THE COMPLEX LANDSLIDE FLOW AND THE METHOD OF INTEGRABLE SYSTEMS

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Abstract. We investigate a connection between the complex landslide flow, defined on a pair of Teichmüller spaces, and the integrable system method's approach to harmonic maps into a symmetric space. We will prove that the holonomy of the complex landslide flow can be derived from the holonomy of the family of flat connections determined by a harmonic map into the hyperbolic two-space.

1. Introduction and the main result

(A) The McMullen's complex earthquake flow [10] is a natural holomorphic extension of the Thurston's earthquake flow, while the *complex landslide flow* introduced in [1] represents a complex cyclic extension of the complex earthquake flow. This flow is defined by the composition of two geometric maps; let M be a differentiable closed oriented surface of genus greater than or equal to two, and let $\mathcal{T}$ the Teichmüller space parametrized by hyperbolic metrics on M. Moreover, let $\mathcal{P}$ be the space of complex projective structures on M. The complex landslide flow P is defined by

$$P : \overline{\mathbb{D}}^{\times} \times \mathcal{T} \times \mathcal{T} \to \mathcal{P}, \quad (q, h, h^{\star}) \mapsto SGr_s(L_\theta(h, h^{\star})), \tag{1.1}$$

where $s = -\log|q|$ and $\theta = -\arg q$, and moreover, $L : [0, 2\pi) \times \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ and $SGr : [0, \infty) \times \mathcal{T} \times \mathcal{T} \to \mathcal{P}$ are the *landslide flow* and *smooth grafting* given in the below. See also Section 2.4 below, [1, Section 5] and [1, Proposition 5.14] for more details.

The landslide: Let $h, h^{\star} \in \mathcal{T}$ be hyperbolic metrics on M. Then there exists a unique bundle morphism $b : TM \to TM$ such that

(1) It is self-adjoint with respect to h.
(2) The determinant is 1.
(3) It satisfies the Codazzi equation $\mathrm{d}^{\nabla} b = 0$, that is, for the Levi-Civita connection ∇ of h and vector fields u, v on M, $\mathrm{d}^{\nabla} b$ is defined by $(\mathrm{d}^{\nabla} b)(u, v) := \nabla_u(bv) - \nabla_v(bu) - b[u, v]$.
(4) The metric given by $h(b\bullet, b\bullet)$ is isometric to $h^{\star}$ by a diffeomorphism isotopic to the identity.

Date: April 23, 2024.
2020 *Mathematics Subject Classification.* Primary 32G15, Seondary 53C43.
Key words and phrases. Constant Gaussian curvature surfaces; complex landslide flow; harmonic maps; integrable system method.
The author is partially supported by Kakenhi 22K03304.

The above bundle morphism b has been called the *Labourie operator* [9]. Then for a given $\theta \in [0, 2\pi)$ the landslide L_θ on $\mathcal{T} \times \mathcal{T}$ is defined by

$$L_\theta(h, h^\star) := (h(\beta_\theta \bullet, \beta_\theta \bullet), h(\beta_{\theta+\pi} \bullet, \beta_{\theta+\pi} \bullet)), \tag{1.2}$$

where β_θ is a family of bundle morphism defined by

$$\beta_\theta := \cos(\theta/2)E + \sin(\theta/2)Jb$$

with $E : TM \to TM$ is the identity and J is the complex structure with respect to h. It is evident that β_θ satisfies the Codazzi equation, and $\det \beta_\theta = 1$. It has been shown that the metric h_θ given by $h_\theta(u, v) = h(\beta_\theta u, \beta_\theta v)$ is a hyperbolic metric on M for all $\theta \in \mathbb{R}$ [1, Proposition 1.7] and L defines an S^1-flow on $\mathcal{T} \times \mathcal{T}$, that is, $L_{\theta'}(L_\theta(h, h^\star)) = L_{\theta'+\theta}(h, h^\star)$ holds. Let L^1 be a map defined by the composition of L with the projection on the first factor. Then $L^1_{e^{i\theta}}$ is analogous to the Thurston's earthquake map $E : \mathcal{T} \times \mathcal{ML} \to \mathcal{T}$, where $\mathcal{ML}$ is the space of measured lamination on M.

The smooth grafting: For a given pair $(h, h^\star) \in \mathcal{T} \times \mathcal{T}$ and a positive number $s > 0$, the *smooth grafting* SGr_s can be defined by a corresponding constant Gaussian curvature (CGC for short) surfaces with $-1 < K < 0$ in the hyperbolic 3-space $\mathbb{H}^3$ and its Gauss map: Let b be the Labourie operator as before, that is, $h^\star = h(b\bullet, b\bullet)$ holds, and consider the metric $\mathrm{I} = \cosh^2(s/2)h$ and the operator $B = -\tanh(s/2)b$. Then the property of the Labourie operator implies that the there exists a unique CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface f in $\mathbb{H}^3$ whose first fundamental form and shape operator are given by I and B, respectively. Moreover, consider a unique geodesic starting from a point $f(p) \in \mathbb{H}^3 (p \in M)$ with the the unit normal normal $n(p)$ of $f(p)$ as initial velocity ending to the boundary of $\mathbb{H}^3 \cong \mathbb{C}P^1$. Then it gives the developing map $\mathrm{dev}_s : \widetilde{M} \to \mathbb{C}P^1$ and defines a complex projective structure on M, where $\widetilde{M}$ denotes the universal cover of M. Thus the smooth grafting $SGr_s : \mathcal{T} \times \mathcal{T} \to \mathcal{P}$ has been defined.

The complex landslide flow $P_q = SGr_s \circ L_\theta$, $(s = -\log|q|,\ \theta = -\arg q)$ shares many properties with the complex earthquake flow, and in particular it converges to the complex earthquake flow, see [1, 2] in details. Moreover, it has been shown in [1, Theorem 5.1] that the complex landslide P is actually a flow and holomorphic with respect to $q \in \overline{\mathbb{D}}^\times$ by showing the holomorphicity of the holonomy of the projective structure.

(B) On the one hand, a classification of weakly complete CGC surfaces with $-1 < K < 0$ in $\mathbb{H}^3$ has been given in [7] in terms of integrable system method, the so-called the *loop group method*. The heart of the classification is based on harmonic maps from a Riemann surface M into the symmetric space, the hyperbolic 2-space $\mathbb{H}^2 = \mathrm{SU}(1, 1)/\mathrm{U}(1)$ and the spectral deformation family in $\mathbb{C}^\times$.

Harmonic maps: Recall that a harmonic map from a Riemann surface M into a symmetric space G/K is characterized as follows, for example see [6]: Let us consider a smooth map g from a Riemann surface M into a symmetric space G/K. Denote the universal cover of M by $\widetilde{M}$, and take a lift $F : \widetilde{M} \to G$ and the Maurer-Cartan form $\alpha = F^{-1}\mathrm{d}F$. Moreover, define a family parametrized by $\lambda \in S^1$

$$\nabla^\lambda = \mathrm{d} + \alpha^\lambda, \quad \text{with} \quad \alpha^\lambda = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{p}}, \tag{1.3}$$

where $\alpha_{\mathfrak{p}}$ and $\alpha_{\mathfrak{k}}$ denote the $\mathfrak{p}$- and the $\mathfrak{k}$-part of the direct decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(K) = \mathfrak{k}$. Moreover, $'$ and $''$ denote the $(1, 0)$- and $(0, 1)$-part with respect to the complex structure of the Riemann surface M. Then ∇^λ defines a family of connections parametrized by $\lambda \in \mathbb{C}^\times$ and $\mathrm{d} + \alpha^\lambda|_{\lambda=1}$ gives a flat connection from the Maurer-Cartan equation $(\mathrm{d}\alpha^\lambda + \alpha^\lambda \wedge \alpha^\lambda)|_{\lambda=1} = 0$ for the frame F. It is well known that $g : M \to G/K$ is harmonic if and only if ∇^λ gives a family of *flat connections* for any $\lambda \in S^1$, that is, $\mathrm{d}\alpha^\lambda + \alpha^\lambda \wedge \alpha^\lambda = 0$ holds for all $\lambda \in S^1$. Then there exists a family of frames

$$F^\lambda : \widetilde{M} \to \Lambda G_\tau$$

such that $\alpha^\lambda = (F^\lambda)^{-1}\mathrm{d}F^\lambda$ and it is called the *extended frame* of the harmonic map g. Here ΛG_τ denote the loop group of G:

$$\Lambda G_\tau = \{\gamma : S^1 \to G \mid \gamma \text{ is smooth and } \gamma(-\lambda) = \tau\gamma(\lambda)\},$$

where τ is the involution according to the symmetric space G/K, that is, $(\mathrm{fix}\,\tau)_0 \subset K \subset \mathrm{fix}\,\tau$, where $\mathrm{fix}\,\tau$ denotes the fixed point set of τ in G and the subscript zero denotes the identity component. Moreover, introducing a suitable topology to ΛG_τ, it becomes a infinite dimensional Banach Lie group, the so-called *loop group* [12]. It has been known the Weierstrass type representation for such harmonic maps [6].

The spectral deformation: In our case, G/K is the hyperbolic two space $\mathbb{H}^2$ and $G = \mathrm{SU}(1, 1)$ is the indefinite special unitary group of degree 2 and $K = \mathrm{U}(1) = \{\mathrm{diag}(e^{it}, e^{-it}) \mid t \in \mathbb{R}\} \subset \mathrm{SU}(1, 1)$. Moreover, the involution τ is explicitly given by $\tau F = \mathrm{Ad}\,\mathrm{diag}(1, -1)F$ for $F \in \mathrm{SU}(1, 1)$. Let us consider a harmonic map from M into $\mathbb{H}^2$ and take the extended frame $F^\lambda : \widetilde{M} \to \Lambda\mathrm{SU}(1, 1)_\tau$ as above. If we evaluate the extended frame F^λ at $\lambda \in \mathbb{C}^\times \setminus S^1$ then it takes values in $\Lambda\mathrm{SL}(2, \mathbb{C})_\tau$ not in $\Lambda\mathrm{SU}(1, 1)_\tau$. In [7, Theorem 2.1], it has been shown that the extended frame F^λ gives a family of CGC surfaces with $-1 < K < 0$ in $\mathbb{H}^3$ as

$$f^\lambda = F^\lambda(F^\lambda)^*|_{\lambda=\lambda_0}, \quad |\lambda_0| \in (0, 1),$$

where the upper subscript $*$ denotes a composition of the complex conjugation and the transpose, and the constant Gaussian curvature is given by

$$K = -\left(\frac{2|\lambda_0|}{|\lambda_0|^2 + 1}\right)^2 \in (-1, 0).$$

Note here that the hyperbolic three-space $\mathbb{H}^3$ is realized by the quadric in the complex Hermitian 2 by 2 matrices, see Section 2.1. The evaluation at $\lambda_0 \in \mathbb{D}^\times$ has been called the *spectral deformation* of the extended frame. This construction was the heart of the classification of weakly complete CGC $-1 < K < 0$ surfaces in $\mathbb{H}^3$, see [7]. Moreover, for a fixed $|\lambda_0| \in (0, 1)$, f^λ gives the S^1-family of CGC surfaces with a fixed $-1 < K < 0$ in $\mathbb{H}^3$ and it becomes the *associated family* of a CGC $-1 < K < 0$ surface in $\mathbb{H}^3$.

(C) From the discussions of (A) and (B), it is natural to expect a certain relation between the above constructions. In the both constructions, the S^1-families and the scaling families naturally appear, and in particular the landslide and the smooth grafting look similar to the associated family and the spectral deformation, respectively. In fact the following theorem is the main result of the paper.

Theorem 1.1. *The holonomy of any complex landslide flow P_q ($q \in \overline{\mathbb{D}}^{\times}$) is given by the holonomy of the family of flat connections ∇^{λ} for a harmonic map into $\mathbb{H}^2$ evaluated at $\lambda = \sqrt{q} \in \overline{\mathbb{D}}^{\times}$.*

We will give a proof in Section 3. The extend frame F^{λ} has a holomorphic dependence on the spectral parameter $\lambda \in \mathbb{C}^{\times}$. In particular it is defined on $\overline{\mathbb{D}}^{\times}$. Then by noting Remark 1.1 (1), the holomorphic dependence of the holonomy of the complex landslide flow follows and the following statement holds.

Corollary 1.1. *The complex landslide flow is holomorphic on $\overline{\mathbb{D}}^{\times}$.*

Remark 1.1.

(1) *Because we utilized the twisted loop group formulation of a harmonic map into a symmetric space, the square root $\lambda = \sqrt{q}$ in* Theorem 1.1 *arises. However, using the untwisted loop group formulation as demonstrated in* [3], *it can be expressed by the evaluation as $\lambda = q$, see the proof in* Section 3. *Then* Corollary 1.1 *follows immediately.*
(2) Corollary 1.1 *has been proved in* [1, Theorem 5.1] *by a direct computation of derivatives of the holonomy and by showing that the Cauchy-Riemann equation holds, however,* Theorem 1.1 *gives a natural correspondence between the complex landslide flow and the family of flat connections.*
(3) *A relation between the harmonic map and the complex landslide has been also discussed in* [1], *and the spectral deformation family in this paper will give a natural understanding of such relation through the integrable systems approach. The key observation is a relation in* Lemma 3.1 *of two complex structures on M, namely, the J and $i = \sqrt{-1}$ induced from the first and second fundamental forms of a CGC surface, respectively.*

The paper is organized as follows: Preliminaries will be given in Section 2 and the proof of Theorem 1.1 will be given in Section 3.

2. Preliminaries

In this section, according to [7] we will recall surfaces in the hyperbolic three-space $\mathbb{H}^3$ and two Gauss maps, that is, the Legendrian and the Lagrangian Gauss maps, respectively. Then by the spectral deformation of the Lagrangian Gauss map, we will relate the extended frame of a harmonic map into hyperbolic two-space $\mathbb{H}^2$ and a constant Gaussian curvature $-1 < K < 0$ surface in $\mathbb{H}^3$. Moreover, we will review the complex landslide and a CGC $-1 < K < 0$ surface in $\mathbb{H}^3$ according to [1].

2.1. Surfaces in the hyperbolic three-space.

Let

$$\boldsymbol{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \boldsymbol{e}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \boldsymbol{e}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The 4-dimensional Minkowski space $\mathbb{E}^{1,3}$ can be identified with the complex Hermitian 2 by 2-matrices

$$\mathrm{Her}(2,\mathbb{C}) = \left\{ \sum_{j=0}^{3} \xi_j \boldsymbol{e}_j \mid \xi_j \in \mathbb{R} \right\} \longleftrightarrow \mathbb{E}^{1,3} = \{(\xi_0,\xi_1,\xi_2,\xi_3) \mid \xi_0,\xi_1,\xi_2,\xi_3 \in \mathbb{R}\}\,.$$

Then the standard inner product of $\mathbb{E}^{1,3}$ is given by $\langle \xi, \eta\rangle = -\frac{1}{2}\operatorname{tr}(\xi \boldsymbol{e}_2 {}^t\eta \boldsymbol{e}_2)$ and $\langle \xi,\xi\rangle = -\det \xi$ under the identification of $\mathbb{E}^{1,3} \cong \mathrm{Her}(2,\mathbb{C})$. The *hyperbolic* 3-*space* $\mathbb{H}^3$ is defined by the unit central hyperquadrics in $\mathbb{E}^{1,3} \cong \mathrm{Her}(2,\mathbb{C})$:

$$\mathbb{H}^3 = \{\xi \in \mathrm{Her}(2,\mathbb{C}) \mid \det \xi = 1,\ \operatorname{tr}\xi > 0\}\,.$$

The special linear group of degree two $\mathrm{SL}(2,\mathbb{C})$ acts isometrically and transitively on $\mathbb{H}^3$ via the action (g,ξ) as $g\xi g^*$, where the subscript $*$ denotes $g^* = {}^t\bar{g}$ for $g \in \mathrm{SL}(2,\mathbb{C})$. The isotropy subgroup of this action at $\boldsymbol{e}_0 = (1,0,0,0) \cong \mathrm{id}$ is the special unitary group $\mathrm{SU}(2)$ and hence $\mathbb{H}^3$ can be represented as a homogeneous Riemannian symmetric space $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. The natural projection $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathbb{H}^3$ is given by $\pi(g) = gg^*$ and thus the hyperbolic 3-space can be represented as $\mathbb{H}^3 = \{gg^* \mid g \in \mathrm{SL}(2,\mathbb{C})\}$.

Let $f : M \to \mathbb{H}^3$ be a surface and assume that its Gaussian curvature satisfies $K > -1$. Then by the formula $K = -1 + \det B$, where B is the shape operator of f, $\det B > 0$ follows. Thus the second fundamental form $\mathrm{I\!I}$ defines a Riemannian metric and thus the unique complex structure follows. Let us denote the complex coordinate induced from the second fundamental form by $z = x + yi$. Then the fundamental forms can be represented as follows: Set $\sigma = \det B$ and the unit normal of f as n. The first, second and third fundamental forms can be computed by

$$\mathrm{I} = \langle \mathrm{d}f, \mathrm{d}f\rangle = Q\,\mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar{z} + \bar{Q}\,\mathrm{d}\bar{z}^2, \tag{2.1}$$

$$\mathrm{I\!I} = \langle \mathrm{d}f, \mathrm{d}n\rangle = \sigma(e^u - |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar{z}, \tag{2.2}$$

$$\mathrm{I\!I\!I} = \langle \mathrm{d}n, \mathrm{d}n\rangle = \sigma^2\left(-Q\,\mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar{z} - \bar{Q}\,\mathrm{d}\bar{z}^2\right), \tag{2.3}$$

where $u : M \to \mathbb{R}$ is a smooth function and $Q\,\mathrm{d}z^2$ is the $(2,0)$-part of the first fundamental form with respect to the conformal structure of the second fundamental form, which will be called the *Klotz differential*. Note that the mean curvature H and the Gaussian curvature K of the surface f can be represented by

$$H = \frac{\sigma}{2}\frac{e^{2u} + |Q|^2}{e^{2u} - |Q|^2} \quad \text{and} \quad K = -1 + \sigma^2, \tag{2.4}$$

see [7] for more details.

2.2. **Natural projections, two Gauss maps and Ruh-Vilms type theorem.** We have several natural projections from the unit tangent sphere bundle of $\mathbb{H}^3$

$$\mathrm{U}\mathbb{H}^3 = \{(\boldsymbol{x},\boldsymbol{v}) \in \mathrm{Her}_2\mathbb{C} \times \mathrm{Her}_2\mathbb{C} \mid \det \boldsymbol{x} = 1, \operatorname{tr}\boldsymbol{x} > 0,\ \det \boldsymbol{v} = -1,\ \langle \boldsymbol{x},\boldsymbol{v}\rangle = 0\},$$

see [5] in detail. A natural projection $\pi_+ : \mathrm{U}\mathbb{H}^3 \to \mathbb{H}^3$ is given by $\pi_+(\boldsymbol{x},\boldsymbol{v}) = \boldsymbol{x}$. We now introduce notion of the space $\mathrm{Geo}(\mathbb{H}^3)$ of all oriented geodesics in $\mathbb{H}^3$ which is identified with the Grassmannian manifold $\mathrm{Gr}_{1,1}(\mathbb{E}^{1,2})$ of all oriented timelike planes in $\mathbb{E}^{1,3} = \mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C}$. It is known that $\mathrm{Geo}(\mathbb{H}^3)$ can be represented as

$$\mathrm{Geo}(\mathbb{H}^3) \cong \mathrm{Gr}_{1,1}(\mathbb{E}^{1,2}) = \mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C} \cong \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \mathrm{diag}\,. \tag{2.5}$$

Then we have a natural projection $\pi_0 : \mathrm{U}\mathbb{H}^3 \to \mathrm{Geo}(\mathbb{H}^3)$ by $\pi_0(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{x} \wedge \boldsymbol{v}$. Finally there is also a natural projection from $\pi_* : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{U}\mathbb{H}^3$ by $\pi_*(g) = (gg^*, g\boldsymbol{e}_1 g^*)$. Note that the projection $\pi : \mathrm{SL}(2, \mathbb{C}) \to \mathbb{H}^3 = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$ is given by the composition $\pi = \pi_+ \circ \pi_*$.

Using these projections, we will recall two Gauss maps for a surface in $\mathbb{H}^3$, which are crucial in this paper. Note that several natural Gauss maps were already defined in see [5, Appendix of the archive version].

Definition 2.1. *Let $f : M \to \mathbb{H}^3$ be a surface with unit normal n, and let $\mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}) = \mathrm{Geo}(\mathbb{H}^3)$. The* Legendrian Gauss map *G_{Le} and the* Lagrangian Gauss map *G_{La} of f are respective defined by*

$$G_{Le} = (f, n) : M \to \mathrm{U}\mathbb{H}^3 \quad \textit{and} \quad G_{La} = f \wedge n : M \to \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}).$$

Remark 2.1. *In* [5]*, using the conformal structure induced from the first fundamental form, constant mean curvature surfaces have been characterized by harmonicity of the Legendrian Gauss map.*

In [7, Theorem A], the following theorem has been proved.

Theorem 2.1. *Let $f : M \to \mathbb{H}^3$ be a surface with Gaussian curvature $K > -1$ and let n be the unit normal of f. Moreover let $G_{Le} = (f, n) : M \to \mathrm{U}\mathbb{H}^3$ and $G_{La} = f \wedge n : M \to \mathrm{Geo}(\mathbb{H}^3)$ be the Legendrian and Lagrangian Gauss maps, respectively. Then the following statements are mutually equivalent*:

(1) *The Gaussian curvature of f is constant.*
(2) *The Legendrian Gauss map is harmonic.*
(3) *The Lagrangian Gauss map is harmonic.*
(4) *The Klotz differential is holomorphic.*

2.3. **The extended frames and CGC surfaces.** Let us consider the structure equations for a CGC $K > -1$ surface:

$$\bar{\partial}\partial u + \frac{K}{2}(e^u - |Q|^2 e^{-u}) = 0 \quad \text{and} \quad \bar{\partial} Q = 0, \tag{2.6}$$

where we set

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Note that, under a normalization $Q = 1$, the elliptic PDE (2.6) for u becomes the famous *sinh-Gordon* equation [11].

If u and Q are solutions to (2.6), then u and $\lambda^{-2}Q$ with any constant $\lambda \in S^1$ are also solutions to (2.6). Thus there exists a family of CGC $K > -1$ surfaces $\{f^\lambda\}_{\lambda \in S^1}$ in $\mathbb{H}^3$. Accordingly, there exists a family of unit normal vectors $\{n^\lambda\}_{\lambda \in S^1}$ such that

$$G^\lambda_{Le} = (f^\lambda, n^\lambda) : M \to \mathrm{U}\mathbb{H}^3 \quad \text{and} \quad G^\lambda_{La} = f^\lambda \wedge n^\lambda : M \to \mathrm{Geo}(\mathbb{H}^3)$$

are a family of Legendrian and Lagrangian harmonic Gauss maps of f^λ, respectively. Then the family of CGC $-1 < K = -1/\cosh^2 s < 0$ surfaces in $\mathbb{H}^3$ has the following fundamental forms:

$$\mathrm{I}^\lambda = \langle \mathrm{d}f^\lambda, \mathrm{d}f^\lambda\rangle = \lambda^{-2}Q\,\mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar z + \lambda^2 \bar Q\,\mathrm{d}\bar z^2, \tag{2.7}$$

$$\mathrm{I\!I}^\lambda = \langle \mathrm{d}f^\lambda, \mathrm{d}n^\lambda\rangle = \sigma(e^u - |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar z, \tag{2.8}$$

$$\mathrm{I\!I\!I}^\lambda = \langle \mathrm{d}n^\lambda, \mathrm{d}n^\lambda\rangle = \sigma^2\left(-\lambda^{-2}Q\,\mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar z - \lambda^2 \bar Q\,\mathrm{d}\bar z^2\right), \tag{2.9}$$

where

$$\sigma = \tanh(s/2), \quad (s > 0), \tag{2.10}$$

and the Gaussian curvature K of f^λ is

$$-1 < K = -1 + \sigma^2 = -\frac{1}{\cosh^2(s/2)} < 0.$$

The S^1-parameter λ has been called the *spectral parameter*, and the family of CGC $-1 < K < 0$ surfaces $\{f^\lambda\}_{\lambda\in S^1}$ in the above has been called the *associated family* of f. Note that $\mathrm{I\!I}^\lambda$ is the same for any $\lambda \in S^1$, that is $\mathrm{I\!I}^\lambda = \mathrm{I\!I}$.

On the other hand, for a fixed positive number $s > 0$, define two metrics parametrized by $\theta = \arg\lambda, (\lambda \in S^1)$ by

$$h_\theta := \frac{1}{\cosh^2(s/2)}\mathrm{I}^\lambda \quad \text{and} \quad h^\star_\theta := \frac{1}{\sinh^2(s/2)}\mathrm{I\!I\!I}^\lambda.$$

Then $(h, h^\star)$ is a pair of hyperbolic metrics. Moreover by using the shape operator $B^\lambda = \left(\mathrm{I}^\lambda\right)^{-1}\mathrm{I\!I}^\lambda$,

$$b_\theta = -\coth(s/2)\,B^\lambda$$

is the Labourie operator for $(h_\theta, h^\star_\theta)$.

In [7, Theorem B (1)], it has been proved that the family $\{G^\lambda_{La}\}_{\lambda\in S^1}$ of Lagrangian Gauss maps of $\{f^\lambda\}_{\lambda\in S^1}$ can be characterized as follows:

Theorem 2.2 (Theorem B (1) in [7])**.** *Let f^λ be the family of CGC $-1 < K < 0$ surfaces in $\mathbb{H}^3$ defined as above, and set $\lambda_0 \in \mathbb{C}\setminus S^1$ such that $|\lambda_0| = \exp(\operatorname{arcosh}\sqrt{-1/K})$ holds. Then the map $G^\lambda_{La}|_{\lambda=\lambda_0}$ is a harmonic local diffeomorphism into $\mathbb{H}^2$.*

Conversely let $g : M \to \mathbb{H}^2$ be a harmonic map and consider the extended frame

$$F^\lambda : \widetilde{M} \to \Lambda\mathrm{SU}(1,1)_\tau,$$

as explained in Introduction. The extended frame is defined not only on S^1 but also on $\mathbb{C}^\times$. Then $F^\lambda|_{\lambda\in\mathbb{C}^\times}$ is a map into $\Lambda\mathrm{SL}(2,\mathbb{C})_\tau$ not $\Lambda\mathrm{SU}(1,1)_\tau$.

Let $\pi : \mathrm{SU}(1,1) \to \mathbb{H}^2$ be a natural projection. For the extended frame F^λ, $\pi\circ F^\lambda|_{\lambda=1}$ is the original harmonic map g up to isometry and moreover, $g^\lambda = \pi\circ F^\lambda|_{\lambda\in S^1}$ gives a family of harmonic maps in $\mathbb{H}^2$, the so-called associated family of g. If $Q\,\mathrm{d}z^2$ is the Hopf differential of f, then the Hopf differential of g^λ can be realized as $\lambda^{-2}Q\,\mathrm{d}z^2$.

Theorem 2.3 (Theorem 2.1 in [7])**.** *Let F^λ be the extended frame of a harmonic map $g : M \to \mathbb{H}^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$ and let $\lambda_0 \in \mathbb{D}^\times$. Then*

$$f^\lambda = F^\lambda(F^\lambda)^*|_{\lambda=\lambda_0}, \quad n^\lambda = F^\lambda \boldsymbol{e}_1 (F^\lambda)^*|_{\lambda=\lambda_0}, \quad \lambda_0 \in \mathbb{D}^\times,$$

is a family of CGC surfaces in $\mathbb{H}^3$ *with the constant Gaussian curvature*

$$K = -\left(\frac{2|\lambda_0|}{|\lambda_0|^2+1}\right)^2 \in (-1,0)$$

and a family of unit normals n^λ. *Moreover the Klotz differential of* f^λ *is given by*

$$Q^\lambda\,\mathrm{d}z^2 = \frac{\lambda_0^2}{|\lambda_0|^2} Q\,\mathrm{d}z^2. \tag{2.11}$$

The evaluation at $|\lambda_0| \in (0,1)$ has been called the *spectral deformation* of the extended frame, and the S^1-family $\lambda_0 = |\lambda_0|e^{i\theta}$ with fixed $|\lambda_0|$ is the associated family.

Moreover, the Lagrangian Gauss map can be represented by F^λ as

$$G^\lambda_{La} = f^\lambda \wedge n^\lambda|_{\lambda=\lambda_0} = F^\lambda \boldsymbol{e}_1 (F^\lambda)^{-1}|_{\lambda=\lambda_0}, \quad \lambda_0 \in \mathbb{D}^\times. \tag{2.12}$$

Remark 2.2. *From* (2.11), *the family of CGC surfaces* f^λ *in* Theorem 2.3 *has a symmetry, that is,* f^λ *and* $f^{-\lambda}$ *are the same CGC surface up to rigid motion.*

2.4. **The complex landslides flow and CGC surfaces.** Recall that the landslide (1.2) is given by

$$L_\theta(h,h^\star) := (h(\beta_\theta,\beta_\theta), h(\beta_{\theta+\pi},\beta_{\theta+\pi})),$$

where $\beta_\theta := \cos(\theta/2)E + \sin(\theta/2)Jb$, and b is the Labourie operator for $(h,h^\star)$, see Introduction. Note that J is the complex structure compatible with respect to h and it is not related to the complex structure given in Section 2.1. We recall the following fundamental lemma, see [1].

Lemma 2.1. *Let* $(h,h^\star) \in \mathcal{T}\times\mathcal{T}$ *and a positive constant* $s > 0$. *Then there exists a unique CGC* $-1 < K = -1/\cosh^2(s/2) < 0$ *surface* f *in* $\mathbb{H}^3$ *such that the first and third fundamental forms are respectively given as*

$$\mathrm{I} = \cosh^2(s/2)h \quad \textit{and} \quad \mathrm{I\!I\!I} = \sinh^2(s/2)h^\star. \tag{2.13}$$

Conversely, for a given CGC $-1 < K = -1/\cosh^2(s/2) < 0$ *surface* f *in* $\mathbb{H}^3$, *define* h *and* $h^\star$ *by* (2.13). *Then* h *and* $h^\star$ *are hyperbolic metrics.*

Proof. For a given $(h,h^\star)$, there exists a unique Labourie operator b such that $h(b\bullet,b\bullet) = h^\star$, see Introduction. Define

$$\mathrm{I} = \cosh^2(s/2)h, \quad B = -\tanh(s/2)b.$$

Note that $\det B = \tanh^2(s/2)$ holds. Then from the properties of the Labourie operator b, the pair (I, B) satisfies

$$\mathrm{d}^\nabla B = 0, \quad K = -1 + \det B, \tag{2.14}$$

where ∇ is the Levi-Civita connection of I and $K = -1/\cosh^2(s/2)$ is the curvature of I. Since $(\mathrm{d}^\nabla B)(u,v) = 0$ is equivalent to

$$\nabla_u(Bv) - \nabla_v(Bu) = B[u,v],$$

which is the Codazzi equation for a surface in $\mathbb{H}^3$, see [4, p.138]. Since the second equation in (2.14) is the Gauss equation, thus there exists a unique surface f such that the first fundamental form is I and the second fundamental form $\mathrm{I\!I}$ is $\mathrm{I\!I}(u,v) = \mathrm{I}(u, Bv)$. Moreover, the third fundamental form $\mathrm{I\!I\!I}$ can be computed as

$$\begin{aligned}\mathrm{I\!I\!I}(u,v) &= \mathrm{I}(Bu, Bv)\\ &= \tanh^2(s/2)\mathrm{I}(bu, bv)\\ &= \sinh^2(s/2)h(bu, bv)\\ &= \sinh^2(s/2)h^\star(u,v),\end{aligned}$$

and the claim follows. The converse statement is clear and this completes the proof. ☐

The complex landslide flow $P : \overline{\mathbb{D}}^\times \times \mathcal{T} \times \mathcal{T} \to \mathcal{P}$ is defined by $P(q, h, h^\star) = SGr_s(L_\theta(h, h^\star))$ with $s = -\log|q|$ and $\theta = -\arg q$, where SGr_s is the smooth grafting, which has been explained in Introduction:

(1) Let the pair of hyperbolic metrics by the landslide $L_\theta(h, h^\star)$.
(2) By Lemma 2.1, there exists a CGC surface f with $-1 < K = -1/\cosh^2(s/2) < 0$.
(3) Take the unit normal normal $n(p)$ of $f(p)$ as initial velocity ending to the boundary of $\mathbb{H}^3 \cong \mathbb{C}P^1$.
(4) It gives the developing map

$$\mathrm{dev}_s : \widetilde{M} \to \mathbb{C}P^1 \tag{2.15}$$

and it defines a complex projective structure on M.

In [1, Theorem 5.1], the complex landslide flow P is holomorphic with respect to $\overline{\mathbb{D}}^\times$ by showing that the holonomy of the developing map is holomorphic.

3. Proof of the main theorem

We now prove Theorem 1.1 in Introduction. The crucial observation is the following lemma about a relation of the complex structure J on the first fundamental form I and the complex structure, which is denoted by $z = x + iy$, on the conformal class of the second fundamental form $\mathrm{I\!I}$.

Lemma 3.1. *Let f be a CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface in $\mathbb{H}^3$ with the fundamental forms $\mathrm{I}, \mathrm{I\!I}$ and $\mathrm{I\!I\!I}$ in* (2.1), (2.2) *and* (2.3), *respectively. Define an endomorphism $J : TM \to TM$ by*

$$\begin{cases} J\partial_z := i\coth(s/2)B\partial_z \\ J\partial_{\bar z} := -i\coth(s/2)B\partial_{\bar z} \end{cases}, \tag{3.1}$$

where B is the shape operator of f. Then J is the unique complex structure on M compatible with the first fundamental form I.

Proof. By a straightforward computation,

$$\mathrm{I}(J\partial_z, J\partial_z) = -\coth^2(s/2)\mathrm{I}(B\partial_z, B\partial_z) = -\coth^2(s/2)\mathrm{I\!I\!I}(\partial_z, \partial_z) = Q = \mathrm{I}(\partial_z, \partial_z),$$
$$\mathrm{I}(J\partial_{\bar z}, J\partial_{\bar z}) = -\coth^2(s/2)\mathrm{I}(B\partial_{\bar z}, B\partial_{\bar z}) = -\coth^2(s/2)\mathrm{I\!I\!I}(\partial_{\bar z}, \partial_{\bar z}) = \bar{Q} = \mathrm{I}(\partial_{\bar z}, \partial_{\bar z}),$$

and

$$\mathrm{I}(J\partial_z, J\partial_{\bar z}) = \coth^2(s/2)\mathrm{I}(B\partial_z, B\partial_{\bar z}) = \coth^2(s/2)\mathrm{III}(\partial_z, \partial_{\bar z}) = \frac{1}{2}(e^u + |Q|^2e^{-u}) = \mathrm{I}(\partial_z, \partial_{\bar z})$$

hold. Thus J is compatible with I. Moreover, a straightforward computation shows that

$$J\partial_z = \frac{2iH}{\tanh(s/2)}\partial_z - \frac{2iQ}{e^u - |Q|^2e^{-u}}\partial_{\bar z},$$
$$J\partial_{\bar z} = \frac{2i\bar Q}{e^u - |Q|^2e^{-u}}\partial_z - \frac{2iH}{\tanh(s/2)}\partial_{\bar z},$$

where $Q\,\mathrm{d}z^2$ is the Klotz differential and H is the mean curvature in (2.4). From the definition of J in (3.1), it is easy to see that

$$J^2\partial_z = -\partial_z \quad \text{and} \quad J^2\partial_{\bar z} = -\partial_{\bar z}.$$

It is also easy to see that J preserves the orientation, and thus J is the unique complex structure on M compatible with I. This completes the proof. □

Corollary 3.1. *Let β_θ be the family of operators by $\beta_\theta = \cos(\theta/2)E + \sin(\theta/2)Jb$, $(\theta \in [0, 2\pi))$ of the landslide flow in* (1.2)*. Moreover, set $\lambda^{1/2} = \cos(\theta/2) + i\sin(\theta/2) \in S^1$. Then the following relations hold*:

$$\beta_\theta\partial_z = \lambda^{-1/2}\partial_z \quad \textit{and} \quad \beta_\theta\partial_{\bar z} = \lambda^{1/2}\partial_{\bar z}. \tag{3.2}$$

Proof. A straightforward computation by using (3.1) shows that

$$\beta_\theta\partial_z = \cos(\theta/2)\partial_z + \sin(\theta/2)Jb\partial_z = \cos(\theta/2)\partial_z - i\sin(\theta/2)\partial_z = \lambda^{-1/2}\partial_z$$

holds. Similarly $\beta_\theta\partial_{\bar z} = \lambda^{1/2}\partial_{\bar z}$ holds. □

Combining the landslide flow and Lemma 3.1, we have the following proposition.

Proposition 3.1. *For a given $(h, h^\star) \in \mathcal{T} \times \mathcal{T}$, let f the corresponding CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface in $\mathbb{H}^3$ given by* Lemma 2.1*, and let $\{f^\lambda\}_{\lambda\in S^1}$ be the associated family of f given in* Section 2.3*. Moreover, for a given the landslide flow $(h_\theta, h^\star_\theta) = L_\theta(h, h^\star) \in \mathcal{T} \times \mathcal{T}$, $(\theta \in [0, 2\pi))$ and a positive constant $s > 0$, consider the corresponding CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface $\tilde f_\theta$ in $\mathbb{H}^3$ given by* Lemma 2.1*. Then $\tilde f_\theta = f^{\sqrt{\lambda}}$ with $\lambda = e^{i\theta}$ holds for any $s > 0$ and $\theta \in [0, 2\pi)$.*

Proof. First recall that $(h_\theta, h^\star_\theta) = L_\theta(h, h^\star)$ is given by

$$(h_\theta, h^\star_\theta) = (h(\beta_\theta, \beta_\theta), h^\star(\beta_{\theta+\pi}, \beta_{\theta+\pi})), \quad \text{where } \beta_\theta = \cos(\theta/2)E + \sin(\theta/2)Jb,$$

and b is the Labourie operator between h and $h^\star$. The first and third fundamental forms of $\tilde f_\theta$ are given by

$$(\mathrm{I}_\theta, \mathrm{III}_\theta) = (\cosh^2(s/2)h_\theta, \sinh^2(s/2)h^\star_\theta).$$

Moreover the second fundamental form $\mathrm{II}_\theta = \mathrm{I}_\theta(\bullet, -\tanh(s/2)b_\theta\bullet)$, where b_θ is the Labourie operator between h_θ and $h^\star_\theta$, that is, $h^\star_\theta = h_\theta(b_\theta\bullet, b_\theta\bullet)$ holds, see Lemma 2.1. For $\theta = 0$, we introduce the coordinates $z = x + iy$ such that the first, second and third fundamental forms $\mathrm{I}_0, \mathrm{II}_0$ and III_0 of $\tilde f_0$ can be represented by (2.1), (2.2) and (2.3), that is,

$$(\mathrm{I}_0, \mathrm{II}_0, \mathrm{III}_0) = \cosh^2(s/2)(h,\ h(\bullet, B\bullet),\ h(B\bullet, B\bullet))$$

holds, where we set $B = -\tanh(s/2)b$ with $b = b_0$. It has been known [1, Lemma 3.5] that the Labourie operator b_θ is explicitly given by $b_\theta = \beta_{-\theta} \circ b \circ \beta_\theta$.

To show that a CGC surface $\tilde{f}_\theta\,(\theta \in [0, 2\pi))$ constructed by Lemma 2.1 is the associated family $f^{\sqrt{\lambda}}\,(\lambda \in S^1)$ in Section 2.3, we compare $\mathrm{I}_\theta, \mathrm{II}_\theta$ and III_θ and I^λ in (2.7), II^λ in (2.8) and III^λ in (2.9), respectively; Set $\lambda = e^{i\theta} \in S^1$. By Corollary 3.1,

$$\begin{aligned}\mathrm{I}_\theta(\partial_z, \partial_z) &:= \cosh^2(s/2)h(\beta_\theta\partial_z, \beta_\theta\partial_z)\\ &= \cosh^2(s/2)h(\lambda^{1/2}\partial_z, \lambda^{1/2}\partial_z)\\ &= \lambda\cosh^2(s/2)h(\partial_z, \partial_z)\\ &= \mathrm{I}^{\sqrt{\lambda}}(\partial_z, \partial_z)\end{aligned}$$

holds. Similarly, $\mathrm{I}_\theta(\partial_{\bar{z}}, \partial_{\bar{z}}) = \mathrm{I}^{\sqrt{\lambda}}(\partial_{\bar{z}}, \partial_{\bar{z}})$ follows. Moreover,

$$\begin{aligned}\mathrm{I}_\theta(\partial_z, \partial_{\bar{z}}) &:= \cosh^2(s/2)h(\beta_\theta\partial_z, \beta_\theta\partial_{\bar{z}})\\ &= \cosh^2(s/2)h(\lambda^{1/2}\partial_z, \lambda^{-1/2}\partial_{\bar{z}})\\ &= \cosh^2(s/2)h(\partial_z, \partial_{\bar{z}})\\ &= \mathrm{I}^{\sqrt{\lambda}}(\partial_z, \partial_{\bar{z}})\end{aligned}$$

holds, therefore $\mathrm{I}_\theta = \mathrm{I}^{\sqrt{\lambda}}$. The same argument can be applied to III_θ, and it follows $\mathrm{III}_\theta = \mathrm{III}^{\sqrt{\lambda}}$. Next the second fundamental form II_θ can be computed as

$$\mathrm{II}_\theta(\partial_z, \partial_z) = \mathrm{I}_\theta(\partial_z, B_\theta\partial_z).$$

Since $b_\theta = \beta_{-\theta} \circ b \circ \beta_\theta$, thus the shape operator B_θ is given by $\beta_{-\theta} \circ B \circ \beta_\theta$, and thus

$$\begin{aligned}\mathrm{II}_\theta(\partial_z, \partial_z) &= \mathrm{I}_\theta(\partial_z, \beta_{-\theta} \circ B \circ \beta_\theta\partial_z)\\ &= \mathrm{I}(\beta_\theta\partial_z, B \circ \beta_\theta\partial_z)\\ &= \lambda\mathrm{I}(\partial_z, B\partial_z)\\ &= 0 = \mathrm{II}^{\sqrt{\lambda}}(\partial_z, \partial_z)\end{aligned}$$

holds. Similarly $\mathrm{II}_\theta(\partial_{\bar{z}}, \partial_{\bar{z}}) = 0 = \mathrm{II}^{\sqrt{\lambda}}(\partial_{\bar{z}}, \partial_{\bar{z}})$ holds. Finally

$$\begin{aligned}\mathrm{II}_\theta(\partial_z, \partial_{\bar{z}}) &= \mathrm{I}_\theta(\partial_z, \beta_{-\theta} \circ B \circ \beta_\theta\partial_{\bar{z}})\\ &= \mathrm{I}(\beta_\theta\partial_z, B \circ \beta_\theta\partial_{\bar{z}})\\ &= \mathrm{I}(\partial_z, B\partial_{\bar{z}})\\ &= \mathrm{II}^{\sqrt{\lambda}}(\partial_z, \partial_{\bar{z}}).\end{aligned}$$

Thus $\mathrm{II}_\theta = \mathrm{II}^{\sqrt{\lambda}}$ also follows, and therefore the two families of CGC surfaces $\{\tilde{f}_\theta\}_{\theta\in[0,2\pi)}$ and $\{f^{\sqrt{\lambda}}\}_{\lambda\in S^1}$ are the same up to isometry. Since $f = \tilde{f}_\theta|_{\theta=0} = f^{\sqrt{\lambda}}|_{\lambda=1}$ thus the two families exactly coincide. This completes the proof. □

We now look at a relation between the complex landslide and the Lagrangian Gauss map.

Corollary 3.2. *Retain the assumptions in* Proposition 3.1. *Then the complex landslide $P(= SGr \circ L)$ is given by the associated family of the Lagrangian Gauss map of f, and vice versa.*

Proof. Let us consider the smooth grafting SGr_s of the landslide $L_\theta(h, h^\star)$. By Proposition 3.1, there exists the family of corresponding CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surfaces, which is the associated family $\{f^{\sqrt{\lambda}}\}_{\lambda\in S^1}$ with $\lambda = e^{-i\theta}$. Then the developing map $\mathrm{dev}_s : \widetilde{M} \to \mathbb{C}P^1$ of SGr_s is given as in (2.15).

On the one hand, the Lagrangian Gauss map $G^\lambda_{La} = f^\lambda \wedge n^\lambda$ takes values in

$$\mathrm{Geo}(\mathbb{H}^3) \cong \mathrm{Gr}_{1,1}(\mathbb{E}^{1,2}) = \mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C} \cong \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \mathrm{diag},$$

and thus dev_s is given by $\mathrm{pr}_1 \circ G^\lambda_{La}$, where pr_1 denotes the projection to the first component. Moreover, G^λ_{La} can be represented by

$$G^\lambda_{La} = (F^\lambda)^{-1}\boldsymbol{e}_1 F^\lambda \tag{3.3}$$

evaluated at $\sqrt{\lambda_0} = e^{-s/2}e^{-i\theta/2} \in \mathbb{C}^\times \setminus S^1$, see (2.12). Therefore the Lagrangian Gauss map evaluated at $\sqrt{\lambda_0} \in \mathbb{C}^\times \setminus S^1$ gives the developing map dev_s and thus the complex landslide. The converse statement is also clear. This completes the proof. □

By Corollary 3.2, the holonomy of the complex landslide $P_q = SGr_s \circ L_\theta$ with $s = -\log|q|, \theta = -\arg q$ is given by the holonomy of the extened frame F^λ evaluated at $\lambda = \sqrt{q}$, and the statement of Theorem 1.1 follows. We are now ready to prove Corollary 1.1 in Introduction.

Proof of Corollary 1.1. To see the holomorphic dependence with respect to λ, we consider the gauged extended frame:

$$F^\lambda \boldsymbol{e}_1 (F^\lambda)^{-1}|_{\lambda=\sqrt{\lambda_0}} = F^\lambda D^\lambda \boldsymbol{e}_1 (F^\lambda D^\lambda)^{-1}|_{\lambda=\sqrt{\lambda_0}} = D^{\sqrt{\lambda}} \hat{F}^\lambda \boldsymbol{e}_1 (\hat{F}^\lambda)^{-1} D^{1/\sqrt{\lambda}}\big|_{\lambda=\lambda_0},$$

where $D^\lambda = \mathrm{diag}(\lambda^{-1/2}, \lambda^{1/2})$ and we set $\hat{F}^\lambda = (D^{1/\lambda}F^\lambda D^\lambda)|_{\lambda=\sqrt{\lambda}}$. Recalling that $\alpha^\lambda = (F^\lambda)^{-1}\mathrm{d}F^\lambda = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{p}}$, it is straightforward to see that

$$\hat{\alpha}^\lambda := (\hat{F}^\lambda)^{-1}\mathrm{d}\hat{F}^\lambda = \lambda^{-1}{\alpha'_{\mathfrak{p}}}_{21} + {\alpha'_{\mathfrak{p}}}_{12} + \alpha_{\mathfrak{k}} + \lambda{\alpha''_{\mathfrak{p}}}_{12} + {\alpha''_{\mathfrak{p}}}_{21}, \tag{3.4}$$

where $x_{ij}(i, j \in \{1, 2\})$ denotes the (ij)-entry for a 2 by 2 matrix valued 1-form x. Therefore $\hat{F}^\lambda$ is holomorphic with respect to $\lambda \in \mathbb{D}^\times$, and moreover

$$\widehat{\mathrm{dev}}_s = \mathrm{pr}_1 \circ \left(\hat{F}^\lambda \boldsymbol{e}_1 (\hat{F}^\lambda)^{-1}\right)\Big|_{\lambda=\lambda_0}$$

defines the same complex projective structure given by dev_s. The gauged frame $\hat{F}^\lambda$ will be called the *untwisted extended frame*, and we set $\widehat{\nabla}^\lambda = \mathrm{d} + \hat{\alpha}^\lambda$.

Then the holonomy of the untwisted extended frame of $\widehat{\nabla}^\lambda$ is given by

$$\hat{H}^\lambda : \pi_1(M) \to \Lambda\mathrm{SL}(2, \mathbb{C})$$

with $\gamma^*\hat{F}^\lambda = \hat{H}^\lambda\hat{F}^\lambda$ for $\gamma \in \pi_1(M)$ and $\lambda \in \mathbb{D}^\times$. Clearly the holonomy $\hat{H}^\lambda$ is holomorphic with respect to the spectral parameter $\lambda \in \mathbb{D}^\times$, and thus the holonomy of the developing map $\widehat{\mathrm{dev}}_s$ is holomorphic with respect to λ in $\mathbb{D}^\times$. The limit $s \to 0$ to the holonomy of the developing map $\widehat{\mathrm{dev}}_s$ gives the holonomy at $\lambda \in S^1$. This completes the proof. □

Remark 3.1. *It is well known* [3] *that the relation*

$$F^\lambda \longleftrightarrow \hat{F}^\lambda = (D^{1/\lambda}F^\lambda D^\lambda)|_{\lambda=\sqrt{\lambda}} \quad \textit{with} \quad D^\lambda = \mathrm{diag}(\lambda^{1/2}, \lambda^{-1/2})$$

is an isomorphism between twisted and untwisted loop groups, that is, F^λ belongs to the twisted loop group $\Lambda\mathrm{SL}(2,\mathbb{C})_\tau$ with respect to τ, that is, $F^{-\lambda} = \tau F^\lambda$ holds, while $\hat{F}^\lambda$ belongs to untwisted loop group $\Lambda\mathrm{SL}(2,\mathbb{C})$ and it does not satisfy such relation.

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