NORMALIZED GROUNDED STATES FOR A COUPLED NONLINEAR SCHRÖDINGER SYSTEM IN $\mathbb{R}^3$

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Abstract. We investigate the existence of normalized ground states to system of coupled Schrödinger equations:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 |u_1|^{p_1-2}u_1 + \beta r_1 |u_1|^{r_1-2}u_1|u_2|^{r_2} & \text{in } \mathbb{R}^3, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 |u_2|^{p_2-2}u_2 + \beta r_2 |u_1|^{r_1}|u_2|^{r_2-2}u_2 & \text{in } \mathbb{R}^3, \end{cases} \tag{0.1}$$

subject to the constraint $\mathcal{S}_{a_1} \times \mathcal{S}_{a_2} = \{u_1 \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} u_1^2 dx = a_1^2\} \times \{u_2 \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} u_2^2 dx = a_2^2\}$, where $\mu_1, \mu_2 > 0$, $r_1, r_2 > 1$, and $\beta \geq 0$. Our focus is on the coupled mass super-critical case, specifically,

$$\frac{10}{3} < p_1, p_2, r_1 + r_2 < 2^* = 6.$$

We demonstrate that there exists a $\tilde{\beta} \geq 0$ such that equation (0.1) admits positive, radially symmetric, normalized ground state solutions when $\beta > \tilde{\beta}$. Furthermore, this result can be generalized to systems with an arbitrary number of components under some assumptions, and the corresponding standing wave is orbitally unstable.

*Keywords:*Nonlinear Schrödinger system; normalized solution; variational methods.
MSC: 35J50, 35J15, 35J60.

1 Introduction

This article investigates the existence of normalized ground state solutions to the coupled nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 |u_1|^{p_1-2}u_1 + \beta r_1 |u_1|^{r_1-2}u_1|u_2|^{r_2} & \text{in } \mathbb{R}^3, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 |u_2|^{p_2-2}u_2 + \beta r_2 |u_1|^{r_1}|u_2|^{r_2-2}u_2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

subject to the constraint $(u_1, u_2) \in \mathcal{S}_{a_1} \times \mathcal{S}_{a_2}$, where λ_1 and λ_2 are interpreted as Lagrange multipliers. Parameters $\mu_1, \mu_2 > 0$, $r_1, r_2 > 1$, $a_1, a_2 > 0$, and $\beta \geq 0$ are defined accordingly. These equations arise from the study of solitary waves for the system of coupled Schrödinger equations:

$$\begin{cases} -\iota \frac{\partial}{\partial t}\Phi_j = \Delta \Phi_j + \mu_j \left|\Phi_j\right|^{p_j-2} \Phi_j + \sum_{i\neq j} \beta_{ij} r_j \left|\Phi_i\right|^{r_i} \left|\Phi_j\right|^{r_j-2}\Phi_j & (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad j = 1, \dots, k, \\ \Phi_j(x,t) \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{1.2}$$

Here, $\mu_j > 0$ represents the self-focusing effect within the j^{th} beam component, while β_{ij} denotes the interaction strength between the i^{th} and j^{th} components, with attractive or repulsive interactions depending on the sign of β_{ij}. The system (1.2) is applicable in various physical contexts, particularly in nonlinear optics. Furthermore, it is analogous to the Hartree–Fock theory describing a binary Bose-Einstein condensate in distinct hyperfine states(cf. [16]).

To obtain solitary wave solutions of system (1.2), we assume $\Phi_j(x,t) = e^{i\lambda_j t}u_j(x)$, which allows us to convert the system into a set of steady-state, N-coupled nonlinear Schrödinger equations given by

$$\begin{cases} \Delta u_j - \lambda_j u_j + \mu_j |u_j|^{p_j-2}u_j + \sum_{i\neq j} r_j\beta_{ij}|u_i|^{r_i}|u_j|^{r_j-2}u_j = 0 \text{ in } \mathbb{R}^3 \\ u_j(x) \in H^1(\mathbb{R}^3) \end{cases} \quad j = 1,\dots,k. \tag{1.3}$$

We make the assumption that $\beta_{ij} = \beta_{ji} \geq 0$ for all $i \neq j$, and fix $\mu_i > 0$. There are two viable approaches: one can either consider the frequencies λ_i as constant or incorporate them as unknown variables, prescribing the masses accordingly. The latter approach is particularly compelling from a physical perspective, as it treats λ_i as Lagrange multipliers associated with the mass constraint.

The issue of fixed λ_i has been extensively researched over the past decade. For systems with two components and the existence of positive solutions (i.e., $u_1, u_2 > 0$ in $\mathbb{R}^3$), the understanding is relatively comprehensive. Interested readers are referred to the literature, including [1, 2, 3, 4, 9, 10, 13, 14, 15, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31] and their citations.

In contrast, fewer studies have addressed the existence of normalized solutions. References on this topic include [5, 6, 7, 8, 11, 17, 18], among others. Specifically, only the works [6, 11, 17, 18] tackle the issue of normalized solutions for system (1.3) with $\beta_{ij} > 0$.

We define the sets $\mathcal{D}_{a_j} = \{u \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} u^2\,dx \leqslant a_j^2\}$, $\partial\mathcal{D}_{a_j} = \mathcal{S}_{a_j} = \{u \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} u^2\,dx = a_j^2\}$, and $\mathbf{u} = (u_1, u_2, \dots, u_k)$ and $\boldsymbol{\beta} = (\beta_{12}, \dots, \beta_{1k}, \dots, \beta_{k-1k})$. Additionally, $(H^1(\mathbb{R}^3))^k$ denotes the Cartesian product of $H^1(\mathbb{R}^3)$ with itself k times.

The constrained energy functional is given by

$$E_{\boldsymbol{\beta}}(\mathbf{u}) = \frac{1}{2}\sum_{j=1}^{k}\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx - \sum_{j=1}^{k}\frac{\mu_j}{p_j}\int_{\mathbb{R}^3}|u_j|^{p_j}\,dx - \sum_{i,j=1,i\neq j}^{k}\beta_{ij}\int_{\mathbb{R}^3}|u_i|^{r_i}|u_j|^{r_j}\,dx. \tag{1.4}$$

Since $E_{\boldsymbol{\beta}}(\mathbf{u})$ is unbounded on $\mathcal{S}_{a_1} \times \dots \times \mathcal{S}_{a_k}$, we utilize the Pohozaev-Nehari identity:

$$\begin{aligned} J_{\boldsymbol{\beta}}(\mathbf{u}) = &\sum_{j=1}^{k}\int_{\mathbb{R}^3}|\nabla u_j|^2\,dx - \sum_{j=1}^{k}\frac{3\mu_j(p_j-2)}{2p_j}\int_{\mathbb{R}^3}|u_j|^{p_j}\,dx \\ &- \sum_{i,j=1,i\neq j}^{k}\frac{3\beta_{ij}(r_i+r_j-2)}{2}\int_{\mathbb{R}^3}|u_i|^{r_i}|u_j|^{r_j}\,dx. \end{aligned} \tag{1.5}$$

We will show that all solutions to (1.3) satisfy (1.5). The set $\mathcal{M}_{\boldsymbol{\beta}}$ is defined as $\{\mathbf{u} \in (H^1(\mathbb{R}^3))^k \backslash \{\mathbf{0}\} | J_{\boldsymbol{\beta}}(\mathbf{u}) = 0\}$, where $\mathbf{u} \neq \mathbf{0}$ indicates that at least one $u_j \neq 0$. A normalized ground state solution to (1.3) is a nontrivial solution that minimizes $E_{\boldsymbol{\beta}}(\mathbf{u})$ among all nontrivial solutions. Specifically, if $\mathbf{u}$ solves (1.3) and $E_{\boldsymbol{\beta}}(\mathbf{u}) = \inf_{\mathcal{S}_{a_1}\times\dots\times\mathcal{S}_{a_k}\cap\mathcal{M}_{\boldsymbol{\beta}}} E_{\boldsymbol{\beta}}$, then $\mathbf{u}$ is a normalized ground state solution.

We examine the scalar problem defined by the following system of equations:

$$\begin{cases} -\triangle w + \lambda w = \mu|w|^{p-2}w \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} w^2\,dx = a^2. \end{cases} \tag{1.6}$$

According to [19], this system has a unique positive solution, denoted by $w_{a,\mu,p}$. Additionally, $w_{a,\mu,p}$ belongs to the set $\mathcal{P}(a,\mu,p)$ and satisfies $I_{a,\mu,p}(w_{a,\mu,p}) = \inf_{w\in\mathcal{P}(a,\mu,p)} I_{a,\mu,p}(w)$, where

$$\mathcal{P}(a,\mu,p) = \{w \in \mathcal{S}_a : \int_{\mathbb{R}^3}|\nabla w|^2\,dx = \frac{3(p-2)\mu}{2p}\int_{\mathbb{R}^3}|w|^p\,dx\}$$

and

$$I_{a,\mu,p}(w) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2\, dx - \frac{\mu}{p}\int_{\mathbb{R}^3} |w|^p\, dx.$$

Furthermore, we define $\ell(a, \mu, p) := I_{a,\mu,p}(w_{a,\mu,p}) > 0$.

1.1 Case: k=2

Firstly, we examine the scenario where $k = 2$, and equation (1.3) is equivalent to (1.1).

For the case $p_1 = p_2 = 4$ and $r_1 = r_2 = 2$, [6] demonstrated the existence of a normalized solution with Morse index 2 when β is less than a certain positive constant. Additionally, they showed the existence of a mountain pass solution on the constraint, which also represents a normalized ground state solution when β exceeds a fixed positive constant. [11] verified the existence of a normalized solution for β across a wide range, employing a novel method based on the fixed point index in cones, bifurcation theory, and the continuation method. However, they did not confirm whether the normalized solution corresponds to a ground state. Under the following conditions:

(H_0) $N \geq 1, 1 < p, q < 2 + \frac{4}{N}, r_1, r_2 > 1, 2 + \frac{4}{N} < r_1 + r_2 < 2^*$;

(H_1) $N \geq 1, 2 + \frac{4}{N} < p, q < 2^*, r_1, r_2 > 1, r_1 + r_2 < 2 + \frac{4}{N}$, [17] obtained a multiplicity result.

Recently, [18] established the existence of normalized ground state solutions for Equation (1.1) in the mass super-critical case, specifically for $2 + \frac{4}{N} < p_1, p_2, r_1 + r_2 < 2^*$, where $1 \leqslant N \leqslant 4$, with 2^* denoting the critical Sobolev index and $2 + \frac{4}{N}$ the mass subcritical index. However, the parameters r_1 and r_2 were required to meet the conditions $1 < r_1 \leqslant 2$ or $1 < r_2 \leqslant 2$. Motivated by [12], this paper introduces a distinct approach for investigating the existence of normalized ground state solutions, and our findings generalize those of [18] for $N = 3$. Furthermore, our approach requires only that $r_1, r_2 > 1$ and $r_1 + r_2 < 6$, indicating a relaxation of the conditions on r_1 and r_2 compared to [18].

We define $\gamma(\beta) = \inf_{\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\cap\mathcal{M}_\beta} E_\beta(u_1, u_2)$. We present the main results below.

Theorem 1.1. *Assume that $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 6$ with $r_1, r_2 > 1$. There exists a non-negative constant $\tilde{\beta}$. The function $\gamma(\beta)$ and $\tilde{\beta}$ possess the following characteristics:*

(i) $\gamma(\beta)$ is continuously defined on $[0, \infty)$;

(ii) For $\beta \in [0, \tilde{\beta}]$, $\gamma(\beta)$ is given by $\min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\}$, and it is strictly decreasing on $(\tilde{\beta}, \infty)$;

(iii) As $\beta \to \infty$, $\gamma(\beta) \to 0$.

Theorem 1.2. *Assume that $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 6$ and $r_1, r_2 > 1$. Then equation 1.1 possesses positive, radially symmetric, normalized ground state solutions $(\lambda_1, \lambda_2, u_1, u_2)$ for $\beta > \tilde{\beta}$.*

Remark 1.3. $\tilde{\beta}$ is a fixed constant. According to [18, Theorem 2.2], we can obtain the concrete value of $\tilde{\beta}$ under some assumptions.

Theorem 1.4. *Assume that $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 6$ and $r_1, r_2 > 1$. Let $\beta > \tilde{\beta}$ and consider a sequence $\beta_n \to \beta$ with $\beta_n > \tilde{\beta}$. There exists $(u_{1,n}, u_{2,n}) \in \mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_{\beta_n}$ that achieves the minimum energy $E_{\beta_n}(u_{1,n}, u_{2,n}) = \gamma(\beta_n)$. Then $(u_{1,n}, u_{2,n})$ is a Palais-Smale sequence for the functional E_β restricted to $\mathcal{S}_{a_1} \times \mathcal{S}_{a_2}$. Moreover, there is a normalized ground state solution (u_1, u_2) of E_β such that $(u_{1,n}, u_{2,n}) \to (u_1, u_2)$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ up to a subsequence.*

We define the set $K_\beta = \{(u_1,u_2)|(u_1,u_2) \in \mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_\beta, E_\beta(u_1,u_2) = \gamma(\beta)\}$. It follows that $K_\beta \subseteq H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)$ and, according to Theorem 1.4, K_β is compact. The expressions for $d_1(\beta)$ and $\hat{d}_1(\beta)$ are given by

$$d_1(\beta) = \inf_{(u_1,u_2)\in K_\beta} \int_{\mathbb{R}^3} |u_1|^{r_1}|u_2|^{r_2}\,dx = \min_{(u_1,u_2)\in K_\beta} \int_{\mathbb{R}^3} |u_1|^{r_1}|u_2|^{r_2}\,dx,$$

$$\hat{d}_1(\beta) = \sup_{(u_1,u_2)\in K_\beta} \int_{\mathbb{R}^3} |u_1|^{r_1}|u_2|^{r_2}\,dx = \max_{(u_1,u_2)\in K_\beta} \int_{\mathbb{R}^3} |u_1|^{r_1}|u_2|^{r_2}\,dx.$$

The functions $d_1(\beta)$ and $\hat{d}_1(\beta)$ are well-defined for $\beta \in (\tilde{\beta}, \infty)$.

Theorem 1.5. *Assume that $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 6$ and $r_1, r_2 > 1$. Then $\gamma(\beta)$ is differentiable at almost everywhere $\beta \in (\tilde{\beta}, +\infty)$. Furthermore, for $\beta_0 \in (\tilde{\beta}, +\infty)$,*

$$\lim_{\beta\to\beta_0^+} \frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0} = -\hat{d}_1(\beta_0), \quad \lim_{\beta\to\beta_0^-} \frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0} = -d_1(\beta_0).$$

In particular, $d_1(\beta_0) = \hat{d}_1(\beta_0) \Leftrightarrow \gamma(\beta)$ is differentiable at β_0 and $\gamma'(\beta_0) = -d_1(\beta_0)$.

Theorem 1.6. *Assume that $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 6$ and $r_1, r_2 > 1$. Let $(u_{1,\beta}, u_{2,\beta})$ be the normalized ground state solution of E_β for $\beta > \tilde{\beta}$. If $\lambda_1(\beta)$ and $\lambda_2(\beta)$ are the Lagrange multipliers associated with $(u_{1,\beta}, u_{2,\beta})$, then*

(i) $\lambda_1(\beta)$ and $\lambda_2(\beta)$ are continuous on $(\tilde{\beta}, \infty)$ and are strictly positive;

(ii) as $\beta \to \infty$, $\lambda_1(\beta) \to 0$ and $\lambda_2(\beta) \to 0$.

1.2 Case: k=3

In the case where $k = 3$, it has been shown by [6] that when $p_1 = p_2 = p_3 = p_4 = 4$ and $r_1 = r_2 = r_3 = 2$, a normalized solution exists for β_{ij}, $1 \leqslant i < j \leqslant 3$, sufficiently small. This result was established using an implicit function argument. Moreover, [6] proved the existence of a mountain pass solution within a certain range of $(\beta_{12}, \beta_{13}, \beta_{23})$. In this work, we build upon these findings from [6] to provide further insights.

For the sake of simplicity, we take $\ell(a_1, \mu_1, p_1) = \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2), \ell(a_3, \mu_3, p_3)\}$. An analogous consideration applies to the other instances. We define

$$\gamma_{123}(\boldsymbol{\beta}) = \inf_{\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}\cap\mathcal{M}_{\boldsymbol{\beta}}} E_{\boldsymbol{\beta}}(\boldsymbol{u}),$$

where $\boldsymbol{\beta} = (\beta_{12}, \beta_{13}, \beta_{23})$ and $\boldsymbol{u} = (u_1, u_2, u_3)$. We introduce the notation

$$\gamma_{ij}(\beta_{ij}) = \inf_{\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\cap\mathcal{M}_{\beta_{ij}}} E_{\beta_{ij}}(u_i, u_j),$$

with $\tilde{\beta}_{ij}$ as defined in Theorem 1.1 for the functional $E_{\beta_{ij}}(u_i, u_j)$, for $1 \leqslant i < j \leqslant 3$. We denote

$$\Xi = \{(\beta_{12}, \beta_{13}, \beta_{23})|\beta_{12}, \beta_{13}, \beta_{23} \geqslant 0\}.$$

According to Theorem 1.1, there is a $\bar{\beta}_{23} \geqslant 0$ for which $\gamma(\bar{\beta}_{23}) = \ell(a_1, \mu_1, p_1)$.

Theorem 1.7. *Suppose* $\frac{10}{3} < p_1, p_2, p_3, r_1 + r_2, r_1 + r_3, r_2 + r_3 < 6$ *and* $r_1, r_2, r_3 > 1$. *Then, the function* $\gamma_{123}(\boldsymbol{\beta})$ *has the following characteristics:*

(i) $\gamma_{123}(\boldsymbol{\beta})$ *is continuous over the set* Ξ*;*

(ii) As $|\boldsymbol{\beta}| \to \infty$, *where* $|\boldsymbol{\beta}| = (\beta_{12}^2 + \beta_{13}^2 + \beta_{23}^2)^{\frac{1}{2}}$, *we have* $\gamma_{123}(\boldsymbol{\beta}) \to 0$*;*

(iii) Let $\ell(a_1, \mu_1, p_1) = \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2), \ell(a_3, \mu_3, p_3)\}$. *Then, the following holds:*

$$\begin{cases} \gamma_{123}(\beta_{12}, 0, 0) = \gamma_{12}(\beta_{12}) \\ \gamma_{123}(0, \beta_{13}, 0) = \gamma_{13}(\beta_{13}) \\ \gamma_{123}(0, 0, \beta_{23}) = \begin{cases} \ell(a_1, \mu_1, p_1) & \text{for } 0 \leqslant \beta_{23} \leqslant \bar{\beta}_{23} \\ \gamma(\beta_{23}) & \text{for } \beta_{23} > \bar{\beta}_{23}. \end{cases} \end{cases}$$

Let $\Omega = \{(\beta_{12}, \beta_{13}, \beta_{23}) | \gamma_{123}(\beta_{12}, \beta_{13}, \beta_{23}) < \gamma_{ij}(\beta_{ij}), 1 \leq i < j \leq 3\}$.

Theorem 1.8. *Suppose* $\frac{10}{3} < p_1, p_2, p_3, r_1 + r_2, r_1 + r_3, r_2 + r_3 < 6$, *with* $r_1, r_2, r_3 > 1$, *and* $\ell(a_1, \mu_1, p_1) = \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2), \ell(a_3, \mu_3, p_3)\}$. *Furthermore, if there exists* $j \in \{1, 2, 3\}$ *such that* $r_j < 2$, *then:*

(i) Ω *is a non-empty, unbounded, simply connected open subset of* Ξ*;*

(ii) Equation 1.3 has positive radially symmetric normalized ground state solutions $(\lambda_1, \lambda_2, \lambda_3, u_1, u_2, u_3)$ *for* $\boldsymbol{\beta} \in \Omega$.

Remark 1.9. Under the same hypotheses as above but with $r_1 = r_2 = r_3 = 2$, and for $(\beta_{12}, \beta_{13}, \beta_{23}) \in (\tilde{\beta}_{12}, +\infty) \times (\tilde{\beta}_{13}, +\infty) \times (\bar{\beta}_{23}, +\infty)$, assume the existence of (u_2, u_3), $(\bar{u}_1, \bar{u}_2)$, and $(\tilde{u}_1, \tilde{u}_3)$ such that $\gamma_{23}(\beta_{23}) = E_{\beta_{23}}(u_2, u_3)$, $\gamma_{12}(\beta_{12}) = E_{\beta_{12}}(\bar{u}_1, \bar{u}_2)$, and $\gamma_{13}(\beta_{13}) = E_{\beta_{13}}(\tilde{u}_1, \tilde{u}_3)$. By analogy with the proof of [18, Theorem 7.2], to establish $\Omega \neq \emptyset$, one must find $(\beta_{12}, \beta_{13}, \beta_{23})$ satisfying the following three inequalities for $h \in H^1(\mathbb{R}^3) \setminus \{0\}$:

(i) $-\frac{1}{2}\Delta h < \beta_{12}|u_2|^2 h - \beta_{13}|u_3|^2 h$ in $\mathbb{R}^3$;

(ii) $-\frac{1}{2}\Delta h < \beta_{12}|\bar{u}_2|^2 h - \beta_{23}|\bar{u}_3|^2 h$ in $\mathbb{R}^3$;

(iii) $-\frac{1}{2}\Delta h < \beta_{12}|\tilde{u}_2|^2 h - \beta_{23}|\tilde{u}_3|^2 h$ in $\mathbb{R}^3$. We indetermination the existence of $(\beta_{12}, \beta_{13}, \beta_{23})$ satisfying three inequations.

Theorem 1.10. *Suppose* $\frac{10}{3} < p_1, p_2, p_3, r_1 + r_2, r_1 + r_3, r_2 + r_3 < 6$, *with* $r_1, r_2, r_3 > 1$, *and* $\ell(a_1, \mu_1, p_1) = \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2), \ell(a_3, \mu_3, p_3)\}$. *Furthermore, there exists* $j \in \{1, 2, 3\}$ *such that* $r_j < 2$. *If* $\boldsymbol{\beta} \in \Omega$ *and* $\boldsymbol{\beta}_n \in \Omega$ *converge to* $\boldsymbol{\beta}$ *in* $\mathbb{R}^3$, *then there exist sequences* $\boldsymbol{u}_n = (u_{1,n}, u_{2,n}, u_{3,n})$ *such that* $E_{\boldsymbol{\beta}_n}(\boldsymbol{u}_n) = \gamma_{123}(\boldsymbol{\beta}_n)$. *Consequently, we can obtain* $\boldsymbol{u} = (u_1, u_2, u_3)$ *in* $(H^1(\mathbb{R}^3))^3$ *such that* $\boldsymbol{u}_n$, *up to a subsequence, converges to* $\boldsymbol{u}$ *and* $E_{\boldsymbol{\beta}}(\boldsymbol{u}) = \gamma(\boldsymbol{\beta})$.

We define the set $K_{\boldsymbol{\beta}} = \{(u_1, u_2, u_3) | (u_1, u_2, u_3) \in \mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \times \mathcal{S}_{a_3} \cap \mathcal{M}_{\boldsymbol{\beta}}, E_\beta(u_1, u_2, u_3) = \gamma(\boldsymbol{\beta})\}$, and which is a subset of $H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3) \times H_r^1(\mathbb{R}^3)$. By Theorem 1.10, it is established that $K_{\boldsymbol{\beta}}$ is compact. The following quantities are defined for $1 \leq i < j \leq 3$:

$$c_{ij}(\boldsymbol{\beta}) = \inf_{\mathbf{u} \in K_{\boldsymbol{\beta}}} \int_{\mathbb{R}^3} |u_i|^{r_i} |u_j|^{r_j} dx = \min_{\mathbf{u} \in K_{\boldsymbol{\beta}}} \int_{\mathbb{R}^3} |u_i|^{r_i} |u_j|^{r_j} dx$$

and

$$\hat{c}_{ij}(\boldsymbol{\beta}) = \sup_{\mathbf{u} \in K_{\boldsymbol{\beta}}} \int_{\mathbb{R}^3} |u_i|^{r_i} |u_j|^{r_j} dx = \max_{\mathbf{u} \in K_{\boldsymbol{\beta}}} \int_{\mathbb{R}^3} |u_i|^{r_i} |u_j|^{r_j} dx.$$

Theorem 1.11. *Suppose $\frac{10}{3} < p_1, p_2, p_3, r_1+r_2, r_1+r_3, r_2+r_3 < 6$, with $r_1, r_2, r_3 > 1$, and $\ell(a_1,\mu_1,p_1) = \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2), \ell(a_3,\mu_3,p_3)\}$. Furthermore, there exists $j \in \{1,2,3\}$ such that $r_j < 2$. The partial derivatives of $\gamma(\boldsymbol{\beta})$ exist almost everywhere $\boldsymbol{\beta} \in \Omega$. Furthermore,*

$$\lim_{\delta\beta_{ij}\to 0^+} \frac{\gamma(\boldsymbol{\beta}+\delta\boldsymbol{\beta}_{ij}) - \gamma(\boldsymbol{\beta})}{\delta\beta_{ij}} = -\hat{c}_{ij}(\boldsymbol{\beta})$$

and

$$\lim_{\delta\beta_{ij}\to 0^-} \frac{\gamma(\boldsymbol{\beta}+\delta\boldsymbol{\beta}_{ij}) - \gamma(\boldsymbol{\beta})}{\delta\beta_{ij}} = -c_{ij}(\boldsymbol{\beta}).$$

Here $\delta\boldsymbol{\beta}_{12} = (\delta\beta_{12}, 0, 0)$, $\delta\boldsymbol{\beta}_{13} = (0, \delta\beta_{13}, 0)$, $\delta\boldsymbol{\beta}_{23} = (0,0,\delta\beta_{23})$ and $1 \leqslant i < j \leqslant 3$.

Specifically, $c_{ij}(\boldsymbol{\beta}) = \hat{c}_{ij}(\boldsymbol{\beta}) \Leftrightarrow$ the partial derivative of $\gamma(\beta)$ with respect to β_{ij} at β_0 exists and $\frac{\partial\gamma(\beta)}{\partial\beta_{ij}} = -c_{ij}(\beta)$.

Theorem 1.12. *Suppose $\frac{10}{3} < p_1, p_2, p_3, r_1+r_2, r_1+r_3, r_2+r_3 < 6$, with $r_1, r_2, r_3 > 1$, and $\ell(a_1,\mu_1,p_1) = \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2), \ell(a_3,\mu_3,p_3)\}$. Furthermore, there exists $j \in \{1,2,3\}$ such that $r_j < 2$. Let $(u_{1,\beta}, u_{2,\beta}, u_{3,\beta})$ be a normalized ground state solution of $E_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \in \Omega$. Let $\lambda_1(\boldsymbol{\beta})$, $\lambda_2(\boldsymbol{\beta})$ and $\lambda_3(\boldsymbol{\beta})$ be Lagrange multipliers of $(u_{1,\beta}, u_{2,\beta}, u_{3,\beta})$. Then*

(i) $\lambda_1(\boldsymbol{\beta})$, $\lambda_2(\boldsymbol{\beta})$ and $\lambda_3(\boldsymbol{\beta})$ are continuous on Ω. Moreover, $\lambda_1(\boldsymbol{\beta}), \lambda_2(\boldsymbol{\beta}), \lambda_3(\boldsymbol{\beta}) > 0$;

(ii) $\lambda_1(\boldsymbol{\beta}), \lambda_2(\boldsymbol{\beta}), \lambda_3(\boldsymbol{\beta}) \to 0$ as $|\boldsymbol{\beta}| \to +\infty$.

1.3 Case: k>3

We can extend Theorems 1.7, 1.8, 1.10, 1.11, and 1.12 to the corresponding versions of Equation (1.3) for $k > 3$. However, the corresponding results are omitted here for simplicity.

Let us proceed to investigate the orbital stability of solitary waves for the system

$$-i\frac{\partial}{\partial t}\Phi_j = \Delta\Phi_j + \mu_j|\Phi_j|^{p_j-2}\Phi_j + \sum_{i\neq j}\beta_{ij}r_j|\Phi_i|^{r_i}|\Phi_j|^{r_j-2}\Phi_j, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}, \quad j=1,\dots,k,$$

which are associated with the solutions obtained in Theorem 1.8, or Theorem 1.2 when $k = 2$. By [5, Theorem 1.8], the following theorem can be established.

Theorem 1.13. *Let $k \geqslant 2$. Suppose $(\lambda_1,\dots,\lambda_k,u_1,\dots,u_k)$ represents the solution obtained from Theorem 1.8 (or Theorem 1.2 when $k = 2$). Then the associated solitary wave is orbitally unstable by blow up in finite time.*

We outline our approach to finding normalized ground states for equation (1.1). We employ the framework from [12] to identify the minimizer (u_1,u_2) of $E_\beta(u_1,u_2)$ over the set $\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\cap\mathcal{M}_\beta$. We then demonstrate that if $\gamma(\beta) < \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2)\}$, the pair (u_1,u_2) resides in $\mathcal{S}_{a_1}\times\mathcal{S}_{a_2}$. According to [18, Lemmas 4.8 and 4.9], $\mathcal{M}_\beta$ is a natural constraint, implying that (u_1,u_2) serves as a normalized ground state solution to (1.1). Furthermore, we investigate the range of β for which $\gamma(\beta) < \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2)\}$. We establish the monotonicity of $\gamma(\beta)$, which guarantees the existence of a $\tilde{\beta} \geq 0$ such that $\gamma(\beta)$ is constant on $[0,\tilde{\beta}]$ and strictly decreasing on $(\tilde{\beta},+\infty)$. Ultimately, we show that $\gamma(0) = \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2)\}$ and $\gamma(\beta) \searrow 0$ as $\beta\to+\infty$, indicating the existence of $\tilde{\beta}$ such that for $\beta > \tilde{\beta}$, $\gamma(\beta) < \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2)\}$.

When $k = 3$, we can employ a similar approach to establish the existence of a minimizer for

$$\inf_{\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}\cap\mathcal{M}_{\boldsymbol{\beta}}} E_{(\beta_{12},\beta_{13},\beta_{23})}(u_{12},u_{13},u_{23}).$$

Subsequently, we investigate the interrelation between $\gamma_{ij}(\beta_{ij})$ and $\gamma_{123}(\boldsymbol{\beta})$. In contrast to the case with $k = 2$, it is necessary to show that Ω is non-empty. However, we merely demonstrate that if there exists an index $j \in \{1, 2, 3\}$ such that $1 < r_j < 2$, then Ω is indeed non-empty. Ultimately, we can derive the corresponding result for $\boldsymbol{\beta} \in \Omega$.

The structure of the paper is as follows: In Section 2, we investigate the properties of $E_\beta(u_1, u_2)$ and $J_\beta(u_1, u_2)$ and demonstrate the existence of $\inf_{\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\cap\mathcal{M}_\beta} E_\beta(u_1, u_2)$. Section 3 is dedicated to the case $k = 2$, where we establish that the minimizers of $E_\beta(u_1, u_2)$ within $\mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \cap \mathcal{M}_\beta$ are exclusively contained in $\mathcal{S}_{a_1} \times \mathcal{S}_{a_1}$ under an assumption of β. Subsequently, we examine the characteristics of $\gamma(\beta)$. In Section 4, we consider the scenario $k = 3$. We generalize some findings from Section 2 and explore the relationship between $\gamma(\beta_{ij})$ and $\gamma(\boldsymbol{\beta})$, for $1 \leqslant i < j \leqslant 3$. Finally, we demonstrate that Ω is non-empty.

Let us establish the following notations. The L^p norm, denoted as $|\cdot|_p$, is considered for $p \geq 1$. We define $H^1(\mathbb{R}^3, \mathbb{R}^k)$ as $(H^1(\mathbb{R}^3))^k$, and its corresponding norm is given by

$$||(u_1, \dots, u_k)||^2_{H^1(\mathbb{R}^3,\mathbb{R}^k)} = \sum_{j=1}^k |u_j|_2^2 + \sum_{j=1}^k |\nabla u_j|_2^2.$$

For $(u_1, \dots, u_k) \in H^1(\mathbb{R}^3, \mathbb{R}^k)$, we simplify the notations as $|(u_1, \dots, u_k)|_2^2 = \sum_{j=1}^k |u_j|_2^2$ and

$$|\nabla(u_1, \dots, u_k)|_2^2 = \sum_{j=1}^k |\nabla u_j|_2^2.$$

Furthermore, $\mathbf{u} = (u_1, \dots, u_k) \neq \mathbf{0}$ signifies that there exists at least one $j \in \{1, \dots, k\}$ such that $u_j \neq 0$. Lastly, $(H^1_r(\mathbb{R}^3), \mathbb{R}^k)$ denotes the product of k copies of $H^1_r(\mathbb{R}^3)$, with $H^1_r(\mathbb{R}^3)$ representing the space of radially symmetric functions in $H^1(\mathbb{R}^3)$.

2 Prelimainaries

In this section, we demonstrate several properties of $E_\beta(u_1, u_2)$ and $J_\beta(u_1, u_2)$ that can be generalized to the scenario where $k > 2$.

Lemma 2.1. *([18, Lemma 4.1]) Let $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 2^*$, $r_1, r_2 > 1$, and $\beta \geq 0$. If $(u_1, u_2) \neq \mathbf{0}$ is a weak solution to Equation 1.1 restricted to $\mathcal{S}_{a_1} \times \mathcal{S}_{a_2}$, then $J_\beta(\mathbf{u}) = 0$.*

Lemma 2.2. *([18, Lemma 4.4]) Assume $\frac{10}{3} < p_1, p_2 < 2^*$ and $r_1 + r_2 < 2^*$, with $r_1, r_2 > 1$ and $\beta \geq 0$. There exists a positive constant C_0, dependent solely on p_1, p_2, r_1, r_2, such that*

$$E_\beta(u_1, u_2) \geq C_0 \left(|\nabla u_1|_2^2 + |\nabla u_2|_2^2\right)$$

holds for any (u_1, u_2) that fulfill $J_\beta(u_1, u_2) = 0$. This indicates that E_β is coercive on the set $\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\cap\mathcal{M}_\beta$.

Lemma 2.3. *([18, Lemma 4.6]) Suppose $\frac{10}{3} < p_1, p_2 < 2^*$ and $r_1, r_2 > 1$ with $r_1 + r_2 < 2^*$, and let $\beta \geq 0$. There exists some $\delta > 0$ such that the following holds:*

$$\inf_{(u_1,u_2)\in\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\cap\mathcal{M}_\beta} \left(|\nabla u_1|_2^2 + |\nabla u_2|_2^2\right) > \delta,$$

implying $\gamma(\beta) > 0$.

We define $t \star (u_1, u_2) = (u_1^t, u_2^t) = (t^{\frac{3}{2}} u_1(tx), t^{\frac{3}{2}} u_2(tx))$ for all $t > 0$. It is straightforward to show that $|\nabla(t \star (u_1, u_2))|_2 = |\nabla(u_1, u_2)|_2$. The following proposition can be derived.

Proposition 2.4. *([18, Lemma 4.2 and Corollary 4.3]) Suppose that* $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 2^*$, $r_1, r_2 > 1$, *and* $\beta \geq 0$. *If* $(u_1, u_2) \in \mathcal{D}_{a_1} \times \mathcal{D}_{a_2}$ *with* $(u_1, u_2) \neq \mathbf{0}$, *there exists a unique* $t_0 > 0$ *such that*

$$\max_{t>0} E_\beta(t \star (u_1, u_2)) = E_\beta(t_0 \star (u_1, u_2))$$

and $t_0 \star (u_1, u_2) \in \mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \cap \mathcal{M}_\beta$.

Lemma 2.5. *([18, Lemma 4.8 and 4.9]) Assume that* $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 2^*$, *with* $r_1, r_2 > 1$ *and* $\beta \geq 0$. *For any critical point of* E_β *on* $\mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \cap \mathcal{M}_\beta$, *there exist* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that*

$$E'_\beta(u_1, u_2) + \lambda_1(u_1, 0) + \lambda_2(0, u_2) = 0.$$

Before demonstrating that the infimum $\inf_{\mathcal{D}_{a_1 \times a_2} \cap \mathcal{M}_\beta} E_\beta(\mathbf{u})$ is achieved, we require a result regarding profile decomposition, which is a generalization of [25, Theorem 1.4].

Lemma 2.6. *Let* $(u_{1,n}, u_{2,n}) \subset H^1(\mathbb{R}^3, \mathbb{R}^2)$ *be bounded. Then, for any* $n \geq 1$, *there exist sequences* $(\tilde{u}_{1,i}, \tilde{u}_{2,i})_{i=0}^\infty \subset H^1(\mathbb{R}^3, \mathbb{R}^2)$ *and* $(y_n^i)_{i=0}^\infty \subset \mathbb{R}^3$ *such that* $y_n^0 = 0$, $|y_n^i - y_n^j| \to \infty$ *as* $n \to \infty$ *for* $i \neq j$, *and, upon passing to a subsequence, the following conditions hold for any* $i \geq 0$:

$$u_{1,n}(\cdot + y_n^i) \rightharpoonup \tilde{u}_{1,i} \text{ in } H^1(\mathbb{R}^3) \text{ as } n \to \infty,$$
$$u_{2,n}(\cdot + y_n^i) \rightharpoonup \tilde{u}_{2,i} \text{ in } H^1(\mathbb{R}^3) \text{ as } n \to \infty,$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_{1,n}|^2 + |u_{2,n}|^2 \, dx = \sum_{j=0}^{i} \int_{\mathbb{R}^3} |\nabla \tilde{u}_{1,j}|^2 + |\nabla \tilde{u}_{2,j}|^2 \, dx + \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla v_{1,n}^i|^2 + |\nabla v_{2,n}^i|^2 \, dx,$$

where $v_{1,n}^i := u_{1,n} - \sum_{j=0}^i \tilde{u}_{1,j}(\cdot - y_n^j)$ *and* $v_{2,n}^i := u_{2,n} - \sum_{j=0}^i \tilde{u}_{2,j}(\cdot - y_n^j)$. *Furthermore, we have*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} H(u_{1,n}, u_{2,n}) \, dx = \sum_{j=0}^{\infty} \int_{\mathbb{R}^3} H(\tilde{u}_{1,j}, \tilde{u}_{2,j}) \, dx,$$

where the functional H *is defined as*

$$H(u_1, u_2) = \frac{\mu_1}{p_1} \int_{\mathbb{R}^3} |u_1|^{p_1} \, dx + \frac{\mu_2}{p_2} \int_{\mathbb{R}^3} |u_2|^{p_2} \, dx + \beta \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} \, dx.$$

We draw upon [12, Lemma 2.7] and [18, Lemma 5.1] to establish the following lemma. The proof is analogous to those provided therein, and thus, it is omitted here.

Lemma 2.7. *Suppose that* $\frac{10}{3} < p_1, p_2 < 2^*$ *and* $r_1 + r_2 < 2^*$, *with* $r_1, r_2 > 1$ *and* $\beta \geq 0$. *The minimizer* $\inf_{\mathcal{D}_{a_1 \times a_2} \cap \mathcal{M}_\beta} E_\beta(\mathbf{u})$ *is achieved by a non-negative, radially symmetric function over the interval* $[0, +\infty)$.

3 case: k=2

In this section, we demonstrate that the minimizers of $E_\beta(u_1, u_2) \in \mathcal{D}_{a_1} \times \mathcal{D}_{a_1} \cap \mathcal{M}_\beta$ only belong to $\mathcal{S}_{a_1} \times \mathcal{S}_{a_1}$ under the assumption that is $\gamma(\beta) < \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\}$. Additionally, we discuss certain properties of the function $\gamma(\beta)$.

Lemma 3.1. *Assume that $\frac{10}{3} < p_1, p_2, r_1 + r_2 < 2^*$, with $r_1, r_2 > 1$ and $\beta > 0$, if*

$$\gamma(\beta) < \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\},$$

then for any $(u_1, u_2) \in (\mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \setminus \mathcal{S}_{a_1} \times \mathcal{S}_{a_2}) \cap \mathcal{M}_\beta$, it follows that

$$\inf_{\mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_\beta} E_\beta < E_\beta(u_1, u_2).$$

Proof. Suppose by contradiction that there is $(u_1, u_2) \in \mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \cap \mathcal{M}_\beta$ such that $\int_{\mathbb{R}^3} u_1^2 dx < a_1$ and $\gamma(\beta) = E(u_1, u_2) \leqslant \inf_{\mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_\beta} E_\beta$. We claim that $u_1 \neq 0$. If not, we would have $u_1 = 0$. Since $(u_1, u_2) \in \mathcal{M}_\beta$, we have $u_2 \neq 0$. We easily have $|u_2|_2 = a_2$ according to [12, Lemma 2.8]. Then $\gamma(\beta) = \ell(a_2, \mu_2, p_2)$, which is a contradiction.

According to Proposition 2.4, there exists a unique $t = t(l)$ such that $(t^{\frac{3}{2}} l u_1(tx), t^{\frac{3}{2}} u_2(tx)) \in \mathcal{M}_\beta$ for any fixed $l > 0$ and $t(1) = 1$. This leads to the following equality:

$$\begin{aligned} l^2 t^2 |\nabla u_1|_2^2 + t^2 |\nabla u_2|_2^2 = &\frac{3\mu_1(p_1 - 2)}{2p_1} l^{p_1} t^{\frac{3}{2}p_1 - 3} |u_1|_{p_1}^{p_1} + \frac{3\mu_2(p_2 - 2)}{2p_2} t^{\frac{3}{2}p_2 - 3} |u_2|_{p_2}^{p_2} \\ &+ \frac{3\beta(r_1 + r_2 - 2)}{2} l^{r_1} t^{\frac{3}{2}(r_1 + r_2) - 3} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx, \end{aligned}$$

from which we derive

$$\frac{dt}{dl}|_{l=1} = \frac{2|\nabla u_1|_2^2 - \frac{3(p_1-2)}{2}\mu_1 |u_1|_{p_1}^{p_1} - \frac{3\beta r_1(r_1+r_2-2)}{2} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx}{\frac{3(p_1-2)}{2p_1}(\frac{3}{2}p_1 - 5)\mu_1 |u_1|_{p_1}^{p_1} + \frac{3(p_2-2)}{2p_2}(\frac{3}{2}p_2 - 5)\mu_2 |u_2|_{p_2}^{p_2} + \frac{3(r_1+r_2-2)}{2}(\frac{3}{2}(r_1 + r_2) - 5)\beta \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx}.$$

Given that $(u_1, u_2) \in \mathcal{M}_\beta$, it follows that

$$\begin{aligned} E_\beta(u_1, u_2) &= E_\beta(u_1, u_2) - \frac{1}{2} J_\beta(u_1, u_2) \\ &= \frac{3(p_1 - 2) - 4}{4p_1} \mu_1 |u_1|_{p_1}^{p_1} + \frac{3(p_2 - 2) - 4}{4p_2} \mu_2 |u_2|_{p_2}^{p_2} + \frac{3(r_1 + r_2 - 2) - 4}{4} \beta \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx. \end{aligned}$$

Since $\gamma(\beta) = E(u_1, u_2)$, we have

$$\frac{\partial E_\beta(t^{\frac{3}{2}} l u_2(tx), t^{\frac{3}{2}} u_2(tx))}{\partial l}|_{l=1} = 0.$$

Direct computation and the fact that $(u_1, u_2) \in \mathcal{M}_\beta$ lead to

$$|\nabla u_1|_2^2 - \mu_1 |u_1|_{p_1}^{p_1} - r_1 \beta \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx = 0. \tag{3.1}$$

Notice that $\gamma(\beta) = \inf_{u \in \mathcal{D}_{a_1} \cap \{u | (u, u_2) \in \mathcal{M}_\beta\}} E_\beta(u, u_2) = E_\beta(u_1, u_2)$ and $u_1 \in \mathcal{D}_{a_1} \backslash \mathcal{S}_{a_1}$, there exists λ such that

$$\begin{aligned} &- \Delta u_1 - \mu_1 |u_1|^{p_1 - 2} u_1 - \beta r_1 |u_1|^{r_1 - 2} u_1 |u_2|^{r_2} \\ &+ \lambda(-2\Delta u_1 - \frac{3(p_1 - 2)}{2} \mu_1 |u_1|^{p_1 - 2} u_1 - \frac{3\beta r_1(r_1 + r_2 - 2)}{2} |u_1|^{r_1 - 2} u_1 |u_2|^{r_2}) = 0 \end{aligned}$$

or equivalently

$$-(1 + 2\lambda)\Delta u_1 - (1 + \frac{3\lambda(p_1 - 2)}{2})\mu_1 |u_1|^{p_1 - 2} u_1 - (1 + \frac{3\lambda(r_1 + r_2 - 2)}{2}) r_1 \beta |u_1|^{r_1 - 2} u_1 |u_2|^{r_2} = 0.$$

This implies that u_1 satisfies the Nehari-type identity

$$(1+2\lambda)|\nabla u_1|_2^2 = (1+\frac{3\lambda(p_1-2)}{2})\mu_1|u_1|_{p_1}^{p_1} + (1+\frac{3\lambda(r_1+r_2-2)}{2})r_1\beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx.$$

In conjunction with (3.1), we can conclude that $\lambda = 0$ or

$$2|\nabla u_1|_2^2 = \frac{3(p_1-2)}{2}\mu_1|u_1|_{p_1}^{p_1} + \frac{3(r_1+r_2-2)}{2}r_1\beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx.$$

Case 1: $\lambda = 0$.

We have

$$-\Delta u_1 - \mu_1|u_1|^{p_1-2}u_1 - \beta r_1|u_1|^{r_2-2}u_1|u_2|^{r_2} = 0.$$

Thus u_1 satisfies the Pohozaev identity:

$$|\nabla u_1|_2^2 = 6(\frac{\mu_1}{p_1}|u_1|_{p_1}^{p_1} + \beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx).$$

From Equation (3.1), it follows that

$$\frac{6-p_1}{p_1}\mu_1|u_1|_{p_1}^{p_1} + (6-r_1)\beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx = 0,$$

which implies a contradiction since $u_1 \neq 0$.

Case 2: $2|\nabla u_1|_2^2 = \frac{3(p_1-2)}{2}\mu_1|u_1|_{p_1}^{p_1} + \frac{3(r_1+r_2-2)}{2}r_1\beta\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx$.

Utilizing Equation (3.1) yields

$$\frac{3p_1-10}{4}\mu_1|u_1|_{p_1}^{p_1} + \frac{3(r_1+r_2)-10}{4}\beta r_1\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx = 0,$$

which again contradicts the assumption $u_1 \neq 0$. Thus $u_1 \in \mathcal{S}_{a_1}$.

An analogous argument shows that $u_2 \in \mathcal{S}_{a_2}$, completing the proof. $\square$

We will now proceed to discuss the characteristics of the function $\gamma(\beta)$.

Proposition 3.2. *Assuming that* $\frac{10}{3} < p_1, p_2 < 2^*$ *and* $r_1, r_2 > 1$ *with* $r_1 + r_2 < 2^*$. $\gamma(\beta)$ *is non-increasing over the interval* $[0, +\infty)$.

Proof. Let us assume that $0 \leqslant \beta_1 < \beta_2$. For any arbitrarily small $\varepsilon > 0$, there exists $(u_1, u_2) \in \mathcal{M}_{\beta_1}$ such that $E_{\beta_1}(u_1, u_2) < \gamma(\beta_1) + \varepsilon$, with u_1 and u_2 both nonzero. Proposition 2.4 ensures the existence of a unique $t > 0$ for which $t\star(u_1, u_2) \in \mathcal{M}_{\beta_2}$ and $E_{\beta_1}(t\star(u_1, u_2)) \leqslant E_{\beta_1}(u_1, u_2)$. It follows from the expression of E_β that the following chain of inequalities holds:

$$\gamma(\beta_2) \leqslant E_{\beta_2}(t\star(u_1, u_2)) \leqslant E_{\beta_1}(t\star(u_1, u_2)) \leqslant E_{\beta_1}(u_1, u_2) < \gamma(\beta_1) + \varepsilon.$$

Consequently, we deduce that $\gamma(\beta_2) \leqslant \gamma(\beta_1)$. $\square$

The proof of Theorem 1.1 (i). Due to Proposition 3.2, we need only demonstrate that for any sequence $\beta_n \to \beta^+$, the inequality holds:

$$\gamma(\beta) \leqslant \lim_{n\to+\infty}\gamma(\beta_n). \tag{3.2}$$

Let $\varepsilon > 0$ be arbitrarily fixed. Suppose there exists a sequence $(u_{1,n}, u_{2,n}) \in \mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \cap \mathcal{M}_{\beta_n}$ such that $E_{\beta_n}(u_{1,n}, u_{2,n}) < \gamma(\beta_n) + \varepsilon$. According to Proposition 2.4, there exists a sequence t_n such that $t_n \star (u_{1,n}, u_{2,n}) \in \mathcal{M}_\beta$. Notice that

$$\gamma(\beta) \leqslant E_\beta(t_n \star (u_{1,n}, u_{2,n})) = E_{\beta_n}(t_n \star (u_{1,n}, u_{2,n})) + (\beta_n - \beta) t_n^{\frac{3(r_1+r_2-2)}{2}} \int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx.$$

By Proposition 2.4, we have $E_{\beta_n}(t_n \star (u_{1,n}, u_{2,n})) \leqslant E_{\beta_n}(u_{1,n}, u_{2,n}) < \gamma(\beta_n) + \varepsilon$. Therefore, it suffices to show that

$$t_n^{\frac{3(r_1+r_2-2)}{2}} \int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx$$

is bounded. Proposition 3.2 and Lemma 2.2, yield

$$C_0(|\nabla u_{1,n}|_2^2 + |\nabla u_{2,n}|_2^2) \leqslant \gamma(\beta_n) + \varepsilon \leqslant \gamma(\beta) + \varepsilon.$$

This implies that $(u_{1,n}, u_{2,n})$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R}^2)$ and $\int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx$ is bounded. Since $t_n \star (u_{1,n}, u_{2,n}) \in \mathcal{M}_\beta$, we obtain

$$\begin{aligned} |\nabla u_{1,n}|_2^2 + |\nabla u_{2,n}|_2^2 = {} & t_n^{\frac{3(p_1-2)}{2}-2} \frac{3\mu_1(p_1-2)}{2p_1} \int_{\mathbb{R}^3} |u_{1,n}|^{p_1} dx + t_n^{\frac{3(p_2-2)}{2}-2} \frac{3\mu_2(p_2-2)}{2p_2} \int_{\mathbb{R}^3} |u_{2,n}|^{p_2} dx \\ & + t_n^{\frac{3(r_1+r_2-2)}{2}-2} \frac{3\beta(r_1+r_2-2)}{2} \int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx. \end{aligned}$$

Assume that $p_1 = \min\{p_1, p_2, r_1+r_2\}$, then by $(u_{1,n}, u_{2,n}) \in \mathcal{M}_{\beta_n}$, we obtain

$$\begin{aligned} |\nabla u_{1,n}|_2^2 + |\nabla u_{2,n}|_2^2 = {} & t_n^{\frac{3(p_1-2)}{2}-2} (|\nabla u_{1,n}|_2^2 + |\nabla u_{2,n}|_2^2 - \frac{3(\beta_n - \beta)(r_1+r_2-2)}{2} \int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx) \\ & + (t_n^{\frac{3(p_2-2)}{2}-2} - t_n^{\frac{3(p_1-2)}{2}-2}) \frac{3\mu_1(p_2-2)}{2p_2} \int_{\mathbb{R}^3} |u_{2,n}|^{p_2} dx \\ & + (t_n^{\frac{3(r_1+r_2-2)}{2}-2} - t_n^{\frac{3(p_1-2)}{2}-2}) \frac{3\beta(r_1+r_2-2)}{2} \int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx. \end{aligned} \tag{3.3}$$

Again by Proposition 3.2, we know that $\gamma(\beta_n)$ has a positive lower bound, which implies $|\nabla u_{1,n}|_2^2 + |\nabla u_{2,n}|_2^2$ possesses a positive lower bound. Notice that $\frac{3(p_1-2)}{2} - 2 > 0$, we infer from Equation (3.3) that for $t_n > 1$, the following inequality holds:

$$|\nabla u_{1,n}|_2^2 + |\nabla u_{2,n}|_2^2 \geqslant t_n^{\frac{3(p_1-2)}{2}-2} (|\nabla u_{1,n}|_2^2 + |\nabla u_{2,n}|_2^2 - \frac{3(\beta_n - \beta)(r_1+r_2-2)}{2} \int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx).$$

This implies that t_n is bounded, thus completing the proof. □

For any $\beta \geqslant 0$, $E_\beta(w_{a_1,\mu_1,p_1}, 0) = \ell(a_1, \mu_1, p_1)$ and $E_\beta(0, w_{a_2,\mu_1,p_2}) = \ell(a_2, \mu_2, p_2)$. So we have

$$\gamma(\beta) \leqslant \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\}.$$

The proof of Theorem 1.1 (iii). We assume that $(u_1, u_2) \in \mathcal{D}_{a_1} \times \mathcal{D}_{a_2}$, where $u_1 \neq 0$ and $u_2 \neq 0$. According to Proposition 2.4, there exists a sequence t_n such that $t_n \star (u_1, u_2) \in \mathcal{M}_{\beta_n}$. Notice that

$$\begin{aligned} t_n^2(|\nabla u_1|_2^2 + |\nabla u_2|_2^2) = {} & t_n^{\frac{3(p_1-2)}{2}} \frac{3\mu_1(p_1-2)}{2p_1} \int_{\mathbb{R}^3} |u_1|^{p_1} dx \\ & + t_n^{\frac{3(p_2-2)}{2}} \frac{3\mu_2(p_2-2)}{2p_2} \int_{\mathbb{R}^3} |u_2|^{p_2} dx \\ & + t_n^{\frac{3(r_1+r_2-2)}{2}} \frac{3\beta_n(r_1+r_2-2)}{2} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx. \end{aligned}$$

which implies $t_n \to 0$ as $\beta_n \to +\infty$. Since Lemma 2.2 and

$$\gamma(\beta_n) \leqslant E_{\beta_n}(t_n \star (u_1, u_2)) \leqslant \frac{1}{2}t_n^2 \int_{\mathbb{R}^3} |\nabla u_1|^2 + |\nabla u_2|^2 dx,$$

we have $\gamma(\beta) \to 0$ as $\beta \to +\infty$. $\square$

The proof of Theorem 1.1 (ii). Step 1: $\gamma(0) = \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\}$.

Notice that

$$E_0(u_1, u_2) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_1|^2 + |\nabla u_2|^2 dx - \int_{\mathbb{R}^3} \frac{\mu_1}{p_1}|u_1|^{p_1} + \frac{\mu_2}{p_2}|u_2|^{p_2} dx$$

and

$$\mathcal{M}_0 = \{(u_1, u_2) \neq \mathbf{0} | J_0(u_1, u_2) = |\nabla u_1|_2^2 + |\nabla u_2|_2^2 - \frac{3\mu_1(p_1 - 2)}{2p_1} \int_{\mathbb{R}^3} |u_1|^{p_1} dx - \frac{3\mu_2(p_2 - 2)}{2p_1} \int_{\mathbb{R}^3} |u_2|^{p_2} dx = 0\}.$$

Consequently,

$$\mathcal{M}_0 = \bigcup_{A\in\mathbb{R}} \{(u_1, u_2) \neq \mathbf{0} | \int_{\mathbb{R}^3} |\nabla u_1|^2 - \frac{3\mu_1(p_1 - 2)}{2p_1}|u_1|^{p_1} dx = A, \int_{\mathbb{R}^3} |\nabla u_2|^2 - \frac{3\mu_2(p_2 - 2)}{2p_2}|u_2|^{p_2} dx = -A\}$$

and $E_0(u_1, u_2) = (\frac{3\mu_1(p_1-2)}{4} - 1)\frac{\mu_1}{p_1}|u_1|_{p_1}^{p_1} + (\frac{3\mu_2(p_2-2)}{4} - 1)\frac{\mu_2}{p_2}|u_2|_{p_2}^{p_2}$ for $(u_1, u_2) \in \mathcal{M}_0$. Furthermore, let

$$\mathfrak{A} = \{(u_1, u_2) \neq \mathbf{0} | \int_{\mathbb{R}^3} |\nabla u_1|^2 - \frac{3\mu_1(p_1 - 2)}{2p_1}|u_1|^{p_1} dx = A, \int_{\mathbb{R}^3} |\nabla u_2|^2 - \frac{3\mu_2(p_2 - 2)}{2p_2}|u_2|^{p_2} dx = -A\},$$

and

$$\mathfrak{o} = \{(u_1, u_2) \neq \mathbf{0} | \int_{\mathbb{R}^3} |\nabla u_1|^2 + \frac{3\mu_1(p_1 - 2)}{2p_1}|u_1|^{p_1} dx = 0, \int_{\mathbb{R}^3} |\nabla u_2|^2 + \frac{3\mu_2(p_2 - 2)}{2p_2}|u_2|^{p_2} dx dx = 0\}.$$

We claim that

$$\inf_{(u_1,u_2)\in\mathfrak{o}\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}} E_0(u_1, u_2) = \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\}$$

and

$$\inf_{(u_1,u_2)\in\mathfrak{A}\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}} E_0(u_1, u_2) \geqslant \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\}.$$

Notice that

$$\inf_{(u_1,u_2)\in\mathfrak{o}\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}} E_0(u_1, u_2) = \inf_{\mathcal{D}_{a_1}\cap\{u_1\in H^1(\mathbb{R}^3)| \int_{\mathbb{R}^3} |\nabla u_1|^2 - \frac{3(p_1-2)\mu_1}{2p_1}|u_1|^{p_1} dx = 0\}} [\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_1|^2 dx - \frac{\mu_1}{p_1} \int_{\mathbb{R}^3} |u_1|^{p_1} dx]$$
$$+ \inf_{\mathcal{D}_{a_2}\cap\{u_2\in H^1(\mathbb{R}^3)| \int_{\mathbb{R}^3} |\nabla u_2|^2 - \frac{3(p_2-2)\mu_2}{2p_2}|u_2|^{p_2} dx = 0\}} [\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_2|^2 dx - \frac{\mu_2}{p_2} \int_{\mathbb{R}^3} |u_2|^{p_2} dx].$$

Since either $u_1 \neq 0$ or $u_2 \neq 0$, it follows that $\inf_{(u_1,u_2)\in\mathfrak{o}} E_0(u_1, u_2) = \min\{\ell(a_1, \mu_1, p_1), \ell(a_2, \mu_2, p_2)\}$ according to [12, Lemma 2.8]. For a given $A > 0$, we define

$$\mathfrak{B} = \{u_2 \in H^1(\mathbb{R}^3)| \int_{\mathbb{R}^3} |\nabla u_2|^2 - \frac{3\mu_2(p_2 - 2)}{2p_2}|u_2|^{p_2} dx = -A\}$$

and

$$\mathfrak{C} = \{u_2 \in H^1(\mathbb{R}^3)\backslash\{0\}| \int_{\mathbb{R}^3} |\nabla u_2|^2 - \frac{3\mu_2(p_2 - 2)}{2p_2}|u_2|^{p_2} dx = 0\}.$$

For any $u_2 \in H^1(\mathbb{R}^3)\backslash\{0\}$, we consider the functional

$$I(u_2^t) = t^2 \int_{\mathbb{R}^3} |\nabla u_2|^2 dx - t^{\frac{3(p_2-2)}{2}} \frac{3\mu_2(p_2-2)}{2p_2} \int_{\mathbb{R}^3} |u_2|^{p_2} dx,$$

which implies that $\mathfrak{B}$ is homomorphic to $\mathfrak{C}$, and there exists $t(u) > 1$ such that $u^{t(u)} \in \mathfrak{B}$ for any $u \in \mathfrak{C}$. Consequently, we have $\inf_{(u_1,u_2)\in\mathfrak{A}\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}} E_0(u_1,u_2) \geqslant \inf_{u_2\in\mathfrak{C}\cap\mathcal{D}_{a_2}} (\frac{3\mu_2(p_2-2)}{4} - 1)t(u)^{\frac{3(p_2-2)}{2}} \frac{\mu_2}{p_2} |u_1|_{p_1}^{p_1} \geqslant \ell(a_2,\mu_2,p_2)$. Similarly, for $A < 0$, we obtain $\inf_{(u_1,u_2)\in\mathcal{D}_{a_1}\times\mathcal{D}_{a_1}\cap\mathfrak{A}} E_0(u_1,u_2) \geqslant \ell(a_1,\mu_1,p_1)$. Therefore, $\gamma(0) = \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2)\}$.

According to Proposition 3.2, Theorem 1.1 (iii) and step 1, there exists $\tilde{\beta}$ such that

$$\gamma(\beta) = \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2)\}$$

for $\beta \in [0,\tilde{\beta}]$ and $\gamma(\beta) < \min\{\ell(a_1,\mu_1,p_1), \ell(a_2,\mu_2,p_2)\}$ when $\beta \in (\tilde{\beta},+\infty)$.

Step 2: $\gamma(\beta)$ is strictly decreasing on $(\tilde{\beta},+\infty)$.

For any $\beta_0 > \tilde{\beta}$, according to Lemma 3.1, there exists $(u_{1,0},u_{2,0}) \in \mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_{\beta_0}$ such that $\gamma(\beta_0) = E(u_{1,0},u_{2,0})$. Similar to Proposition 3.2, we can infer $\gamma(\beta_0) > \gamma(\beta)$ for any $\beta > \beta_0$. □

The proof of Theorem 1.2. From Lemma 2.7, Theorem 1.1 (ii) and Lemma 3.1, We have $\gamma(\beta)$ can be obtained by $(u_{1,\beta},u_{2,\beta})$ in the set $\mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_\beta$ in the interval $\beta \in (\tilde{\beta},+\infty)$. Furthermore, it follows from Lemma 2.5 that $\mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_\beta$ is a natural constraint. The combination of Lemma 3.1 and the maximum principles leads to the completion of the proof. □

The proof of Theorem 1.4. Since $E_{\beta_n}(u_{1,n},u_{2,n}) = \gamma(\beta_n)$ and $(u_{1,n},u_{2,n}) \in \mathcal{S}_{a_1} \times \mathcal{S}_{a_2} \cap \mathcal{M}_{\beta_n}$, it follows that $(u_{1,n},u_{2,n})$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R}^2)$. Notice that

$$E_\beta(u_{1,n},u_{2,n}) = E_{\beta_n}(u_{1,n},u_{2,n}) + (\beta_n - \beta) \int_{\mathbb{R}^3} |u_{1,n}|^{r_1} |u_{2,n}|^{r_2} dx.$$

We derive $E_\beta(u_{1,n},u_{2,n}) \to \gamma(\beta)$ by Theorem 1.1 (i). Furthermore,

$$E_\beta|'_{\mathcal{S}_{a_1}\times\mathcal{S}_{a_2}}(u_{1,n},u_{2,n}) = E_{\beta_n}|'_{\mathcal{S}_{a_1}\times\mathcal{S}_{a_2}}(u_{1,n},u_{2,n}) + (\beta-\beta_n)(r_1|u_{1,n}|^{r_1-2}u_{1,n}|u_{2,n}|^{r_2} + r_2|u_{1,n}|^{r_1}|u_{2,n}|^{r_2-2}u_{2,n}).$$

Consequently, $E_\beta|'_{\mathcal{S}_{a_1}\times\mathcal{S}_{a_2}}(u_{1,n},u_{2,n}) \to 0$ in $H^{-1}(\mathbb{R}^3)$. Notice that

$$J_\beta(u_{1,n},u_{2,n}) = J_{\beta_n}(u_{1,n},u_{2,n}) + \frac{3(\beta_n-\beta)(r_1+r_2-2)}{2} \int_{\mathbb{R}^3} |u_{1,n}|^{r_1}|u_{2,n}|^{r_2} dx,$$

which means $J_\beta(\mathbf{u}_n) \to 0$.

Hence, $(u_{1,n},u_{2,n})$ is a Palais-Smale consequence of E_β constrained $\mathcal{S}_{a_1} \times \mathcal{S}_{a_2}$ with $J_\beta(u_{1,n},u_{2,n}) \to 0$. According to [18, Lemma 8.1], there exists a normalized ground state solution (u_1,u_2) satisfying $(u_{1,n},u_{2,n}) \to (u_1,u_2)$, up to a subsequence. This completes the proof. □

The proof of Theorem 1.6. We have

$$\lambda_1(\beta) = \frac{\mu_1 \int_{\mathbb{R}^3} |u_{1,\beta}|^{p_1} dx + \beta r_1 \int_{\mathbb{R}^3} |u_{1,\beta}|^{r_1}|u_{2,\beta}|^{r_2} dx - \int_{\mathbb{R}^3} |\nabla u_{1,\beta}|^2 dx}{a_1^2}$$

and

$$\lambda_2(\beta) = \frac{\mu_2 \int_{\mathbb{R}^3} |u_{2,\beta}|^{p_2} dx + \beta r_2 \int_{\mathbb{R}^3} |u_{2,\beta}|^{r_2}|u_{1,\beta}|^{r_1} dx - \int_{\mathbb{R}^3} |\nabla u_{2,\beta}|^2 dx}{a_2^2}.$$

Hence,

$$\lambda_1(\beta)a_1^2+\lambda_2(\beta)a_2^2=\mu_1|u_{1,\beta}|_{p_1}^{p_1}+\mu_2|u_{2,\beta}|_{p_2}^{p_2}+\beta(r_1+r_2)\int_{\mathbb{R}^3}|u_{1,\beta}|^{r_1}|u_{2,\beta}|^{r_2}dx-|\nabla u_{1,\beta}|_2^2-|\nabla u_{2,\beta}|_2^2,$$

where $(u_{1,\beta},u_{2,\beta})\in\mathcal{M}_\beta$. we obtain

$$\begin{aligned}\lambda_1(\beta)a_1^2+\lambda_2(\beta)a_2^2=&(1-\frac{3(p_1-2)}{2p_1})\mu_1|u_{1,\beta}|_{p_1}^{p_1}+(1-\frac{3(p_2-2)}{2p_2})\mu_2|u_{2,\beta}|_{p_2}^{p_2}\\&+(r_1+r_2-\frac{3(r_1+r_2-2)}{2})\beta\int_{\mathbb{R}^3}|u_{1,\beta}|^{r_1}|u_{2,\beta}|^{r_2}dx.\end{aligned}$$

Given that $\frac{10}{3}<p_1,p_2,r_1+r_2<6$, it follows that $\lambda_1(\beta)a_1^2+\lambda_2(\beta)a_2^2>0$, indicating that $\lambda_1(\beta)>0$ or $\lambda_2(\beta)>0$. Assume by contradiction that $\lambda_1(\beta)>0$ and $\lambda_2(\beta)\leqslant 0$. We then have

$$-\triangle u_{2,\beta}=-\lambda_2u_{2,\beta}+\mu_2|u_{2,\beta}|^{p_2-2}u_{2,\beta}+\beta r_2|u_{1,\beta}|^{r_1}|u_{2,\beta}|^{r_2-2}u_{2,\beta}.$$

According to Lemma 3.1, both $u_{1,\beta}$ and $u_{2,\beta}$ are non-negative, leading to $-\triangle u_{2,\beta}\geqslant 0$, which would imply $u_{2,\beta}=0$. This is a contradiction. Consequently, $\lambda_1(\beta),\lambda_2(\beta)>0$.

We can directly deduce (i) from Theorem 1.4. Moreover, Theorem 1.1 (iii) and Lemma 2.2 imply that $\gamma(\beta)\to 0$ as $\beta\to+\infty$, which in turn indicates that $|\nabla u_{1,\beta}|_2^2\to 0$ and $|\nabla u_{2,\beta}|_2^2\to 0$ as $\beta\to+\infty$. Notice that

$$\beta\int_{\mathbb{R}^3}|u_{2,\beta}|^{r_2}|u_{1,\beta}|^{r_1}dx=\int_{\mathbb{R}^3}|\nabla u_{1,\beta}|^2+|\nabla u_{2,\beta}|^2dx-\frac{3\mu_1(p_1-2)}{2p_1}\int_{\mathbb{R}^3}|u_{1,\beta}|^{p_1}dx-\frac{3\mu_2(p_2-2)}{2p_2}\int_{\mathbb{R}^3}|u_{2,\beta}|^{p_2}dx,$$

we can establish (ii) by the Garliardo-Nirenberg inequality. □

Lemma 3.3. *Assume that $(u_{1,\beta},u_{2,\beta})\in K_\beta$ and $(u_{1,\beta_0},u_{2,\beta_0})\in K_{\beta_0}$, where $\beta,\beta_0>\tilde{\beta}$. Let $t_0\star(u_{1,\beta},u_{2,\beta})\in K_{\beta_0}$ and $t\star(u_{1,\beta_0},u_{2,\beta_0})\in K_\beta$. We have*

$$E_\beta(u_{1,\beta},u_{2,\beta})\leqslant(\beta_0-\beta)\int_{\mathbb{R}^3}|u_{1,\beta_0}^t|^{r_1}|u_{2,\beta_0}^t|^{r_2}dx+E_{\beta_0}(u_{1,\beta_0},u_{2,\beta_0}),$$

$$E_{\beta_0}(u_{1,\beta_0},u_{2,\beta_0})\leqslant(\beta-\beta_0)\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx+E_\beta(u_{1,\beta},u_{2,\beta}).$$

Proof. Notice that

$$\begin{aligned}E_\beta(u_{1,\beta},u_{2,\beta})&\leqslant E_\beta(t\star(u_{1,\beta_0},u_{2,\beta_0}))\\&=E_{\beta_0}(t\star(u_{1,\beta_0},u_{2,\beta_0}))+(\beta_0-\beta)\int_{\mathbb{R}^3}|u_{1,\beta_0}^t|^{r_1}|u_{2,\beta_0}^t|^{r_2}dx\\&\leqslant(\beta_0-\beta)\int_{\mathbb{R}^3}|u_{1,\beta_0}^t|^{r_1}|u_{2,\beta_0}^t|^{r_2}dx+E_{\beta_0}(u_{1,\beta_0},u_{2,\beta_0}),\end{aligned}$$

and

$$\begin{aligned}E_{\beta_0}(u_{1,\beta_0},u_{2,\beta_0})&\leqslant E_{\beta_0}(u_{1,\beta}^{t_0},u_{2,\beta}^{t_0})\\&=E_\beta(u_{1,\beta}^{t_0},u_{2,\beta}^{t_0})+(\beta-\beta_0)\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx\\&\leqslant(\beta-\beta_0)\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx+E_\beta(u_{1,\beta},u_{2,\beta}).\end{aligned}$$

□

The proof of theorem 1.5. According to Lemma 3.3, we deduce that

$$(\beta_0-\beta)\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx \leqslant \gamma(\beta)-\gamma(\beta_0)\leqslant(\beta_0-\beta)\int_{\mathbb{R}^3}|u_{1,\beta_0}^{t}|^{r_1}|u_{2,\beta_0}^{t}|^{r_2}dx,$$

where $t_0\star(u_{1,\beta},u_{2,\beta})$ and $t\star(u_{1,\beta_0},u_{2,\beta_0})$ are given in Lemma 3.3. By the definition of $\mathcal{M}_\beta$ and Proposition 2.4, it follows that $t\to 1$ as $\beta\to\beta_0$.

First, assuming $\beta>\beta_0$, we obtain

$$-\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx\leqslant\frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0}\leqslant-\int_{\mathbb{R}^3}|u_{1,\beta_0}^{t}|^{r_1}|u_{2,\beta_0}^{t}|^{r_2}dx.$$

On the one hand,

$$\begin{aligned}\limsup_{\beta\to\beta_0^+}\frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0}&\leqslant-\liminf_{\beta\to\beta_0^+}\int_{\mathbb{R}^3}|u_{1,\beta_0}^{t}|^{r_1}|u_{2,\beta_0}^{t}|^{r_2}dx\\&=-\liminf_{\beta\to\beta_0^+}\int_{\mathbb{R}^3}t^{\frac{3(r_1+r_2-2)}{2}}|u_{1,\beta_0}|^{r_1}|u_{2,\beta_0}|^{r_2}dx\\&=-\int_{\mathbb{R}^3}|u_{1,\beta_0}|^{r_1}|u_{2,\beta_0}|^{r_2}dx.\end{aligned}$$

From the arbitrariness of $(u_{1,\beta_0},u_{2,\beta_0})$, we conclude

$$\limsup_{\beta\to\beta_0^+}\frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0}\leqslant-\hat{d}_1(\beta_0).$$

On the other hand,

$$\liminf_{\beta\to\beta_0^+}\frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0}\geqslant\liminf_{\beta\to\beta_0^+}-\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx=-\limsup_{\beta\to\beta_0^+}\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx.$$

By Theorem 1.4, there exists $(u_{1,0},u_{2,0})\in K_{\beta_0}$ such that

$$\limsup_{\beta\to\beta_0^+}\int_{\mathbb{R}^3}|u_{1,\beta}^{t_0}|^{r_1}|u_{2,\beta}^{t_0}|^{r_2}dx=\int_{\mathbb{R}^3}|u_{1,0}|^{r_1}|u_{2,0}|^{r_2}dx.$$

Therefore,

$$\liminf_{\beta\to\beta_0^+}\frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0}\geqslant-\int_{\mathbb{R}^3}|u_{1,0}|^{r_1}|u_{2,0}|^{r_2}dx\geqslant-\hat{d}_1(\beta_0),$$

yielding $\lim_{\beta\to\beta_0^+}\frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0}=-\hat{d}_1(\beta_0)$.

Analogously, one can demonstrate that

$$\lim_{\beta\to\beta_0^-}\frac{\gamma(\beta)-\gamma(\beta_0)}{\beta-\beta_0}=-d_1(\beta_0).$$

Since $\gamma(\beta)$ is continuous and strictly decreasing, $\gamma(\beta)$ is differentiable almost everywhere in $\beta\in(\tilde{\beta},+\infty)$. Consequently, we have $d_1(\beta)=\hat{d}_1(\beta)\Leftrightarrow\gamma(\beta)$ is differentiable at β and $\gamma'(\beta)=-d_1(\beta)$. $\square$

4 case: k=3

In this section, we demonstrate the existence of a normalized ground state solution for Equation (1.3) in the case where $k=3$. We define $t\star(u_1,u_2,u_3)=(u_1^t,u_2^t,u_3^t)=(t^{\frac{3}{2}}u_1(tx),t^{\frac{3}{2}}u_2(tx),t^{\frac{3}{2}}u_3(tx))$ for $t>0$. Following a similar argument to that of Proposition 3.2, we establish the following proposition.

Proposition 4.1. *Suppose that* $\frac{10}{3} < p_1, p_2, r_1 + r_2, r_1 + r_3, r_2 + r_3 < 6$ *and* $r_1, r_2, r_3 > 1$. *Then,* $\gamma_{123}(\boldsymbol{\beta})$ *is monotonically non-increasing on the* $\beta_{ij}^{+} -$ *axis for* $1 \leqslant i < j \leqslant 3$ *where* $\beta_{ij}^{+} = \{\beta_{ij} | \beta_{ij} > 0\}$.

The proof of the Theorem 1.7 (i). We choose $\boldsymbol{\beta}_0 = \{\beta_{12,0}, \beta_{13,0}, \beta_{23,0}\} \in \Xi$. $\forall \varepsilon > 0$ and $\boldsymbol{\beta} = \{\beta_{12}, \beta_{13}, \beta_{23}\} \in B_\varepsilon(\boldsymbol{\beta}_0) \cap \Xi$, there exists $\mathbf{u} = \{u_1, u_2, u_3\} \in \mathcal{M}_{\boldsymbol{\beta}}$ such that $E_{\boldsymbol{\beta}}(\mathbf{u}) < \gamma_{123}(\boldsymbol{\beta}) + \varepsilon$, where $B_\varepsilon(\boldsymbol{\beta}_0) = \{\boldsymbol{\beta} \in \mathbb{R}^3 || \boldsymbol{\beta}_0 - \boldsymbol{\beta}| < \varepsilon|\}$.

Step 1: we demonstrate $\gamma_{123}(\boldsymbol{\beta}_0) < \gamma_{123}(\boldsymbol{\beta}) + \varepsilon$.

According to Proposition 2.4, there exists $t > 0$ such that $t \star \mathbf{u} \in \mathcal{M}_{\boldsymbol{\beta}_0}$ and $\gamma_{123}(\boldsymbol{\beta}_0) \leqslant E_{\boldsymbol{\beta}_0}(t \star \mathbf{u})$, with $t = t(\mathbf{u})$. Notice that

$$
\begin{aligned}
E_{\boldsymbol{\beta}_0}(t \star \mathbf{u}) = E_{\boldsymbol{\beta}}(t \star \mathbf{u}) &+ (\beta_{12} - \beta_{12,0}) t^{\frac{3(r_1+r_2-2)}{2}} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx \\
&+ (\beta_{13} - \beta_{13,0}) t^{\frac{3(r_1+r_3-2)}{2}} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_3|^{r_3} dx \\
&+ (\beta_{23} - \beta_{23,0}) t^{\frac{3(r_2+r_3-2)}{2}} \int_{\mathbb{R}^3} |u_2|^{r_2} |u_3|^{r_3} dx
\end{aligned} \tag{4.1}
$$

and $E_{\boldsymbol{\beta}}(t \star \mathbf{u}) < \gamma_{123}(\boldsymbol{\beta}) + \varepsilon$.

Subsequently, we verify that $t(\mathbf{u})$, $\int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx$, $\int_{\mathbb{R}^3} |u_1|^{r_1} |u_3|^{r_3} dx$ and $\int_{\mathbb{R}^3} |u_2|^{r_2} |u_3|^{r_3} dx$ are bounded on $\mathbb{R}$ for any $\boldsymbol{\beta} \in B_\varepsilon(\boldsymbol{\beta}_0) \cap \Xi$ and $\mathbf{u} \in \{\mathbf{u} \in \mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \times \mathcal{D}_{a_3} \cap \mathcal{M}_\beta | E_{\boldsymbol{\beta}}(\mathbf{u}) < \gamma_{123}(\boldsymbol{\beta}) + \varepsilon\}$. Notice that $\gamma_{123}(\boldsymbol{\beta}) \leqslant \ell(a_1, \mu_1, p_1)$ and

$$
\begin{aligned}
E_{\boldsymbol{\beta}}(\mathbf{u}) &= E_{\boldsymbol{\beta}}(\mathbf{u}) - \frac{1}{2} J_{\boldsymbol{\beta}}(\mathbf{u}) \\
&= (\frac{3(p_1-2)}{4} - 1) \frac{\mu_1}{p_1} \int_{\mathbb{R}^3} |u_1|^{p_1} dx + (\frac{3(p_2-2)}{4} - 1) \frac{\mu_2}{p_2} \int_{\mathbb{R}^3} |u_2|^{p_2} dx \\
&+ (\frac{3(p_3-2)}{4} - 1) \frac{\mu_3}{p_3} \int_{\mathbb{R}^3} |u_3|^{p_3} dx + (\frac{3(r_1+r_2-2)}{4} - 1) \beta_{12} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx \\
&+ (\frac{3(r_1+r_3-2)}{4} - 1) \beta_{13} \int_{\mathbb{R}^3} |u_1|^{r_1} |u_3|^{r_3} dx + (\frac{3(r_2+r_3-2)}{4} - 1) \beta_{23} \int_{\mathbb{R}^3} |u_2|^{r_2} |u_3|^{r_3} dx.
\end{aligned} \tag{4.2}
$$

Thus, $\int_{\mathbb{R}^3} |u_1|^{r_1} |u_2|^{r_2} dx$, $\int_{\mathbb{R}^3} |u_1|^{r_1} |u_3|^{r_3} dx$ and $\int_{\mathbb{R}^3} |u_2|^{r_2} |u_3|^{r_3} dx$ are bounded due to $\frac{3(p_j-2)}{4} - 1, \frac{3(r_i+r_j-2)}{4} - 1 > 0$, where $1 \leqslant i < j \leqslant 3$. On the other hand, similar to the argument with Theorem 1.1 (i), we can prove $t(u)$ is bounded.

Finally, we conclude that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\gamma_{123}(\boldsymbol{\beta}_0) < \gamma_{123}(\boldsymbol{\beta}) + \varepsilon$ for any $\boldsymbol{\beta} \in B_\delta(\boldsymbol{\beta}_0) \cap \Xi$.

Step 2: we prove $\gamma_{123}(\boldsymbol{\beta}) < \gamma_{123}(\boldsymbol{\beta}_0) + \varepsilon$.

$\forall \varepsilon > 0$, there exist $\mathbf{u}_0 = \{u_{1,0}, u_{2,0}, u_{3,0}\}$ and $t > 0$ such that $E_{\boldsymbol{\beta}_0}(\mathbf{u}_0) < \gamma_{123}(\boldsymbol{\beta}_0) + \varepsilon$ and $\mathbf{u}_0^t \in \mathcal{M}_{\boldsymbol{\beta}}$. Notice that

$$
\begin{aligned}
\gamma_{123}(\boldsymbol{\beta}) \leqslant E_{\boldsymbol{\beta}}(\mathbf{u}_0^t) = E_{\boldsymbol{\beta}_0}(\mathbf{u}_0^t) &+ (\beta_{12,0} - \beta_{12}) t^{\frac{3(r_1+r_2-2)}{2}} \int_{\mathbb{R}^3} |u_{1,0}|^{r_1} |u_{2,0}|^{r_2} dx \\
&+ (\beta_{13,0} - \beta_{13}) t^{\frac{3(r_1+r_3-2)}{2}} \int_{\mathbb{R}^3} |u_{1,0}|^{r_1} |u_{3,0}|^{r_3} dx \\
&+ (\beta_{23,0} - \beta_{23}) t^{\frac{3(r_2+r_3-2)}{2}} \int_{\mathbb{R}^3} |u_{2,0}|^{r_2} |u_{3,0}|^{r_3} dx.
\end{aligned} \tag{4.3}
$$

Similar to the argument with Theorem 1.1 (i), we can prove $t = t(\boldsymbol{\beta})$ is bounded. Thus we complete the proof. $\square$

The proof of Theorem 1.7 (ii). The proof resembles that of Theorem 1.1 (iii), and thus, we omit the details here. $\square$

The proof of Theorem 1.7 (iii). Firstly, we demonstrate that $\gamma_{123}(\beta_{12},0,0)=\gamma_{12}(\beta_{12})$. We have

$$E_{\beta_{12},0,0}=\sum_{j=1}^{3}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_j|^2dx-\sum_{j=1}^{3}\frac{\mu_j}{p_j}\int_{\mathbb{R}^3}|u_j|^{p_j}dx-\beta_{12}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx$$

and

$$\begin{aligned}\mathcal{M}_{\beta_{12},0,0}=\{(u_1,u_2,u_3)\in H^1(\mathbb{R}^3,\mathbb{R}^3)\backslash\{\mathbf{0}\}|\sum_{j=1}^{3}\int_{\mathbb{R}^3}|\nabla u_j|^2dx-\sum_{j=1}^{3}\frac{3\mu_j(p_j-2)}{2p_j}\int_{\mathbb{R}^3}|u_j|^{p_j}dx\\-\frac{3\beta_{12}(r_1+r_2-2)}{2}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx=0\}.\end{aligned}$$

We define

$$\begin{aligned}\mathcal{N}_A=&\{(u_1,u_2,u_3)\in H^1(\mathbb{R}^3,\mathbb{R}^3)\backslash\{\mathbf{0}\}|\int_{\mathbb{R}^3}|\nabla u_1|^2+|\nabla u_2|^2dx-\int_{\mathbb{R}^3}\frac{3\mu_1(p_1-2)}{2p_1}|u_1|^{p_1}-\frac{3\mu_2(p_2-2)}{2p_2}|u_2|^{p_2}dx\\&-\frac{3\beta_{12}(r_1+r_2-2)}{2}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx=A,\int_{\mathbb{R}^3}|\nabla u_3|^2-\frac{3\mu_3(p_3-2)}{2p_3}|u_3|^{p_3}dx=-A\}.\end{aligned}$$

Therefore, we have

$$\mathcal{M}_{(\beta_{12},0,0)}=\bigcup_{A\in\mathbb{R}}\mathcal{N}_A,$$

and

$$\begin{aligned}E_{(\beta_{12},0,0)}(u_1,u_2,u_3)=&(\frac{3(p_1-2)}{4}-1)\frac{\mu_1}{p_1}\int_{\mathbb{R}^3}|u_1|^{p_1}dx+(\frac{3(p_2-2)}{4}-1)\frac{\mu_2}{p_2}\int_{\mathbb{R}^3}|u_2|^{p_2}dx\\&+(\frac{3(p_3-2)}{4}-1)\frac{\mu_3}{p_3}\int_{\mathbb{R}^3}|u_3|^{p_3}dx+(\frac{3(r_1+r_2-2)}{4}-1)\beta_{12}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx\end{aligned}$$

for $(u_1,u_2,u_3)\in\mathcal{M}_{(\beta_{12},0,0)}$. We claim that

$$\inf_{(u_1,u_2,u_3)\in\mathcal{N}_0\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}}E_{(\beta_{12},0,0)}(u_1,u_2,u_3)=\gamma_{12}(\beta_{12})$$

and

$$\inf_{(u_1,u_2,u_3)\in\mathcal{N}_A\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}}E_{(\beta_{12},0,0)}(u_1,u_2)\geqslant\gamma_{12}(\beta_{12}).$$

We define

$$\begin{aligned}\mathcal{Q}_0=\{(u_1,u_2)\in H^1(\mathbb{R}^3,\mathbb{R}^2)\setminus\{\mathbf{0}\}|\int_{\mathbb{R}^3}|\nabla u_1|^2+|\nabla u_2|^2dx-\int_{\mathbb{R}^3}\frac{3\mu_1(p_1-2)}{2p_1}|u_1|^{r_1}-\frac{3\mu_2(p_2-2)}{2p_2}|u_2|^{r_2}dx\\-\frac{3\beta_{12}(r_1+r_2-2)}{2}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx=0\}\end{aligned}$$

and

$$\mathcal{P}_0=\{u_3\in H^1(\mathbb{R}^3)\setminus\{0\}|\int_{\mathbb{R}^3}|\nabla u_3|^2-\frac{3\mu_2(p_3-2)}{2p_3}|u_3|^{p_3}dx=0\}$$

Notice that

$$
\begin{aligned}
\inf_{(u_1,u_2,u_3)\in\mathcal{N}_0\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}} E_{(\beta_{12},0,0)}(u_1,u_2,u_3) = [&\inf_{\mathcal{Q}_0\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}}(\frac{3(p_1-2)}{4}-1)\frac{\mu_1}{p_1}\int_{\mathbb{R}^3}|u_1|^{p_1}dx\\
&+(\frac{3(p_2-2)}{4}-1)\frac{\mu_1}{p_1}\int_{\mathbb{R}^3}|u_2|^{p_2}dx\\
&+(\frac{3(r_1+r_2-2)}{4}-1)\beta_{12}\int_{\mathbb{R}^3}|u_1|^{r_1}|u_2|^{r_2}dx]\\
&+\inf_{\mathcal{P}_0\cap\mathcal{D}_{a_3}}(\frac{3(p_3-2)}{4}-1)\frac{\mu_3}{p_3}\int_{\mathbb{R}^3}|u_3|^{p_3}dx.
\end{aligned}
$$

By considering the cases when either $(u_1,u_2)\neq\mathbf{0}$ or $u_3\neq 0$, we obtain

$$
\inf_{(u_1,u_2,u_3)\in\mathcal{N}_0\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}} E_{(\beta_{12},0,0)}(u_1,u_2,u_3)=\gamma_{12}(\beta_{12})
$$

since $\ell(a_3,\mu_3,p_3)\geqslant\ell(a_1,\mu_1,p_1)$.

For any $A\in\mathbb{R}$, we assume $A>0$. We can make same argument with Theorem 1.1 (ii) to obtain

$$
\inf_{(u_1,u_2,u_3)\in\mathcal{N}_A\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}} E_{\beta_{12},0,0}(u_1,u_2,u_3)\geqslant\inf_{u\in\mathcal{P}_0\cap\mathcal{D}_{a_3}} t(u)^{\frac{3(p_3-2)}{2}}\frac{3\mu_3(p_3-2)}{2p_3}\int_{\mathbb{R}^3}|u_3|^{p_3}dx\geqslant\ell(a_3,\mu_3,p_3),
$$

where $t(u)\geqslant 1$. For any $A<0$, we have

$$
\inf_{(u_1,u_2,u_3)\in\mathcal{N}_A\cap\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}} E_{\beta_{12},0,0}(u_1,u_2,u_3)\geqslant\gamma_{12}(\beta_{12}).
$$

Similarly, The other two formulas also can be showed. We complete the proof. □

The proof of Theorem 1.8 (i). We assume, for simplicity, that $1<r_1<2$. For a fixed $\beta_{12}>0$, it follows from Theorem 1.7 (i), (ii), (iii) and Proposition 4.1 that there exists unbounded, simply connected open subset $\Gamma_1(\beta_{12})=\{(\beta_{13},\beta_{23})|\gamma_{123}(\beta_{12},\beta_{13},\beta_{23})<\gamma_{12}(\beta_{12})\}$ with respect to $\mathbb{R}^+\times\mathbb{R}^+$. Analogously, for a fixed $\beta_{13}>0$, there exists unbounded simply connected open subset $\Gamma_2(\beta_{13})=\{(\beta_{12},\beta_{23})|\gamma_{123}(\beta_{12},\beta_{13},\beta_{23})<\gamma_{13}(\beta_{13})\}$ with respect to $\mathbb{R}^+\times\mathbb{R}^+$. By imitating the proof of [18, Theorem 7.2], we obtain $\gamma_{123}(\beta_{12},\beta_{13},\beta_{23})<\gamma_{23}(\beta_{23})$ provided $\beta_{23}>\bar{\beta}_{23}$ and $(\beta_{12},\beta_{13})\neq(0,0)$. Consequently, for any $(\beta_{12},\beta_{13})\neq(0,0)$, there exists some $\hat{\beta}_{23}$ such that $(\beta_{12},\beta_{13},\beta_{23})\in\Omega$ for $\beta_{23}>\hat{\beta}_{23}$. Furthermore, according to Theorem 1.7 (i), (ii), Ω is demonstrated to be an unbounded, simply connected open subset of Ξ. We complete the proof. □

Lemma 4.2. *Assume that $\frac{10}{3}<p_1,p_2,p_3,r_1+r_2,r_1+r_3,r_2+r_3<6$, $r_1,r_2,r_3>1$ and $\ell(a_1,\mu_1,p_1)=\min\{\ell(a_1,\mu_1,p_1),\ell(a_2,\mu_2,p_2),\ell(a_3,\mu_3,p_3)\}$. If $\gamma_{123}(\boldsymbol{\beta})\in\Omega$, then for any $\boldsymbol{u}\in(\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}\backslash\mathcal{S}_{a_1}\times\mathcal{S}_{a_2}\times\mathcal{S}_{a_3})\cap\mathcal{M}_{\boldsymbol{\beta}}$, there holds*

$$
\inf_{\mathcal{S}_{a_1}\times\mathcal{S}_{a_2}\times\mathcal{S}_{a_3}\cap\mathcal{M}_{\boldsymbol{\beta}}} E_{\boldsymbol{\beta}}<E_{\boldsymbol{\beta}}(\boldsymbol{u}).
$$

Proof. Suppose by contradiction that there exists $\mathbf{u}=(u_1,u_2,u_3)\in\mathcal{D}_{a_1}\times\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}\cap\mathcal{M}_{\boldsymbol{\beta}}$ such that $\int_{\mathbb{R}^3}u_1^2dx<a_1$ and $\gamma_{123}(\boldsymbol{\beta})=E_{\boldsymbol{\beta}}(\mathbf{u})\leqslant\inf_{\mathcal{S}_{a_1}\times\mathcal{S}_{a_2}\times\mathcal{S}_{a_3}\cap\mathcal{M}_{\boldsymbol{\beta}}}E_{\boldsymbol{\beta}}$. We assert that $u_1\neq 0$. If not, we could assume $u_1=0$. Then $\gamma_{123}(\boldsymbol{\beta})=\inf_{\mathcal{D}_{a_2}\times\mathcal{D}_{a_3}\cap\mathcal{M}_{\beta_{23}}}E_{\beta_{23}}(u_2,u_3)\geqslant\gamma_{23}(\beta_{23})$, which is a contradiction.

Aualogous to the argument in Lemma 3.1, we can conclude that $u_1\in\mathcal{S}_{a_1}$. Similarly, $(u_2,u_3)\in\mathcal{S}_{a_2}\times\mathcal{S}_{a_3}$. This completes the proof. □

The proof of Theorem 1.8 (ii). Combining Theorem 1.7 with Lemma 4.2 and be similar to Theorem 1.2, we can derive the result. □

The proof of Theorem 1.10. The proof is similar to the one of Theorem 1.4, so we omit details. □

The proof of Theorem 1.11. The proof is quite similar to the one of Theorem 1.5 and we ignore details. □

The proof of Theorem 1.12 . The proof is similar to the one of Theorem 1.6, so we omit details. □

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