PROJECTIVITY CRITERIA FOR KÄHLER MORPHISMS

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Abstract. In this short note we prove two projectivity criteria for fibrations between mildly singular compact Kähler spaces. They are the relative versions of the celebrated criteria of Kodaira and Moishezon. As an application we obtain that the MRC fibration always has a model that is a projective morphism.

1. Introduction

Given a compact Kähler manifold X, one of the most basic questions is to decide whether X is (the analytification of) a projective manifold. There are two well-known cases where this always holds:

- Kodaira's criterion: X is projective if $H^0(X, \Omega_X^2) = 0$.
- Moishezon's theorem: X is projective if there exists a line bundle $L \to X$ that is big.

In view of the recent progress on the MMP for projective morphisms [Fuj22, DHP24], it is interesting to find sufficient conditions for a Kähler morphism to be projective. In this short note we generalise the classical criteria to the relative situation. Our first result is:

1.1. Theorem. *Let $f\colon X \to Y$ be a fibration between compact Kähler manifolds. Assume one of the following:*

- *The natural map $f^* : H^0(Y, \Omega_Y^2) \longrightarrow H^0(X, \Omega_X^2)$ is an isomorphism.*
- *The morphism f is Moishezon, i.e. there exists a line bundle $L \to X$ that is f-big.*

Then f is a projective morphism.

The first assumption gives a global version of [Bin83, Cor.1.2] [Nak02, Prop.3.3.1] and is verified by every fibration that is cohomologically constant (cf. [DS21] for the terminology). In particular it holds for any smooth model of the MRC fibration of a compact Kähler manifold, see below. The second statement improves [CP00, Thm.10.1] where it was shown that f is locally projective. In the case where Y is a point Boucksom's version of Moishezon's theorem [Bou02, Thm.1.2.13] gives a more precise result: if X is Kähler and Moishezon, then the projection of a Kähler class on the real Neron-Severi space remains Kähler. It is not clear to us if this still holds in the relative setting.

Date: April 23, 2024.

2020 *Mathematics Subject Classification.* 32J25, 32Q15, 14E30.

Key words and phrases. Kähler morphisms, projectivity criteria, MMP for projective morphisms.

The proofs of the two statements are quite different. For the first case we use the Hodge decomposition to construct a relatively ample line bundle, whereas in the second case we use a relative MMP to show that the projectiveness of the fibration is preserved under suitably chosen bimeromorphic modifications. The latter technique also works for mildly singular spaces, allowing us to prove a variant of the main statement that is suitable for applications in the MMP, cf. Theorem 3.1. In particular we obtain:

1.2. Theorem. *Let X be a normal compact Kähler space with klt singularities. Then there exists a model of the MRC fibration that is a projective morphism.*

This answer a question asked to us by Juanyong Wang. In fact our proof shows that every holomorphic model $X' \to Y'$ with X' and Y' smooth is a projective morphism, cf. also Remark 4.4.

In Section 4 we give some applications of our main results, for example to bimeromorphic morphism between strongly $\mathbb{Q}$-factorial klt spaces.

Acknowledgements. We are very grateful to C. Voisin for a discussion that lead us to the proof of the first case of Theorem 1.1. We thank F. Campana, O. Fujino, J. Kollár and T. Peternell for some very helpful communications.

Both authors would like to thank Institut Universitaire de France for providing excellent working conditions. BC benefits from the support of the French government "Investissements d'Avenir" program integrated to France 2030, bearing the following reference ANR-11-LABX-0020-01.

2. Basic facts and notation

All complex spaces are supposed to be separated and of finite dimension, a complex manifold is a smooth irreducible complex space. An analytic variety is a complex space that is irreducible and reduced. A fibration is a proper surjective morphism with connected fibres between complex spaces. We refer to [Gra62, Fuj79, Dem85] for basic definitions about (p,q)-forms and Kähler forms in the singular case.

We use the standard terminology of the MMP as explained in [KM98, Deb01], cf. also [Nak87] for foundational material in the case of projective morphisms.

2.A. **Relatively ample line bundles.** Let X be a normal compact complex space with at most rational singularities. Suppose that X is in the Fujiki class, i.e. X is bimeromorphic to a compact Kähler manifold. A $(1,1)$-class on X is an element of $N^1(X) := H^{1,1}_{\mathrm{BC}}(X)$, the Bott-Chern group of $(1,1)$-currents that are locally $\partial\bar{\partial}$-exact (cf. [HP16, Sect.2] for details). Even if X is projective, the inclusion

$$\mathrm{NS}(X) \otimes \mathbb{R} \subset N^1(X)$$

is typically not an equality. However if $H^2(X, \mathcal{O}_X) = 0$, we can apply Kodaira's criterion to a desingularisation to see that X is projective and every $(1,1)$-class is an $\mathbb{R}$-divisor class.

2.1. Definition. *Let $f\colon X \to Y$ be a fibration between analytic varieties. The fibration f is projective (resp. Moishezon) if there exists a line bundle $L \to X$ that is relatively ample (resp. relatively big).*

The fibration f is locally projective (resp. locally Moishezon), if there exists a covering of Y by open sets U_i such that $f^{-1}(U_i) \to U_i$ is projective (resp. Moishezon).

2.2. Remark. In the literature, e.g. [GPR94, VII, §6] a compact analytic variety is said to be Moishezon if its dimension is equal to its algebraic dimension, i.e. the transcendence degree of its field of meromorphic functions. If X is smooth this is equivalent to the existence of a big line bundle [GPR94, VII, §6.3].

Our definition of projective, resp. Moishezon morphism is more general than in parts of the literature, e.g. [Kol22].

2.3. Remark. The composition of two projective morphisms is in general not projective, but it is straightforward to show that this holds if the base of the fibrations are compact.

The relative ampleness of a line bundle can be read off positivity properties of its Chern class:

2.4. Lemma. *Let $f\colon X \to Y$ be a fibration between compact Kähler manifolds. A line bundle $L \in \mathrm{Pic}(X)$ is f-ample if and only if there exists $[\omega_Y] \in H^2(Y, \mathbb{R})$ a Kähler class such that $c_1(L) + f^*(\omega_Y)$ is a Kähler class on X.*

We have not been able to locate this statement in the literature, cp. [FS90, Sect.1] for similar considerations. We thus describe a sketch of proof for the reader's convenience.

Sketch of proof. If $L \in \mathrm{Pic}(X)$ is f-ample, we can just apply [Fuj79, Lemma 4.4].

Conversely, assume that for some smooth metric h on L and some Kähler form ω_Y on Y, the $c_1(L, h) + f^*(\omega_Y)$ form is Kähler. We can then consider $U \subset Y$ a Stein open subset where $(\omega_Y)|_U = i\partial\overline{\partial}\varphi_U$ for some smooth strictly psh function $\varphi_U\colon U \to \mathbb{R}$. Over $X_U := f^{-1}(U)$, the line bundle L can be endowed with the metric $he^{-\varphi_U\circ f}$ that has positive curvature. Being proper over a Stein manifold, X_U is holomorphically convex hence weakly pseudoconvex. We can then resort to the usual L^2 estimates over X_U for the Hermitian line bundle $(L, he^{-\varphi_U\circ f})$ [Dem82] and produce sections of $L^{\otimes m}$ defined over X_U which separate points and tangent vectors (for some integer m depending on U). The manifold Y being compact, we cover it with finitely many open subsets as above and we then choose an m uniformly such that $L^{\otimes m}$ is f-very ample. □

We recall the following application of Chow's lemma:

2.5. Lemma. *Let $f\colon X \to Y$ be a Moishezon fibration between compact analytic varieties. Then there exists a projective log-resolution $\mu_0\colon X_0 \to X$ such that the composition $f \circ \mu_0$ is a projective morphism.*

Proof. It is shown in [DH20, Lem.2.18] that we can find a log-resolution μ_0 such that $f \circ \mu_0$ is a projective morphism. The property that μ_0 is also projective is not claimed in [DH20], but can be achieved by applying the analytic version of Chow's lemma [Hir75, Cor.2] to the bimeromorphic map μ_0. □

2.B. **Preliminaries on MMP.** A normal projective variety X is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier, or more formally the natural map $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ induces an isomorphism

$$\mathrm{Pic}(X) \otimes \mathbb{Q} \longrightarrow \mathrm{Cl}(X) \otimes \mathbb{Q}.$$

Following [DH20] we extend the definition to the Kähler setting:

2.6. Definition. *Let X be a normal complex space. We denote by $W(X)$ the group of divisorial sheaves, i.e. the group of isomorphism classes of reflexive sheaves of rank one, endowed with the group operation $\mathcal{F} \circ \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{**}$.*

2.7. Remark. We have a natural inclusion

$$\begin{cases} \mathrm{Cl}(X) & \hookrightarrow & W(X) \\ \quad D & \longmapsto & \mathcal{O}_X(D). \end{cases}$$

If X is not the projective this inclusion is not necessarily surjective: let S be a K3 surface that is very general in its deformation space, so $\mathrm{Pic}(S) = 0$. Let $X = \mathbb{P}(\Omega_S)$, then we have $\mathrm{Pic}(X) \simeq \mathbb{Z}K_X \neq 0$. However X does not contain any divisor, cf. [AH21, Cor.3.5], so $\mathrm{Cl}(X) = 0$.

2.8. Definition. [DH20, Defn.2.2(ix)] *A normal complex space X is strongly $\mathbb{Q}$-factorial if every divisorial sheaf is $\mathbb{Q}$-Cartier, i.e. for every $\mathcal{F} \in W(X)$ there exists $m \in \mathbb{N}$ such that $\mathcal{F}^{[m]}$ is locally free.*

2.9. Remark. Since a line bundle defines a locally free sheaf, we have an inclusion $\mathrm{Pic}(X) \hookrightarrow W(X)$. The complex space is strongly $\mathbb{Q}$-factorial if the inclusion induces an isomorphism

$$\mathrm{Pic}(X) \otimes \mathbb{Q} \longrightarrow W(X) \otimes \mathbb{Q}.$$

In this case we have a well-defined morphism

$$\begin{cases} W(X) & \longrightarrow & N^1(X) \\ \quad \mathcal{F} & \longmapsto & c_1(\mathcal{F}) := \frac{1}{m} c_1(\mathcal{F}^{[m]}) \end{cases}$$

where we choose $m \in \mathbb{N}^*$ such that $\mathcal{F}^{[m]}$ is locally free.

Let us restate the results on MMP for projective morphisms in the form that we will use in the sequel:

2.10. Theorem. [Nak87, DHP24, Fuj22] *Let X be a normal compact strongly $\mathbb{Q}$-factorial Kähler space. Assume that there exists a boundary divisor Δ on X such that the pair (X, Δ) is klt. Let $f\colon X \to Y$ be a projective morphism onto a normal compact Kähler space Y.*

(1) *If $K_X + \Delta$ is not f-nef, there exists a countable collection of rational curves $l_i \subset X$ such that $f(l_i)$ is a point and*

$$\overline{\mathrm{NE}}(X/Y) = \overline{\mathrm{NE}}(X/Y)_{K_X+\Delta \geq 0} + \sum_{i \in I} \mathbb{R}^+[l_i].$$

One has

$$0 < -(K_X + \Delta) \cdot l_i \leq 2 \dim X$$

and for every extremal ray $\mathbb{R}^+[l_i] \subset \overline{\mathrm{NE}}(X/Y)$ there exists a line bundle $L \to X$ such that $\mathbb{R}^+[l_i] = \overline{\mathrm{NE}}(X/Y) \cap c_1(L)^\perp$.

(2) *For every extremal ray* $\mathbb{R}^+[l_i] \subset \overline{\mathrm{NE}}(X/Y)$ *as above there exists a projective morphism* $g\colon Z \to Y$ *and a contraction morphism*

$$\varphi\colon X \to Z$$

onto a normal compact Kähler space Z *such that* $-(K_X + \Delta)$ *is* φ*-ample and for any curve* $C \subset X$ *such that* $f(C)$ *is a point we have*

$$\varphi(C) = pt. \ \Leftrightarrow \ [C] \in \mathbb{R}^+[l_i].$$

(3) *We can run a* $K_X + \Delta$*-MMP over* Y*, i.e. there exists a sequence of bimeromorphic maps* $\varphi_j\colon X_j \dashrightarrow X_{j+1}$ *over* Y *such that* φ_j *is either the divisorial contraction of a* $K_{X_j} + \Delta_j$*-negative extremal ray or its flip (if the contraction is small). Moreover the pair* $(X_{j+1}, (\varphi_j)_*(\Delta_j))$ *is klt, the normal space* X_{j+1} *is strongly* $\mathbb{Q}$*-factorial and the natural morphism* $f_{j+1}\colon X_{j+1} \to Y$ *is projective.*

(4) *If* $K_X + \Delta$ *is* f*-pseudoeffective and* Δ *or* $K_X + \Delta$ *are* f*-big, any MMP with scaling by an* f*-ample line bundle terminates with a relative minimal model, i.e. a projective morphism* $f_m\colon X_m \to Y$ *such that* $K_{X_m} + \Delta_m$ *is* f_m*-nef.*

(5) *If* $K_X + \Delta$ *is not* f*-pseudoeffective, any MMP with scaling by an* f*-ample line bundle terminates with a relative Mori fibre space, i.e. a Mori fibre space* $\varphi_m\colon X_m \to Z$ *onto a normal compact Kähler space* Z *of dimension at most* $\dim X - 1$ *and a projective morphism* $g\colon Z \to Y$ *such that* $f_m = g \circ \varphi_m$*.*

Proof. The first statement is [Nak87, Thm.4.12(1)] (applied in the case $Y = W$), the statement on the length of the extremal ray is [DHP24, Thm.2.44(3)b)], [Fuj22, Thm.9.1].

The existence of the contraction in the second statement is [Nak87, Thm.4.12(2)] (applied in the case $Y = W$), note that any open neighbourhood $W \subset U \subset Y$ is equal to Y. While [Nak87] does not state that the morphism g is projective, this is clear from the first item: the supporting nef line bundle descends to Z [Nak87, Thm.4.12(3,b)] and by construction is strictly positive on $\overline{\mathrm{NE}}(Z/Y)$. Thus it is relatively ample by [Nak87, Prop.4.7]. It is clear that Z is Kähler since Y is Kähler by assumption and the morphism g is Kähler (even projective).

The existence of MMP in the third statement is [Fuj22, Thm.1.7], [DHP24, Thm.1.4(1)]. Strong $\mathbb{Q}$-factoriality is shown in [DH20, Lemma 2.5], the klt property is shown as in the projective case [KM98, Cor.3.42, Cor.3.43]

The two last statements are stated in [Fuj22, Thm.1.7], [DHP24, Thm.1.4] for a MMP with a scaling by a relatively ample *divisor*, but the arguments work if we scale by the first Chern class of a relatively ample line bundle. Alternatively note that given an f-ample line bundle L, we can take a finite open cover Y_k of the base such that each Y_k satisfies the condition (P) in [DHP24, Fuj22] and $c_1(L)|_{f^{-1}(Y_k)}$ is represented by an $\mathbb{R}$-divisor. □

2.11. Lemma. *Let* X *be a normal compact Kähler space, and let* Δ *be a boundary divisor such that* (X, Δ) *is klt. Then there exists a small projective modification* $\nu\colon X' \to X$ *such that* X' *is strongly* $\mathbb{Q}$*-factorial.*

Proof. This is shown for threefolds in [DH20, Lem.2.27], by running a relative MMP for some projective log-resolution. As shown by Theorem 2.10, the results of [Fuj22, DHP24] allow to run this MMP in any dimension. Thus the proof works without changes. □

We prove a variant of [DH20, Lem.2.32], following their proof:

2.12. Lemma. *Let $\mu\colon X' \to X$ be a bimeromorphic morphism between normal compact Kähler spaces. Assume that X has strongly $\mathbb{Q}$-factorial klt singularities. Let $\tau\colon X_0 \to X'$ be a log-resolution of X' and the morphism μ such that $\mu_0 := \mu \circ \tau$ is a projective morphism (cf. Lemma 2.5) with SNC exceptional locus.*

Then there exists a boundary divisor Δ_0 such that (X_0, Δ_0) is klt and a decomposition of the morphism $\mu_0\colon X_0 \to X$ into a finite sequence of $K_{X_\bullet} + \Delta_\bullet$-negative bimeromorphic Mori contractions and flips

$$\varphi_i\colon X_i \dashrightarrow X_{i+1} \qquad i = 0, \ldots, m-1$$

between normal compact Kähler spaces over X such that $X_m \simeq X$.

Proof. We denote by $E_1, \ldots, E_k \subset X_0$ the μ_0-exceptional divisors. Since X has klt singularities we have

$$K_{X_0} \sim_{\mathbb{Q}} \mu_0^* K_X + \sum_{i=1}^{k} a_i E_i$$

with $a_i > -1$ for all $i = 1, \ldots, k$. Thus we can choose $0 < \epsilon < 1$ such that $a_i - \epsilon > -1$ for all $i = 1, \ldots, k$. Since X_0 is smooth and the divisor $\sum_{i=1}^k E_i$ has SNC support, the pair $(X_0, (1-\epsilon)\sum_{i=1}^k E_i)$ is klt. Since the morphism μ_0 is projective and the boundary is big on the general fibre (an empty condition for bimeromorphic maps), we can run by Theorem 2.10 a terminating directed MMP

$$\varphi\colon X_0 \dashrightarrow X_m$$

over X such that $K_{X_m} + (1-\epsilon)\sum_{i=1}^k \varphi_* E_i$ is relatively nef for the bimeromorphic morphism $\mu_m\colon X_m \to X$.

Since

$$K_{X_m} + (1-\epsilon)\sum_{i=1}^{k} \varphi_* E_i \sim_{\mathbb{Q}} \mu_m^* K_X + \sum_{i=1}^{k} (1 - \epsilon + a_i)\varphi_* E_i,$$

the effective divisor $\sum_{i=1}^k (1-\epsilon+a_i)\varphi_* E_i$ is μ_m-nef and μ_m-exceptional. Thus the $\mathbb{Q}$-divisor $-\sum_{i=1}^k (1-\epsilon+a_i)\eta_* E_i$ is effective by the negativity lemma [Wan21, Lem.1.3] and therefore the MMP φ contracts all the divisors E_i. Since the MMP does not extract any divisors, the exceptional locus of μ_m has codimension at least two. By assumption X is strongly $\mathbb{Q}$-factorial, so μ_m is an isomorphism [DH20, Lem.2.4]. □

3. Proof of the main theorem

We will prove the following singular variant, which obviously implies Theorem 1.1.

3.1. Theorem. *Let $f\colon X \to Y$ be a fibration between normal compact Kähler spaces. Assume that X has strongly $\mathbb{Q}$-factorial klt singularities. Assume one of the following:*

- *The normal space Y has klt singularities and the natural map* [KS21, Thm.1.9] $f^* : H^0(Y, \Omega_Y^{[2]}) \longrightarrow H^0(X, \Omega_X^{[2]})$ *is an isomorphism.*
- *The morphism f is Moishezon.*

Then f is a projective morphism.

Namikawa showed in [Nam02] that a normal compact Kähler spaces with 1-rational singularities that is Moishezon is even projective. Since klt singularities are rational this is more general than our statement (in the case where Y is a point). The technique is quite different: while Namikawa's proof aims at constructing an ample line bundle on X, we obtain the polarisation as a consequence of running a suitable MMP.

Proof of Theorem 3.1. We divide the proof into three steps.

Step 1. We prove the theorem under the first assumption, assuming moreover that X and Y are smooth. Let us denote $d := \dim X - \dim Y$, and let ω be a Kähler class on X.

Let $\eta \in H^2(X, \mathbb{Q})$ be any class, then according to the assumption we can write

$$\eta = f^*(\gamma) + \beta + f^*(\bar{\gamma}) \tag{1}$$

with $\gamma \in H^0(Y, \Omega_Y^2)$ and β of type $(1,1)$. In particular for every $y \in Y$ the restriction of η to the fibre X_y is of type $(1,1)$. Since Y is compact we can choose $\eta \in H^2(X, \mathbb{Q})$ sufficiently close to ω such that β is still a relative Kähler class.

We claim that we can replace η with $\widetilde{\eta}$ so that $\widetilde{\eta} - \eta \in f^* H^2(Y, \mathbb{R})$ and $\widetilde{\eta}$ is also of type $(1,1)$. Then we know by the Lefschetz $(1,1)$-theorem that, up to replacing $\widetilde{\eta}$ by $m\widetilde{\eta}$ with $m \in \mathbb{N}$ sufficiently divisible, there exists a line bundle $L \in \operatorname{Pic}(X)$ such that $\widetilde{\eta} = c_1(L)$. By construction $c_1(L)$ is a relative Kähler class, so by [Fuj79, Lemma 4.4] we can find a Kähler class $[\omega_Y]$ on Y such that $c_1(L) + f^*[\omega_Y]$ is Kähler. Therefore L is f-ample by Lemma 2.4.

Proof of the claim. Note that (1) implies that

$$\begin{aligned} f_*(\eta^d) &= f_*(\beta^d) \qquad \text{and} \\ f_*(\eta^{d+1}) &= f_*(\beta^{d+1}) + (d+1) f_*(\eta^d)\,(\gamma + \bar{\gamma})\,. \end{aligned} \tag{2}$$

This is indeed a consequence of the projection formula

$$f_*\left(\alpha \wedge f^*(\delta)\right) = f_*(\alpha) \wedge \delta$$

and of the fact that f_* is a morphism of Hodge structures of bidegree $(-d, -d)$. Let us note that the equality (2) is nothing but the Hodge decomposition of $f_*(\eta^{d+1})$ in $H^2(Y, \mathbb{R})$.

Since f_* is defined over $\mathbb{Q}$ and η is a rational class we have $f_*(\eta^k) \in H^{2k-2d}(Y, \mathbb{Q})$ for every $k \in \mathbb{N}$. Moreover

$$f_*(\eta^d) = \int_F \beta^d \in H^0(Y, \mathbb{Q})$$

is a positive rational number, since by construction the restriction of β to the general fibre F is a Kähler class. With this in mind, we can set

$$\eta_Y := \frac{1}{(d+1) f_*(\eta^d)} f_*(\eta^{d+1}) \in H^2(Y, \mathbb{Q})$$

and can consider

$$\widetilde{\eta} := \eta - f^*(\eta_Y) \in H^2(X, \mathbb{Q}).$$

Using the equality (2), we see that

$$\widetilde{\eta} = \beta - \frac{1}{(d+1) f_*(\eta^d)} f^* \left(f_*(\beta^{d+1}) \right)$$

is of type $(1,1)$. This is the sought rational $(1,1)$-class and it proves the claim.

Step 2. We prove the theorem under the second assumption. By Lemma 2.5 we can find a log-resolution $\mu_0 \colon X_0 \to X$ such that μ_0 and

$$f \circ \mu_0 : X_0 \to Y$$

are projective morphisms. Applying Lemma 2.12 we obtain that the bimeromorphic map μ_0 decomposes into a sequence

$$\varphi_i \colon X_i \dashrightarrow X_{i+1}$$

of divisorial contractions and flips over X. Denote by $\mu_i \colon X_i \to X$ the natural morphisms. Since $f_0 := f \circ \mu_0$ is projective and $f_m \simeq f$ we are done if we show that projectiveness of $f_i = f \circ \mu_i$ is invariant under every step of our MMP. We will show this for the first contraction, the statement then follows by induction.

Denote by $\varphi \colon X_0 \to Z$ the elementary Mori contraction of the extremal ray $\mathbb{R}^+[C]$ in $\overline{\mathrm{NE}}(X_0/X)$ (so φ is small if φ_0 is a flip, and $\varphi = \varphi_0$ in the divisorial case), and let $g \colon Z \to Y$ be the natural map. We have natural inclusions

$$\overline{\mathrm{NE}}(X_0/X) \subset \overline{\mathrm{NE}}(X_0/Y) \subset \overline{\mathrm{NE}}(X_0)$$

and we claim that $\mathbb{R}^+[C] \in \overline{\mathrm{NE}}(X_0/Y)$ is still an extremal ray. Indeed if $l_1, l_2 \in \overline{\mathrm{NE}}(X_0/Y)$ are pseudoeffective classes such that $l_1 + l_2 \in \mathbb{R}^+[C]$, then

$$0 = \varphi_*(l_1 + l_2) = \varphi_* l_1 + \varphi_* l_2.$$

Since Z is Kähler by Theorem 2.10(2), and $\varphi_* l_j \in \overline{\mathrm{NE}}(Z)$ we have $\varphi_* l_j = 0$ for $j = 1, 2$. Yet the kernel of

$$\varphi_* \colon N_1(X_0) \to N_1(Z)$$

is exactly $\mathbb{R}[C]$. Thus we obtain $l_j \in \mathbb{R}^+[C]$.

Since $\mathbb{R}^+[C] \in \overline{\mathrm{NE}}(X_0/Y)$ is an extremal ray we know by Theorem 2.10 (2) that there exists a projective contraction morphism $\eta \colon X_0 \to \tilde{X}_1$ such that $\tilde{X}_1 \to Y$ is projective. Yet φ_0 and η contract exactly the same curves, so by the rigidity lemma [BS95, Lem.4.1.13] we have an isomorphism $\tilde{X}_1 \to Z$ that identifies φ and η.

If φ is divisorial we have $Z \simeq X_1$, so this proves the claim. If φ is small, note that the flip $\varphi^+ \colon X_1 \to Z$ is a projective morphism polarised by the $\mathbb{Q}$-line bundle K_{X_1}, so $g \circ \varphi^+$ is projective by Remark 2.3.

Step 3. We prove the theorem under the first assumption. Let $\mu_Y : Y' \to Y$ and $\mu_X : X' \to X$ be projective modifications by compact Kähler manifolds such that we have an induced fibration $f' : X' \to Y'$. By [KS21, Thm.1.2] we have isomorphisms

$$H^0(Y', \Omega^2_{Y'}) \simeq H^0(Y, \Omega^{[2]}_Y), \qquad H^0(X', \Omega^2_{X'}) \simeq H^0(X, \Omega^{[2]}_X)$$

and therefore the injection

$$(f')^* : H^0(Y', \Omega^2_{Y'}) \to H^0(X', \Omega^2_{X'})$$

is an isomorphism. By Step 1 the fibration f' is projective, and therefore the fibration f is Moishezon (note that X' is strongly $\mathbb{Q}$-factorial, so the push-forward of a relatively ample line bundle induces a relatively big line bundle). Yet by Step 2 this implies that f is projective. $\square$

4. Applications of the main result

For lack of reference let us state the Kähler version of [Kol86, Thm.7.1]:

4.1. Theorem. [Kol86, Tak95] *Let $f\colon X \to Y$ be a fibration between normal compact Kähler spaces with rational singularities, and let F be a general fibre. Then the following statements are equivalent:*

- *$R^i f_* \mathcal{O}_X = 0$ for all $i > 0$;*
- *$h^i(F, \mathcal{O}_F) = 0$ for all $i > 0$.*

Proof. Since X has rational singularities we can replace it with a desingularisation without changing the statement. Now we follow the proof of [Kol86, Thm.7.1]: this proof is based on [Kol86, Thm.2.1] and general duality theory. The Kähler case of [Kol86, Thm.2.1] is shown in [Tak95, Thm. II and IV], duality theory in the analytic setting is established in [RR70]. $\square$

The assumption in Theorem 3.1 can easily be verified:

4.2. Corollary. *Let $f\colon X \to Y$ be a fibration between normal compact Kähler spaces with klt singularities. Assume that X is strongly $\mathbb{Q}$-factorial. If*

$$R^1 f_* \mathcal{O}_X = R^2 f_* \mathcal{O}_X = 0,$$

then f is projective. In particular

- *if $h^i(F, \mathcal{O}_F) = 0$ for all $i > 0$, or*
- *if f is bimeromorphic*

the morphism f is projective.

Proof. Since $R^1 f_* \mathcal{O}_X = R^2 f_* \mathcal{O}_X = 0$, the Leray spectral sequence shows that $H^2(X, \mathcal{O}_X) \simeq H^2(Y, \mathcal{O}_Y)$. Since klt singularities are rational, Hodge duality holds (see e.g. [Kir15, Cor.B.2.8]), so we obtain the isomorphism in the assumption of Theorem 3.1. The last statement is now clear by Theorem 4.1 (and since f bimeromorphic corresponds to the case $\dim(F) = 0$). $\square$

4.3. Remark. It was proven by Nakayama [Nak02, Prop.3.3.1] that a Kähler morphism $f\colon X \to Y$ with $R^2 f_* \mathcal{O}_X = 0$ is locally projective. It is not possible to improve this statement, i.e. Corollary 4.2 does not hold if we only assume $R^2 f_* \mathcal{O}_X = 0$: let X be a compact complex torus of dimension 2 and algebraic dimension $a(X) = 1$. Then the algebraic reduction is a fibration $f\colon X \to Y$ onto an elliptic curve Y. For dimension reasons we have $R^2 f_* \mathcal{O}_X = 0$ (but $R^1 f_* \mathcal{O}_X \simeq \mathcal{O}_Y \neq 0$). The morphism f is not projective, since otherwise X would also be projective.

Proof of Theorem 1.2. The MRC fibration is an almost holomorphic fibration $f\colon X \dashrightarrow Y$ such that the general fibre F is rationally chain connected. Since F has klt singularities, a desingularisation $F' \to F$ is rationally chain connected [HM07] and therefore $h^i(F, \mathcal{O}_F) = h^i(F', \mathcal{O}_{F'}) = 0$ for all $i > 0$. Choose now any modification $Y' \to Y$ such that Y' is a compact Kähler space with klt singularities, and choose a resolution of the indeterminacies $X' \to X$ such that X' is a compact Kähler space with strongly $\mathbb{Q}$-factorial klt singularities (e.g. X' is smooth). Then the morphism $f' : X' \to Y'$ satisfies the assumptions of Corollary 4.2 and is therefore projective. □

4.4. Remark. In general the MRC fibration is not holomorphic, so a priori it has no distinguished bimeromorphic model. However we can choose a model which is canonical in many ways and a projective morphism: let X be a normal compact Kähler space with klt singularities, and let F be a general fibre of the MRC fibration. Since the MRC fibration is almost holomorphic, the class of F is contained in a unique irreducible component of the cycle space $\mathcal{C}(X)$, and we denote its normalisation by Z. The universal family $\Gamma \to Z$ defines a canonically defined meromorphic MRC fibration $f\colon X \dashrightarrow Z$. Denote by $Z_c \to Z$ the canonical modification (this exists by [Fuj22, Thm.1.16]). Let X_c be the canonical modification of the unique component of $\Gamma \times_Z Z_c$ that dominates Z_c. By Lemma 2.11 we can assume after a small modification that X_c is strongly $\mathbb{Q}$-factorial. Thus we can apply Corollary 4.2 to obtain that

$$f_c\colon X_c \longrightarrow Z_c$$

is a projective morphism.

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