MAXWELL'S AND STOKES' OPERATORS ASSOCIATED WITH ELLIPTIC DIFFERENTIAL COMPLEXES

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ABSTRACT. We propose a new technique to generate reasonable systems of partial differential equations (PDE) that could be potential candidates for depicting models in natural sciences related to quasi-linear equations. Such systems appear within typical constructions of the Homological Algebra as complexes of differential operators describing compatibility conditions for overdetermined systems of PDE's. The related models can be both steady and evolutionary. Additional assumptions on the ellipticity of the differential complex provide a wide class of elliptic, parabolic and hyperbolic operators that could be generated in this way. In particular, it appears that an essentially large amount of equations related to the modern Mathematical Physics is generated by the de Rham complex of differentials on the exterior differential forms. These includes the elliptic Laplace and Lamé type operators; the parabolic heat transfer equation; the Euler type and Navier-Stokes type equations in Hydrodynamics; the hyperbolic wave equation and the Maxwell equations in Electrodynamics; the Klein-Gordon equation in Relativistic Quantum Mechanics; and so on. Our model generation method covers a broad class of generating systems, especially in higher spatial dimensions, due to different basic algebraic structures at play.

INTRODUCTION

The vast majority of differential equations of modern Mathematical Physics was constructed with the use of the standard time derivative $\partial_t = \partial/\partial t$, gradient operator ∇, the divergence operator div and the infinitesimal circulation operator curl, known since Hamilton [14] and Maxwell [17]. The operators satisfy familiar relations

$$\mathrm{curl} \circ \nabla = 0,\ \mathrm{div} \circ \mathrm{curl} = 0, \tag{0.1}$$

generating the elliptic Laplace operator

$$\Delta = \mathrm{div} \circ \nabla,$$

which is used in the parabolic heat transfer and duffusion equations (operator $\partial_t - \Delta$) and in the hyperbolic wave equations (operator $\partial_t^2 - \Delta$).

Advanced algebraic concepts, such as Dirac matrix algebra, Pauli matrix algebra, Clifford algebra, quaternionic (octonionic, sedenionic) constructions and so on, were used within Mathematical Physics in order to express physical laws in more clear and compact ways, see, for instance, [6], [26], [21], [16] and many others.

In this paper, using Stokes' system of Hydrodynamics as a model example, we propose a more general algebraic construction related to Homological Algebra that helps to describe physical laws in a unified form with the use of elliptic differential

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complexes, see, instance, [34]. The approach does not give a precise description of the related models, but it suggest dimensions of the corresponding known and unknown vectors and type of equations up to (both linear and non-linear) perturbations. The other details depend usually on the particular type of the processes and symmetries behind them.

Of course, there are other ways to generate mathematical models in standardised ways, for instance, in the frame of General Relativity Theory, see, for example, [28]. But in the present paper, instead explaining how considerations in Physics involve partial differential equations, we illustrate how a system of PDE's may generate more extensive mathematical model within Mathematical Physics.

We also indicate simple conditions, providing (Petrovskii or Douglis-Nirenberg) ellipticity of the related steady Maxwell's and Stokes' type systems and, consequently, parabolicity or hyperbolicity of the related time dependent systems. Actually, this opens a way to construct easily parametrices and fundamental solutions to Maxwell' and Stokes' type operators under the considerations.

1. Differential complexes

Let us shortly recall the notion of differential complex and related matters.

1.1. **Differential operators.** Let X be a C^∞-smooth Riemannian manifold of dimension $n \geq 2$ with a smooth (possibly, empty) boundary ∂X. We tacitly assume that it is enclosed into a smooth manifold $\tilde{X}$ of the same dimension. Let also $\overset{\circ}{X}$ denotes the interior of X.

For any smooth $\mathbb{C}$-vector bundles E and F of rangs k and l, respectively, over X, we write $\mathrm{Diff}_m(X; E \to F)$ for the space of all the linear partial differential operators of order $\leq m \in \mathbb{Z}_+$ between sections of the bundles E and F. Then, for an open set $\mathcal{O} \subset \overset{\circ}{X}$ over which the bundles and the manifold are trivial, the sections over $\mathcal{O}$ may be interpreted as (vector-) functions and $A \in \mathrm{Diff}_m(X; E \to F)$ is given as $(l \times k)$-matrix of scalar differential operators, i.e. we have

$$A = A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

where $a_\alpha(x)$ are $(l \times k)$-matrices of $C^\infty(\mathcal{O})$-functions, $\partial_j = \frac{\partial}{\partial x_j}$, $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_1^{\alpha_n}$. Denote by I_k the identity operator on sections of the bundle E (a unit $(k \times k)$-matrix in the local situation) and by E^* the conjugate bundle of E. Any Hermitian metric $(.,.)_{E,x}$ on E gives rise to a sesquilinear bundle isomorphism (the Hodge operator) $\star_E : E \to E^*$ by the equality $\langle \star_E v, u \rangle_{E,x} = (u, v)_{E,x}$ for all sections u and v of E; here $\langle .,. \rangle_{E,x}$ is the natural pairing in the fibers of E^* and E. Pick a volume form dx on X, thus identifying the dual bundle, the conjugate bundle and the Lebesgue space $L^2(E)$ with the inner product induced by $(.,.)_{E,x}$. Then for $A \in \mathrm{Diff}_m(X; E \to F)$ denote by $A^* \in \mathrm{Diff}_m(X; F \to E)$ the corresponding formal adjoint operator. Let also $\mathcal{D}$ be a bounded domain (i.e. open connected set) in $\overset{\circ}{X}$.

1.2. **Compatibility differential complexes.** We recall that a differential operator A is called overdetermined on X if there is a non-zero differential operator B over X such that

$$B \circ A \equiv 0. \tag{1.1}$$

An operator B, satisfying (1.1), is called a compatibility operator for A if for any operator $\tilde{B}$ satisfying $\tilde{B} \circ A \equiv 0$ there is an operator C such that $\tilde{B} = C \circ B$. Clearly, a compatibility operator is not uniquely defined; however it gives necessary solvability conditions to the operator equation

$$Au = f \text{ in } \mathcal{D}$$

in a domain $\mathcal{D} \subset X$, i.e. $Bf = 0$. However, a compatibility operator may also contain addition information on a physical model where the operator A appeared. Of course, the operator B can also be overdetermined.

Thus, our principal object to discuss will be a complex $\{A_q, E_q\}_{q=0}^N$ of partial differential operators over X (see, for instance, [32], [34]),

$$0 \to C^\infty(E_0) \overset{A_0}{\to} C^\infty(E_1) \overset{A_1}{\to} C^\infty(E_2) \to \dots \overset{A_{N-1}}{\to} C^\infty(E_N) \to 0, \tag{1.2}$$

where E_q are bundles of rangs k_q, respectively, over X and A_q are differential operators from $\mathrm{Diff}_{m_q}(X; E_q \to E_{q+1})$ with

$$A_{q+1} \circ A_q \equiv 0; \tag{1.3}$$

we assume that $A_q = 0$ for both $q < 0$ and $q \geq N$. Actually, it is often convenient to consider complex $\{A_q, E_q\}_{q=0}^N$ as a graduated operator $A^\cdot$ of degree 1 over a graduated topological vector space $\mathfrak{S}^\cdot = \oplus_{q=0}^N \mathfrak{S}^q(E_q)$ in such a way that $A^\cdot u = A_q u$ for a section $u \in \mathfrak{S}^q(E_q)$ of the bundle E_q.

It may happens, see examples below, that orders m_q of the differential operators A_q are different; so we set $m = \max\limits_{0 \leq q \leq N-1} m_q$. But the most simple constructions corresponds to the cases where

$$m_j = m \text{ for all } 0 \leq j \leq N-1. \tag{1.4}$$

As above, we fix Hermitian metrics $(\cdot,\cdot)_{q,x} = (\cdot,\cdot)_{E_q,x}$ in each fiber $E_{q,x}$.

The differential complex $\{A_q, E_q\}$ is called a compatibility complex for A_0 if for each $q \geq 0$ the differential operator A_{q+1} is a compatibility operator for A_q. As the compatibility operator is not unique, the compatibility complex is not unique, too. The notions of homotopical equivalence of complexes ([34, Definition 1.1.17]) and equivalent operators [34, Definition 1.2.5]) help to improve the situation. In particular, homotopically equivalent complexes have isomorphic cohomologies over many standard functional classes, see [34, Proposition 1.24]. According to [34, Propositions 1.2.7 and 1.2.8], if differential operators A_0 and $\tilde{A}_0$ are equivalent and the operator A_0 is included into a compatibility complex $\{A_q, E_q\}_{i=0}^N$ then for the operator $\tilde{A}_0$ there is a compatibility complex $\{\tilde{A}_q, \tilde{E}_q\}_{q=0}^N$ and, moreover, the corresponding complexes are homotopicaly equivalent.

The algebraic structures lying at the bottom of the theory of differential complexes are rather natural, see [34, Ch. 1], though this depends on the class of considered operators. Namely, let us consider the two typical cases.

If $X = \mathbb{R}^n$ and $A = A(D)$ is an $(l \times k)$-matrix differential operator with constant coefficients then one may use $\mathcal{P}$-modules of the ring $\mathcal{P}$ of all the polynomials with complex coefficients, see [25], [34, §1.2], or elsewhere. Let us denote by $\mathcal{P}^k$ the direct sum of k copies of the ring $\mathcal{P}$ and denote by $A(\zeta)$ the polynomial matrix

$$A(\zeta) = \sum_{|\alpha| \leq m} a_\alpha (\iota\zeta)^\alpha, \ \zeta \in \mathbb{C}^n.$$

Then the transposed matrix $A'(\zeta)$ naturally defines a mapping $A'(\zeta) : \mathcal{P}^l \to \mathcal{P}^k$. As the ring $\mathcal{P}$ is Noetherian, then the $\mathcal{P}$-module $\mathcal{P}^k / A'(\zeta)\mathcal{P}^l$ is finitely generated, i.e. there are natural numbers N, $k_1, \dots k_N$ and polynomial $(k_{i+1} \times k_i)$-matrices

$$A_q(\zeta) = \sum_{|\alpha| \leq m_q} a^{(q)}_\alpha (\iota\zeta)^\alpha, \ 0 \leq q \leq N-1,$$

such that $k_0 = k$, $k_1 = l$, $A_0(\zeta) = A(\zeta)$, $A'_q(\zeta) \circ A'_{q+1}(\zeta) = 0$ for all $\zeta \in \mathbb{C}$, and the following sequence is exact, see Hilbert Syzygies Theorem, [3, §8],

$$0 \leftarrow \mathcal{P}^k / A'(\zeta)\mathcal{P}^l \leftarrow \mathcal{P}^{k_0} \xleftarrow{A'_0(z)} \mathcal{P}^{k_1} \xleftarrow{A'_1(z)} \mathcal{P}^{k_2} \leftarrow \dots \xleftarrow{A'_{N-1}(z)} \mathcal{P}^{k_N} \leftarrow 0.$$

Then the related operators

$$A_q(D) = \sum_{|\alpha| \leq m_q} a^{(q)}_\alpha \partial^\alpha$$

with constant coefficients form the desired compatibility differential complex (1.2) for $A = A_0$; it is called the Hilbert complex associated with the $\mathcal{P}$-module for A. Actually, if $\mathcal{D}$ is a convex domain in $\mathbb{R}^n$ then for any section $f \in C^\infty(\mathcal{D}, E_{q+1})$ satisfying $A_{q+1} f = 0$ in $\mathcal{D}$ there is $u \in C^\infty(\mathcal{D}, E_q)$ satisfying $A_q u = f$ in $\mathcal{D}$, i.e. the Hilbert complex gives both necessary and sufficient conditions for the solvability of the related operator equations in this particular situation, see, for instance, [25].

In the general case of differential operators with variable coefficients (or even operators on manifold), to construct a compatibility complex for an operator A is a more delicate procedure, see [12], [29], [32]. In particular, the related complex might be not finite. D.C. Spencer [32] granted existence of a (finite) compatibility differential complex for any "sufficiently regular" differential operator with infinitely smooth coefficients, see also [34, §1.3] for a more advanced discussion. To define the concept "sufficient regularity" one should consider jets j^s of sections E and F over X of finite length s and the prolongations $j^s \circ A$ of the differential operator A to the spaces of jets $\mathcal{J}^s(E)$, see [34, §1.3.2]. More precisely, let $\eta(A) : J^m(E) \to F$ be the bundle homomorphism satisfying $\eta(A) \circ j^m = A$ and

$$\mathfrak{R}^s(x) = \ker\{\eta(j^{s-m} \circ A) : \mathcal{J}^s(E)_x \to \mathcal{J}^{s-m}(F)_x\}, \ x \in X.$$

The operator A is called "sufficiently regular" if 1) the dimensions $d(s,x)$ of the spaces $\mathfrak{R}^s(x)$ do not depend on $x \in X$ for $s \geq m$ and 2) the natural "projections" $\pi^{s_2,s_1} : \mathfrak{R}^{s_2}(x) \to \mathfrak{R}^{s_1}(x)$ have constant rank for all $s_2 \geq s_1 \geq m$. Of course, the operators with constant coefficients are "sufficiently regular".

1.3. **Elliptic differential complexes.** Let $\pi : T^*X \to X$ be the (real) cotangent bundle of X and let π^*E be a induced bundle for the bundle E (i.e. the fiber of π^*E over the point $(x,\zeta) \in T^*X$ coincides with E_x). We write $\sigma(A) : \pi^*E \to \pi^*F$ for the principal homogeneous symbol of the order m of the operator A, see, for instance, [34, §1.1.9]. Of course, in a suitable local chart we have

$$\sigma(A)(x,\zeta) = \sum_{|\alpha|=m} a_\alpha(x)(\iota\zeta)^\alpha, \ x \in \mathcal{O}, \ \zeta \in \mathbb{R}^n,$$

where ι is the imaginary unit. We recall that A is called elliptic on X if $k = l$ and the mapping $\sigma(A)(x,\zeta) : \pi^*E_x \to \pi^*F_x$ is invertible for $(x,\zeta) \in T^*X$ with $\zeta \neq 0$, see, for instance, [10, Ch 1, §3, Ch. 2, §2]. Sometimes A is called *overdetermined elliptic* if $k < l$ and the mapping $\sigma(A)(x,\zeta) : \pi^*E \to \pi^*F$ is injective for all $(x,\zeta) \in T^*X$

with $\zeta \neq 0$, but not surjective for some (x,ζ). A typical operator with injective symbol is a suitable connection ∇_E related to a bundle E, i.e. a first differential operator of the type $E \to E \otimes T^* X$, compatible with Hermitian metric $(\cdot,\cdot)_{E,x}$ in each fiber E_x, see, for instance, [37, Ch. III]. In particular, for a trivial vector bundle $E = \mathbb{R}^n \times \mathbb{C}^k$ we have $\nabla_E = I_k \otimes \nabla$ with the usual gradient operator ∇ in $\mathbb{R}^n$ where $M_1 \otimes M_2$ stands for the tensor product of matrices M_1 and M_2.

Recall that an operator A of an even order $m = 2p$ and of type $E \to E$ is called strongly elliptic, if

$$\Re(\sigma(A)(x,\zeta)\, w, w)_{E,x} > 0 \text{ for all } (x,\zeta) \in T^*X \setminus \{0\}, w \in E_x \setminus \{0\},$$

where $\Re\, a$ is the real part of a complex number a. A typical strongly elliptic operator of the second order is given by the 'Laplacian' $\nabla_E^* \nabla_E$. For a trivial vector bundle $E = \mathbb{R}^n \times \mathbb{C}^k$ we have $\nabla_E^* \nabla_E = -I_k \otimes \Delta$ with the usual Laplace operator Δ in $\mathbb{R}^n$.

A more general notion of ellipticity was introduced by A. Douglis and L. Nirenberg, [9] (see also, for instance, [10, Ch. 1, §3] or, [38, §9.2]). Namely, let the entries of an $(k \times k)$-matrix linear operator A be scalar differential operators $A^{(p,r)} = \sum_{|\alpha| \leq m} a_\alpha^{(p,r)}(x) \partial^\alpha$ with $a_\alpha^{(p,r)}(x)$ being the components of the functional $(k \times k)$-matrix $a_\alpha^{(p,r)}(x)$. Given two vectors $\vec{s}, \vec{t} \in \mathbb{R}^k$, the $(\vec{s},\vec{t})$-principal part of the operator A is the $(k \times k)$-matrix linear operator $\tilde{A}$ with components

$$\tilde{A}^{(p,r)} = \begin{cases} \sum_{|\alpha| = s_p - t_r} a_\alpha^{(p,r)}(x) \partial^\alpha, & s_p \geq t_r, \\ 0, & s_p < t_r. \end{cases}$$

Then $(\vec{s},\vec{t})$-principal symbol of A is the $(k \times k)$-matrix $\sigma_{\vec{s},\vec{t}}(X)(x,\zeta)$ with the components

$$\Big(\sum_{|\alpha| = s_p - t_r} a_\alpha^{(p,r)}(x) \zeta^\alpha \Big).$$

The operator A is called Douglis-Nirenberg elliptic, if there are two vectors $\vec{s}, \vec{t} \in \mathbb{Z}^k$ such that

$$\det \sigma_{\vec{s},\vec{t}}(X)(x,\zeta) \neq 0 \text{ for all } x \in X, \zeta \in \mathbb{R}^n \setminus \{0\}.$$

Next, for the *principal symbols* of the operators from complex (1.2), we have

$$\sigma(A_{q+1}) \circ \sigma(A_q) \equiv 0. \tag{1.5}$$

Complex (1.2) is called elliptic, if the corresponding symbolic complex,

$$0 \to \pi^* E_0 \overset{\sigma(A_0)}{\to} \pi^* E_1 \overset{\sigma(A_1)}{\to} \pi^* E_2 \to \dots \overset{\sigma(A_{N-1})}{\to} \pi^* E_N \to 0,$$

is exact for all $(x,z) \in T^*X \setminus \{0\}$, i.e. the range of the mapping $\sigma(A_q)$ coincides with the kernel of the mapping $\sigma(A_{q+1})$. In particular, $\sigma(A_0)$ is injective and $\sigma(A_{N-1})$ is surjective for all $(x,z) \in T^*X \setminus \{0\}$. Of course, an operator A_0 is elliptic if and only if the following complex is elliptic:

$$0 \to \pi^* E_0 \overset{\sigma(A_0)}{\to} \pi^* E_1 \to 0. \tag{1.6}$$

There is also a Douglis-Nirenberg type ellipticity for elliptic complexes, see [2].

For the sake of notations, we set $\sigma_q = \sigma(A_q)$ and $\delta_q = \sigma_q^* \sigma_q + \sigma_{q-1} \sigma_{q-1}^*$; then $\sigma_q^* = \sigma(A_q^*)$ and according to (1.5), for all $0 \leq q \leq N-1$ we have

$$\sigma_q^*\, \sigma_{q+1}^* = 0,\ \delta_{q+1}\, \sigma_q = \sigma_q\, \delta_q = \sigma_j\, \sigma_q^*\, \sigma_q,\ \sigma_q^*\, \delta_{q+1} = \delta_q\, \sigma_j^* = \sigma_q^*\, \sigma_q\, \sigma_q^*. \tag{1.7}$$

Lemma 1.1. *Complex* (1.2) *is elliptic if and only if the mappings* $\delta_q : \pi^* E_q \to \pi^* E_q$ *are bijective for all* $(x,z) \in T^*X \setminus \{0\}$ *and all* $0 \le q \le N$.

Denote by Δ_q the Hodge's Laplacians of complex (1.2):

$$\Delta_q = A_q^* A_q + A_{q-1} A_{q-1}^*, \, 0 \le q \le N.$$

If $m_q = m_{q-1}$ then $\delta_q = \sigma(A_q^* A_q + A_{q-1} A_{q-1}^*)$. According to Lemma 1.1, if complex (1.2) satisfies (1.4) then, for the complex to be elliptic, it is necessary and sufficient that the Laplacians Δ_q of the complex are strongly elliptic differential operators of order $2m$ for all $0 \le q \le N$.

Next, given a pair $\boldsymbol{\mu}_q$ consisting of formally non-negative self-adjoint differential operators $\boldsymbol{\mu}_q^{(0)} \in \mathrm{Diff}_{2\tilde{m}_q}(X, E_{q+1} \to E_{q+1})$ and $\boldsymbol{\mu}_q^{(1)} \in \mathrm{Diff}_{2\hat{m}_q}(X, E_{q-1} \to E_{q-1})$, $0 \le q \le N$, with some numbers $\tilde{m}_q, \hat{m}_q \in \mathbb{Z}_+$, satisfying $0 \le \tilde{m}_q \le m - m_q$, $0 \le \hat{m}_q \le m - m_{q-1}$, we denote by $\Delta_{q,\boldsymbol{\mu}}$ the steady Lamé type operators

$$\Delta_{q,\boldsymbol{\mu}} = A_q^* \boldsymbol{\mu}_q^{(0)} A_q + A_{q-1} \boldsymbol{\mu}_q^{(1)} A_{q-1}^*.$$

If orders $\tilde{m}_q$ and $\hat{m}_q$ equal to zero then strong ellipticity means that $\boldsymbol{\mu}_q^{(0)}$, $\boldsymbol{\mu}_{q+2}^{(1)}$ are bijective self-adjoint non-negative mappings. In general, we may produce the operators $\boldsymbol{\mu}_q^{(0)}$, $\boldsymbol{\mu}_q^{(1)}$ with the use of connections over E_{q+1} and E_{q-1}, respectively:

$$\boldsymbol{\mu}_q^{(0)} = (\nabla^*_{E_{q+1}} \nabla_{E_{q+1}})^{\tilde{m}_q}, \, \boldsymbol{\mu}_q^{(0)} = (\nabla^*_{E_{q-1}} \nabla_{E_{q-1}})^{\hat{m}_q}.$$

On this way, taking $\tilde{m} = m - m_q$, $\hat{m} = m - m_{q-1}$ we may achieve that all the operators $\Delta_{q,\boldsymbol{\mu}}$ have the same order $2m$. For this reason we will often use the following assumption.

Assumption 1.2. *The formally self-adjoint non-negative operators* $\boldsymbol{\mu}_j^{(0)}$ *and* $\boldsymbol{\mu}_j^{(1)}$ *are strongly elliptic.*

Set $\delta_{q,\boldsymbol{\mu}} = \sigma_q^* \, \sigma(\boldsymbol{\mu}_q^{(0)}) \, \sigma_q + \sigma_{q-1} \, \sigma(\boldsymbol{\mu}_q^{(1)}) \, \sigma_{q-1}^*$.

Lemma 1.3. *Let complex* (1.2) *be elliptic. If* $0 \le j \le N$ *then, under Assumption* 1.2*, the mapping* $\delta_{j,\boldsymbol{\mu}} : \pi^* E_j \to \pi^* E_j$ *is bijective for all* $(x,z) \in T^*X \setminus \{0\}$*; in particular, the operator* $\Delta_{j,\boldsymbol{\mu}}$ *is strongly elliptic self-adjoint non-negative, too, if we additionally have* $m_j + \tilde{m}_j = m_{j-1} + \hat{m}_j$.

Remark 1.1. Clearly, the generalized Laplacians can be factorized as follows

$$\Delta_{q,\boldsymbol{\mu}} = \begin{pmatrix} A_q^*, & A_{q-1} \boldsymbol{\mu}_q^{(1)} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_q^{(0)} A_q \\ A_{q-1}^* \end{pmatrix}, 0 \le q \le N. \tag{1.8}$$

As we have noted above, a compatibility complex $\{A_q, E_q\}$ for an operator A_0 is not uniquely defined. For this reason, formula (1.8) suggests the following natural conditions for the operators $\boldsymbol{\mu}_q^{(k)}$:

$$A_{q+1} \, \boldsymbol{\mu}_{q+2}^{(1)} \, \boldsymbol{\mu}_q^{(0)} \, A_q \equiv 0 \text{ for all } 0 \le q \le N-1. \tag{1.9}$$

On the symbolic level this means

$$\sigma_{q+1} \, \sigma(\boldsymbol{\mu}_{q+2}^{(1)}) \, \sigma(\boldsymbol{\mu}_q^{(0)}) \, \sigma_q \equiv 0 \text{ for all } 0 \le q \le N-1. \tag{1.10}$$

If the complex $\{A_q, E_q\}$ consists of the operators with constant coefficients then we may set $\boldsymbol{\mu}_q^{(0)} = (-1)^{\tilde{m}_q} I_{k_q+1} \otimes \Delta^{\tilde{m}_q}$, $\boldsymbol{\mu}_q^{(1)} = (-1)^{\hat{m}_q} I_{k_q-1} \otimes \Delta^{\hat{m}_q}$; in this case,

$$\boldsymbol{\mu}_q^{(0)} A_q = (-1)^{\tilde{m}_q} A_q (I_{k_q} \otimes \Delta^{\tilde{m}_q}), \, A_{q+2} \boldsymbol{\mu}_{q+2}^{(1)} = (-1)^{\hat{m}_q} (I_{k_{q+3}} \otimes \Delta^{\hat{m}_q}) A_{q+2},$$

and hence (1.9), (1.10) hold true. In general case we should look for suitable commutative relations between $\boldsymbol{\mu}^{(0)}_q$, $\boldsymbol{\mu}^{(1)}_{q+2}$, A_q and A_{q+2} that is not a trivial task.

Finally, if $m_q+\tilde{m}_q = m_{q-1}+\hat{m}_q$, we may consider the Helmholtz-Lamé operators

$$\mathfrak{D}_{q,\boldsymbol{\mu}} = \Delta_{q,\boldsymbol{\mu}} + \sum_{|\alpha|\leq 2(m_q+\tilde{m}_q)-1} d_\alpha(x)\partial^\alpha \tag{1.11}$$

that are strongly elliptic if the operators $\boldsymbol{\mu}^{(0)}_q$, $\boldsymbol{\mu}^{(1)}_q$ are strongly elliptic ($\mathfrak{D}_q$ corresponds to the case where $\boldsymbol{\mu}^{(0)}_q = I_{k_q+1}$, $\boldsymbol{\mu}^{(1)}_q = I_{k_q-1}$).

Note that the most natural part of the low order perturbation of the Laplacian $\Delta_{q,\boldsymbol{\mu}}$ is usually given as follows:

$$C_qA_q + \tilde{C}_qA^*_{q-1} + M_q \tag{1.12}$$

with a formally self-adjoint operator $M_q \in \mathrm{Diff}_0(X, E_q \to E_q)$, and operators $C \in \mathrm{Diff}_{m_q+2\tilde{m}_q-1}(X, E_{q+1} \to E_q)$, $\tilde{C} \in \mathrm{Diff}_{m_{q-1}+2\hat{m}_q-1}(X, E_{q-1} \to E_q)$.

1.4. **The induced complexes and time dependent processes.** The constructions considered in the previous subsection are fit for steady models of Mathematical Physics. To use the differential complexes for time dependent models, one may introduce the so-called induced complex

$$0 \to C^\infty(X, E_0(t)) \overset{A_0}{\to} \cdots \to \ldots \overset{A_{N-1}}{\to} C^\infty(X, E_N(t)) \to 0,$$

where sections of the induced bundles $E_q(t)$ and the coefficients $a^{(q)}_\alpha$ of the differential operators A_q depend on both x and the real parameter t. The induced complex $\{A_q, E_q(t)\}$ is elliptic on $X \times [0, T)$ with a (possibly, infinite) time T, if the corresponding symbolic complex,

$$0 \to \pi^*E_0(t) \overset{\sigma(A_0)}{\to} \pi^*E_1(t) \overset{\sigma(A_1)}{\to} \pi^*E_2(t) \to \ldots \overset{\sigma(A_{N-1})}{\to} \pi^*E_N(t) \to 0,$$

is exact for all $(x, z) \in T^*X \setminus \{0\}$ and each $t \in [0, T)$. Note that for some kind of problems one needs a more subtle notion of ellipticity with a parameter, see, for instance, [1] or [10, Ch 2, §2].

In any case, one may easily introduce the operators

$$\mathcal{L}_{q,\boldsymbol{\mu}} = \partial_t + \mathfrak{D}_{q,\boldsymbol{\mu}},\ \mathcal{H}_{q,\boldsymbol{\mu}} = \partial^2_t + \mathfrak{D}_{q,\boldsymbol{\mu}}$$

over $X \times [0, T)$. Note that $\mathcal{L}_{q,\boldsymbol{\mu}}$ is strongly parabolic, if $\mathfrak{D}_{q,\boldsymbol{\mu}}$ is strongly elliptic, see [11], [10, Ch 1, §3, Ch. 2, §5] and $\mathcal{H}_{q,\boldsymbol{\mu}}$ has the 'hyperbolicity' properties if $\mathfrak{D}_{q,\boldsymbol{\mu}}$ is strongly elliptic, see, for instance, [10, Ch 1, §3, Ch. 2, §4].

2. Maxwell's and Stokes' type operators for differential complexes

It is well known that Stokes' system $S = S_{1,\mu}(d, \partial_t)$,

$$S\begin{pmatrix} \vec{v} \\ p \end{pmatrix} = \begin{pmatrix} (\partial_t - \mu\Delta)I_n & \nabla \\ \mathrm{div} & 0 \end{pmatrix}\begin{pmatrix} \vec{v} \\ p \end{pmatrix} = \begin{pmatrix} \vec{f} \\ 0 \end{pmatrix} \tag{2.1}$$

plays an essential role in mathematical models for incompressible fluid with given the dynamical viscosity $\mu > 0$ of the fluid under the consideration, the density vector of outer forces $\vec{f}$, the search-for velocity vector field $\vec{v}$ and pressure p of the flow, see, for instance, [33], [15], [36]. Actually, Stokes' system S and its steady version give the principal linear parts of the steady and evolutionary Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$, endowed with the non-linear perturbation given by

$$N_1(\vec{v}) = \begin{pmatrix} (\vec{v}\cdot\nabla)\vec{v} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Big(\sum\limits_{j=1}^{n} v_j \partial_j\Big)\vec{v} & 0 \\ 0 & 0 \end{pmatrix}.$$

It appears, that S can be easily written in the context of the de Rham complex $\{\nabla, \mathrm{curl}, \mathrm{div}\}$ in $\mathbb{R}^3$ at the step $q = 1$ with

$$\Delta_1 = -\Delta I_3 = \mathrm{curl}^*\mathrm{curl} + \mathrm{div}^*\mathrm{div} = \mathrm{curl}\,\mathrm{curl} - \nabla\,\mathrm{div}.$$

A generalisation of Stokes' type operators for elliptic complex (1.2) was proposed in [19, formula (1.2)] (cf. [31] for the de Rham complex):

$$\tilde{S}_q(A, \partial_t) = \begin{pmatrix} \mathcal{L}_{q,\boldsymbol{\mu}} & A_{q-1} \\ A^*_{q-1} & 0 \end{pmatrix}, \; 0 \leq q \leq N.$$

It was noted in [31, Proposition 2.2] and [27, formula (0.4)] that for $q \geq 2$ one more natural line should be added to Stokes' type operator associated with complex (1.2) that is missed for $q = 1$:

$$\hat{S}_q(A, \partial_t) = \begin{pmatrix} \mathcal{L}_{q,\boldsymbol{\mu}} & A_{q-1} \\ A^*_{q-1} & 0 \\ 0 & A^*_{q-2} \end{pmatrix}, \; 0 \leq q \leq N.$$

Actually the additional operator equation provides some uniqueness for Stokes' type equations. However, adding this natural equation we see that new operators becomes overdetermined. This fact may complicate essentially the theory of Stokes' type equations for the complex (1.2) at the degrees $q > 1$. To overcome this difficulty, let us introduce slightly different generalisations of Stokes' type operators.

2.1. **Steady Maxwell's and Stokes' type operators for elliptic complexes.** Consider the following steady Maxwell's and Stokes' type operators related to complex (1.2) at degrees $0 \leq q \leq N$. Namely, set $r_q = (\sum_{j=0}^{q} k_j)$, $0 \leq q \leq N$. By B^N we denote $(N+1)\times(N+1)$-block matrix (in fact, it is a $(r_N \times r_N)$-matrix), such that each its block b^N_{ij} is a $(k_{N-i+1} \times k_{N-j+1})$-matrix. Let B_j be such a matrix with $b^N_{jj} = I_{k_j}$ and $B^N_{pq} = 0$ for $p \neq j$ or $q \neq j$. Clearly,

$$B_j B_j = B_j, \; B_i B_j = 0 \text{ if } j \neq i \tag{2.2}$$

and all the blocks of the matrix $B_i B^N B_j$ equal to zero except the block

$$(B_i B^N B_j)_{ij} = b^N_{ij}.$$

For this reason, if P is an operator of type $E_{N-i+1} \to E_{N-j+1}$ then we denote by $B_i P B_j$ the $(N+1)\times(N+1)$-block matrix with all the block being zero except the block $b^N_{ij} = P$.

Next, set $\mathfrak{E}_q = \oplus_{j=0}^{q} E_j$, $0 \leq q \leq N$. Given set $\boldsymbol{\mu} = (\boldsymbol{\mu}_0, \ldots, \boldsymbol{\mu}_q)$ of pairs of differential operators, we introduce the following Maxwell's type operators acting on sections $U_N = (u_0, \ldots, u_N)$ of the bundle $\mathfrak{E}_N$: $\mathcal{M}^{(0)}_{0,\boldsymbol{\mu}}(A) = 0$, $\mathcal{M}^{(1)}_{0,\boldsymbol{\mu}}(A) = 0$,

$$\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A) = \sum_{j=0}^{q-1} \Big(B_{j+1} \boldsymbol{\mu}^{(0)}_j A_j B_j + B_j A^*_j B_{j+1} \Big), \; 1 \leq q \leq N, \tag{2.3}$$

$$\mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A) = \sum_{j=0}^{q-1} \Big(B_{j+1} A_j \boldsymbol{\mu}^{(1)}_{j+1} B_j + B_j A_j^* B_{j+1} \Big), \; 1 \le q \le N. \tag{2.4}$$

Obviously, $\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A) = \mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A)$, if

$$A_j \boldsymbol{\mu}^{(1)}_{j+1} = \boldsymbol{\mu}^{(0)}_j A_j \text{ for all } 0 \le j \le q; \tag{2.5}$$

consequently, we will write $\mathcal{M}_{q,\boldsymbol{\mu}}(A)$ for $\mathcal{M}^{(j)}_{q,\boldsymbol{\mu}}(A)$ in this case. In particular, we will use the notation $\mathcal{M}_q(A)$ in the simplest case where $\boldsymbol{\mu}^{(0)}_j = I_{k_{j+1}}$, $\boldsymbol{\mu}^{(1)}_j = I_{k_{j-1}}$ for all $0 \le j \le q$.

Similarly, we introduce Stokes' type operators

$$S_{q,a}(A, \mathfrak{D}_{\boldsymbol{\mu}}) = \sum_{j=0}^{q} B_j \mathfrak{D}_{j,\boldsymbol{\mu}} B_j + a \sum_{j=0}^{q-1} \Big(B_{j+1} A_j B_j + B_j A_j^* B_{j+1} \Big), \tag{2.6}$$

with $a = a_q$ being equal to 0 or 1.

In fact, by the construction, the operators $\mathcal{M}^{(i)}_{q,\boldsymbol{\mu}}(A)$, $S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ act on sections $U_q = (u_0, \dots, u_q)$ of the bundle $\mathfrak{E}_q$ and hence we will identify them with the lower right $(r_q \times r_q)$-minors of the related full $(r_N \times r_N)$-matrices. For instance, in a more bulky matrix form the operator $S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ may be written as

$$\begin{pmatrix}
\mathfrak{D}_{q,\boldsymbol{\mu}} & A_{q-1} & 0 & 0 & 0 & 0 & \dots & 0 \\
A^*_{q-1} & \mathfrak{D}_{q-1,\boldsymbol{\mu}} & A_{q-2} & 0 & 0 & 0 & \dots & 0 \\
0 & A^*_{q-2} & \mathfrak{D}_{q-2,\boldsymbol{\mu}} & A_{q-3} & 0 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & \dots & 0 & A^*_2 & \mathfrak{D}_{2,\boldsymbol{\mu}} & A_1 & 0 \\
0 & \dots & \dots & \dots & 0 & A^*_1 & \mathfrak{D}_{1,\boldsymbol{\mu}} & A_0 \\
0 & \dots & \dots & \dots & \dots & 0 & A^*_0 & \mathfrak{D}_{0,\boldsymbol{\mu}}
\end{pmatrix}.$$

More compact notations (2.3), (2.4), (2.6) echo with the sedeonic form of equations of Mathematical Physics proposed in [20], [21], [22], [23].

Let's explain the connection between Maxwell's and Stokes' type operators.

Lemma 2.1. *If* (1.9) *is fulfilled for all* $0 \le j \le q$, *then we have*

$$\mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A) \mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A) = B_q A_{q-1} \boldsymbol{\mu}^{(1)}_{q-1} A^*_{q-1} B_q + \sum_{j=0}^{q-1} B_j \Delta_{j,\boldsymbol{\mu}} B_j. \tag{2.7}$$

In particular, if $\mathfrak{D}_{j,\boldsymbol{\mu}_j} = \Delta_{j,\boldsymbol{\mu}_j}$ *for all* $0 \le j \le N$, *then*

$$S_{q,a}(A, \mathfrak{D}_{\boldsymbol{\mu}}) = \mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A) \mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A) + B_q A^*_q \boldsymbol{\mu}^{(0)}_q A_q B_q + a \, \mathcal{M}_q(A), \; 0 \le q \le N.$$

Next, we note that Stokes' type operator $S_0(A, \mathfrak{D}_{0,\boldsymbol{\mu}}) = \mathfrak{D}_{0,\boldsymbol{\mu}}$ is elliptic and strongly elliptic on X if the operator $\boldsymbol{\mu}^{(0)}_0$ is strongly elliptic. As it is known, see, for instance, [18, Ch. II, §4, Example 1], Stokes' operator (2.1) is Douglis-Nirenberg elliptic over $\mathbb{R}^n$. Then the following two statements are rather expectable. To formulate the statements, we set

$$\tilde{\sigma}(\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A)) = \sum_{j=0}^{q-1} \Big(B_{j+1} \, \sigma(\boldsymbol{\mu}^{(0)}_j) \, \sigma_j \, B_j + B_j \, \sigma^*_j \, B_{j+1} \Big),$$

$$\tilde{\sigma}(\mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A)) = \sum_{j=0}^{q-1} \Big(B_{j+1} \, \sigma_j \, \sigma(\boldsymbol{\mu}^{(1)}_{j+1}) \, B_j + B_j \, \sigma^*_j \, B_{j+1} \Big).$$

Proposition 2.2. *Let complex* (1.2) *be elliptic, Assumption* 1.2 *be fulfilled for all* $0 \leq j \leq N$ *and* (1.10) *be true for all* $0 \leq j \leq N$. *Then the symbolic matrices* $\tilde{\sigma}(\mathcal{M}^{(i)}_{N,\boldsymbol{\mu}}(A))$, $i = 0, 1$, *are invertible for all* $(x, \zeta) \in T^*X \setminus \{0\}$. *In particular,*

- *the operators* $\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)$, $\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)$, $N \geq 1$, *are elliptic if* (1.4) *is true for all* $0 \leq j \leq N$;
- *the operator* $S_{N,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})$, $N \geq 1$, *is elliptic for any* a *if for all* $0 \leq j \leq N$

$$m_j + \tilde{m}_j = m_{j-1} + \hat{m}_j = m; \tag{2.8}$$

- *under* (1.4) *the operator* $S_{N,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})$, $N \geq 1$, *is elliptic if*

$$\boldsymbol{\mu}^{(0)}_j = 0 \text{ for all } 0 \leq j \leq N-1, \ \boldsymbol{\mu}^{(1)}_j = 0 \text{ for all } 1 \leq j \leq N. \tag{2.9}$$

- *the operators* $\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)$, $\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)$, $N \geq 1$, *are Douglis-Nirenberg elliptic;*
- *the operator* $S_{N,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})$, $N \geq 1$, *is Douglis-Nirenberg elliptic.*

Proof. Indeed, similarly to (2.7), under condition (1.10), for all $0 \leq q \leq N$ we have

$$\tilde{\sigma}(\mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A))\tilde{\sigma}(\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A)) = B_q \sigma_{q-1} \boldsymbol{\mu}^{(1)}_q \sigma^*_{q-1} B_q + \sum_{j=0}^{q-1} B_j \delta_{j,\boldsymbol{\mu}_j} B_j. \tag{2.10}$$

In particular,

$$\tilde{\sigma}(\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A))\tilde{\sigma}(\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)) = \sum_{j=0}^{N} B_j \delta_{j,\boldsymbol{\mu}_j} B_j. \tag{2.11}$$

Hence, as the symbolic matrices $\delta_{j,\boldsymbol{\mu}_j}$ are invertible for all $(x, \zeta) \in T^*X$ with $\zeta \neq 0$ and all $0 \leq j \leq N$ (see Lemma 1.3), then the matrices $\tilde{\sigma}(\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A))$, $\tilde{\sigma}(\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A))$ are invertible for such (x, ζ), too.

If all the operators A_q, $0 \leq q \leq N-1$, have the same order m then the orders of the operators $\boldsymbol{\mu}^{(i)}_j$ equal to zero and hence

$$\tilde{\sigma}(\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)) = \sigma(\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)), \ \tilde{\sigma}(\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)) = \sigma(\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)),$$

i.e. the operators $\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)$ $\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)$ are elliptic.

Moreover, under (2.8),

$$\sigma(S_{N,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})) = \sum_{j=0}^{N} B_j \delta_{j,\boldsymbol{\mu}} B_j$$

and then $S_{N,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ is elliptic for any a because of Lemma 1.3.

If (1.4) is fulfilled then

$$\sigma(S_{N,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})) = a\,\sigma(\mathcal{M}_{N,\boldsymbol{\mu}}(A)).$$

Thus, in this case the operator $S_{N,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ is elliptic, too, if $a = 1$.

If $N = 1$ (that corresponds to an elliptic operator with symbolic complex (1.6)) then operators $\mathcal{M}^{(0)}_{1,\boldsymbol{\mu}}(A)$, $\mathcal{M}^{(1)}_{1,\boldsymbol{\mu}}(A)$ are always elliptic because the orders of the operators $\boldsymbol{\mu}^{(i)}_0$ equal to zero.

If $N \geq 2$ and the orders m_q of the operators A_q are different, then, we may solve the following system of $4N$ equations with respect to $4(N+1)$ unknown numbers

$s_1^{(i)}, \dots s_{N+1}^{(i)}, t_1^{(i)}, \dots t_{N+1}^{(i)}$, $i = 0, 1$:

$$\left\{ \begin{array}{ll} s_j^{(0)} - t_{j+1}^{(0)} = m_{N-j} + \tilde{m}_{N-j}, & s_{j+1}^{(0)} - t_j^{(0)} = m_{N-j}, \quad 1 \leq j \leq N, \\ s_{j+1}^{(1)} - t_j^{(1)} = m_{N-j} + \hat{m}_{N-j}, & s_j^{(1)} - t_{j+1}^{(1)} = m_{N-j}, \quad 1 \leq j \leq N, \end{array} \right. \tag{2.12}$$

As $4(N+1) - 4N = 4$ we set $t_1^{(0)} = t_2^{(0)} = t_1^{(1)} = t_2^{(1)} = 0$ and then

$$s_1^{(0)} = m_{N-1} + \tilde{m}_{N-1},\ s_2^{(0)} = m_{N-1},\ s_1^{(1)} = m_{N-1},\ s_2^{(1)} = m_{N-1} + \hat{m}_{N-1},$$

and we obtain a recurrent formula:

$$\left\{ \begin{array}{lll} t_{j+1}^{(0)} = s_j^{(0)} - m_{N-j} - \tilde{m}_{N-j}, & s_{j+1}^{(0)} = m_{N-j} + t_j^{(0)}, & 2 \leq j \leq N, \\ t_{j+1}^{(1)} = s_j^{(1)} - m_{N-j}, & s_{j+1}^{(1)} = m_{N-j} + t_j^{(1)} - \hat{m}_{N-j}, & 2 \leq j \leq N. \end{array} \right.$$

Hence we obtain a solution $\vec{s}^{(0)}$, $\vec{t}^{(0)}, \vec{s}^{(1)}, \vec{t}^{(1)}$ to system (2.12) with integer components. Then there is a non-negative integer c such that the vectors $\vec{s}^{(j)} = (s_1^{(j)} + c, \dots s_{N+1}^{(j)} + c)$, $\vec{t}^{(j)} = (t_1^{(j)} + c, \dots t_{N+1}^{(j)} + c)$, $j = 0, 1$, are solutions to (2.12) with non negative components. Assigning the values $s_p^{(i)}$, $t_r^{(i)}$ for each component of the block $\mathcal{M}_{N,\boldsymbol{\mu}}^{(i)}(A, p, r)$ in the block matrix $\mathcal{M}_{N,\boldsymbol{\mu}}^{(i)}(A)$ we see that

$$\tilde{\sigma}(\mathcal{M}_{N,\boldsymbol{\mu}}^{(0)} A)) = \sigma_{s^{(\vec{0})}, \vec{t}^{(0)}}(\mathcal{M}_{N,\boldsymbol{\mu}}^{(0)}(A)),\ \tilde{\sigma}(\mathcal{M}_{N,\boldsymbol{\mu}}^{(1)} A)) = \sigma_{\vec{s}^{(1)}, \vec{t}^{(1)}}(\mathcal{M}_{N,\boldsymbol{\mu}}^{(1)}(A)).$$

Thus, Lemma 1.3 and formula (2.11) imply that the operators $\mathcal{M}_{N,\boldsymbol{\mu}}^{(0)}$, $\mathcal{M}_{N,\boldsymbol{\mu}}^{(1)}$ are Douglis-Nirenberg elliptic. In particular, $\mathcal{M}_{N,\boldsymbol{\mu}}$ is Douglis-Nirenberg elliptic, too.

And, finally, under (2.9) we have

$$\sigma_{\vec{s}^{(0)}, \vec{t}^{(0)}}(S_{N,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})) = \tilde{\sigma}(\mathcal{M}_N(A)).$$

Therefore the operator $S_{N,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ is always Douglis-Nirenberg elliptic. □

Proposition 2.3. *Let complex* (1.2) *be elliptic,* (1.10) *be true,* $N \geq 2$ *and* $0 \leq q \leq N-1$. *If* (2.8) *and Assumption* 1.2 *are fulfilled for all* $0 \leq j \leq q$, *then the Stokes operator* $S_{q,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ *is (Petrovskii) elliptic. If* $m_q + \tilde{m}_q = m_{q-1} + \hat{m}_q$, *and Assumption* 1.2 *is fulfilled for* $j = q$ *then* $S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ *is a Douglis-Nirenberg elliptic operator.*

Proof. For $q = 0$ we always have $S_0(A, \mathfrak{D}_{\boldsymbol{\mu}}) = \mathfrak{D}_{0,\boldsymbol{\mu}}$, i.e. it is strongly elliptic if the differential operator $\boldsymbol{\mu}_0^{(0)}$ is strongly elliptic on X. Moreover, under the hypothesis of the first part of this proposition we have

$$\sigma(S_{q,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})) = \sum_{j=0}^{q} B_j \delta_{j,\boldsymbol{\mu}} B_j.$$

As in this particular case, $\Delta_{j,\boldsymbol{\mu}}$ are strongly elliptic operators, see Lemma 1.3, we conclude that the operator $S_{q,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ is elliptic, too.

Let us prove the second statement of the proposition. With this purpose, let us solve the following system of $(2q+1)$ equations with respect to $2(q+1)$ unknown numbers $s_1, \dots s_{q+1}, t_1, \dots t_{q+1}$:

$$\left\{ \begin{array}{l} s_1 - t_1 = 2(m_q + \tilde{m}_q) = 2(m_{q-1} + \hat{m}_q), \\ s_j - t_{j+1} = m_{q-j},\ s_{j+1} - t_j = m_{q-j}, \qquad 1 \leq j \leq q, \end{array} \right. \tag{2.13}$$

As $2(q+1) - 2q + 1 = 1$, we set $t_1 = 0$ and then

$$s_1 = 2(m_q + \tilde{m}_q),\ t_2 = 2(m_q + \tilde{m}_q) - m_{q-1},$$

and we again obtain a recurrent formula:

$$t_{j+1} = s_j - m_{N-j},\ s_{j+1} = m_{N-j} + t_j,\ 2 \le j \le N.$$

Thus, system (2.13) has a solution $\vec{s}$, $\vec{t}$ with integer components. Then there is a non-negative integer c such that the numbers $s_1 + c, \dots s_{q+1} + c, t_1 + c, \dots t_{q+1} + c$ are solutions to (2.13) with non negative components. Again, assigning values s_p, t_r for each component of the block $S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}}), p, r)$ in the block matrix $S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}}))$ we see that

$$\sigma_{\vec{s},\vec{t}}(S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}}))) = B_q \delta_{q,\boldsymbol{\mu}_q} B_q + \tilde{\sigma}(\mathcal{M}_q(A)).$$

Next, using (1.7) we conclude that

$$\delta_{j,\boldsymbol{\mu}} \sigma_j^* \sigma(\boldsymbol{\mu}_j^{(0)}) \sigma_j = \sigma_j^* \sigma(\boldsymbol{\mu}_j^{(0)}) \sigma_j \sigma_j^* \sigma(\boldsymbol{\mu}_j^{(0)}) \sigma_j = \sigma_j^* \sigma(\boldsymbol{\mu}_j^{(0)}) \sigma_j \delta_{j,\boldsymbol{\mu}}$$

and, if the matrix $\delta_{j,\boldsymbol{\mu}}$ is invertible, then

$$\sigma_j^* \sigma(\boldsymbol{\mu}_j^{(0)}) \sigma_j \delta_{j,\boldsymbol{\mu}}^{-1} = \delta_{j,\boldsymbol{\mu}}^{-1} \sigma_j^* \sigma(\boldsymbol{\mu}_j^{(0)}) \sigma_j. \tag{2.14}$$

Consider the following matrix:

$$\mathcal{N}_\sigma^{(q)} = B_q \delta_{q,\boldsymbol{\mu}}^{-1} \sigma_q^* \sigma(\boldsymbol{\mu}_q^{(0)}) \sigma_q B_q + \tag{2.15}$$

$$B_q \sigma_{q-1} B_{q-1} + B_{q-1} \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* B_q - B_{q-1} \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* \sigma_{q-1} B_{q-1}.$$

Then, properties (2.2) of matrices B_j and formulae (2.10), (2.14) imply

$$B_q \delta_{q,\boldsymbol{\mu}} B_q \left(\mathcal{N}_\sigma^{(q)} + \tilde{\sigma}(\mathcal{M}_{q-1}(A)) \right) = B_q \sigma_q^* \sigma(\boldsymbol{\mu}_q^{(0)}) \sigma_q B_q + B_q \sigma_{q-1} \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* \sigma_{q-1} B_{q-1},$$

$$\tilde{\sigma}(\mathcal{M}_q(A)) \tilde{\sigma}(\mathcal{M}_{q-1}(A)) = \tilde{\sigma}(\mathcal{M}_{q-1}(A)) \tilde{\sigma}(\mathcal{M}_{q-1}(A)),$$

$$\tilde{\sigma}(\mathcal{M}_q(A)) \mathcal{N}_\sigma^{(q)} = B_q \sigma_{q-1} \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* B_q + B_{q-1} \sigma_{q-1}^* \left(\delta_{q,\boldsymbol{\mu}}^{-1} \sigma_q^* \sigma(\boldsymbol{\mu}_q^{(0)}) \sigma_q \right) B_q +$$

$$B_{q-1} \sigma_{q-1}^* \sigma_{q-1} B_{q-1} - B_q \sigma_{q-1} \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* \sigma_{q-1} B_{q-1} +$$

$$B_{q-2} \sigma_{q-2}^* \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* B_q - B_{q-2} \sigma_{q-2}^* \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* \sigma_{q-1} B_{q-1} =$$

$$B_q \sigma_{q-1} \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* B_q + B_{q-1}^q \sigma_{q-1}^* \sigma_{q-1} B_{q-1} - B_q \sigma_{q-1} \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* \sigma_{q-1} B_{q-1}.$$

Thus, we arrive at the following identity:

$$\sigma_{\vec{s},\vec{t}}(S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})) \left(\mathcal{N}_\sigma^{(q)} + \tilde{\sigma}(\mathcal{M}_{q-1}(A)) \right) = B_q \delta_{q,\boldsymbol{\mu}} B_q + \tag{2.16}$$

$$\sum_{j=0}^{q-1} B_j \delta_j B_j + B_{q-2} \sigma_{q-2}^* \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* B_q - B_{q-2} \sigma_{q-2}^* \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* \sigma_{q-1} B_{q-1}.$$

Now, if $U = (u_1, \dots u_q)$ satisfies

$$\sigma_{\vec{s},\vec{t}}(S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})) \left(\mathcal{N}_\sigma^{(q)} + \tilde{\sigma}(\mathcal{M}_{q-1}(A)) \right) U = 0,$$

then

$$\delta_{q,\boldsymbol{\mu}} u_q = 0,\ \delta_{q-1} u_{q-1} = 0,\ \delta_j u_j = 0,\ 0 \le j \le q-3,$$

$$\sigma_{q-2}^* \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* u_q - \sigma_{q-2}^* \sigma(\boldsymbol{\mu}_q^{(1)}) \sigma_{q-1}^* \sigma_{q-1} u_{q-1} + \delta_{q-2} u_{q-2} = 0.$$

Immediately we see that $u_j = 0$ for all $0 \le j \le q$, $j \ne q-2$, because symbolic matrices $\delta_{q,\boldsymbol{\mu}}$, δ_j are invertible for all $(x,\zeta) \in T^*X$ with $\zeta \ne 0$. Therefore $\delta_{q-2} u_{q-2} = 0$ and then, again $u_{q-2} = 0$ for the same reason. Hence the matrices

$$\sigma_{\vec{s},\vec{t}}(S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})) \left(\mathcal{N}_\sigma^{(q)} + \tilde{\sigma}(\mathcal{M}_{q-1}(A)) \right) \text{ and } \sigma_{\vec{s},\vec{t}}(S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}}))$$

are invertible for all $(x,\zeta) \in T^*X$ with $\zeta \neq 0$, too, i.e. the operator $S_{q,1}(A,\mathfrak{D}_{\boldsymbol{\mu}})$ is Douglis-Nirenberg elliptic. $\square$

2.2. **Maxwell's and Stokes' type operators for induced elliptic complexes.** For $N \geq 1$ introduce non-steady Maxwell's type operators

$$\mathcal{M}^{(i)}_{q,\boldsymbol{\mu}}(A,\mathbf{b}\partial_t) = \sum_{j=0}^{q} B_j b_j B_j \partial_t + \mathcal{M}^{(i)}_{q,\boldsymbol{\mu}}(A), i = 1,2,$$

with a vector $\mathbf{b} = (b_1, \dots b_q)$ consisting of complex entries.

Lemma 2.4. *Let the coefficients of the operators A_j, $0 \leq j \leq N-1$, do not depend on the time variable t. If* (1.9) *and* (2.5) *are fulfilled for all $0 \leq j \leq q$ then for any real vector $\mathbf{b}$ we have*

$$\mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A,-\mathbf{b}\partial_t)\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A,\mathbf{b}\partial_t) =$$

$$B_q(A_{q-1}\boldsymbol{\mu}^{(1)}_q A^*_{q-1} - b_q^2\partial_t^2)B_q + \sum_{j=1}^{q-1} B_j \Big(\Delta_{j,\boldsymbol{\mu}} - b_j^2\partial_t^2\Big) B_j + B_0(\Delta_{0,\boldsymbol{\mu}} - b_0^2\partial_t^2)B_0,$$

$1 \leq q \leq N$. In particular, Maxwell's type operators $\mathcal{M}^{(j)}_{N,\boldsymbol{\mu}}(A,\pm\mathbf{b}\partial_t)$ are elliptic for $N \geq 1$, if $\mathbf{b} \in \mathbb{R}^q$, $m_j = 1$, $|b_j| > 0$, and Assumption 1.2 is fulfilled for all $0 \leq j \leq N$.

Next, for $\mathbf{b} \in \mathbb{C}^q$ we set

$$S_{q,a}(A,\mathbf{b}\mathcal{L}_{\boldsymbol{\mu}}) = \sum_{j=0}^{q} B_j b_j^2 \big(\partial_t + \mathfrak{D}_{j,\boldsymbol{\mu}}\big) B_j + a\mathcal{M}_q(A),$$

representing models with the leading 'parabolic' part, and

$$S_{q,a}(A,\mathbf{b}\mathcal{H}_{\boldsymbol{\mu}}) = \sum_{j=0}^{q} B_j b_j \big(\partial_t^2 + \mathfrak{D}_{j,\boldsymbol{\mu}}\big) B_j + a\,\mathcal{M}_q(A)$$

corresponding to models with the leading 'hyperbolic' part (as before, $a = a_q$ equals to 1 or 0). It is worth to note that, similarly to steady case, the main part of the Stokes operator $S_{N,0}(A,\mathfrak{D}_{\boldsymbol{\mu}},\mathbf{b}\partial_t^2)$ could be easily factorized if one chooses suitable operators $\mathfrak{D}_{j,\boldsymbol{\mu}}$ and numbers b_j.

Lemma 2.5. *Let the coefficients of the operators A_j, $0 \leq j \leq N-1$, do not depend on the time variable t. If for all $0 \leq j \leq q$ identities* (1.9) *and* (2.5) *are fulfilled then for any real vector $\mathbf{b} \in \mathbb{R}^q$ we have*

$$\mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A,-\iota\mathbf{b}\partial_t)\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A,\iota\mathbf{b}\partial_t) =$$

$$B_q b_q^2\big(\partial_t^2 + A_{q-1}\boldsymbol{\mu}^{(1)}_{q-1}A^*_{q-1}\big)B_q + \sum_{j=0}^{q-1} B_j b_j^2(\partial_t^2 + \Delta_{j,\boldsymbol{\mu}_j})B_j.$$

In particular, if $\mathfrak{D}_{j,\boldsymbol{\mu}} = \Delta_{j,\boldsymbol{\mu}}$ for all $0 \leq j \leq q$, then

$$S_{N,0}(A,\mathbf{b}\mathcal{H}_{\boldsymbol{\mu}}) = \mathcal{M}^{(1)}_{q,\boldsymbol{\mu}}(A,-\iota\mathbf{b}\partial_t)\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A,\iota\mathbf{b}\partial_t).$$

Thus, for the first order complex (1.2) one may treat operator $\mathcal{M}^{(0)}_{q,\boldsymbol{\mu}}(A,\iota\mathbf{b}\partial_t)$ as the first order 'wave operator'.

2.3. **Parametrices for steady Maxwell' and Stokes' operators.** At this point we note that the (both Petrovskii and Douglis-Nirenberg) ellipticity implies the regularity property for solutions to the operators $\mathcal{M}_{N,\boldsymbol{\mu}}(A)$, $S_{q,1}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ and existence of parametrices (and even fundamental solutions) for them, see, for instance, [1], [34, §2.3], [35, §2.2.9, §4.4], [10, Ch. 2], [38, Theorem 8.69]. This actually results in many useful integral formulae for solutions to related systems of differential equations, see for instance, [34, §2.4, §2.5].

Let us indicate a way to construct parametrices for Maxwell's and Stokes' operators using suitable kernels for the generalized Laplacians $\Delta_{j,\boldsymbol{\mu}}$ of elliptic complex (1.2). Namely, if $\Delta_{j,\boldsymbol{\mu}}$ are strongly elliptic operators, each of them admits a parametrix, say, $\Phi_{j,\boldsymbol{\mu}}$, i.e. such a pseudo-differential operator on X^0 that on $C^\infty_0(X, E_j)$ we have

$$\Phi_{j,\boldsymbol{\mu}}\Delta_{j,\boldsymbol{\mu}} + \Pi^L_{j,\boldsymbol{\mu}} = I, \ \Delta_{j,\boldsymbol{\mu}}\Phi_{j,\boldsymbol{\mu}} + \Pi^R_{j,\boldsymbol{\mu}} = I, \tag{2.17}$$

with pseudo-differential operators $\Pi^R_{j,\boldsymbol{\mu}}$, $\Pi^L_{j,\boldsymbol{\mu}}$ of negative orders, where $0 \leq j \leq N$ and I is the identity operator; in some situation one needs smoothing operators $\Pi^R_{j,\boldsymbol{\mu}}$, $\Pi^L_{j,\boldsymbol{\mu}}$, i.e. the pseudo-differential operators of order minus infinity. If $\Pi^L_{j,\boldsymbol{\mu}} = 0$ in (2.17) then $\Phi_{j,\boldsymbol{\mu}}$ is a left fundamental solution for $\Delta_{j,\boldsymbol{\mu}}$ on X; similarly, if $\Pi^R_{j,\boldsymbol{\mu}} = 0$ then Φ_j is a right fundamental solution for $\Delta_{j,\boldsymbol{\mu}}$ on X. In particular, if $\Delta_{j,\boldsymbol{\mu}}$ satisfy the so-called Uniqueness Condition in small on X then $\Pi^L_{j,\boldsymbol{\mu}} = \Pi^R_{j,\boldsymbol{\mu}_j} = 0$, i.e. $\Phi_{j,\boldsymbol{\mu}}$ is the bilateral fundamental solution for $\Delta_{j,\boldsymbol{\mu}_j}$ on X, see, for instance [35, §4.4].

Theorem 2.6. *Let complex* (1.2) *be elliptic, Assumption* 1.2 *be fulfilled,* (1.10) *and* $m_j + \tilde{m}_j = m_{j-1} + \hat{m}_j = m$ *for all* $0 \leq j \leq N$*. Then the operator*

$$\mathcal{F}^{(1)}_{N,\boldsymbol{\mu}}(A) = \mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)\Big(\sum_{j=0}^{N-1} B_j\Phi_{j,\boldsymbol{\mu}}B_j\Big),$$

is a parametrix for $\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)$*. Moreover, if* $\Phi_{j,\boldsymbol{\mu}}$*,* $0 \leq j \leq N$*, are right fundamental solutions for* $\Delta_{j,\boldsymbol{\mu}}$ *then* $\mathcal{F}^{(1)}_{N,\boldsymbol{\mu}}(A)$ *is a right fundamental solution to* $\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)$*.*

Proof. It follows from (2.7) that the operator $\mathcal{F}^{(0)}_{N,\boldsymbol{\mu}}(A)$ is a right parametrix for $\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)$ if $\Phi_{j,\boldsymbol{\mu}}$ are parametrices for $\Delta_{j,\boldsymbol{\mu}}$, respectively (similarly, a right fundamental solution if $\Pi^R_{j,\boldsymbol{\mu}} = 0$), $0 \leq j \leq N$. Finally, we note that for 'elliptic' operators a right parametrix is a left parametrix, too, see, for instance, [35, §2.2.9]. □

Similarly, we obtain the following statement.

Theorem 2.7. *Let complex* (1.2) *be elliptic, Assumption* 1.2 *be fulfilled,* (1.10) *be true and* $m_j + \tilde{m}_j = m_{j-1} + \hat{m}_j = m$ *for all* $0 \leq j \leq N$*. Then the operator*

$$\mathcal{F}^{(0)}_{N,\boldsymbol{\mu}}(A) = \Big(\sum_{j=0}^{N-1} B_j\Phi_{j,\boldsymbol{\mu}}B_j\Big)\mathcal{M}^{(1)}_{N,\boldsymbol{\mu}}(A)$$

is a parametrix for $\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)$*. Moreover, if* $\Phi_{j,\boldsymbol{\mu}}$*,* $0 \leq j \leq N$*, are left fundamental solutions for* $\Delta_{j,\boldsymbol{\mu}}$ *then* $\mathcal{F}^{(0)}_{N,\boldsymbol{\mu}}(A)$ *is a left fundamental solution to* $\mathcal{M}^{(0)}_{N,\boldsymbol{\mu}}(A)$*.*

Finally, let us write down a fundamental solution to Stokes' operator $S_{q,1}(A,\mathfrak{D}_{\boldsymbol{\mu}})$ in a particular case. With this purpose, let $\mathcal{N}^{(q)}(A)$ be given by

(2.18)
$$B_q\Phi_{q,\boldsymbol{\mu}}A^*_q\boldsymbol{\mu}^{(0)}_q A_qB_q+B_qA_{q-1}B_{q-1}+B_{q-1}\boldsymbol{\mu}^{(1)}_q A^*_{q-1}B_q-B_{q-1}\boldsymbol{\mu}^{(1)}_q A^*_{q-1}A_{q-1}B_{q-1}.$$

Theorem 2.8. *Let complex* (1.2) *be elliptic,* (1.9) *be true,* $N\geq 2$ *and* $1\leq q\leq N-1$*. Let also* $m_j=m$ *and* $\boldsymbol{\mu}^{(i)}_j=0$ *for all* $i=1,2$ *and all* $0\leq j\leq q-1$*,* $m_q+\tilde{m}_q=m$*, Assumption* 1.2 *be fulfilled for* $j=q$ *and*

$$A^*_{q-2}\boldsymbol{\mu}^{(1)}_q A^*_{q-1}=0. \tag{2.19}$$

If Φ_j *are right fundamental solutions for* Δ_j*,* $0\leq j\leq q-1$*, and* $\Phi_{q,\boldsymbol{\mu}}$ *is a bilateral fundamental solution to* $\Delta_{q,\boldsymbol{\mu}}$*, then the operator*

$$\mathfrak{F}_{q,\boldsymbol{\mu}}(A)=\Big(\mathcal{N}^{(q)}(A)+\mathcal{M}_{q-1}(A)\Big)\Big(B_q\Phi_{q,\boldsymbol{\mu}}B_q+\sum_{j=0}^{q-1}B_j\Phi_jB_j\Big)$$

is a right fundamental solution to $S_{q,1}(A,\Delta_{\boldsymbol{\mu}})$.

Proof. Indeed, as $A_{i+1}\circ A_i\equiv 0$ then we have

$$A^*_i\circ A^*_{i+1}\equiv 0,\ \Delta_{i+1}A_i=A_i\Delta_i=A_iA^*_iA_i,\ A^*_i\Delta_{i+1}=\Delta_iA^*_i=A^*_iA_iA^*_i. \tag{2.20}$$

Next, using (2.20) we conclude that

$$\Delta_{j,\boldsymbol{\mu}}A^*_j\boldsymbol{\mu}^{(0)}_jA_j=A^*_j\boldsymbol{\mu}^{(0)}_jA_jA^*_j\boldsymbol{\mu}^{(0)}_jA_j=A^*_j\boldsymbol{\mu}^{(0)}_jA_j\Delta_{j,\boldsymbol{\mu}}$$

and, if $\Phi_{j,\boldsymbol{\mu}}$ is a bilateral fundamental solution for $\Delta_{j,\boldsymbol{\mu}}$, then

$$A^*_j\boldsymbol{\mu}^{(0)}_jA_j\Phi_{j,\boldsymbol{\mu}}=\Phi_{j,\boldsymbol{\mu}}A^*_j\boldsymbol{\mu}^{(0)}_jA_j. \tag{2.21}$$

Hence calculating as in the proof of Proposition 2.3 (see formulae (2.15), (2.16)) and applying (2.19), we obtain

$$S_{q,1}(A,\Delta_{\boldsymbol{\mu}})\Big(\mathcal{N}^{(q)}(A)+\mathcal{M}_{q-1}(A)\Big)=B_q\Delta_{q,\boldsymbol{\mu}}B_q+\sum_{j=0}^{q-1}B_j\Delta_jB_j, \tag{2.22}$$

i.e. $\mathfrak{F}_{q,\boldsymbol{\mu}}(A)$ is a right fundamental solution for $S_{q,1}(A,\Delta_{\boldsymbol{\mu}})$. □

Note that for the de Rham complex the pseudo-differential operator

$$\mathfrak{F}_{1,\mu}(d)=\begin{pmatrix}\mu^{-1}\varphi\,\mathrm{rot}\,\mathrm{rot}\,\varphi & -\varphi\,\nabla\\ \mathrm{div}\,\varphi & -\mu\, I\end{pmatrix}$$

is closely related to the so-called steady Ozeen tensor for Stokes' system $S_{q,1}(d,\Delta_{\boldsymbol{\mu}})$ in $3D$-Hydrodynamics, see [24], where φ is the standard fundamental solution of the Laplace operator in $\mathbb{R}^3$, $\boldsymbol{\mu}^{(0)}_1=\mu I_n$, $\boldsymbol{\mu}^{(1)}_1=\mu$, $\mu>0$.

2.4. **Parametrices for evolutionary operators.** Under very mild assumptions on the coefficients of the operator $\mathcal{L}_{j,\boldsymbol{\mu}}$, $0\leq j\leq N$, it admits the (unique) fundamental solution $\Psi_{j,\boldsymbol{\mu}}$ on $X\times[0,T]$ solving the Cauchy problem for $\mathcal{L}_{j,\boldsymbol{\mu}}$, with initial data on the plane $t=0$, see, for instance, [11], [13, Ch 1, §7 and Ch. 9].

Again, let us write down a fundamental solution to Stokes' operator $S_{q,1}(A,\mathbf{b}\mathcal{L}_{\boldsymbol{\mu}})$ in a particular case. With this purpose let $\mathcal{N}^{(q)}(A,t)$ be given by

$$B_q\Psi_{q,\boldsymbol{\mu}}A^*_q\boldsymbol{\mu}^{(0)}_qA_qB_q+B_qA_{q-1}B_{q-1}+B_{q-1}\boldsymbol{\mu}^{(1)}_qA^*_{q-1}B_q-B_{q-1}(\boldsymbol{\mu}^{(1)}_qA^*_{q-1}A_{q-1}+\partial_t)B_{q-1}.$$

Theorem 2.9. *Let complex* (1.2) *be elliptic,* (1.9) *be true,* $N \geq 2$ *and* $1 \leq q \leq N-1$*, and the coefficients of the operators* A_j *do not depend on* t*. Let also* $\mathbf{b} = (0,0,\dots,0,1)$*,* $m_j = m$ *and* $\boldsymbol{\mu}_j^{(i)} = 0$ *for all* $i = 1,2$ *for all* $0 \leq j \leq q-1$*,* $m_q + \tilde{m}_q = m$*, Assumption* 1.2 *be fulfilled for* $j = q$ *and* (2.19) *hold true. If* Φ_j *are right fundamental solutions for* Δ_j*,* $0 \leq j \leq q-1$*, and* $\Phi_{q,\boldsymbol{\mu}}$*,* $\Psi_{q,\boldsymbol{\mu}}$ *are bilateral fundamental solution to* $\Delta_{q,\boldsymbol{\mu}}$*,* $(\partial_t + \Delta_{q,\boldsymbol{\mu}})$*, respectively, then the operator*

$$\mathfrak{F}_{q,\boldsymbol{\mu}}(A,t) = \Big(\mathcal{N}^{(q)}(A,t) + \mathcal{M}_{q-1}(A)\Big)\Big(B_q \Phi_{q,\boldsymbol{\mu}} B_q + \sum_{j=0}^{q-1} B_j \Phi_j B_j\Big)$$

is a right fundamental solution to $S_{q,1}(A, \mathbf{b}(\partial_t + \Delta_{\boldsymbol{\mu}}))$*.*

Proof. Indeed, similarly to (2.21), if the coefficients of the operators A_j do not depend on t and $\Psi_{j,\boldsymbol{\mu}}$ is a bilateral fundamental solution to $(\partial_t + \Delta_{j,\boldsymbol{\mu}})$, then

$$A_j^* \boldsymbol{\mu}_j^{(0)} A_j \Psi_{j,\boldsymbol{\mu}} = \Psi_{j,\boldsymbol{\mu}} A_j^* \boldsymbol{\mu}_j^{(0)} A_j. \tag{2.23}$$

Hence calculating as in the proof of Theorem 2.8 (see formulae (2.18), (2.22)) and applying (2.19) and (2.23), we obtain

$$S_{q,1}(A, \partial_t + \Delta_{\boldsymbol{\mu}})\Big(\mathcal{N}^{(q)}(A,t) + \mathcal{M}_{q-1}(A)\Big) = B_q \Delta_{q,\boldsymbol{\mu}} B_q + \sum_{j=0}^{q-1} B_j \Delta_j B_j,$$

i.e. $\mathfrak{F}_{q,\boldsymbol{\mu}}(A,t)$ is a right fundamental solution for $S_{q,1}(A, \partial_t + \Delta_{\boldsymbol{\mu}})$. □

2.5. **Natural perturbations.** In order to define suitable perturbations of Stokes' type systems $\tilde{S}_q(A)$ (and $\hat{S}_q(A)$) related to the Navier-Stokes equations, [27] proposed to introduce two bilinear mappings $\mathcal{Q}_{q,j}$, satisfying

$$\mathcal{Q}_{q,1,x} : E_{q+1,x} \otimes E_{q,x} \to E_{q,x}, \ \mathcal{Q}_{q,2,x} : E_{q,x} \otimes E_{q,x} \to E_{q-1,x}, \tag{2.24}$$

at each point $x \in X$. Then one may set for a sufficiently differentiable section v of the vector bundle E_q:

$$\mathcal{N}_q(v)(x) = \mathcal{Q}_{q,1,x}((A_q v)(x), v(x)) + A_{q-1} \mathcal{Q}_{q,2,x}(v(x), v(x)). \tag{2.25}$$

For the de Rham complex at the degree $q = 1$, corresponding to the classical Navier-Stokes equations, this leads to the so-called Lamb form (see [15, §15] for $n = 3$) of the related non-linear part:

$$\mathcal{N}_1(v) = \star(\star d_1 v \wedge v) + d_0 \star (v \wedge \star v)/2 = ((\operatorname{curl} \vec{v}) \times \vec{v} + (1/2)\nabla |\vec{v}|^2 \text{ for } n = 3),$$

where $\star : \Lambda^q \to \Lambda^{n-q}$ is the Hodge $\star$-operator, $\wedge$ is the exterior product of the differential forms (and $\vec{c} \times \vec{d}$ stands for the vector product of 3-vectors $\vec{c}$ and $\vec{d}$).

Apparently Stokes' type systems $S_{q,a}(A, \mathfrak{D}_{\boldsymbol{\mu}})$ have a more complicated structure. In the simplest case where all the orders m_j are the same for all $0 \leq j \leq q$ we suggest to introduce multi-linear differential operators of order $(m-1)$:

$$Q_{q,1} : (\oplus_{i=1}^{q+1} E_i) \otimes (\oplus_{i=0}^{q} E_i) \to \oplus_{i=0}^{q} E_i, \ Q_{q,2} : (\oplus_{i=0}^{q} E_i) \otimes (\oplus_{i=0}^{q} E_i) \to \oplus_{i=0}^{q-1} E_i,$$

and to set the following natural non-linear perturbations

$$\mathcal{N}_q(U_q) = Q_{q,1}(A^{\cdot} U_q, U_q) + A^{\cdot} Q_{q,2}(U_q, U_q),$$

of the differential operator $\sum_{j=0}^{q} B_j \mathfrak{D}_{j,\boldsymbol{\mu}} B_j$ acting on sections $U_q = (u_0, \dots, u_q)$ of the bundle $\mathfrak{E}_q = \oplus_{j=0}^{q} E_j$, $0 \leq q \leq N$.

3. Maxwell's and Stokes' type operators for the de Rham complex

3.1. **The de Rham complex.** Let $T^*_{\mathbb{C}}X$ be the complexified tangential bundle of X and let $\Lambda^q = \Lambda^q T^*_{\mathbb{C}}X$ be the bundle of complex valued exterior differential forms of degree q, $0 \leq q \leq n$, over X. In each coordinate neighbourhood $\mathcal{O}$ on X any differential form u admits local representation

$$u_{|\mathcal{O}}(x) = \sum_{\#I=q} u_I(x) dx_I$$

where, for $\#I = q$, we consider $I = (i_1, \dots i_q)$ as a multi-index with $1 \leq i_1 < \dots < i_q \leq n$, $dx_I = dx_{i_1} \wedge \dots \wedge dx_n$, $\{dx_j\}_{j=1}^n$ is a basis in $T^*_{\mathbb{C}}X$ and $\wedge$ is the exterior product of differentials satisfying

$$dx_i \wedge dx_j = -dx_j \wedge dx_i. \tag{3.1}$$

Then the exterior differential operator d_q is defined by local representations

$$d_q u_{|\mathcal{O}}(x) = \sum_{i=1}^n \sum_{\#I=q} \partial_i u_I(x) dx_i \wedge dx_I.$$

Using (3.1) we easily conclude that

$$d_{q+1} \circ d_q = 0 \tag{3.2}$$

and then we obtain the de Rham complex

$$0 \to C^\infty(X, \Lambda^0) \overset{d_0}{\to} C^\infty(X, \Lambda^1) \overset{d_1}{\to} C^\infty(X, \Lambda^2) \to \dots \overset{d_{n-1}}{\to} C^\infty(X, \Lambda^n) \to 0, \tag{3.3}$$

of exterior differentials on the differential forms, see, for instance, [4, Ch 3, §2.5], [5], [34, §1.2.6]. Of course, for $X = \mathbb{R}^n$ the bundle Λ^q may be identified with $\mathbb{R}^n \times \mathbb{C}^{k_q}$ with $k_q = \left(\begin{array}{c} n \\ q \end{array} \right)$ and its sections can be treated as vector-columns of functions with k_q components. Then the de Rham complex is the Hilbert compatibility complex degenerated by the gradient operator $\nabla = d_0$. In this case, in addition to (3.2), we also have

$$\Delta_q = d_q^* d_q + d_{q-1} d_{q-1}^* = -\Delta I_{k_q}, \ 0 \leq q \leq n,$$

where Δ is the usual Laplace operator in $\mathbb{R}^n$.

For three dimensional space, more familiar within classical Physics, we may interprete the de Rham complex as follows:

$$d_0 = \nabla, \ d_1 = \mathrm{curl} = \left(\begin{array}{ccc} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{array} \right), \ d_2 = \mathrm{div} = \Big(\partial_1, \partial_2, \partial_3\Big). \tag{3.4}$$

However we still may define the compatibility de Rham complex in $\mathbb{R}^3$ with the use of the classical algebraic constructions:

$$d_1 u = \nabla \times \vec{u}, \ d_2 v = \nabla \cdot \vec{v}$$

for vector fields $\vec{u}$, $\vec{v}$ (here $\vec{c} \cdot \vec{d}$ means the inner product of vectors $\vec{c}$ and $\vec{d}$). In the higher dimensions the standard algebraic constructions do not work in general.

As $N = n$, Stokes's operator $S_q(d, \Delta)$ over the field $\mathbb{R}$ has dimension $r_q = \sum_{j=0}^q \left(\begin{array}{c} n \\ j \end{array} \right)$; in particular, $r_0 = 1$, $r_1 = 1 + n$, $r_{n-1} = 2^n - 1$, $r_n = 2^n$. Of course, the dimension $r_{q,\mathbb{C}}$ over the field $\mathbb{C}$ equals to $2r_q$; in particular, $r_{n,\mathbb{C}} = 2^{n+1}$.

3.2. **Maxwell's and Stokes' type systems in $\mathbb{R}^3$.** For the de Rham complex over $\mathbb{R}^3$ we have: $r_0 = 1$, $r_1 = 4$, $r_2 = 7$, $r_3 = 8$; in particular, $r_{n,\mathbb{C}} = 16$. Thus, our method echoes with the sedeonic approach by V.L. Mironov and S.V. Mironov, see [20], [21], [22], [23], for compact symmetric formulations of the Laws of classical Physics in $\mathbb{R}^3$.

The typical form of Stokes' system $S_0(d, \mathcal{L}_{\boldsymbol{\mu}})$ is the following:

$$S_0(d, \mathcal{L}_{\boldsymbol{\mu}}) = \partial_t + \mathfrak{D}_{0,\boldsymbol{\mu}} = \partial_t - \mathrm{div}\boldsymbol{\mu}_0^{(0)}\nabla + \vec{a}_0 \cdot \nabla + M_0,$$

with a self-adjoint functional matrix $\boldsymbol{\mu}_0^{(0)}$, a scalar function M_0 and a functional vector $\vec{a}_0$, see (1.11), (1.12), that is a standard second order equation of the Mathematical Physics.

Next, according to (1.11), (1.12), we have

$$\mathfrak{D}_{1,\boldsymbol{\mu}} = \mathrm{curl}\, \boldsymbol{\mu}_1^{(0)}\, \mathrm{curl} - \nabla\, \boldsymbol{\mu}_1^{(1)}\, \mathrm{div} + C_1\, \mathrm{curl} + \vec{a}_1\, \mathrm{div} + M_1$$

with self-adjoint functional (3×3) matrices $\boldsymbol{\mu}_1^{(0)}$, M_1, a functional (3×3)- matrix C_1, a functional 3-vector $\vec{a}$ and a scalar function $\boldsymbol{\mu}_1^{(1)}$. Then the typical steady Stokes' type system $S_1(d, \mathfrak{D}_{\boldsymbol{\mu}})$ is given by

$$S_{1,1}(d, \mathfrak{D}_{\boldsymbol{\mu}}) = \begin{pmatrix} \mathfrak{D}_{1,\boldsymbol{\mu}} & \nabla \\ -\mathrm{div} & \mathfrak{D}_{0,\boldsymbol{\mu}} \end{pmatrix}. \tag{3.5}$$

Similarly, at the second step of the de Rham complex over $\mathbb{R}^3$ we have:

$$\mathfrak{D}_{2,\boldsymbol{\mu}} = -\nabla\, \boldsymbol{\mu}_2^{(0)}\, \mathrm{div} + \mathrm{curl}\, \boldsymbol{\mu}_2^{(1)}\, \mathrm{curl} + C_2\, \mathrm{curl} + \vec{a}_2\, \mathrm{div} + M_2$$

with self-adjoint functional (3×3) matrices $\boldsymbol{\mu}_2^{(1)}$, M_2, a functional (3×3)-matrix C_2, a functional 3-vector $\vec{a}_2$ and a scalar function $\boldsymbol{\mu}_2^{(0)}$. Consequently,

$$S_{2,1}(d, \mathfrak{D}_{\boldsymbol{\mu}}) = \begin{pmatrix} \mathfrak{D}_{2,\boldsymbol{\mu}} & \mathrm{curl} & 0 \\ \mathrm{curl} & \mathfrak{D}_{1,\boldsymbol{\mu}} & \nabla \\ 0 & -\mathrm{div} & \mathfrak{D}_{0,\boldsymbol{\mu}} \end{pmatrix}.$$

Finally, at the third step we have

$$\mathfrak{D}_{3,\boldsymbol{\mu}} = -\mathrm{div}\boldsymbol{\mu}_3^{(1)}\nabla + \vec{a}_3 \cdot \nabla + M_3,$$

with a self-adjoint functional matrix $\boldsymbol{\mu}_3^{(1)}$, a scalar function M_3 and a functional vector $\vec{a}_3$. Then

$$S_3(d, \mathfrak{D}_{\boldsymbol{\mu}}) = \begin{pmatrix} \mathfrak{D}_{3,\boldsymbol{\mu}} & \mathrm{div} & 0 & 0 \\ -\nabla & \mathfrak{D}_{2,\boldsymbol{\mu}} & \mathrm{curl} & 0 \\ 0 & \mathrm{curl} & \mathfrak{D}_{1,\boldsymbol{\mu}} & \nabla \\ 0 & 0 & -\mathrm{div} & \mathfrak{D}_{0,\boldsymbol{\mu}} \end{pmatrix}. \tag{3.6}$$

Of course, we may double the dimension of the related matrices $S_q(d, \mathfrak{D}_{\boldsymbol{\mu}})$ and change signs of entries outside the diagonal with the use of the imaginary unit.

Let us interprete these Stokes' type operators within classical models of the Mathematical Physics. We consider three model examples only, because much more equations, that fit into our scheme, could be found in [20], [21], [22], [23].

Example 3.1. We begin with the equations of electromagnetic field. Let $\mathfrak{c}$ stand for the speed of light. Then the Maxwell's type operator $\iota\mathcal{M}_3(A, \mathbf{b}\partial_t)$ gives us the classical Maxwell equations for electromagnetic field in a vacuum:

$$\iota \begin{pmatrix} \mathfrak{c}^{-1}\partial_t & \mathrm{div} & 0 & 0 \\ \nabla & \mathfrak{c}^{-1}\partial_t & \mathrm{curl} & 0 \\ 0 & -\mathrm{curl} & \mathfrak{c}^{-1}\partial_t & \nabla \\ 0 & 0 & \mathrm{div} & \mathfrak{c}^{-1}\partial_t \end{pmatrix} \begin{pmatrix} 0 \\ \vec{H} \\ \vec{E} \\ 0 \end{pmatrix} = \iota \begin{pmatrix} 0 \\ 0 \\ -4\pi\mathfrak{c}^{-1}\vec{j}_e \\ 4\pi\rho_e \end{pmatrix}, \tag{3.7}$$

where $\vec{E}$, $\vec{H}$ represent the electric and magnetic field strengths, ρ_e is the volume density of electric charge and $\vec{j}_e$ is the volume density of electric current (here the number ι is introduced for the related matrices to be self-adjoint).

Taking into account the magnetic charges and currents in the Dirac monopoles [7], [8] and Schwinger dyons [30] models, the equation (3.7) can be rewritten in more symmetric form

$$\iota \begin{pmatrix} \mathfrak{c}^{-1}\partial_t & \mathrm{div} & 0 & 0 \\ \nabla & \mathfrak{c}^{-1}\partial_t & \mathrm{curl} & 0 \\ 0 & -\mathrm{curl} & \mathfrak{c}^{-1}\partial_t & \nabla \\ 0 & 0 & \mathrm{div} & \mathfrak{c}^{-1}\partial_t \end{pmatrix} \begin{pmatrix} 0 \\ \vec{H} \\ \vec{E} \\ 0 \end{pmatrix} = \iota \begin{pmatrix} 4\pi\rho_m \\ -4\pi\mathfrak{c}^{-1}\vec{j}_m \\ -4\pi\mathfrak{c}^{-1}\vec{j}_e \\ 4\pi\rho_e \end{pmatrix}, \tag{3.8}$$

where ρ_m is the volume density of magnetic charge and $\vec{j}_m$ is the volume density of magnetic current. Significantly, Lemma 2.5 immediately gives us the so-called wave equations for the field's strength related to model (3.8). Indeed, if we denote the d'Alembert operator as

$$\hat{D} = \left(\frac{1}{\mathfrak{c}^2} \frac{\partial^2}{\partial t^2} - \Delta \right),$$

then we get

$$\begin{pmatrix} \hat{D} & 0 & 0 & 0 \\ 0 & \hat{D} & 0 & 0 \\ 0 & 0 & \hat{D} & 0 \\ 0 & 0 & 0 & \hat{D} \end{pmatrix} \begin{pmatrix} 0 \\ \vec{H} \\ \vec{E} \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \tag{3.9}$$

where

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \mathfrak{c}^{-1}\partial_t & -\mathrm{div} & 0 & 0 \\ -\nabla & \mathfrak{c}^{-1}\partial_t & -\mathrm{curl} & 0 \\ 0 & \mathrm{curl} & \mathfrak{c}^{-1}\partial_t & -\nabla \\ 0 & 0 & -\mathrm{div} & \mathfrak{c}^{-1}\partial_t \end{pmatrix} \begin{pmatrix} 4\pi\rho_m \\ -4\pi\mathfrak{c}^{-1}\vec{j}_m \\ -4\pi\mathfrak{c}^{-1}\vec{j}_e \\ 4\pi\rho_e \end{pmatrix}.$$

The matrix equation (3.9) is equivalent to the following system

$$\begin{cases} \left(\dfrac{1}{\mathfrak{c}^2} \dfrac{\partial^2}{\partial t^2} - \Delta \right) \vec{E} = -4\pi \nabla \rho_e - \dfrac{4\pi}{\mathfrak{c}^2} \dfrac{\partial \vec{j}_e}{\partial t} - \dfrac{4\pi}{\mathfrak{c}} \left[\nabla \times \vec{j}_m \right], \\ \left(\dfrac{1}{\mathfrak{c}^2} \dfrac{\partial^2}{\partial t^2} - \Delta \right) \vec{H} = -4\pi \nabla \rho_m - \dfrac{4\pi}{\mathfrak{c}^2} \dfrac{\partial \vec{j}_m}{\partial t} + \dfrac{4\pi}{\mathfrak{c}} \left[\nabla \times \vec{j}_e \right], \\ \dfrac{\partial \rho_m}{\partial t} + \left(\nabla \cdot \vec{j}_m \right) = 0, \ \dfrac{\partial \rho_e}{\partial t} + \left(\nabla \cdot \vec{j}_e \right) = 0, \end{cases}$$

where the last two relations are the laws of conservation of electric and magnetic charges.

Example 3.2. Let us discuss the matrix representation of the hydrodynamic equations. As we noted at the beginning of §2, the Euler and the Navier-Stokes' equations for incompressible fluid fit perfectly to this scheme with Maxwell's type operator operator $\mathcal{M}_1(d, \partial_t)$ and Stokes' type operator $S_1(d, \Delta, \partial_t)$ as principal linear parts, respectively, for instance, [15],

$$\begin{cases} \partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \rho^{-1}\nabla p = \vec{f}, \\ \operatorname{div} \vec{v} = 0, \end{cases} \qquad \begin{cases} (\partial_t - \mu\Delta)\vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \rho^{-1}\nabla p = \vec{f}, \\ \operatorname{div} \vec{v} = 0, \end{cases}$$

where $\vec{v}$ is a local flow velocity, p is a pressure, ρ is a fluid density, μ is viscosity and $\vec{f}$ is the vector of outer forces. Next, as is known [15], the vortex-less free fluid is described by the following system of equations

$$\begin{cases} \partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \rho^{-1}\nabla p = 0, \\ \partial_t \rho + (\vec{v} \cdot \nabla)\rho + \rho \operatorname{div} \vec{v} = 0, \\ \operatorname{curl} \vec{v} = 0, \end{cases} \tag{3.10}$$

with the same entries as above.

Let us assume that the flow is isentropic (i.e., the entropy s is a constant). Using the thermodynamic relation for enthalpy h per unit mass

$$d\,h = T d s + \rho^{-1} dp$$

we can introduce new function u according to the following relations

$$du = \frac{1}{\mathfrak{c}_s} dh = \frac{1}{\mathfrak{c}_s \rho} dp = \frac{\mathfrak{c}_s}{\rho} d\rho,$$

where $\mathfrak{c}_s$ is the speed of sound ($\mathfrak{c}_s^2 = (\partial p / \partial \rho)_s = const$). Then taking into account that the total time derivative is given as

$$\mathfrak{d}_t = \partial_t + (\vec{v} \cdot \nabla),$$

we rewrite the system (3.10) in the following symmetric form

$$\begin{cases} \mathfrak{c}_s^{-1} \mathfrak{d}_t \vec{v} + \nabla u = 0, \\ \mathfrak{c}_s^{-1} \mathfrak{d}_t u + \operatorname{div} \vec{v} = 0, \\ \operatorname{curl} \vec{v} = 0, \end{cases}$$

or in the following matrix form

$$\iota \begin{pmatrix} \mathfrak{c}_s^{-1}\mathfrak{d}_t & \operatorname{div} & 0 \\ \nabla & \mathfrak{c}_s^{-1}\mathfrak{d}_t & \operatorname{curl} \\ 0 & -\operatorname{curl} & \mathfrak{c}_s^{-1}\mathfrak{d}_t \end{pmatrix} \begin{pmatrix} u \\ \vec{v} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.11}$$

where the principal linear part matches with the introduced above Maxwell's type operator $\iota \mathcal{M}_2(d, \mathfrak{c}_s^{-1}\partial_t)$.

In order to interprete Maxwell's type operator $\iota \mathcal{M}_3(d, \mathfrak{c}_s^{-1}\partial_t)$, we rewrite (3.11):

$$\iota \begin{pmatrix} \mathfrak{c}_s^{-1}\mathfrak{d}_t & \operatorname{div} & 0 & 0 \\ \nabla & \mathfrak{c}_s^{-1}\mathfrak{d}_t & \operatorname{curl} & 0 \\ 0 & -\operatorname{curl} & \mathfrak{c}_s^{-1}\mathfrak{d}_t & \nabla \\ 0 & 0 & \operatorname{div} & \mathfrak{c}_s^{-1}\mathfrak{d}_t \end{pmatrix} \begin{pmatrix} u \\ \vec{v} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.12}$$

Next, to consider the vortex flow we introduce two new functions $\xi(\vec{r}, t)$ and $\vec{w}(\vec{r}, t)$ which describe the field of vortex tubes. The value $\vec{w}$ is proportional to the angle vector of tube rotation, while ξ characterizes the twisting of vortex tubes [22]. Then the equation (3.12) for vortex flow is written as

$$\begin{pmatrix} \mathfrak{c}_s^{-1}\mathfrak{d}_t & \mathrm{div} & 0 & 0 \\ \nabla & \mathfrak{c}_s^{-1}\mathfrak{d}_t & \mathrm{curl} & 0 \\ 0 & -\mathrm{curl} & \mathfrak{c}_s^{-1}\mathfrak{d}_t & \nabla \\ 0 & 0 & \mathrm{div} & \mathfrak{c}_s^{-1}\mathfrak{d}_t \end{pmatrix} \begin{pmatrix} u \\ \vec{v} \\ \vec{w} \\ \xi \end{pmatrix} = 0. \tag{3.13}$$

Finally, taking into the account the dissipation we obtain the related equations of viscous vortex flow, including the Stokes' type operator $\iota S_{3,1}(d, \mathfrak{c}_s^{-1}\partial_t + \Delta_{\boldsymbol{\mu}})$ with $\boldsymbol{\mu}_0^{(0)} = \mu$, $\mu_1^{(0)} = \boldsymbol{\mu}_1^{(1)} = \mu I_3$, $\boldsymbol{\mu}_2^{(1)} = \mu$ as the principal linear part:

$$\iota \begin{pmatrix} \mathfrak{c}_s^{-1}(\mathfrak{d}_t - \mu\Delta) & \mathrm{div} & 0 & 0 \\ \nabla & \mathfrak{c}_s^{-1}(\mathfrak{d}_t - \mu\Delta) & \mathrm{curl} & 0 \\ 0 & -\mathrm{curl} & \mathfrak{c}_s^{-1}(\mathfrak{d}_t - \mu\Delta) & \nabla \\ 0 & 0 & \mathrm{div} & \mathfrak{c}_s^{-1}(\mathfrak{d}_t - \mu\Delta) \end{pmatrix} \begin{pmatrix} u \\ \vec{v} \\ \vec{w} \\ \xi \end{pmatrix} = 0,$$

where the parameter μ represents kinematic viscosity. This matrix equation is equivalent to the following system, see [22]:

$$\begin{cases} \mathfrak{c}_s^{-1}\left(\partial_t + (\vec{v}\cdot\nabla) - \mu\Delta\right)\vec{v} + \mathrm{curl}\,\vec{w} + \nabla u = 0, \\ \mathfrak{c}_s^{-1}\left(\partial_t + (\vec{v}\cdot\nabla) - \mu\Delta\right)u + \mathrm{div}\,\vec{v} = 0, \\ \mathfrak{c}_s^{-1}\left(\partial_t + (\vec{v}\cdot\nabla) - \mu\Delta\right)\vec{w} - \mathrm{curl}\,\vec{v} + \nabla\xi = 0, \\ \mathfrak{c}_s^{-1}\left(\partial_t + (\vec{v}\cdot\nabla) - \mu\Delta\right)\xi + \mathrm{div}\,\vec{w} = 0. \end{cases}$$

Example 3.3. Consider the quadrupling of the imaginary de Rham complex over $\mathbb{R}^3$: $A_0 = \iota\, I_4 \otimes \nabla$, $A_1 = \iota\, I_4 \otimes \mathrm{curl}$, $A_2 = \iota\, I_4 \otimes \mathrm{div}$. Then, for a real number M, we have $A_0^* = \iota\, I_4 \otimes \mathrm{div}$, $A_1^* = -\iota\, I_4 \otimes \mathrm{curl}$, $A_2^* = \iota\, I_4 \otimes \nabla$, $(\iota\, M)^* = -\iota\, M$.

Set $\boldsymbol{\mu}_0^{(0)} = 0$, $\boldsymbol{\mu}_1^{(0)} = 0$, $\boldsymbol{\mu}_1^{(1)} = 0$,

$$\mathfrak{D}_{0,\boldsymbol{\mu}} = \iota \begin{pmatrix} 0 & 0 & 0 & -M \\ 0 & 0 & M & 0 \\ 0 & -M & 0 & 0 \\ M & 0 & 0 & 0 \end{pmatrix}, \ \mathfrak{D}_{1,\boldsymbol{\mu}} = \iota \begin{pmatrix} 0 & -\mathrm{curl} & 0 & -M \\ \mathrm{curl} & 0 & M & 0 \\ 0 & -M & 0 & \mathrm{curl} \\ M & 0 & -\mathrm{curl} & 0 \end{pmatrix};$$

in particular, the operators $\mathfrak{D}_{0,\boldsymbol{\mu}}$, $\mathfrak{D}_{1,\boldsymbol{\mu}}$ are formally self-adjoint. Then Stokes' operator $S_{0,\iota\mathfrak{c}^{-1}}(A, \mathfrak{D}_{\boldsymbol{\mu}}, \partial_t)$ coincides with $\iota\mathfrak{c}^{-1}(\partial_t + \mathfrak{D}_{0,\boldsymbol{\mu}})$.

Next, Stokes' system $S_{1,\iota\mathfrak{c}^{-1}}(A, \mathcal{L}_{\boldsymbol{\mu}})$ matches with the operator related to equations for the vector and scalar field's strengths in sedeonic field theory for the case of fields with non-zero mass of quantum m_0 [20],[23]:

$$\begin{pmatrix} \iota\mathfrak{c}^{-1}\partial_t + \mathfrak{D}_{1,\boldsymbol{\mu}} & \iota\, I_4 \otimes \nabla \\ \iota\, I_4 \otimes \mathrm{div} & \iota\mathfrak{c}^{-1}\partial_t + \mathfrak{D}_{0,\boldsymbol{\mu}} \end{pmatrix}.$$

This leads us to the following matrix equation

$$\begin{pmatrix} \hat{\partial}_t & 0 & 0 & -M & \mathrm{div} & 0 & 0 & 0 \\ 0 & \hat{\partial}_t & M & 0 & 0 & \mathrm{div} & 0 & 0 \\ 0 & -M & \hat{\partial}_t & 0 & 0 & 0 & \mathrm{div} & 0 \\ M & 0 & 0 & \hat{\partial}_t & 0 & 0 & 0 & \mathrm{div} \\ \nabla & 0 & 0 & 0 & \hat{\partial}_t & -\mathrm{curl} & 0 & M \\ 0 & \nabla & 0 & 0 & \mathrm{curl} & \hat{\partial}_t & -M & 0 \\ 0 & 0 & \nabla & 0 & 0 & M & \hat{\partial}_t & \mathrm{curl} \\ 0 & 0 & 0 & \nabla & -M & 0 & -\mathrm{curl} & \hat{\partial}_t \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ \vec{G}_1 \\ \vec{G}_2 \\ \vec{G}_3 \\ \vec{G}_4 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \vec{J}_1 \\ \vec{J}_2 \\ \vec{J}_3 \\ \vec{J}_4 \end{pmatrix},$$

which is equivalent to the following system (see [20]):

$$\begin{cases} \mathfrak{c}^{-1}\partial_t g_1 + \mathrm{div}\vec{G}_1 - Mg_4 = 4\pi\rho_1, \\ \mathfrak{c}^{-1}\partial_t g_2 + \mathrm{div}\vec{G}_2 + Mg_3 = 4\pi\rho_2, \\ \mathfrak{c}^{-1}\partial_t g_3 + \mathrm{div}\vec{G}_3 - Mg_2 = 4\pi\rho_3, \\ \mathfrak{c}^{-1}\partial_t g_4 + \mathrm{div}\vec{G}_4 + Mg_1 = 4\pi\rho_4, \\ \mathfrak{c}^{-1}\partial_t \vec{G}_1 + \nabla g_1 - \mathrm{curl}\vec{G}_2 + M\vec{G}_4 = -\dfrac{4\pi}{\mathfrak{c}}\vec{j}_1, \\ \mathfrak{c}^{-1}\partial_t \vec{G}_2 + \nabla g_2 + \mathrm{curl}\vec{G}_1 - M\vec{G}_3 = -\dfrac{4\pi}{\mathfrak{c}}\vec{j}_2, \\ \mathfrak{c}^{-1}\partial_t \vec{G}_3 + \nabla g_3 + \mathrm{curl}\vec{G}_4 + M\vec{G}_2 = -\dfrac{4\pi}{\mathfrak{c}}\vec{j}_3, \\ \mathfrak{c}^{-1}\partial_t \vec{G}_4 + \nabla g_4 - \mathrm{curl}\vec{G}_3 - M\vec{G}_1 = -\dfrac{4\pi}{\mathfrak{c}}\vec{j}_4. \end{cases}$$

Here g_i and $\vec{G}_i$ are scalar and vector field strengths; $q_i = 4\pi\rho_i$ (where ρ_i are volume densities of charges); $\vec{J}_i = -\frac{4\pi}{\mathfrak{c}}\vec{j}_i$ (where $\vec{j}_i$ are volume densities of currents); $(i \in \{1,\ 2,\ 3,\ 4\})$; $\hat{\partial}_t = \mathfrak{c}^{-1}\partial_t$; $M = m_0\mathfrak{c}/\hbar$, see [23].

The approach predicts two more Stokes' operators related to this mathematical model. One may conjecture that $\boldsymbol{\mu}_2^{(i)} = 0$, $\boldsymbol{\mu}_3^{(1)} = 0$ and $\mathfrak{D}_{2,\boldsymbol{\mu}}$, $\mathfrak{D}_{3,\boldsymbol{\mu}}$ are given by

$$\mathfrak{D}_{2,\boldsymbol{\mu}} = \begin{pmatrix} 0 & \gamma_2\mathrm{curl} & 0 & \alpha_2 M \\ \overline{\gamma}_2\mathrm{curl} & 0 & \beta_2 M & 0 \\ 0 & \overline{\beta}_2 M & 0 & \delta_2\mathrm{curl} \\ \overline{\alpha}_2 M & 0 & \overline{\delta}_2\mathrm{curl} & 0 \end{pmatrix}, \quad \mathfrak{D}_{3,\boldsymbol{\mu}} = \begin{pmatrix} 0 & 0 & 0 & \alpha_3 M \\ 0 & 0 & \beta_3 M & 0 \\ 0 & \overline{\beta}_3 M & 0 & 0 \\ \overline{\alpha}_3 M & 0 & 0 & 0 \end{pmatrix},$$

respectively, with complex numbers $\alpha_j, \beta_j, \gamma_j, \delta_j$ (highly likely, $\pm\iota$) and then

$$S_{2,\mathfrak{c}^{-1}\iota}(A, \partial_t + \mathfrak{D}_{\boldsymbol{\mu}}) = \begin{pmatrix} \iota\mathfrak{c}^{-1}\partial_t + \mathfrak{D}_{2,\boldsymbol{\mu}} & \iota\, I_4 \otimes \mathrm{curl} & 0 \\ -\iota\, I_4 \otimes \mathrm{curl} & \iota\partial_t + \mathfrak{D}_{1,\boldsymbol{\mu}} & \iota\, I_4 \otimes \nabla \\ 0 & \iota\, I_4 \otimes \mathrm{div} & \iota\partial_t + \mathfrak{D}_{0,\boldsymbol{\mu}} \end{pmatrix},$$

$$S_{3,\iota}(A, \mathcal{L}_{\boldsymbol{\mu}}) = \iota \begin{pmatrix} \partial_t + \mathfrak{D}_{3,\boldsymbol{\mu}} & I_4 \otimes \mathrm{div} & 0 & 0 \\ I_4 \otimes \nabla & \mathfrak{c}^{-1}\partial_t + \mathfrak{D}_{2,\boldsymbol{\mu}} & I_4 \otimes \mathrm{curl} & 0 \\ 0 & -I_4 \otimes \mathrm{curl} & \mathfrak{c}^{-1}\partial_t + \mathfrak{D}_{1,\boldsymbol{\mu}} & I_4 \otimes \nabla \\ 0 & 0 & I_4 \otimes \mathrm{div} & \mathfrak{c}^{-1}\partial_t + \mathfrak{D}_{0,\boldsymbol{\mu}} \end{pmatrix}.$$

4. Some typical elliptic differential complexes

Consider some other typical examples of elliptic complexes.

Example 4.1. A large part of complexes represents the so-called Koszul complexes. Namely, let A_0 be a column of scalar differential operators $(Q_1, \dots Q_N)^T$ over an open set $X \subset \mathbb{R}^n$, $1 \le N \le n$, satisfying the following commutation assumptions:

$$Q_i Q_j = Q_j Q_i \text{ for all } 1 \le i < j \le N. \tag{4.1}$$

Setting $E_q = X \times \mathbb{C}^{k_q}$ with $k_q = \begin{pmatrix} N \\ q \end{pmatrix}$ we may define differential operators

$$A_q = \sum_{\# I = q} Q_i(x) dy_i \wedge dy_I, \, x \in X \subset \mathbb{R}^n,$$

where $y = (y_1, \dots, y_N)$ are coordinates in $\mathbb{R}^N$. Again $A_{q+1} \circ A_q = 0$ and hence we obtain a differential complex $\{A_q, E_q\}_{q=0}^N$, see [34, §1.2.8], that is usually called Koszul complex associated with the set $(Q_1, \dots Q_N)$.

According to [34, Proposition 1.2.51] this complex is elliptic if and only if the principal symbol of the operator $A_0 = \sum_{i=1}^N Q_i(x) dy_i$ is injective; of course we may interpret A_j as matrix differential operators:

$$A_0 = \begin{pmatrix} Q_1 \\ \dots \\ Q_N \end{pmatrix}, \dots, A_{N-1} = \begin{pmatrix} Q_1, \dots, Q_N \end{pmatrix}$$

In particular, for operators with constant coefficients we have $Q^*_j Q_i = Q_i Q^*_j$ and

$$\Delta_q = \Big(\sum_{i=1}^N Q^*_i Q_i \Big) I_{k_q}.$$

If the operators Q_i has the same order then the Laplacians Δ_q are strongly elliptic.

Unfortunately the Koszul complexes are not always compatibility complexes. For operators with constant coefficients one may use [34, Proposition 1.2.52] giving a simple sufficient condition providing the compatibility property: the dimension of the algebraic variety

$$\mathcal{N}(A_0) = \{ z \in \mathbb{C}^n : Q_1(z) = \dots = Q_N(z) = 0 \}$$

is no more than $(n - N)$. Again, for $N = 3$ with the use of the classical algebraic constructions we obtain:

$$A_0 h = (Q_1 h, Q_2 h, Q_3 h)^T, \, A_1 \vec{u} = A_0 \times \vec{u}, \, A_2 \vec{v} = A_0 \cdot \vec{v},$$

for (vector-)functions $h(x)$, $\vec{u}(x)$, $\vec{v}(x)$ of n variables $x = (x_1, \dots x_n)$ with $n \ge 3$.

Of course, the initial operator A_0 may be non-homogeneous.

Example 4.2. Let $X = \mathbb{R}^n$ and $p \in \mathbb{N}$, $p \ge 2$. For a q-differential form u we set

$$A^{(p)}_q u(x) = \sum_{i=1}^n \sum_{\# I = q} \partial^p_i u_I(x) dx_i \wedge dx_I.$$

Using (3.1) we easily conclude that

$$A^{(p)}_{q+1} \circ A^{(p)}_q = 0 \tag{4.2}$$

and then we obtain a Koszul complex $\{A^{(p)}_q, \Lambda^q\}$. In this case $N = n$ and the algebraic variety $\mathcal{N}(A^{(p)}_0)$ is trivial, i.e. $\{A^{(p)}_q, \Lambda^q\}$ is a compatibility complex for the operator $A^{(p)}_0 = \begin{pmatrix} \partial_1^p \\ \dots \\ \partial_n^p \end{pmatrix}$, see Example 4.1 above. Of course, in addition to (4.2), we also have $\Delta^{(p)}_q = (A^{(p)}_q)^* A^{(p)}_q + A^{(p)}_{q-1}(A^{(p)}_{q-1})^* = (-1)^p \Big(\sum_{j=1}^n \partial_j^{2p}\Big) I_{k_q}$, $0 \le q \le n$. Again we still may define this compatibility complex in $\mathbb{R}^3$ with the use of the classical algebraic constructions: $A^{(p)}_1 u = A^{(p)}_0 \times u$, $A^{(p)}_2 v = A^{(p)}_0 \cdot v$.

Example 4.3. Consider the differential operator

$$A = \begin{pmatrix} 0 & -\partial_3 \\ \partial_3 & 0 \\ -\partial_2 & \partial_1 \\ -\partial_1 & -\partial_2 \end{pmatrix}$$

in $\mathbb{R}^3$, that is closely related to the de Rham complex on the plane Ox_1x_2:

$$d_0 = \nabla_2 = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix},\ d_1 = \mathrm{curl}_2 = (-\partial_2, \partial_1),\ d_0^* = -\mathrm{div}_2 = -(\partial_1, \partial_2).$$

Alledgedly, the related system of equations

$$\begin{cases} \partial_3 v_1 = 0, \\ \partial_3 v_2 = 0, \\ \mathrm{curl}_2 \vec{v} = -\partial_2 v_1 + \partial_1 v_2 = f_1, \\ -\mathrm{div}_2 \vec{v} = -\partial_1 v_1 - \partial_2 v_2 = f_2, \end{cases} \tag{4.3}$$

was pointed out by L. Euler for the description of the velocity $\vec{v} = (v_1, v_2)$ of a plane-parallel flow on layers $\{x_3 = const\}$ with given 'plane rotation' f_1 and 'plane source' f_2 in the case where the flow does not depend on the layer. Taking a complex valued functions $\tilde{v}(x_1, x_2) = -\iota(v_1 - \iota v_2)$, $\tilde{f}(x_1, x_2) = f_1 - \iota f_2$ one easily reduces the last two equations in (4.3) to the non- homogeneous Cauchy-Riemann system $\overline{\partial}\tilde{v} = \tilde{f}$ on the plane Ox_1x_2.

Clearly, for the differential operator

$$B = \begin{pmatrix} \partial_2 & -\partial_1 & 0 & -\partial_3 \\ \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix}$$

we have $B \circ A \equiv 0$. Passing to the polynomial matrices, we see that if a differential operator $\tilde{B}$ satisfies $\tilde{B} \circ A \equiv 0$ then for $\tilde{B}(\zeta) = (b_1(\zeta), b_2(\zeta), b_3(\zeta), b_4(\zeta))$ we have

$$\zeta_3 b_2 - \zeta_2 b_3 - \zeta_1 b_4 = 0,\ -\zeta_3 b_1 + \zeta_1 b_3 - \zeta_2 b_4 = 0, \tag{4.4}$$

and hence

$$|\zeta|^2 b_3 = \zeta_1\zeta_3 b_1 + \zeta_2\zeta_3 b_2,\ |\zeta|^2 b_4 = -\zeta_2\zeta_3 b_1 + \zeta_1\zeta_3 b_2.$$

In particular, this means that there are polynomials $c_1(\zeta)$, $c_2(\zeta)$ such that

$$b_3(\zeta) = \zeta_3 c_1(\zeta),\ b_4(\zeta) = \zeta_3 c_2(\zeta)$$

and then, taking into account (4.4),

$$\tilde{B}(\zeta) = (\zeta_1 c_1 - \zeta_2 c_2, \zeta_1 c_2 + \zeta_1 c_2, \zeta_3 c_1, \zeta_3 c_2) = \big(-c_2,\ c_1\big) B(\zeta).$$

Thus, $\{A_0 = A, A_1 = B\}$ is a compatibility complex. Moreover, as the related Laplacians have the following form: $\Delta_0 = -\Delta I_2$, $\Delta_1 = -\Delta I_4$, $\Delta_2 = -\Delta I_2$, where Δ is the usual Laplace operator in $\mathbb{R}^3$, we conclude that this complex is elliptic.

For system (4.3) the compatibility conditions induced by the operator B just mean that the data f_1, f_2 do not depend on the variable x_3:

$$\partial_3 f_1 = \partial_3 f_2 = 0.$$

Situation becomes more complicated if we consider the system

$$\left\{\begin{array}{l} \partial_3 v_1 = g_1(x_1,x_2,x_3), \\ \partial_3 v_2 = g_2(x_1,x_2,x_3), \\ \mathrm{curl}_2 \vec{v} = \partial_1 v_2 - \partial_2 v_1 = f_1(x_1,x_2,x_3), \\ -\mathrm{div}_2 \vec{v} = -\partial_1 v_1 - \partial_2 v_2 = f_2(x_1,x_2,x_3), \end{array}\right. \tag{4.5}$$

i.e. we are looking for 'planar' velocity $\vec{v} = (v_1, v_2)$ of the flow on each layer $\{x_3 = const\}$ with given 'plane rotation' f_1 and 'plane source' f_2 in the case where the flow *depends* on the layers. Then the compatibility conditions are the following:

$$\left\{\begin{array}{l} \mathrm{curl}_2\, \vec{g} = \partial_1 g_2 - \partial_2 g_1 = \partial_3 f_1(x_1,x_2,x_3), \\ -\mathrm{div}_2\, \vec{g} = -\partial_1 g_1 - \partial_2 g_2 = \partial_3 f_2(x_1,x_2,x_3). \end{array}\right.$$

The related linear Maxwell' type operators $\mathcal{M}_j(A,\partial_t)$ are the following:

$$\mathcal{M}_1(A,\partial_t) = \begin{pmatrix} \partial_t & 0 & 0 & 0 & 0 & -\partial_3 \\ 0 & \partial_t & 0 & 0 & \partial_3 & 0 \\ 0 & 0 & \partial_t & 0 & -\partial_2 & \partial_1 \\ 0 & 0 & 0 & \partial_t & -\partial_1 & -\partial_2 \\ 0 & -\partial_3 & \partial_2 & \partial_1 & \partial_t & 0 \\ \partial_3 & 0 & -\partial_1 & \partial_2 & 0 & \partial_t \end{pmatrix},$$

$$\mathcal{M}_2(A,\partial_t) = \begin{pmatrix} \partial_t & 0 & \partial_2 & -\partial_1 & 0 & -\partial_3 & 0 & 0 \\ 0 & \partial_t & \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 \\ -\partial_2 & -\partial_1 & \partial_t & 0 & 0 & 0 & 0 & -\partial_3 \\ \partial_1 & -\partial_2 & 0 & \partial_t & 0 & 0 & \partial_3 & 0 \\ 0 & -\partial_3 & 0 & 0 & \partial_t & 0 & -\partial_2 & \partial_1 \\ \partial_3 & 0 & 0 & 0 & 0 & \partial_t & -\partial_1 & -\partial_2 \\ 0 & 0 & 0 & -\partial_3 & \partial_2 & \partial_1 & \partial_t & 0 \\ 0 & 0 & \partial_3 & 0 & -\partial_1 & \partial_2 & 0 & \partial_t \end{pmatrix}.$$

Now, taking into the account the following relation between the two-dimensional and the three-dimensional rotation operators, $\mathrm{curl}_2(v_1,v_2) = \mathrm{curl}_3(v_1,v_2,0)$, we see that the related Maxwells' system $\mathcal{M}_1(A,\partial_t)$ is reduced to system $\mathcal{M}_2(d,\partial_t)$ for the de Rham complex with $\vec{v} = (v_1,v_2,v_3)$ truncated to $(v_1,v_2,0)$ that corresponds to one line missing in (3.11):

$$-\partial_2 w_1 + \partial_1 w_2 + \partial_3 u + \mathfrak{d}_t v_3 = 0.$$

The similar fact is valid for Stokes' operators $S_1(A,\mathfrak{D}_{\boldsymbol{\mu}},\partial_t)$ and $S_2(d,\mathfrak{D}_{\boldsymbol{\mu}},\partial_t)$ with the missing line in the corresponding non-linear perturbation of $S_2(d,\mathfrak{D}_{\boldsymbol{\mu}},\partial_t)$:

$$-\partial_2 w_1 + \partial_1 w_2 + \partial_3 u + (\mathfrak{d}_t - \mu\Delta) v_3 = 0.$$

Significantly, $\mathcal{M}_2(A,\partial_t)$ coincides with $\mathcal{M}_3(d,\partial_t)$ up to the order of lines and columns, and similarly for $S_2(A,\mathfrak{D}_{\boldsymbol{\mu}},\partial_t)$ and $S_3(d,\mathfrak{D}_{\boldsymbol{\mu}},\partial_t)$, i.e. the missing component v_3 and the missing lines are restored automatically on the last step.

Example 4.4. Let $X = \mathbb{R}^2$ and

$$A = \begin{pmatrix} \partial_1 & 0 \\ \partial_2 & \partial_1 \\ 0 & \partial_2 \end{pmatrix}.$$

Then its principal symbol is injective because

$$A^*A = -\begin{pmatrix} \Delta & \partial_1\partial_2 \\ \partial_1\partial_2 & \Delta \end{pmatrix}, \det(\sigma(A^*A)(\zeta)) = |\zeta|^4 - \zeta_1^2\zeta_2^2 > 0 \text{ for all } \zeta \in \mathbb{R}^2 \setminus \{0\}.$$

Clearly, for the polynomial vector $B(\zeta) = (\zeta_2^2, -\zeta_1\zeta_2, \zeta_1^2)$ we have $B(\zeta)A(\zeta) \equiv 0$. If a vector $\tilde{B}(\zeta) = (b_1(\zeta), b_2(\zeta), b_3(\zeta))$ satisfies $\tilde{B}(\zeta)A(\zeta) \equiv 0$ then

$$b_1(\zeta)\zeta_1 + b_2(\zeta)\zeta_2 = b_2(\zeta)\zeta_1 + b_3(\zeta)\zeta_2 = 0.$$

Hence $b_1\zeta_1^2 = b_3\zeta_2^2$, and for the polynomial $p(\zeta) = b_3/\zeta_1^2$ we have $\tilde{B}(\zeta) = p(\zeta)B(\zeta)$, i.e. the differential operator $B = (\partial_2^2, -\partial_1\partial_2, \partial_1^2)$ is a compatibility operator for A. As the mapping $B(\zeta) : \mathcal{P}^3 \to \mathcal{P}$ is surjective for $\zeta \in \mathbb{R}^n \setminus \{0\}$, we see that any polynomial $C(\zeta)$, satisfying $C(\zeta)B(\zeta) = 0$, is identically zero. Thus operators A and B form a compatibility differential complex.

Moreover, if $w = (w_1, w_2, w_3)^T$ is a complex vector, satisfying

$$B(\zeta)w = \zeta_2^2 w_1 - \zeta_1\zeta_2 w_2 + w_3\zeta_1^2 = 0$$

then there is a complex vector $v = (v_1, v_2)^T$ such that $w = A(\zeta)v$ if $\zeta \in \mathbb{R}^2 \setminus \{0\}$:

$$w_1 = 0, v_1 = \frac{w_2}{\zeta_2}, v_2 = \frac{w_3}{\zeta_2} \text{ if } \zeta_1 = 0, \zeta_2 \neq 0,$$

$$w_3 = 0, v_1 = \frac{w_1}{\zeta_1}, v_2 = \frac{w_2}{\zeta_1} \text{ if } \zeta_1 \neq 0, \zeta_2 = 0,$$

$$w_2 = \frac{\zeta_2^2 w_1 + w_3\zeta_1^2}{\zeta_1\zeta_2}, v_1 = \frac{w_1}{\zeta_1}, v_2 = \frac{w_3}{\zeta_2} \text{ if } \zeta_1 \neq 0, \zeta_2 \neq 0.$$

Thus, the range of the mapping $A(\zeta) : \mathbb{C}^2 \to \mathbb{C}^3$ coincides with the kernel of the mapping $B(\zeta) : \mathbb{C}^3 \to \mathbb{C}^1$ if $\zeta \in \mathbb{R}^2 \setminus \{0\}$. As the mapping $A(\zeta)$ is injective and the mapping $B(\zeta)$ is surjective, we conclude that the related complex is elliptic.

However, the operators in the complex have different orders and, unfortunately, we can not decrease the order of B. Of course, $\Delta_2 = \Delta^2 - \partial_1^2\partial_2^2$ is a strongly elliptic operator, but taking $\boldsymbol{\mu}_0^{(0)} = -\Delta I_2$, $\boldsymbol{\mu}_1^{(1)} = -\Delta I_3$ we obtain

$$\Delta_{0,\boldsymbol{\mu}_0} = \begin{pmatrix} \Delta^2 & \partial_1\Delta\partial_2 \\ \partial_1\Delta\partial_2 & \Delta^2 \end{pmatrix}, \Delta_{1,\boldsymbol{\mu}_1} = \begin{pmatrix} \Delta^2 - \partial_1^2\partial_2^2 & \partial_1^3\partial_2 & \partial_1^2\partial_2^2 \\ \partial_1^3\partial_2 & \Delta^2 + \partial_1^2\partial_2^2 & \partial_1\partial_2^3 \\ \partial_1^2\partial_2^2 & \partial_1\partial_2^3 & \Delta^2 - \partial_1^2\partial_2^2 \end{pmatrix},$$

strongly elliptic non-negative self-adjoint operators.

Example 4.5. Consider the compatibility complex for the multidimensional Cauchy-Riemann operator $\overline{\partial}$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, $n > 1$, i.e. for n-vector column with the components $\frac{\partial}{\partial \overline{z}_j}$, $1 \leq j \leq n$, where, as usual, $z_j = x_{2j-1} + \iota x_{2j}$, $\overline{z}_j = x_{2j-1} - \iota x_{2j}$,

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - \iota\frac{\partial}{\partial x_{2j}}\right), \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} + \iota\frac{\partial}{\partial x_{2j}}\right), 1 \leq j \leq n.$$

Then the Dolbeault complex is the elliptic compatibility Koszul complex related to the column $\overline{\partial}$, see [34, §1.2.7] or Example 4.1 above. As $\overline{\partial}^* = -\left(\frac{\partial}{\partial z_1}, \ \ldots, \ \frac{\partial}{\partial z_n} \right)$, then we easily calculate the related Laplacians $\Delta_q = (-1/4)\Delta I_{k_q}$, $0 \leq q \leq n$, where Δ is the usual Laplace operator in $\mathbb{R}^{2n}$ and $k_q = \begin{pmatrix} n \\ q \end{pmatrix}$.

The simplest related Stokes' type operators arising in $\mathbb{C}^2 \cong \mathbb{R}^4$ with the coordinates $(z_1, z_2) = (x_1 + \iota x_2, x_3 + \iota x_4) \cong (x_1, y_1, x_2, y_2)$ can be written as follows:

$$S_1(\overline{\partial}, \partial_t + \Delta_{\boldsymbol{\mu}}) = \begin{pmatrix} \partial_t - \mu\Delta & 0 & 0 & 0 & \mathrm{curl}_{x_1,x_2} \\ 0 & \partial_t - \mu\Delta & 0 & 0 & \mathrm{div}_{x_1,x_2} \\ 0 & 0 & \partial_t - \mu\Delta & 0 & \mathrm{curl}_{x_3,x_4} \\ 0 & 0 & 0 & \partial_t - \mu\Delta & \mathrm{div}_{x_3,x_4} \\ \mathrm{curl}^*_{x_1,x_2} & -\nabla_{x_1,x_2} & \mathrm{curl}^*_{x_3,x_4} & -\nabla_{x_3,x_4} & (\partial_t - \mu\Delta)\, I_2 \end{pmatrix},$$

where μ is a positive real number and

$$\mathrm{curl}_{x_{2j-1},x_{2j}} = (-\partial_{2j}, \partial_{2j-1}),\ \mathrm{div}_{x_{2j-1},x_{2j}} = (\partial_{2j-1}, \partial_{2j}).$$

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References

[1] Agranovich, M.S. Vishik, M.I. *Elliptic problems with a parameter and parabolic problems of general form*, Russ. Math. Surv. **19**:3 (1964), 53–157.

[2] Andreotti, A., Nacinovich, M. *Analytic convexity*, Ann. Sc. Norm. Sup., Classse di Sc., Ser. 4, V. 7, no. 2 (1984), 287–372.

[3] Bourbaki, N. Algebre. Ch. X: Algebre homologique. Nauka, Moscow, 1987.

[4] Cartan, H. Calcul différentiel. Hermann&C, Éditeurs, Paris, 1967.

[5] De Rham, G. *Variétés Différentiables*, Hermann&C, Éditeurs, Paris, 1955.

[6] Dirac, P.A.M. The Principles of Quantum Mechanics, Oxford University Press, Amen House, London, 1948.

[7] Dirac, P.A.M. *Quantised singularities in the electromagnetic field*. Proceedings of Royal Society at London, Ser. A., **133** (1931), 60–72.

[8] Dirac, P.A.M. *The theory of magnetic poles*. Physical Review, **74** (1948), 817.

[9] Douglis, A., Nirenberg, L. *Interior estimates for elliptic systems of partial differential equations*, Comm. Pure Appl. Math. 8 (1955), 503–538.

[10] Egorov, Yu.V., Shubin, M.A. Partial Differential Equations, I. Foundations of the Classical Theory, Springer, Berlin, 2001.

[11] Eidel'man, S.D. *Parabolic equations*, Partial differential equations – 6, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 63, VINITI, Moscow, 1990, 201–313.

[12] Finikov S.P. Method of Cartan's exterior differential forms in differential geometry. Moscow, OGIZ, 1948. (in Russian).

[13] Friedman, A. Partial differential equations of parabolic type, Englewood Cliffs, NJ, Prentice-Hall, Inc., 1964.

[14] Hamilton, W.R. Elements of quaternions, London, Longmans, Green & Co, 1866. (London, Cambridge University Press, 2010).

[15] Landau, L. D., and Lifshitz, E. M. Fluid Mechanics (V. 6 of A Course of Theoretical Physics), Pergamon Press, 1959.

[16] Majernik, V. *Quaternionic formulation of the classical fields*, Advances in Applied Clifford Algebras, **9**:1 (1999), 119–130.

[17] Maxwell, J.C. *A Dynamical Theory of the Electromagnetic Field*, Philosophical Transactions of the Royal Society of London, **155** (1865), 459–512.

[18] Maslennikova, V.N. Partial Differential Equations, M., RUDN, 1997.

[19] Mera A., Shlapunov A.A., Tarkhanov N. *Navier-Stokes Equations for Elliptic Complexes*, Journal of Siberian Federal University, Math. and Phys., **12**:1 (2019), 3–27.

[20] Mironov, S.V., Mironov, V.L. *Sedeonic Equations of Massive Fields*, Int. J. Theor. Phys. **54** (2015), 153–168.

[21] Mironov, V.L., Mironov, S.V., Spatio-temporal sedeons and their applications in Relativistic Quantum Mechanics and Field Theory, Sib. branch of RAS, Novosibirsk, 2018 (in Russian).
[22] Mironov, V.L., Mironov, S.V. *Generalized sedeonic equations of hydrodynamics*, Eur. Phys. J. Plus, **135**:9 (2020), 708.
[23] Mironov, V.L., Mironov, S.V. *Sedeonic equations in field theory*, Advances in Applied Clifford Algebras, **30**:3 (2020), 44.
[24] Oseen, C. W., *Über die Stokes'sche formel, und über eine verwandte Aufgabe in der Hydrodynamik*, Arkiv för matematik, astronomi och fysik, VI (29), 1910.
[25] Palamodov, V.P. Linear Differential Operators with Constant Coefficients, Grundlehren der Mathematischen Wissenschaften 168. Springer, Berlin, 1970.
[26] Pauli, W. *Zur Quantenmechanik des Magnetischen Elektrons*, Zeitschrift für Physik, **43**:9-10 (1927), 601–623.
[27] Parfenov A.A., Shlapunov A.A. *On the Fredholm property for the steady Navier-Stokes equations in weighted Hölder spaces*, Journal of Siberian Federal University, Math. and Phys., **11**:5 (2018), 659–662.
[28] Rendall, A. Partial differential equations in General Relativity, Oxford University Press, Oxford, 2008.
[29] Riquier, Ch. Les systèmes d'équations aux dérivées partielles. Paris, Gauthier-Villars. 1910
[30] Schwinger, J. *A magnetic model of matter*, Science, **165**:3895 (1969), 757—761.
[31] Shlapunov, A.A., Tarkhanov, N. *An open mapping theorem for the Navier-Stokes type equations associated with the de Rham complex over* $\mathbb{R}^n$, Siberian Electronic Math. Reports, **18**:2 (2021), 1433–1466.
[32] Spencer, D.C. *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc. **75** (1969), no. 2, 179–239.
[33] Stokes, G.G., *On some cases of fluid motion*. Transactions of the Cambridge Philosophical Society, **8** (1843), 105–137.
[34] Tarkhanov, N. Complexes of differential operators, Kluwer Academic Publishers, Dordrecht, NL, 1995.
[35] Tarkhanov, N. The Cauchy Problem for Solutions of Elliptic Equations, Akademie-Verlag, Berlin, 1995.
[36] Temam, R. Navier-Stokes Equations. Theory and Numerical Analysis, North Holland Publ. Comp., Amsterdam, 1979.
[37] Wells, R. Differential analysis on complex manifolds, Prentice Hall, Englewood Cliffs, N.J., 1973.
[38] Wloka, J. T., Rowley, B., and Lawruk, B. Boundary Value Problems for Elliptic Systems, Cambridge Univ. Press, Cambridge, 1995.

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