1.5, which proves Theorem 1.5.

2. A reformulation of the Andoni-Naor-Neiman inequalities

In this section, we slightly reformulate and generalize the Andoni-Naor-Neiman inequalities (2). We first recall the definition of CAT(0) space.

Definition 2.1. A complete metric space (X, d_X) is called a CAT(0) *space* if for any $x, y \in X$ and any $t \in [0,1]$, there exists $z \in X$ that satisfies

$$d_X(w,z)^2 \leq (1-t)d_X(w,x)^2 + t d_X(w,y)^2 - t(1-t)d_X(x,y)^2 \tag{3}$$

for any $w \in X$.

Remark 2.2. *If* d_X *is a semimetric on* X*, or in other words, if* (X, d_X) *satisfies the axioms of metric space except for the requirement that* d_X *satisfies the triangle inequalities, then for any* $x, y, w \in X$*, the triangle inequality*

$$d_X(x,y) \leq d_X(w,x) + d_X(w,y)$$

holds if and only if the quadratic metric inequality

$$t(1-t)d_X(x,y)^2 \leq (1-t)d_X(w,x)^2 + t d_X(w,y)^2 \tag{4}$$

holds for any $t \in [0,1]$*. We remark that this inequality* (4) *follows from the inequality* (3) *in Definition 2.1.*

Let (X, d_X) be a CAT(0) space, and let $x_1, \dots, x_n \in X$. Suppose $p_1, \dots, p_n$ are positive real numbers with $\sum_{i=1}^n p_i = 1$. Then it follows from Definition 2.1 that there exists a unique point $z \in X$ that satisfies

$$d_X(w,z)^2 + \sum_{i=1}^n p_i d_X(z,x_i)^2 \leq \sum_{i=1}^n p_i d_X(w,x_i)^2 \tag{5}$$

for any $w \in X$ (see Lemma 4.4 and Theorem 4.9 in [5]). The point $z \in X$ with the above property is called the *barycenter* of the probability measure $\mu = \sum_{i=1}^n p_i \delta_{x_i}$ on X, and denoted by $\mathrm{bar}(\mu)$, where δ_{x_i} is the Dirac measure at x_i. It is easily seen that the point z in Definition 2.1 is no other than $\mathrm{bar}\left((1-t)\delta_x + t\delta_y\right)$.

Although the following proposition and its corollary are just a slight reformulation of Lemma 27 in [2], they will play a key role in the proof of Theorem 1.5. We denote by $\binom{[n]}{2}$ the set of all 2-element subsets of $[n]$.

Proposition 2.3 (cf. Lemma 27 in [2]). *Fix a positive integer n. Suppose $p_1, \dots, p_n$, $q_1, \dots, q_n$ are nonnegative real numbers that satisfy $\sum_{i=1}^n p_i = \sum_{j=1}^n q_j = 1$. Suppose (π_{ij}) is an $n \times n$ matrix with nonnegative real entries that satisfies*

$$\sum_{j=1}^{n} \pi_{ij} = p_i + q_i$$

for every $i \in [n]$. Suppose (X, d_X) is a CAT(0) *space and $x_1, \dots, x_n \in X$. We set*

$$z = \mathrm{bar}\left(\sum_{i=1}^{n} p_i \delta_{x_i}\right), \quad z_{ij} = \mathrm{bar}\left(\frac{\pi_{ij}}{\pi_{ij}+\pi_{ji}}\delta_{x_i} + \frac{\pi_{ji}}{\pi_{ij}+\pi_{ji}}\delta_{x_j}\right)$$

for any $i, j \in [n]$ with $\pi_{ij} + \pi_{ji} > 0$. Then we have

$$\sum_{i,j\in[n] \text{ s.t.} \pi_{ij}+\pi_{ji}>0} \pi_{ij} d_X(z, z_{ij})^2 + \frac{1}{2} \sum_{i,j\in[n] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_X(x_i, x_j)^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j d_X(x_i, x_j)^2.$$

Proof. It follows from the inequality (5) that

$$d_X(x_j, z)^2 + \sum_{i=1}^{n} p_i d_X(x_i, z)^2 \le \sum_{i=1}^{n} p_i d_X(x_i, x_j)^2$$

holds for each $j \in [n]$. By multiplying this inequality by q_j and summing over $j \in [n]$, we obtain

$$\sum_{j=1}^{n} q_j d_X(x_j, z)^2 + \sum_{i=1}^{n} p_i d_X(x_i, z)^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j d_X(x_i, x_j)^2.$$

Since we have $z_{ij} = z_{ji}$ for any $i, j \in [n]$ with $\pi_{ij} + \pi_{ji} > 0$, it follows from (3) that

$$\begin{aligned}
&\sum_{j=1}^{n} q_j d_X(x_j, z)^2 + \sum_{i=1}^{n} p_i d_X(x_i, z)^2 = \sum_{k=1}^{n} (p_k + q_k) d_X(x_k, z)^2 = \sum_{k=1}^{n}\sum_{l=1}^{n} \pi_{kl} d_X(x_k, z)^2 \\
&\quad = \sum_{k=1}^{n} \pi_{kk} d_X(x_k, z)^2 + \sum_{\{k,l\}\in\binom{[n]}{2} \text{ s.t. } \pi_{kl}+\pi_{lk}>0} (\pi_{kl}+\pi_{lk}) \left(\frac{\pi_{kl}}{\pi_{kl}+\pi_{lk}} d_X(x_k, z)^2 + \frac{\pi_{lk}}{\pi_{kl}+\pi_{lk}} d_X(x_l, z)^2\right) \\
&\quad \ge \sum_{k\in[n] \text{ s.t.} \pi_{kk}>0} \pi_{kk} d_X(z_{kk}, z)^2 + \sum_{\{k,l\}\in\binom{[n]}{2} \text{ s.t. } \pi_{kl}+\pi_{lk}>0} (\pi_{kl}+\pi_{lk}) \left(d_X(z, z_{kl})^2 + \frac{\pi_{kl}\pi_{lk}}{(\pi_{kl}+\pi_{lk})^2} d_X(x_k, x_l)^2\right) \\
&\quad = \sum_{i,j\in[n] \text{ s.t.} \pi_{ij}+\pi_{ji}>0} \pi_{ij} d_X(z, z_{ij})^2 + \frac{1}{2} \sum_{i,j\in[n] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_X(x_i, x_j)^2,
\end{aligned}$$

which proves the proposition. □

Remark 2.4. *Because the statement of Proposition 2.3 is symmetric with respect to $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ except for the definition of z, the same statement holds true even if we set $z = \mathrm{bar}(\sum_{j=1}^n q_j \delta_{x_j})$ instead of $z = \mathrm{bar}(\sum_{i=1}^n p_i \delta_{x_i})$.*

The following corollary follows immediately from Proposition 2.3.

Corollary 2.5 (cf. Lemma 27 in [2]). *Fix a positive integer n. Suppose $p_1, \dots, p_n$, $q_1, \dots, q_n$, and (π_{ij}) are as in the statement of Proposition 2.3. Suppose (X, d_X) is a* CAT(0) *space and*

$x_1, \dots, x_n \in X$. *Then we have*

$$\frac{1}{2} \sum_{i,j \in [n] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_X(x_i, x_j)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j d_X(x_i, x_j)^2. \tag{6}$$

We define the following condition.

Definition 2.6. Fix a positive integer n. Let (X, d_X) be a semimetric space. We say that X satisfies the $\mathrm{ANN}(n)$ *inequalities* if any points $x_1, \dots, x_n \in X$ satisfy the inequality (6) for any $p_1, \dots, p_n, q_1, \dots, q_n$, and (π_{ij}) as in the statement of Proposition 2.3.

Remark 2.7. *It is easily seen that for every $t \in [0,1]$, the inequality* (4) *belongs to the* $\mathrm{ANN}(n)$ *inequalities for any $n \geq 3$. Therefore, a semimetric space is a metric space whenever it satisfies the* $\mathrm{ANN}(n)$ *inequalities for some integer $n \geq 3$.*

To set up some basic facts about the $\mathrm{ANN}(n)$ inequalities, we recall the following elementary facts, which follows from straightforward computation.

Proposition 2.8. *Suppose a, a_1, a_2, b, b_1, b_2 are nonnegative real numbers such that $a = a_1 + a_2 > 0$ and $b = b_1 + b_2 > 0$. Then we have*

$$\frac{a_1 b_1}{a_1 + b_1} + \frac{a_2 b_2}{a_2 + b_2} \leq \frac{ab}{a+b}.$$

We will use the following proposition in Section 4.

Proposition 2.9. *Let n be a positive integer, and let $(X = \{x_1, \dots, x_n\}, d_X)$ be a semimetric space with $|X| = n$. Assume that the inequality*

$$\frac{1}{2} \sum_{\{i,j\} \in [n] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_X(x_i, x_j)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j d_X(x_i, x_j)^2$$

holds true for any $p_1, \dots, p_n, q_1, \dots, q_n$, and (π_{ij}) as in the statement of Proposition 2.3. Then X satisfies the $\mathrm{ANN}(m)$ *inequalities for all positive integers m.*

Proof. Fix a positive integer m. Fix nonnegative real numbers $p_1, \dots, p_m$, $q_1, \dots, q_m$ and an $m \times m$ matrix (π_{ij}) with nonnegative real entries such that

$$\sum_{k=1}^{m} p_k = \sum_{k=1}^{m} q_k = 1, \quad \sum_{k=1}^{m} \pi_{ik} = p_i + q_i$$

for every $i \in [m]$. Choose a map $\varphi : [m] \to [n]$ arbitrarily. For each $k, l \in [n]$, we set

$$\tilde{p}_k = \sum_{i \in \varphi^{-1}(\{k\})} p_i, \qquad \tilde{q}_k = \sum_{i \in \varphi^{-1}(\{k\})} q_i, \qquad \tilde{\pi}_{kl} = \sum_{(i,j) \in \varphi^{-1}(\{k\}) \times \varphi^{-1}(\{l\})} \pi_{ij}.$$

Then it is easily seen that we have

$$\sum_{k=1}^{n} \tilde{p}_k = \sum_{k=1}^{n} \tilde{q}_k = 1, \quad \sum_{k=1}^{n} \tilde{\pi}_{ik} = \tilde{p}_i + \tilde{q}_i$$

for every $i \in [n]$. It follows from Proposition 2.8 and the hypothesis that

$$\begin{aligned}
&\frac{1}{2} \sum_{i,j\in[m] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_X(x_{\varphi(i)}, x_{\varphi(j)})^2 \\
&= \frac{1}{2} \sum_{k,l\in[n]} \left(\sum_{(i,j)\in\varphi^{-1}(\{k\})\times\varphi^{-1}(\{l\}) \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} \right) d_X(x_k, x_l)^2 \\
&\leq \frac{1}{2} \sum_{k,l\in[n] \text{ s.t. } \tilde{\pi}_{kl}+\tilde{\pi}_{lk}>0} \left(\frac{\tilde{\pi}_{kl}\tilde{\pi}_{lk}}{\tilde{\pi}_{kl}+\tilde{\pi}_{lk}} \right) d_X(x_k, x_l)^2 \\
&\leq \sum_{k=1}^{n} \sum_{l=1}^{n} \tilde{p}_k \tilde{q}_l d_X(x_k, x_l)^2 \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} \left(\sum_{i\in\varphi^{-1}(\{k\})} p_i \right) \left(\sum_{j\in\varphi^{-1}(\{l\})} q_j \right) d_X(x_k, x_l)^2 \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} \left(\sum_{i\in\varphi^{-1}(\{k\})} \sum_{j\in\varphi^{-1}(\{l\})} p_i q_j d_X(x_{\varphi(i)}, x_{\varphi(j)})^2 \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} p_i q_j d_X(x_{\varphi(i)}, x_{\varphi(j)})^2,
\end{aligned}$$

which proves the proposition. □

The following corollary follows from Proposition 2.9 immediately.

Corollary 2.10. *Suppose m and n are positive integers with $m \leq n$, and X is a semimetric space that satisfies the* ANN(n) *inequalities. Then X satisfies the* ANN(m) *inequalities.*

The following lemma clarifies the relation between the ANN(n) inequalities and the inequalities (2) established by Andoni-Naor-Neiman [2].

Lemma 2.11. *Fix a positive integer n and nonnegative real numbers $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ that satisfy $\sum_{i=1}^n p_i = \sum_{j=1}^n q_j = 1$. Let (X, d_X) be a semimetric space. Suppose $x_1, \ldots, x_n \in X$ are points that satisfy the inequality* (6) *for any n by n matrix (π_{ij}) with nonnegative real entries that satisfies $\sum_{j=1}^n \pi_{ij} = p_i + q_i$ for every $i \in [n]$. Then the inequality*

$$\sum_{i,j\in[n] \text{ s.t. } a_{ij}+b_{ij}>0} \frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}} d_X(x_i, x_j)^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} p_i q_j d_X(x_i, x_j)^2.$$

holds true for any n by n matrices $A = (a_{ij})$ and $B = (b_{ij})$ with nonnegative real entries that satisfy $\sum_{k=1}^n a_{ik} + \sum_{k=1}^n b_{kj} = p_i + q_j$ for any $i, j \in [n]$.

Proof. Fix $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ with nonnegative real entries that satisfy

$$\sum_{k=1}^{n} a_{ik} + \sum_{k=1}^{n} b_{kj} = p_i + q_j$$

for any $i, j \in [n]$. Set $\pi_{ij} = a_{ij} + b_{ji}$ for any $i, j \in [n]$. Then we have $\sum_{j=1}^{n} \pi_{ij} = p_i + q_i$ for every $i \in [n]$ clearly. It follows from Proposition 2.8 that

$$\frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}} + \frac{a_{ji}b_{ji}}{a_{ji}+b_{ji}} \leq \frac{(a_{ij}+b_{ji})(a_{ji}+b_{ij})}{(a_{ij}+b_{ji})+(a_{ji}+b_{ij})} = \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}}$$

whenever $a_{ij} + b_{ij} > 0$ and $a_{ji} + b_{ji} > 0$. Therefore, we have

$$\sum_{i,j\in[n] \text{ s.t. } a_{ij}+b_{ij}>0} \frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}} d_X(x_i, x_j)^2 \leq \frac{1}{2} \sum_{i,j\in[n] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_X(x_i, x_j)^2.$$

Combining this with the hypothesis that the inequality (6) holds true, we obtain the desired inequality. □

The following corollary is an immediate consequence of Lemma 2.11 and Corollary 2.10.

Corollary 2.12. *Let n_0 be a positive integer, and let $n \in [n_0]$. Let X be a semimetric space that satisfies the* ANN(n_0) *inequalities. Then any $x_1, \ldots, x_n \in X$ satisfy the inequality* (2) *for any m, $c_1, \ldots, c_m$, $(p_i^1)_{i\in[n]}, \ldots, (p_i^m)_{i\in[n]}$, $\left(q_j^1\right)_{j\in[n]}, \ldots, \left(q_j^m\right)_{j\in[n]}$, $A_1, \ldots, A_m$, $B_1, \ldots, B_m$ as in the statement of Theorem 1.4.*

3. Lebedeva's 6-point metric space

In this section, we recall the 6-point metric space constructed by Nina Lebedeva, which was appeared in [1, §7.2].

For any two points a and b in the Euclidean space $\mathbb{R}^3$, we denote by $[a, b]$ the line segment joining a and b, and by (a, b) the set $[a, b] \setminus \{a, b\}$. We denote by $\mathrm{conv}(S)$ the convexhull of a subset S of $\mathbb{R}^3$. Suppose $x_1, \ldots, x_6$ are distinct six points in $\mathbb{R}^3$ that satisfy the following conditions:

$$|(x_1, x_3) \cap (x_2, x_4)| = 1; \tag{7}$$

$$\left|(x_5, x_6) \cap \left(\mathrm{conv}(\{x_1, x_2, x_3, x_4\}) \setminus \cup_{\{i,j\}\in\binom{[4]}{2}} [x_i, x_j]\right)\right| = 1. \tag{8}$$

It follows from the conditions (7) and (8) that the points $x_1, \ldots, x_4$ form the vertices of the convex quadrilateral in a 2-dimensional affine subspace of $\mathbb{R}^3$, and that the points $x_1, \ldots, x_6$ form the vertices of a nonregular convex octahedron in $\mathbb{R}^3$. We set $L = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. For any $\varepsilon \geq 0$, we define a map $d_\varepsilon : L \times L \to [0, \infty)$ by

$$d_\varepsilon(a, b) = \begin{cases} \|a - b\| + \varepsilon, & \text{if } \{a, b\} = \{x_5, x_6\}, \\ \|a - b\|, & \text{if } \{a, b\} \neq \{x_5, x_6\}, \end{cases} \tag{9}$$

where $\|a - b\|$ is the Euclidean norm of $a - b$.

Theorem 3.1 (Lebedeva (see §7.2 of [1]))**.** *Fix distinct six points $x_1, \ldots, x_6 \in \mathbb{R}^3$ that satisfy* (7) *and* (8)*. Set $L = \{x_1, \ldots, x_6\}$. Then there exists $C \in (0, \infty)$ such that for any $\varepsilon \in (0, C]$, (L, d_ε) defined by* (9) *is a metric space that satisfies the* $(2+2)$*-point comparison and the* $(4+2)$*-point comparison, but does not admit an isometric embedding into any* CAT(0) *space.*

For the definition of the $(2+2)$-point comparison and the $(4+2)$-point comparison, see [1, §6.2]. It is known that a metric space satisfies the $(2+2)$-comparison if and only if it satisfies the $\boxtimes$-inequalities.

4. Proof of Theorem 1.5

In this section, we prove the following theorem, which implies Theorem 1.5. For any subset M of $\mathbb{R}^3$ and any $R \in (0,\infty)$, we denote by $B(M,R)$ the open R-neighborhood of M in $\mathbb{R}^3$. For any $x \in X$, we denote $B(\{x\},R)$ simply by $B(x,R)$.

Theorem 4.1. *Fix distinct six points $x_1,\dots,x_6 \in \mathbb{R}^3$ that satisfy (7) and (8). Set $L = \{x_1,\dots,x_6\}$. Then there exists $C \in (0,\infty)$ such that for any $\varepsilon \in (0,C]$, (L,d_ε) defined by (9) satisfies the* ANN(n) *inequalities for all positive integers n.*

Proof. We first define four positive real constants h, H, θ and δ that depend only on the choice of $x_1,\dots,x_6 \in \mathbb{R}^3$. We set

$$\begin{aligned}
h &= \min\left\{ \min_{y\in\operatorname{conv}(L\setminus\{x_5\})} \|x_5 - y\|, \quad \min_{y\in\operatorname{conv}(L\setminus\{x_6\})} \|x_6 - y\| \right\},\\
H &= \max\left\{ \max_{y\in\operatorname{conv}(L\setminus\{x_5\})} \|x_5 - y\|, \quad \max_{y\in\operatorname{conv}(L\setminus\{x_6\})} \|x_6 - y\| \right\},\\
\theta &= \min\left\{ \min_{k\in[4]} \measuredangle x_k x_5 x_6, \quad \min_{k\in[4]} \measuredangle x_k x_6 x_5 \right\}
\end{aligned}$$

where we denote by $\measuredangle x_k x_i x_j \in [0,\pi]$ the interior angle measure at x_i of the triangle with vertices x_k, x_i and x_j. It follows from (7) and (8) that $h > 0$, $H > 0$ and $\theta > 0$. It also follows from (7) and (8) that we can choose $\delta \in (0, h/2)$ such that we have

$$\begin{aligned}
&B([x_5,x_6],\delta) \cap B([x_i,x_j],\delta) = \emptyset,\\
&B([x_5,x_6],\delta) \cap B([x_5,x_i],\delta) \subseteq B\left(x_5, \frac{h}{2}\right), \quad B([x_5,x_6],\delta) \cap B([x_6,x_i],\delta) \subseteq B\left(x_6, \frac{h}{2}\right)
\end{aligned}$$

for any $i,j \in [4]$.

Fix nonnegative real numbers $p_1,\dots,p_6,q_1,\dots,q_6$ and a 6×6 matrix (π_{ij}) with nonnegative real entries that satisfy

$$\sum_{k=1}^{6} p_k = \sum_{k=1}^{6} q_k = 1, \quad \sum_{k=1}^{6} \pi_{ik} = p_i + q_i \tag{10}$$

for every $i \in [6]$. If $\pi_{56} = 0$ or $\pi_{65} = 0$, then the desired inequality

$$\frac{1}{2} \sum_{i,j\in[6] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_\varepsilon(x_i,x_j)^2 \le \sum_{i=1}^{6}\sum_{j=1}^{6} p_i q_j d_\varepsilon(x_i,x_j)^2$$

holds true for every $\varepsilon > 0$ by definition of d_ε. So henceforth we assume that $\pi_{56} > 0$ and $\pi_{65} > 0$.

For each $\varepsilon \ge 0$, we define $F(\varepsilon) \in \mathbb{R}$ by

$$F(\varepsilon) = \sum_{i=1}^{6}\sum_{j=1}^{6} p_i q_j d_\varepsilon(x_i,x_j)^2 - \frac{1}{2} \sum_{i,j\in[6] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}} d_\varepsilon(x_i,x_j)^2.$$

Then it follows from the definition of d_ε that

$$F(\varepsilon) \ge \left(p_5 q_6 + p_6 q_5 - \frac{\pi_{56}\pi_{65}}{\pi_{56}+\pi_{65}} \right) f(\varepsilon) + F(0), \tag{11}$$

where $f : [0,\infty) \to [0,\infty)$ is the strictly increasing function defined by

$$f(\varepsilon) = 2\|x_5 - x_6\|\varepsilon + \varepsilon^2, \quad \varepsilon \in [0,\infty).$$

We define $z_p \in \mathbb{R}^3$, $z_q \in \mathbb{R}^3$ and $z_{ij} \in \mathbb{R}^3$ for any $i, j \in [6]$ with $\pi_{ij} + \pi_{ji} > 0$ by

$$z_p = \sum_{k=1}^{6} p_k x_k, \quad z_q = \sum_{k=1}^{6} q_k x_k, \quad z_{ij} = \frac{\pi_{ij} x_i + \pi_{ji} x_j}{\pi_{ij} + \pi_{ji}}.$$

In other words, z_p, z_q and z_{ij} are the barycenters of the probability measures $\sum_{k=1}^{6} p_k \delta_{x_k}$, $\sum_{k=1}^{6} q_k \delta_{x_k}$ and $(\pi_{ij}\delta_{x_i} + \pi_{ji}\delta_{x_j})/(\pi_{ij} + \pi_{ji})$, respectively. It follows from Proposition 2.3 that

$$F(0) \geq \sum_{i,j \in [6] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \pi_{ij} \|z_p - z_{ij}\|^2, \qquad F(0) \geq \sum_{i,j \in [6] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \pi_{ij} \|z_q - z_{ij}\|^2.$$

Combining these inequalities with (11), we obtain

$$\begin{aligned} F(\varepsilon) \geq &\left(p_5 q_6 + p_6 q_5 - \frac{\pi_{56}\pi_{65}}{\pi_{56} + \pi_{65}} \right) f(\varepsilon) \\ &+ \max \left\{ \sum_{i,j \in [6] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \pi_{ij} \|z_p - z_{ij}\|^2, \sum_{i,j \in [6] \text{ s.t. } \pi_{ij}+\pi_{ji}>0} \pi_{ij} \|z_q - z_{ij}\|^2 \right\}. \end{aligned} \tag{12}$$

Set $S = \sum_{i=1}^{4} \sum_{j=1}^{6} \pi_{ij}$. Then we have

$$p_6 \geq 1 - p_5 - S, \quad q_6 \geq 1 - q_5 - S, \quad \pi_{56} \leq p_5 + q_5, \quad \pi_{65} \leq 2 - p_5 - q_5.$$

It follows that

$$\begin{aligned} p_5 q_6 + p_6 q_5 - \frac{\pi_{56}\pi_{65}}{\pi_{56} + \pi_{65}} &\geq p_5(1 - q_5 - S) + (1 - p_5 - S) q_5 - \frac{(p_5 + q_5)(2 - p_5 - q_5)}{2} \\ &= \frac{(p_5 - q_5)^2}{2} - (p_5 + q_5) S \geq -(p_5 + q_5) S \end{aligned} \tag{13}$$

since the function $f(x, y) = xy/(x + y)$, $x, y \in (0, \infty)$ is increasing with respect to both x and y.

We will prove that there exists a constant $C > 0$ such that C depends only on the choice of the points $x_1, \ldots, x_6$ in $\mathbb{R}^3$ and the inequality $F(\varepsilon) \geq 0$ holds for all $\varepsilon \in [0, C]$. We consider several cases separately. We first consider the following two cases.

Case 1: *The inequality $\|z_p - z_{56}\| \geq \delta$ holds, or the inequality $\|z_q - z_{56}\| \geq \delta$ holds.* In this case, it follows from (12) that

$$F(\varepsilon) \geq -\frac{\pi_{56}\pi_{65}}{\pi_{56} + \pi_{65}} f(\varepsilon) + (\pi_{56} + \pi_{65})\delta^2 \geq (\pi_{56} + \pi_{65})(\delta^2 - f(\varepsilon)).$$

Therefore, we have $F(\varepsilon) \geq 0$ for any $\varepsilon \in (0, f^{-1}(\delta^2)]$.

Case 2: *The inequality $\|z_p - z_{ij}\| \geq \delta$ holds for all $i, j \in [6]$ with $\{i, j\} \not\subseteq \{5, 6\}$ and $\pi_{ij} + \pi_{ji} > 0$, or the inequality $\|z_q - z_{ij}\| \geq \delta$ holds for all $i, j \in [6]$ with $\{i, j\} \not\subseteq \{5, 6\}$ and $\pi_{ij} + \pi_{ji} > 0$.* In this case, it follows from (12) and (13) that

$$F(\varepsilon) \geq -(p_5 + q_5) S f(\varepsilon) + S\delta^2 \geq S\left(\delta^2 - 2f(\varepsilon)\right).$$

Therefore, we have $F(\varepsilon) \geq 0$ for any $\varepsilon \in (0, f^{-1}(\delta^2/2)]$.

Henceforth we assume that it is neither Case 1 nor Case 2. Then it follows from the definition of δ that we have

$$\{z_p, z_q\} \subseteq B\left(x_5, \frac{h}{2}\right) \tag{14}$$

or

$$\{z_p, z_q\} \subseteq B\left(x_6, \frac{h}{2}\right). \tag{15}$$

Because it is easily seen that we can obtain the same estimate of $F(\varepsilon)$ in the same way as we describe below whether (14) holds or (15) holds, we assume that (15) holds.

We set

$$t_i = \frac{\pi_{i6}}{\pi_{i6} + \pi_{6i}}.$$

for each $i \in [5]$ with $\pi_{i6} + \pi_{6i} > 0$, and define $K_0 \in [0, \infty)$ to be the nonnegative real number that satisfies

$$\min\{p_6, q_6\} = 1 - K_0 t_5.$$

Then we have

$$\begin{aligned} p_5 q_6 + p_6 q_5 - \frac{\pi_{56}\pi_{65}}{\pi_{56} + \pi_{65}} &\geq (p_5 + q_5)\min\{p_6, q_6\} - t_5(1 - t_5)(\pi_{56} + \pi_{65}) \\ &\geq \pi_{56}(1 - K_0 t_5) - t_5(1 - t_5)(\pi_{56} + \pi_{65}) = (\pi_{56} + \pi_{65})(1 - K_0)t_5^2, \qquad (16) \end{aligned}$$

which implies that the inequality $F(\varepsilon) \geq 0$ holds for any $\varepsilon > 0$ whenever $K_0 \leq 1$. So henceforth we assume that $K_0 > 1$. We define $z \in \mathbb{R}^3$ and $\tilde{z} \in \mathbb{R}^3$ by

$$z = \begin{cases} z_p, & \text{if } p_6 \leq q_6, \\ z_q, & \text{if } q_6 < p_6, \end{cases} \qquad \tilde{z} = \begin{cases} \frac{1}{1-p_6}\sum_{k=1}^5 p_k x_k, & \text{if } p_6 \leq q_6, \\ \frac{1}{1-q_6}\sum_{k=1}^5 q_k x_k, & \text{if } q_6 < p_6. \end{cases}$$

To complete the proof, we fix a constant $\gamma \in (0, \infty)$, and consider three cases.

Case 3: *The inequality $K_0 \geq (1+\gamma)\|x_5 - x_6\|/h$ holds.* In this case, we have

$$\|z - z_{56}\| \geq \|z - x_6\| - \|z_{56} - x_6\| = K_0 t_5 \|\tilde{z} - x_6\| - t_5\|x_5 - x_6\| \geq (K_0 h - \|x_5 - x_6\|)\, t_5 > 0.$$

Together with (12) and (16), this implies that

$$F(\varepsilon) > -(\pi_{56} + \pi_{65})(K_0 - 1)t_5^2 f(\varepsilon) + (\pi_{56} + \pi_{65})\left(K_0 h - \|x_5 - x_6\|\right)^2 t_5^2.$$

It follows that we have $F(\varepsilon) > 0$ whenever

$$f(\varepsilon) \leq \frac{(K_0 h - \|x_5 - x_6\|)^2}{K_0 - 1}.$$

Since $\|x_5 - x_6\|/h \geq 1$, it is easily seen that the function

$$\varphi_0(K) = \frac{(Kh - \|x_5 - x_6\|)^2}{K - 1}, \quad K \in \left[\frac{(1+\gamma)\|x_5 - x_6\|}{h}, \infty\right)$$

is increasing. Therefore, we have $F(\varepsilon) > 0$ whenever

$$\varepsilon \leq f^{-1}\left(\frac{h\gamma^2\|x_5 - x_6\|^2}{(1+\gamma)\|x_5 - x_6\| - h}\right).$$

Case 4: *The inequalities $1 < K_0 < (1+\gamma)\|x_5 - x_6\|/h$ and $\measuredangle z x_6 x_5 \geq \theta/2$ hold.* In this case, we have

$$\|z - z_{56}\| \geq \left(\sin\frac{\theta}{2}\right)\|x_5 - x_6\| t_5.$$

Together with (12) and (16), this implies that

$$F(\varepsilon) > (\pi_{56} + \pi_{65})\left(-\left(\frac{(1+\gamma)\|x_5 - x_6\|}{h} - 1\right)f(\varepsilon) + \left(\sin\frac{\theta}{2}\right)^2\|x_5 - x_6\|^2\right)t_5^2.$$

Therefore, we have $F(\varepsilon) > 0$ whenever

$$\varepsilon \leq f^{-1}\left(\frac{h\left(\sin\frac{\theta}{2}\right)^2\|x_5 - x_6\|^2}{(1+\gamma)\|x_5 - x_6\| - h}\right).$$

Case 5: *The inequalities* $1 < K_0 < (1+\gamma)\|x_5 - x_6\|/h$ *and* $\measuredangle z x_6 x_5 < \theta/2$ *hold.* In this case, we have $\measuredangle z x_6 x_k > \theta/2$ for any $k \in [4]$. Therefore we have

$$\|z - z_{k6}\| > \left(\sin\frac{\theta}{2}\right)\|z - x_6\| = \left(\sin\frac{\theta}{2}\right)\|\tilde{z} - x_6\| K_0 t_5 > \left(\sin\frac{\theta}{2}\right) h t_5 \tag{17}$$

for any $k \in [4]$. Suppose l and m are positive integers at most 4 such that

$$\pi_{l6} > 0, \quad \pi_{m6} > 0, \quad t_l\|x_l - x_6\| \leq \frac{(1+\gamma)^2\|x_5 - x_6\|Ht_5}{h}, \quad t_m\|x_m - x_6\| > \frac{(1+\gamma)^2\|x_5 - x_6\|Ht_5}{h}.$$

Then we have

$$0 < t_l \leq \frac{(1+\gamma)^2\|x_5 - x_6\|Ht_5}{h^2}.$$

Together with (17), this implies that

$$(\pi_{l6} + \pi_{6l})\|z - z_{l6}\|^2 = \frac{1}{t_l}\pi_{l6}\|z - z_{l6}\|^2 > \frac{h^4\left(\sin\frac{\theta}{2}\right)^2}{(1+\gamma)^2\|x_5 - x_6\|H}\pi_{l6}t_5. \tag{18}$$

We also have

$$\begin{aligned}\|z - z_{m6}\| &\geq \|z_{m6} - x_6\| - \|z - x_6\| = t_m\|x_m - x_6\| - K_0 t_5\|\tilde{z} - x_6\| \\ &> t_m\|x_m - x_6\| - \frac{(1+\gamma)\|x_5 - x_6\|Ht_5}{h} > \frac{\gamma(1+\gamma)\|x_5 - x_6\|Ht_5}{h} > 0.\end{aligned}$$

We define $K_1 \in (\gamma(1+\gamma)\|x_5 - x_6\|H/h, \infty)$ to be the positive real number that satisfies

$$K_1 t_5 = t_m\|x_m - x_6\| - \frac{(1+\gamma)\|x_5 - x_6\|Ht_5}{h}.$$

Then we have

$$\|z - z_{m6}\| > K_1 t_5, \quad \pi_{m6} + \pi_{6m} = \frac{1}{t_m}\pi_{m6} \geq \frac{h^2}{\left(hK_1 + (1+\gamma)\|x_5 - x_6\|H\right)t_5}\pi_{m6},$$

and therefore

$$(\pi_{m6} + \pi_{6m})\|z - z_{m6}\|^2 > \frac{h^2K_1^2}{hK_1 + (1+\gamma)\|x_5 - x_6\|H}\pi_{m6}t_5.$$

Because the function

$$\varphi_1(K) = \frac{h^2K^2}{hK + (1+\gamma)\|x_5 - x_6\|H}, \quad K \in \left[\frac{\gamma(1+\gamma)\|x_5 - x_6\|H}{h}, \infty\right)$$

is increasing, we obtain

$$(\pi_{m6} + \pi_{6m})\|z - z_{m6}\|^2 > \gamma^2\|x_5 - x_6\|H\pi_{m6}t_5. \tag{19}$$

It follows from (15) that we have

$$\|z_p - z_{ij}\| > \frac{h}{2}, \quad \|z_q - z_{ij}\| > \frac{h}{2}$$

for any $i \in [4]$ and any $j \in [5]$. Therefore, we have

$$\pi_{ij}\|z - z_{ij}\|^2 > \frac{h^2}{4}\pi_{ij} \geq \frac{h^2}{4}\pi_{ij}t_5 \tag{20}$$

for any $i \in [4]$ and any $j \in [5]$. Set

$$c = \min\left\{\frac{h^4\left(\sin\frac{\theta}{2}\right)^2}{(1+\gamma)^2\|x_5 - x_6\|H}, \quad \gamma^2\|x_5 - x_6\|H, \quad \frac{h^2}{4}\right\}.$$

Then it follows from (18), (19) and (20) that

$$\sum_{i,j\in[n]\ \text{s.t.}\pi_{ij}+\pi_{ji}>0} \pi_{ij}\|z-z_{ij}\|^2 > cSt_5. \tag{21}$$

On the other hand, it follows from (13) that

$$\begin{aligned} p_5q_6+q_5p_6-\frac{\pi_{56}\pi_{65}}{\pi_{56}+\pi_{65}} &\geq -(p_5+q_5)S \\ &\geq -(2-p_6-q_6)S \geq -2K_0St_5 > -\frac{2(1+\gamma)\|x_5-x_6\|}{h}St_5. \end{aligned} \tag{22}$$

It follows from (12), (21) and (22) that

$$F(\varepsilon) \geq -\frac{2(1+\gamma)\|x_5-x_6\|}{h}St_5f(\varepsilon)+cSt_5.$$

Therefore we have $F(\varepsilon)\geq 0$ whenever

$$\varepsilon \leq f^{-1}\left(\frac{ch}{2(1+\gamma)\|x_5-x_6\|}\right).$$

So far, we have proved that there exists a constant $C>0$ such that C depends only on the choice of the points $x_1,\dots,x_6\in\mathbb{R}^3$, and any $\varepsilon\in[0,C]$ satisfies

$$\frac{1}{2}\sum_{i,j\in[6]\ \text{s.t.}\ \pi_{ij}+\pi_{ji}>0}\frac{\pi_{ij}\pi_{ji}}{\pi_{ij}+\pi_{ji}}d_\varepsilon(x_i,x_j)^2 \leq \sum_{i=1}^{6}\sum_{j=1}^{6}p_iq_jd_\varepsilon(x_i,x_j)^2$$

for any nonnegative real numbers $p_1,\dots,p_6,q_1,\dots,q_6$ and any 6×6 matrix (π_{ij}) with nonnegative entries that satisfy (10). Therefore, it follows from Proposition 2.9 that (L,d_ε) satisfies the ANN(n) inequalities for all positive integers n whenever $\varepsilon\in[0,C]$. □

Theorem 1.5 and Corollary 1.6 follow from Theorem 4.1 immediately.

Proof of Theorem 1.5. Theorem 3.1 and Theorem 4.1 imply Theorem 1.5. □

Proof of Corollary 1.6. Theorem 1.5 and Corollary 2.12 imply Corollary 1.6. □

References

[1] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov meets Kirszbraun. *Proceedings of the Gökova Geometry-Topology Conference 2010.* Int. Press, Somerville, MA, 2011, 88–109.

[2] A. Andoni, A. Naor, and O. Neiman. Snowflake universality of Wasserstein spaces. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):657–700, 2018.

[3] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[4] M. Gromov. CAT(κ)-spaces: construction and concentration. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 280(Geom. i Topol. 7):100–140, 299–300, 2001.

[5] K.-T. Sturm. Probability measures on metric spaces of nonpositive curvature. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 357–390. Amer. Math. Soc., Providence, RI, 2003.

[6] T. Toyoda. An intrinsic characterization of five points in a CAT(0) space. *Anal. Geom. Metr. Spaces*, 8(1):114–165, 2020.

Kogakuin University, 2665-1, Nakano, Hachioji, Tokyo, 192-0015 Japan

Email address: toyoda@cc.kogakuin.ac.jp