LIPSCHITZ STABILITY FOR AN ELLIPTIC INVERSE PROBLEM WITH A SINGLE MEASUREMENT

MOURAD CHOULLI AND HIROSHI TAKASE

ABSTRACT. We consider the problem of determining the unknown boundary values of a solution of an elliptic equation outside a bounded domain B from the knowledge of the values of this solution on a boundary of an arbitrary bounded domain surrounding B. We obtain for this inverse problem Lipschitz stability under an additional hypothesis on the unknown boundary function. This result can be also interpreted as quantitative uniqueness of continuation from the Cauchy data on the boundary of the domain surrounding B. Our analysis also applies to an interior problem.

1. INTRODUCTION AND MAIN RESULTS

Let n be a positive integer. Throughout this text, we use the Einstein summation convention for quantities with indices. If in any term appears twice, as both an upper and lower index, that term is assumed to be summed from 1 to n.

Let $(g_{ij}) \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n})$ be a symmetric matrix-valued function satisfying, for some $\theta > 0$

$$g_{ij}(x)\xi^i\xi^j \geq \theta|\xi|^2 \quad x,\xi \in \mathbb{R}^n. \tag{1.1}$$

Note that (g^{ij}) the matrix inverse to g is uniformly positive definite as well. Let $p \in L^\infty(\mathbb{R}^n)$ and recall that the Laplace-Beltrami operator associated to the metric tensor $g = g_{ij}dx^i \otimes dx^j$ is given by

$$\Delta_g u := \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j u\right),$$

where $|g| = \det(g)$.

Let $B \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary. Set $\overline{B}^c := \mathbb{R}^n \setminus \overline{B}$. Consider the exterior boundary value problem

$$\begin{cases} Pu := -\Delta_g u + pu = 0 & \text{in } \overline{B}^c, \\ u = u_0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

where $u_0 \in H^{\frac{3}{2}}(\partial B)$. For the sake of generality, we do not assume existence and uniqueness of solutions of the boundary value problem (1.2). Note however that, under the assumption $p \geq 0$, (1.2) admits a unique solution $u = u(u_0) \in H^1(\overline{B}^c)$. This follows by reducing first (1.2) to a boundary value problem with homogeneous boundary condition and then applying Lax-Milgram lemma to the bilinear form associated with P. After transforming again (1.2) to a boundary value problem

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with homogeneous boundary condition, we apply [4, Theorem 9.25] to obtain that $u \in H^2(\overline{B}^c)$.

Let $\Omega \Supset B$ be a bounded domain. We aim to establish Lipschitz stability inequality of the determination of the unknown function u_0 from the measurement $u(u_0)_{|\partial\Omega}$. This type of inverse problem setup is common in satellite gravitational gradiometry, where the gravitational potential of the Earth's surface is determined from observations of the gravitational potential and gravity on the satellite orbit (e.g., Pereverzev-Schock [12] and Freeden-Nashed [8, Chapter 7]).

Besides the fact that ∂B and $\partial\Omega$ are disjoint, the main assumption on u_0 is $|\nabla_\tau u_0| \le M|u_0|$ on ∂B, for some constant $M > 0$, where ∇_τ stands for the tangential gradient. In the general case the best we can expect is a logarithmic stability (e.g., Choulli [7] and Bellassoued-Choulli [3, Appendix A]). Indeed, it has been known that the Cauchy problem for elliptic equations is ill-posed in the sense that there is no hope of obtaining a Lipschitz stability estimate and Hadamard [9] gave an example in which the stability is exactly logarithmic.

We have similar result for the interior boundary value problem

$$\begin{cases} Pu = 0 & \text{in } B, \\ u = u_0 & \text{on } \partial B, \end{cases} \tag{1.3}$$

when the measurement is made on $\partial\Omega$, with $\Omega \Subset B$.

In other settings, Alessandrini-Beretta-Rosset-Vessella [1] and Alessandrini-Rondi-Rosset-Vessella [2] discuss the logarithmic stability of Cauchy problems for elliptic equations, which are widely found in inverse boundary value problems modeled by elliptic equations. Also, Lin-Nakamura-Wang [11] and Lin-Nagayasu-Wang [10] discuss the quantitative uniqueness of continuation for elliptic equations with singular coefficients.

In the following subsection, we state the main theorem for each problem, and its proof is given in section 3. The proofs are obtained by applying a global Carleman estimate (see Proposition 2.1) to the boundary value problems. It was originally developed by Carleman [5] and has a wide range of applications from quantification of uniqueness of continuation, inverse problems, and control theory for not only elliptic equations but also evolution equations. Although we do not list all the references here, see, for example, Choulli [6] for elliptic Carleman estimates and their applications to quantification of uniqueness of continuation and inverse problems.

1.1. **Main results.** Fix $M > 0$ and define the conditional subset $\mathcal{C}$ by

$$\mathcal{C} := \left\{ v \in H^{\frac{3}{2}}(\partial B) \middle| |\nabla_\tau v(x)| \le M|v(x)| \text{ a.e. on } \partial B \right\}. \tag{1.4}$$

Theorem 1.1. *Let $B \subset \mathbb{R}^n$, $\Omega \Supset B$ be bounded domains with C^4 boundaries and set $\zeta_0 = (g, p, B, \Omega, M)$. Then there exists $C = C(\zeta_0) > 0$ such that for any $u_0 \in \mathcal{C}$ we have*

$$\|u\|_{H^1(\Omega\setminus\overline{B})} + \|u_0\|_{H^1(\partial B)} \le C \left(\|u\|_{H^1(\partial\Omega)} + \|\partial_\nu u\|_{L^2(\partial\Omega)} \right),$$

where $u = u(u_0) \in H^2(\overline{B}^c)$ is a solution of the boundary value problem (1.2) and ν denotes the outer unit normal to $\partial\Omega$.

Theorem 1.2. *Let $B \subset \mathbb{R}^n$, $\Omega \Subset B$ be bounded domains with C^4 boundaries and set $\zeta_0 = (g, p, B, \Omega, M)$. Then there exists $C = C(\zeta_0) > 0$ such that for any $u_0 \in \mathcal{C}$*

we have

$$\|u\|_{H^1(B\setminus\overline{\Omega})} + \|u_0\|_{H^1(\partial B)} \le C\left(\|u\|_{H^1(\partial\Omega)} + \|\partial_\nu u\|_{L^2(\partial\Omega)}\right),$$

where $u = u(u_0) \in H^2(B)$ is a solution of the boundary value problem (1.3) *and ν denotes the outer unit normal to $\partial\Omega$.*

It is worth noting that, under the assumption that 0 is not the eigenvalue of the operator $Au := Pu$ with domain $D(A) = H^2(B) \cap H_0^1(B)$, (1.3) possesses a unique solution $u = u(u_0) \in H^2(B)$.

2. Global Carleman estimate

In this section, we prove a global Carleman estimate with a second large parameter that can be applied to both exterior problem (1.2) and interior problem (1.3). As it was noted by many authors, the role of the second large parameter is to ensure the so-called Hörmander's pseudo-convexity condition. The proof is similar to Choulli [6] in the estimate of terms inside a domain, but the estimate of boundary terms is original to this paper. For the reader's convenience, we provide in this section the detailed proof of our global Carleman inequality.

Let $D \subset \mathbb{R}^n$ be a bounded domain with C^4 boundary ∂D and $\Gamma \subset \partial D$ be a nonempty subboundary. Assume that there exists $\phi \in C^4(\overline{D})$ satisfying

$$\begin{cases} \phi > 0, \quad \text{in } D, \\ \phi_{|\Gamma} = 0, \\ \delta := \min_{\overline{D}} |\nabla\phi| > 0. \end{cases} \tag{2.1}$$

Let ν be the outer unit normal to ∂D and recall that the tangential gradient ∇_τ is defined by $\nabla_\tau w := \nabla w - (\partial_\nu w)\nu$. The surface element on ∂D will be denoted by dS.

Proposition 2.1. *Let $\zeta_1 = (g, p, D, \Gamma, \phi, \delta)$, set $\varphi := e^{\gamma\phi}$ and $\sigma := s\gamma\varphi$. There exist $\gamma_* = \gamma_*(\zeta_1) > 0$, $s_* = s_*(\zeta_1) > 0$ and $C = C(\zeta_1) > 0$ such that for any $\gamma \ge \gamma_*$, $s \ge s_*$ and $u \in H^2(D)$ we have*

$$C\left(\int_D e^{2s\varphi}\sigma(\gamma|\nabla u|^2 + \gamma\sigma^2|u|^2)dx + \int_\Gamma e^{2s\varphi}\sigma^3|u|^2 dS\right)$$
$$\le \int_D e^{2s\varphi}|Pu|^2 dx + \int_{\partial D\setminus\Gamma} e^{2s\varphi}\sigma(|\nabla u|^2 + \sigma^2|u|^2)dS + \int_\Gamma e^{2s\varphi}\sigma|\nabla_\tau u|^2 dS.$$

Proof. As usual, it suffices to show the inequality when $p = 0$. For convenience, we recall the following usual notations

$$\begin{aligned}
&\langle X, Y\rangle = g_{ij}X^iY^j, \quad X = X^i\frac{\partial}{\partial x_i},\ Y = Y^i\frac{\partial}{\partial x_i},\\
&\nabla_g w = g^{ij}\partial_i w\frac{\partial}{\partial x_j}, \quad w \in H^1(D),\\
&|\nabla_g w|_g^2 = \langle\nabla_g w, \nabla_g w\rangle = g^{ij}\partial_i w\partial_j w, \quad w \in H^1(D),\\
&\nu_g = (\nu_g)^i\frac{\partial}{\partial x_j}, \quad (\nu_g)^i = \frac{g^{ij}\nu_j}{\sqrt{g^{k\ell}\nu_k\nu_\ell}}\\
&\partial_{\nu_g} w = \langle\nu_g, \nabla_g w\rangle, \quad w \in H^1(D).
\end{aligned}$$

Also, define the tangential gradient $\nabla_{\tau_g} w$ with respect to g by

$$\nabla_{\tau_g} w := \nabla_g w - (\partial_{\nu_g} w)\nu_g.$$

We find that $|\nabla_{\tau_g} w|_g^2 = |\nabla_g w|_g^2 - |\partial_{\nu_g} w|^2$ holds.

Let $u \in H^2(D)$, $z := e^{s\varphi}u$ and $P_s z := e^{s\varphi}L(e^{-s\varphi}z)$, where we set $L = \Delta_g$. A direct calculation yields $Pz = P_s^+ z + P_s^- z$, where

$$\begin{cases} P_s^+ z := Lz + s^2|\nabla_g\varphi|_g^2 z, \\ P_s^- z := -2s\langle \nabla_g\varphi, \nabla_g z\rangle - sL\varphi z. \end{cases}$$

Let $dV_g = \sqrt{|g|}dx$ and endow $L^2(D)$ with following inner product

$$(v, w)_g := \int_D uv dV_g.$$

The norm associated to this inner product is denoted by $\|\cdot\|_g$.

Hereinafter, the integrals on D are with respect to the measure dV_g and those on ∂D are with respect to the surface measure $dS_g = \sqrt{|g|}dS$. Integrations by parts yield

$$\begin{aligned}
&(P_s^+ z, P_s^- z)_g \\
&= -2\int_D sLz\langle \nabla_g\varphi, \nabla_g z\rangle - \int_D sLzL\varphi z \\
&\qquad - 2\int_D s^3|\nabla_g\varphi|_g^2 z\langle \nabla_g\varphi, \nabla_g z\rangle - \int_D s^3 L\varphi|\nabla_g\varphi|_g^2|z|^2 \\
&= 2\int_D s\nabla_g^2\varphi(\nabla_g z, \nabla_g z) + \int_D s\langle \nabla_g\varphi, \nabla_g|\nabla_g z|_g^2\rangle + \int_D sL\varphi|\nabla_g z|_g^2 \\
&\qquad + \frac{1}{2}\int_D s\langle \nabla_g L\varphi, \nabla_g|z|^2\rangle + \int_D s^3 \operatorname{div}(|\nabla_g\varphi|_g^2\nabla_g\varphi)|z|^2 - \int_D s^3 L\varphi|\nabla_g\varphi|_g^2|z|^2 \\
&\qquad\quad - 2\int_{\partial D} s\partial_{\nu_g} z\langle \nabla_g\varphi, \nabla_g z\rangle - \int_{\partial D} s\partial_{\nu_g} zL\varphi z - \int_{\partial D} s^3|\nabla_g\varphi|_g^2\partial_{\nu_g}\varphi|z|^2 \\
&= 2\int_D s\nabla_g^2\varphi(\nabla_g z, \nabla_g z) - \frac{1}{2}\int_D sL^2\varphi|z|^2 + 2\int_D s^3\nabla_g^2\varphi(\nabla_g\varphi, \nabla_g\varphi)|z|^2 \\
&\qquad - 2\int_{\partial D} s\partial_{\nu_g} z\langle \nabla_g\varphi, \nabla_g z\rangle - \int_{\partial D} s\partial_{\nu_g} zL\varphi z - \int_{\partial D} s^3|\nabla_g\varphi|_g^2\partial_{\nu_g}\varphi|z|^2 \\
&\qquad\quad + \int_{\partial D} s\partial_{\nu_g}\varphi|\nabla_g z|_g^2 + \frac{1}{2}\int_{\partial D} s\partial_{\nu_g} L\varphi|z|^2,
\end{aligned}$$

and

$$
\begin{aligned}
&(P_s^+ z, -sL\varphi z)_g \\
&\qquad = -\int_D sLzL\varphi z - \int_D s^3 L\varphi|\nabla_g\varphi|_g^2|z|^2 \\
&\qquad = \int_D sL\varphi|\nabla_g z|_g^2 + \frac{1}{2}\int_D s\langle\nabla_g L\varphi, \nabla_g|z|^2\rangle \\
&\qquad\qquad - \int_D s^3 L\varphi|\nabla_g\varphi|_g^2|z|^2 - \int_{\partial D} s\partial_{\nu_g} zL\varphi z \\
&\qquad = \int_D sL\varphi|\nabla_g z|_g^2 - \frac{1}{2}\int_D sL^2\varphi|z|^2 - \int_D s^3 L\varphi|\nabla_g\varphi|_g^2|z|^2 \\
&\qquad\qquad - \int_{\partial D} s\partial_{\nu_g} zL\varphi z + \frac{1}{2}\int_{\partial D} s\partial_{\nu_g} L\varphi|z|^2
\end{aligned}
$$

Adding the above equalities yields

$$
\begin{aligned}
&(P_s^+ z, P_s^- z)_g + (P_s^+ z, -sL\varphi z)_g \\
&\quad = \int_D s\left[2\nabla_g^2\varphi(\nabla_g z, \nabla_g z) + L\varphi|\nabla_g z|_g^2\right] - \int_D sL^2\varphi|z|^2 \\
&\qquad + \int_D s^3\left[2\nabla_g^2\varphi(\nabla_g\varphi, \nabla_g\varphi) - L\varphi|\nabla_g\varphi|_g^2\right]|z|^2 \\
&\qquad\quad - 2\int_{\partial D} s\partial_{\nu_g} z\langle\nabla_g\varphi, \nabla_g z\rangle - 2\int_{\partial D} s\partial_{\nu_g} zL\varphi z - \int_{\partial D} s^3|\nabla_g\varphi|_g^2\partial_{\nu_g}\varphi|z|^2 \\
&\qquad\qquad + \int_{\partial D} s\partial_{\nu_g}\varphi|\nabla_g z|_g^2 + \int_{\partial D} s\partial_{\nu_g} L\varphi|z|^2.
\end{aligned}
$$

Henceforth, $C = C(\zeta_1) > 0$, $\gamma_* = \gamma_*(\zeta_1) > 0$ and $s_* = s_*(\zeta_1) > 0$ denote generic constants. By (2.1), we have

$$
|\nabla_g\phi|_g \geq C|\nabla\phi| \geq C\delta.
$$

Hence

$$
\begin{aligned}
&2\nabla_g^2\varphi(\nabla_g z, \nabla_g z) + L\varphi|\nabla_g z|_g^2 \\
&\qquad = \gamma\varphi\left(2\nabla_g^2\phi(\nabla_g z, \nabla_g z) + 2\gamma|\langle\nabla_g\phi, \nabla_g z\rangle|^2 + L\phi|\nabla_g z|_g^2 + \gamma|\nabla_g\phi|_g^2|\nabla_g z|_g^2\right) \\
&\qquad \geq \gamma\varphi\left(2\nabla_g^2\phi(\nabla_g z, \nabla_g z) + L\phi|\nabla_g z|_g^2 + \gamma|\nabla_g\phi|_g^2|\nabla_g z|_g^2\right) \\
&\qquad \geq C\gamma^2\varphi|\nabla_g z|_g^2, \quad \gamma \geq \gamma_*,
\end{aligned}
$$

and

$$
\begin{aligned}
&2\nabla_g^2\varphi(\nabla_g\varphi, \nabla_g\varphi) - L\varphi|\nabla_g\varphi|_g^2 \\
&\qquad = \gamma^3\varphi^3(2\nabla_g^2\phi(\nabla_g\phi, \nabla_g\phi) + \gamma|\nabla_g\phi|_g^4 - L\phi|\nabla_g\phi|_g^2) \\
&\qquad \geq C\gamma^4\varphi^3, \quad \gamma \geq \gamma_*.
\end{aligned}
$$

In consequence, we get

$$
\begin{aligned}
&(P_s^+ z, P_s^- z)_g + (P_s^+ z, -sL\varphi z)_g \\
&\qquad \geq C\int_D s\gamma^2\varphi|\nabla_g z|_g^2 + C\int_D s^3\gamma^4\varphi^3|z|^2 - \mathcal{B}, \quad \gamma \geq \gamma_*,
\end{aligned}
$$

where

$$\begin{aligned}\mathcal{B} &:= 2\int_{\partial D} s\partial_{\nu_g} z\langle \nabla_g\varphi, \nabla_g z\rangle + 2\int_{\partial D} s\partial_{\nu_g} zL\varphi z\\ &\qquad + \int_{\partial D} s^3|\nabla_g\varphi|_g^2\partial_{\nu_g}\varphi|z|^2 - \int_{\partial D} s\partial_{\nu_g}\varphi|\nabla_g z|_g^2 - \int_{\partial D} s\partial_{\nu_g} L\varphi|z|^2.\end{aligned}$$

Set

$$\begin{aligned}\mathcal{B}_{\partial D\setminus\Gamma} := \int_{\partial D\setminus\Gamma} s\Big[&2\partial_{\nu_g} z\langle \nabla_g\varphi, \nabla_g z\rangle + 2\partial_{\nu_g} zL\varphi z\\ &+ s^2|\nabla_g\varphi|_g^2\partial_{\nu_g}\varphi|z|^2 - \partial_{\nu_g}\varphi|\nabla_g z|_g^2 - \partial_{\nu_g} L\varphi|z|^2\Big]\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_{\Gamma} := \int_{\Gamma} s\Big[&2\partial_{\nu_g} z\langle \nabla_g\varphi, \nabla_g z\rangle + 2\partial_{\nu_g} zL\varphi z\\ &+ s^2|\nabla_g\varphi|_g^2\partial_{\nu_g}\varphi|z|^2 - \partial_{\nu_g}\varphi|\nabla_g z|_g^2 - \partial_{\nu_g} L\varphi|z|^2\Big]\end{aligned}$$

so that

$$\mathcal{B} = \mathcal{B}_{\partial D\setminus\Gamma} + \mathcal{B}_{\Gamma}.$$

We check that

$$\mathcal{B}_{\partial D\setminus\Gamma} \le C\int_{\partial D\setminus\Gamma} s\gamma\varphi(|\nabla_g z|_g^2 + s^2\gamma^2\varphi^2|z|^2).$$

On the other hand, according to (2.1), we have

$$|\nabla_g z|_g^2 = |\nabla_{\tau_g} z|_g^2 + |\partial_{\nu_g} z|^2, \quad \langle \nabla_g\varphi, \nabla_g z\rangle = -\gamma\varphi|\nabla_g\phi|_g\partial_{\nu_g} z \quad \text{on } \Gamma$$

and $\nu_g = -\frac{\nabla_g\phi}{|\nabla_g\phi|_g}$ on Γ. Whence

$$\begin{aligned}\mathcal{B}_{\Gamma} &= \int_{\Gamma} s\left[-2\gamma\varphi|\nabla_g\phi|_g|\partial_{\nu_g} z|^2 + 2\partial_{\nu_g} zL\varphi z\right.\\ &\qquad\qquad \left.-s^2\gamma^3\varphi^3|\nabla_g\phi|_g^3|z|^2 + \gamma\varphi|\nabla_g\phi|_g|\nabla_g z|_g^2 - \partial_{\nu_g} L\varphi|z|^2\right]\\ &= \int_{\Gamma} s\left[-\gamma\varphi|\nabla_g\phi|_g|\partial_{\nu_g} z|^2 + 2\gamma\varphi\partial_{\nu_g} z(L\phi + \gamma|\nabla_g\phi|_g^2)z\right.\\ &\qquad\qquad \left.-s^2\gamma^3\varphi^3|\nabla_g\phi|_g^3|z|^2 + \gamma\varphi|\nabla_g\phi|_g|\nabla_{\tau_g} z|_g^2 - \partial_{\nu_g} L\varphi|z|^2\right]\\ &\le \int_{\Gamma} \sigma\left[-C|\partial_{\nu_g} z|^2 + 2(L\phi + \gamma|\nabla_g\phi|_g^2)\partial_{\nu_g} zz\right]\\ &\qquad\qquad + C\int_{\Gamma} \sigma|\nabla_{\tau_g} z|_g^2 + \int_{\Gamma}\left[-C\sigma^3 + O(s\gamma^3\varphi)\right]|z|^2\end{aligned}$$

as $s\to\infty$. We note that

$$\begin{aligned}-C|\partial_{\nu_g} z|^2 + 2(L\phi + \gamma|\nabla_g\phi|_g^2)\partial_{\nu_g} zz &\le -C\left|\partial_{\nu_g} z - \frac{L\phi + \gamma|\nabla_g\phi|_g^2}{C}z\right|^2 + C\gamma^2|z|^2\\ &\le C\gamma^2|z|^2.\end{aligned}$$

Therefore, we obtain

$$\mathcal{B}_{\Gamma} + C\int_{\Gamma}\sigma^3|z|^2 \le C\int_{\Gamma}(\sigma|\nabla_{\tau_g} z|_g^2 + s\gamma^3\varphi|z|^2)$$

and then

$$\mathcal{B}_\Gamma + C\int_\Gamma \sigma^3|z|^2 \le C\int_\Gamma \sigma|\nabla_{\tau_g} z|_g^2, \quad s \ge s_*.$$

Combining the estimates above, we get

$$\begin{aligned}
&C\left(\int_D \sigma(\gamma|\nabla_g z|_g^2 + \gamma\sigma^2|z|^2) + \int_\Gamma \sigma^3|z|^2\right)\\
&\qquad \le (P_s^+ z, P_s^- z)_g + (P_s^+ z, -sL\varphi z)_g + \mathcal{B}_{\partial D\setminus\Gamma} + \int_\Gamma \sigma|\nabla_{\tau_g} z|_g^2\\
&\qquad \le \|P_s z\|_g^2 + \int_D s^2\gamma^4\varphi^2|z|^2\\
&\qquad\qquad\qquad + \int_{\partial D\setminus\Gamma} \sigma(|\nabla_g z|_g^2 + \sigma^2|z|^2) + \int_\Gamma \sigma|\nabla_{\tau_g} z|_g^2, \quad s \ge s_*.
\end{aligned}$$

As the second term in the right-hand side can absorbed by the left-hand side, we find

$$\begin{aligned}
&C\left(\int_D \sigma(\gamma|\nabla_g z|_g^2 + \gamma\sigma^2|z|^2) + \int_\Gamma \sigma^3|z|^2\right)\\
&\qquad \le \|P_s z\|_g^2 + \int_{\partial D\setminus\Gamma} \sigma(|\nabla_g z|_g^2 + \sigma^2|z|^2) + \int_\Gamma \sigma|\nabla_{\tau_g} z|_g^2, \quad s \ge s_*.
\end{aligned}$$

Since $u = e^{-s\varphi}z$ and $\nabla_{\tau_g} u = \nabla_{\tau_g} z$ holds by $\nabla_{\tau_g}\phi = 0$, we end up getting

$$\begin{aligned}
&C\left(\int_D e^{2s\varphi}\sigma(\gamma|\nabla u|^2 + \gamma\sigma^2|u|^2)dx + \int_\Gamma e^{2s\varphi}\sigma^3|u|^2 dS\right)\\
&\qquad \le \int_D e^{2s\varphi}\sigma(\gamma|\nabla_g u|_g^2 + \gamma\sigma^2|u|^2) + \int_\Gamma e^{2s\varphi}\sigma^3|u|^2\\
&\qquad \le \int_D e^{2s\varphi}|Lu|^2 + \int_{\partial D\setminus\Gamma} e^{2s\varphi}\sigma(|\nabla_g u|_g^2 + \sigma^2|u|^2) + \int_\Gamma e^{2s\varphi}\sigma|\nabla_{\tau_g} u|_g^2, \quad s \ge s_*.
\end{aligned}$$

Equivalently, we have

$$\begin{aligned}
&C\left(\int_D e^{2s\varphi}\sigma(\gamma|\nabla u|^2 + \gamma\sigma^2|u|^2)dx + \int_\Gamma e^{2s\varphi}\sigma^3|u|^2 dS\right)\\
&\qquad \le \int_D e^{2s\varphi}|Lu|^2 dx + \int_{\partial D\setminus\Gamma} e^{2s\varphi}\sigma(|\nabla u|^2 + \sigma^2|u|^2)dS\\
&\qquad\qquad\qquad + \int_\Gamma e^{2s\varphi}\sigma|\nabla_\tau u|^2 dS, \quad s \ge s_*.
\end{aligned}$$

The proof is then complete. □

3. Proofs of main results

Before proving Theorem 1.1 and Theorem 1.2, we show that a weight function ϕ satisfying (2.1) can be constructed for each problem in order to apply Proposition 2.1.

3.1. **Construction of a weight function.**

Lemma 3.1. *Let $B \subset \mathbb{R}^n$ and $\Omega \Supset B$ be bounded domains with C^4 boundaries. Then there exists $\phi \in C^4(\overline{\Omega \setminus B})$ satisfying*

$$\begin{cases} \phi > 0, \quad in\, \Omega \setminus \overline{B} \\ \phi_{|\partial B} = 0, \\ \delta := \min_{\overline{\Omega\setminus B}} |\nabla \phi| > 0. \end{cases}$$

Proof. Let $B_R \Supset \Omega$ be an open ball centered at 0 with radius $R > 0$. Applying [13, Theorem 9.4.3] for $\mathcal{O} := B_R \setminus \overline{\Omega} \subset B_R \setminus \overline{B}$, we obtain the desired function ϕ. □

Lemma 3.2. *Let $B \subset \mathbb{R}^n$ and $\Omega \Subset B$ be bounded domains with C^4 boundaries. Then there exists $\phi \in C^4(\overline{B})$ satisfying*

$$\begin{cases} \phi > 0, \quad in\, B \\ \phi_{|\partial B} = 0, \\ \delta := \min_{\overline{B\setminus \Omega}} |\nabla \phi| > 0. \end{cases}$$

Proof. As in the preceding lemma, we get ϕ with the required properties by applying [13, Theorem 9.4.3] with $\mathcal{O} := \Omega \subset B$. □

3.2. **Proof of Theorem 1.1 and Theorem 1.2.**

Proof of Theorem 1.1. By Lemma 3.1, there exist $\phi \in C^4(\overline{\Omega \setminus B})$ such that (2.1) is satisfied for $D := \Omega \setminus \overline{B}$ and $\Gamma := \partial B$. Fix $\gamma > \gamma_*$, where γ_* is given by Proposition 2.1. Henceforth, $C = C(\zeta_0) > 0$ denotes a generic constant. Let $u_0 \in \mathcal{C}$ and $u = u(u_0)$. Applying Proposition 2.1 to $u \in H^2(D)$, we get

$$\begin{aligned} C &\left(\int_D e^{2s\varphi} s|\nabla u|^2 dx + \int_D e^{2s\varphi} s^3 |u|^2 dx + \int_{\partial B} e^{2s\varphi} s^3 |u_0|^2 dS \right) \\ &\qquad \le \int_{\partial\Omega} e^{2s\varphi} (s|\nabla u|^2 + s^3|u|^2) dS + \int_{\partial B} e^{2s\varphi} s |\nabla_\tau u_0|^2 dS \\ &\qquad \le e^{Cs}(\|u\|^2_{H^1(\partial\Omega)} + \|\partial_\nu u\|^2_{L^2(\partial\Omega)}) + M \int_{\partial B} e^{2s\varphi} s |u_0|^2 dS, \quad s \ge s_*, \end{aligned}$$

where $s_* = s_*(\zeta_0) > 0$ is a constant.

Upon modifying s_*, we may and do assume that $Cs^3 - Ms > \frac{Cs^3}{2}$. In this case we have

$$\begin{aligned} \int_D e^{2s\varphi} s(|\nabla u|^2 + |u|^2) dx + \frac{1}{2} \int_{\partial B} e^{2s\varphi} s^3 |u_0|^2 dS \\ \le e^{Cs}(\|u\|_{H^1(\partial\Omega)} + \|\partial_\nu u\|_{L^2(\partial\Omega)})^2, \quad s \ge s_*, \end{aligned}$$

which implies

$$\|u\|^2_{H^1(D)} + \|u_0\|^2_{L^2(\partial B)} \le e^{Cs_*}(\|u\|_{H^1(\partial\Omega)} + \|\partial_\nu u\|_{L^2(\partial\Omega)})^2.$$

We complete the proof by using $\|\nabla_\tau u_0\|^2_{L^2(\partial B)} \le C\|u_0\|^2_{L^2(\partial B)}$. □

Proof of Theorem 1.2. By Lemma 3.2, there exist $\phi \in C^4(\overline{B})$ such that (2.1) is satisfied for $D := B \setminus \overline{\Omega}$ and $\Gamma := \partial B$. The proof completes by applying Proposition 2.1 as well as the proof of Theorem 1.1. □

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Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

Data availability. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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Université de Lorraine, 34 cours Léopold, 54052 Nancy cedex, France
Email address: mourad.choulli@univ-lorraine.fr

Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
Email address: htakase@imi.kyushu-u.ac.jp