Three Simple Reduction Formulas for the Denumerant Functions

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April 22, 2024

Abstract

Let A be a nonempty set of positive integers. The restricted partition function $p_A(n)$ denotes the number of partitions of n with parts in A. When the elements in A are pairwise relatively prime positive integers, Ehrhart, Sertöz-Özlük, and Brown-Chou-Shiue derived three reduction formulas for $p_A(n)$ for A with three parameters. We extend their findings for general A using the Bernoulli-Barnes polynomials.

Mathematic subject classification: Primary 05A17; Secondary 05A15, 11P81.

Keywords: Denumerant; Restricted partition function; Bernoulli-Barnes polynomial.

1 Introduction

Let $A = \{a_1, a_2, ..., a_k\}$ be a set of positive integers with $k \geq 1$. Furthermore, let $p_A(n)$ denote the number of nonnegative integer solutions to the equation

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = n.$$

The $p_A(n)$ is called the *restricted partition function* of the set A. Some scholars also refer to it as *Sylvester's denumerant* [14] when $\gcd(A) = 1$.

Sylvester [14] and Bell [4] proved that $p_A(n)$ is a quasi-polynomial of degree $k-1$, and the period is a common multiple of $a_1, a_2, \ldots, a_k$. Beck, Gessel, and Komatsu [3] found an

*This work is partially supported by the National Natural Science Foundation of China (No.12071311).

expression for the polynomial part of $p_A(n)$. Nathanson [11] gave an asymptotic formula of $p_A(n)$. Cimpoeas [6] proved that the $p_A(n)$ can be reduced to solving a linear congruence formula. Some relevant references can be found in [1, 2, 10, 16].

For $k = 2$, Sertöz [12] and Tripathi [15] independently obtained an explicit formula for $p_A(n)$. For $k = 3$, Ehrhart [7, 8] and Sertöz and Özlük [13] gave recursive formulae for $p_A(n)$. In this paper, we first extend the results of Ehrhart [7, 8] (the case $k = 2, 3$ in Theorem 1.1) as follows.

Theorem 1.1. *Let $A = \{a_1, a_2, \ldots, a_k\}$, where $a_1, a_2, \ldots, a_k$ are pairwise relatively prime positive integers. Let $n = q \cdot a_1a_2\cdots a_k + r$ with $0 \le r < a_1a_2\cdots a_k$. Then*

$$p_A(n) = p_A(r) + (-1)^k(n-r)\sum_{i=0}^{k-2}\frac{(r-n)^i}{(i+1)!(k-i-2)!}\mathcal{B}_{k-i-2}(-r; a_1, a_2, \ldots, a_k),$$

where $\mathcal{B}_i(x; a_1, a_2, \ldots, a_k)$ is the Bernoulli-Barnes polynomials (defined by Equation (2)).

Secondly, we generalize the results of Sertöz and Özlük [13] (the case $k = 3$ in Theorem 1.2) as follows.

Theorem 1.2. *Let $A = \{a_1, a_2, \ldots, a_k\}$, where $a_1, a_2, \ldots, a_k$ are pairwise relatively prime positive integers. Let $1 \le x \le a_1 + a_2 + \cdots + a_k - 1$. Then*

$$p_A(a_1a_2\cdots a_k - x) = (-1)^k(a_1a_2\cdots a_k)\sum_{i=0}^{k-2}\frac{(-a_1a_2\cdots a_k)^i}{(i+1)!(k-i-2)!}\mathcal{B}_{k-i-2}(x; a_1, a_2, \ldots, a_k).$$

Thirdly, we extend the results of Brown, Chou, and Shiue [5] (the case $k = 3$ and $x = a_1 + a_2 + a_3$ (and $x = a_1 + a_2 + a_3 + 1$) in Theorem 1.3) as follows.

Theorem 1.3. *Let $A = \{a_1, a_2, \ldots, a_k\}$, where $a_1, a_2, \ldots, a_k$ are pairwise relatively prime positive integers. Let $a_1 + a_2 + \cdots + a_k \le x \le a_1a_2\cdots a_k$. Then*

$$\begin{aligned} p_A(a_1a_2\cdots a_k - x) &+ (-1)^k p_A(x - a_1 - a_2 - \cdots - a_k) \\ &= (-1)^k(a_1a_2\cdots a_k)\sum_{i=0}^{k-2}\frac{(-a_1a_2\cdots a_k)^i}{(i+1)!(k-i-2)!}\mathcal{B}_{k-i-2}(x; a_1, a_2, \ldots, a_k). \end{aligned}$$

This paper is organized as follows. In Section 2, we introduce some necessary notations and provide the proof of Theorem 1.1. In Section 3, we give a recursive formula for $p_A(n) - p_A(r)$, where $0 \le r < a_1a_2\cdots a_k$. Sections 4 and 5 give the proofs of Theorems 1.2 and 1.3, respectively. Throughout this paper, $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{P}$ denote the set of all complex numbers, all nonnegative integers, and all positive integers, respectively.

2 The Proof of Theorem 1.1

Before obtaining the main results of this section, we need to introduce some definitions and conclusions. Let $f(\lambda)$ be a rational function in $\mathbb{C}((\lambda))$. The $\mathrm{CT}_\lambda f(\lambda)$ denotes the constant term of the Laurent series expansion of $f(\lambda)$ at $\lambda = 0$. The $\mathrm{Res}_{\lambda=\lambda_0} f(\lambda)$ denotes the residue of $f(\lambda)$ when expanded as a Laurent series at $\lambda = \lambda_0$. More precisely, we have

$$\operatorname*{Res}_{\lambda=\lambda_0} \sum_{i\geq i_0} c_i(\lambda-\lambda_0)^i = c_{-1}.$$

For the denumerant $p_A(n)$ with $A = \{a_1, a_2, ..., a_k\}$, we have

$$p_A(n) = \sum_{x_i\geq 0} \operatorname*{CT}_{\lambda} \lambda^{x_1a_1+x_2a_2+\cdots+x_ka_k-n} = \operatorname*{CT}_{\lambda} \frac{\lambda^{-n}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}. \tag{1}$$

Lemma 2.1 ([9])**.** *Let c be a complex number. Suppose $g(s)$ is holomorphic in a neighborhood of $s = c$ and suppose $f(\lambda)$ is meromorphic in a neighborhood of $\lambda = g(c)$. If $g'(c) \neq 0$, then*

$$\operatorname*{Res}_{\lambda=g(c)} f(\lambda) = \operatorname*{Res}_{s=c} f(g(s))g'(s).$$

Lemma 2.2. *Let $r_1, r_2, \ldots, r_k \in \mathbb{P}$ and $b \leq r_1 + r_2 + \cdots + r_k - 1$. Suppose*

$$f(z) = \frac{z^{b-1}}{(z-\xi_1)^{r_1}(z-\xi_2)^{r_2}\cdots(z-\xi_k)^{r_k}}.$$

Then

$$\operatorname*{Res}_{z=0} f(z) = -\sum_{i=1}^{k} \operatorname*{Res}_{z=\xi_i} f(z).$$

Proof. A well-known result in residue computation asserts that

$$\operatorname*{Res}_{z=\infty} f(z) + \operatorname*{Res}_{z=0} f(z) + \sum_{i=1}^{k} \operatorname*{Res}_{z=\xi_i} f(z) = 0.$$

The lemma then follows by showing that $\operatorname*{Res}_{z=\infty} f(z) = 0$. Direct computation gives

$$\begin{aligned}\operatorname*{Res}_{z=\infty} f(z) &= \operatorname*{Res}_{z=0} f(z^{-1})\cdot(-z^{-2}) \\ &= \operatorname*{Res}_{z=0} \frac{-z^{-b-1}}{(z^{-1}-\xi_1)^{r_1}\cdots(z^{-1}-\xi_k)^{r_k}} = \operatorname*{Res}_{z=0} \frac{-z^{r_1+\cdots+r_k-b-1}}{(1-z\xi_1)^{r_1}\cdots(1-z\xi_k)^{r_k}}.\end{aligned}$$

Since $b \leq r_1 + \cdots + r_k - 1$, the expansion of $\frac{-z^{r_1+\cdots+r_k-b-1}}{(1-z\xi_1)^{r_1}\cdots(1-z\xi_k)^{r_k}}$ is a power series in z. Therefore, its residue at $z = 0$ is 0. This completes the proof. □

For $a_1, a_2, \ldots, a_k \in \mathbb{P}$, the *Bernoulli-Barnes polynomials* $\mathcal{B}_i(x; a_1, a_2, \ldots, a_k)$ are polynomials in x defined by

$$\frac{s^k e^{xs}}{(e^{a_1s}-1)(e^{a_2s}-1)\cdots(e^{a_ks}-1)} = \sum_{i\geq 0} \mathcal{B}_i(x; a_1, a_2, \ldots, a_k)\frac{s^i}{i!}. \tag{2}$$

Proof of Theorem 1.1. By Equation (1), we have

$$\begin{aligned}
p_A(n) - p_A(r) &= \operatorname*{CT}_{\lambda} \frac{\lambda^{-n}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})} - \operatorname*{CT}_{\lambda} \frac{\lambda^{-r}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})} \\
&= \operatorname*{CT}_{\lambda} \frac{\lambda^{-r}(\lambda^{-qa_1a_2\cdots a_k}-1)}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.
\end{aligned}$$

For convenience, let

$$F(\lambda) = \frac{\lambda^{-r-1}(\lambda^{-qa_1a_2\cdots a_k}-1)}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.$$

Then

$$p_A(n) - p_A(r) = \operatorname*{CT}_{\lambda} \lambda F(\lambda) = \operatorname*{Res}_{\lambda=0} F(\lambda) = -\sum_{\xi} \operatorname*{Res}_{\lambda=\xi} F(\lambda), \quad \text{(By Lemma 2.2.)}$$

where ξ ranges over all nonzero poles of $F(\lambda)$. We claim that $\mathrm{Res}_{\lambda=\xi} F(\lambda) = 0$ unless $\xi = 1$. Since $a_1, a_2, \dots, a_k$ are pairwise relatively prime positive integers, each $\xi \neq 1$ appears exactly once in the denominator, but the numerator also vanishes at these ξ's. Therefore, we obtain

$$\begin{aligned}
p_A(n) - p_A(r) &= -\operatorname*{Res}_{\lambda=1} F(\lambda) = -\operatorname*{Res}_{s=0} F(e^s)e^s \qquad \text{(By Lemma 2.1.)} \\
&= -\operatorname*{CT}_{s} F(e^s)e^s s \qquad (3) \\
&= -\operatorname*{CT}_{s} \frac{e^{-rs}(e^{-qa_1a_2\cdots a_k s}-1)s}{(1-e^{a_1 s})(1-e^{a_2 s})\cdots(1-e^{a_k s})} \\
&= (-1)^k q a_1 a_2 \cdots a_k \operatorname*{CT}_{s} \frac{1}{s^{k-2}} \cdot \frac{e^{-qa_1a_2\cdots a_k s}-1}{-qa_1a_2\cdots a_k s} \cdot \frac{s^k e^{-rs}}{\prod_{i=1}^{k}(e^{a_i s}-1)} \\
&= (-1)^k q a_1 a_2 \cdots a_k [s^{k-2}] \sum_{i\ge 0} \frac{(-qa_1a_2\cdots a_k)^i}{(i+1)!} s^i \cdot \sum_{j\ge 0} \mathcal{B}_j(-r; a_1, a_2, \dots, a_k) \frac{s^j}{j!} \\
&= (-1)^k (n-r) \sum_{i=0}^{k-2} \frac{(r-n)^i}{(i+1)!(k-i-2)!} \mathcal{B}_{k-i-2}(-r; a_1, a_2, \dots, a_k).
\end{aligned}$$

This completes the proof. □

Corollary 2.3 ([7, 8]). *Following the notation in Theorem 1.1. If $A = \{a_1, a_2\}$, then*

$$p_A(n) = p_A(r) + \frac{n-r}{a_1 a_2}.$$

If $A = \{a_1, a_2, a_3\}$, then

$$p_A(n) = p_A(r) + \frac{q(n+r+a_1+a_2+a_3)}{2}.$$

Proof. By

$$\frac{s^2 e^{-rs}}{(e^{a_1 s}-1)(e^{a_2 s}-1)} = \frac{1}{a_1 a_2} + o(s)$$

and

$$\frac{s^3e^{-rs}}{(e^{a_1s}-1)(e^{a_2s}-1)(e^{a_3s}-1)}=\frac{1}{a_1a_2a_3}-\frac{2r+a_1+a_2+a_3}{2a_1a_2a_3}s+o(s^2),$$

the corollary follows from Theorem 1.1. □

Corollary 2.4. *Following the notation in Theorem 1.1. If* $A=\{a_1,a_2,a_3,a_4\}$*, then*

$$p_A(n)=p_A(r)+\frac{q}{12}\Big(3(n+r)(a_1+a_2+a_3+a_4)+2(n+r)^2-2nr+(a_1+a_2+a_3+a_4)^2 \\ +a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4\Big).$$

If $A=\{a_1,a_2,a_3,a_4,a_5\}$*, then*

$$p_A(n)=p_A(r)+\frac{q}{24}\bigg((n+r)(n^2+r^2)+(2n^2+2nr+2r^2)\Big(\sum_{i=1}^{5}a_i\Big)+(n+r)\Big(\sum_{i=1}^{5}a_i^2\Big) \\ +\sum_{i=1}^{5}a_i^2\Big(\sum_{j=1}^{5}a_j-a_i\Big)+3(n+r)\sum_{1\le i<j\le 5}a_ia_j+3\sum_{1\le i<j\le 5}\frac{a_1a_2a_3a_4a_5}{a_ia_j}\bigg).$$

The proof of Corollary 2.4 is analogous to that of Corollary 2.3 and is left to the reader.

3 A Recursive Formula for $p_A(n)-p_A(r)$

Readers familiar with symmetric functions may find that the formulas in Corollary 2.4 are related to the power sum symmetric function. This does not occur occasionally. We will describe this connection in general. We will also give a recursive formula for $p_A(n)-p_A(r)$ when $n-r=qa_1\cdots a_k$ as in Section 2.

We need the following definitions. The *m-th power sum symmetric function* is

$$p_m(x_1,x_2,\ldots)=\sum_{i\ge 1}x_i^m.$$

We only use the symmetric functions on a finite number of variables, say $x_1,\ldots,x_k$. One can treat $x_i=0$ for $i>k$. We have $p_m(x_1,x_2,\ldots,x_k)=\sum_{i=1}^k x_i^m$. The *Bernoulli numbers* $\mathcal{B}_i$ are defined by

$$\frac{s}{e^s-1}=1-\mathcal{B}_1s+\sum_{i\ge 2}\mathcal{B}_i\frac{s^i}{i!}=1-\frac{1}{2}s+\frac{1}{12}s^2-\frac{1}{720}s^4+\cdots.$$

Then

$$\ln\frac{s}{e^s-1}=-\sum_{i\ge 1}\frac{\mathcal{B}_i}{i!\cdot i}s^i=-\frac{1}{2}s-\frac{1}{24}s^2+\frac{1}{2880}s^4+\cdots.$$

By the proof of Theorem 1.1, we have

$$p_A(n) - p_A(r) = (-1)^k q \operatorname*{CT}_s \frac{1}{s^{k-2}} \cdot e^{-rs} \cdot \frac{e^{-qa_1a_2\cdots a_k s} - 1}{-qa_1a_2\cdots a_k s} \cdot \prod_{i=1}^{k} \frac{a_i s}{e^{a_i s} - 1}.$$

We consider the following formula. The technique for taking logarithms below comes from [17].

$$\begin{aligned}
h(s) &= \ln\left(e^{-rs} \cdot \frac{e^{-qa_1a_2\cdots a_k s} - 1}{-qa_1a_2\cdots a_k s} \cdot \prod_{i=1}^{k} \frac{a_i s}{e^{a_i s} - 1}\right) \\
&= -rs + \ln\left(\frac{-qa_1a_2\cdots a_k s}{e^{-qa_1a_2\cdots a_k s} - 1}\right)^{-1} + \sum_{i=1}^{k} \ln \frac{a_i s}{e^{a_i s} - 1} \\
&= -rs + \sum_{i\geq 1} \frac{\mathcal{B}_i}{i! \cdot i}(-qa_1a_2\cdots a_k)^i s^i + \sum_{j\geq 1} \frac{-\mathcal{B}_j p_j(a_1, a_2, \ldots, a_k)}{j! \cdot j} s^j \\
&= -rs + \sum_{i\geq 1} \frac{\mathcal{B}_i}{i! \cdot i}((r-n)^i - p_i(a_1, a_2, \ldots, a_k))s^i.
\end{aligned}$$

Then

$$p_A(n) - p_A(r) = (-1)^k q \operatorname*{CT}_s \frac{1}{s^{k-2}} \cdot e^{h(s)} = (-1)^k q [s^{k-2}] e^{h(s)}.$$

Let $f(s) = e^{h(s)} = \sum_{i\geq 0} f_i s^i$. By

$$f'(s) = e^{h(s)} \cdot h'(s) = f(s) \cdot h'(s),$$

we have

$$\sum_{i\geq 0} i f_i s^{i-1} = \sum_{i\geq 0}(f_i s^i) \cdot \sum_{i\geq 0}(h'_i s^i),$$

that is

$$f_0 = 1, \quad f_i = \frac{1}{i} \cdot \sum_{j=1}^{i} f_{i-j} h'_{j-1}, \quad (i \geq 1).$$

Therefore, we have

$$p_A(n) - p_A(r) = (-1)^k q \cdot f_{k-2}.$$

We summarize the above discussion as follows.

Theorem 3.1. *Let $A = \{a_1, a_2, \ldots, a_k\}$, where $a_1, a_2, \ldots, a_k$ are pairwise relatively prime positive integers. Let $n = q \cdot a_1a_2\cdots a_k + r$ with $0 \leq r < a_1a_2\cdots a_k$. Suppose*

$$h(s) = -rs + \sum_{i\geq 1} \frac{\mathcal{B}_i}{i! \cdot i}((r-n)^i - p_i(a_1, a_2, \ldots, a_k))s^i, \qquad h'(s) = \sum_{i\geq 0} h'_i s^i.$$

Then

$$p_A(n) = p_A(r) + (-1)^k q \cdot f_{k-2},$$

where f_i can be recursively obtained by

$$f_0 = 1, \qquad f_i = \frac{1}{i} \cdot \sum_{j=1}^{i} f_{i-j} h'_{j-1}, \quad (i \geq 1).$$

4 The Proof of Theorem 1.2

Proof of Theorem 1.2. By Equation (1), we have

$$p_A(a_1a_2\cdots a_k - x) = \operatorname*{CT}_{\lambda} \frac{\lambda^{-a_1a_2\cdots a_k+x}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.$$

The rational function

$$\frac{\lambda^x}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}$$

is a proper rational function since $1 \le x \le a_1 + a_2 + \cdots + a_k - 1$. Obviously, its constant term is 0. We have

$$p_A(a_1a_2\cdots a_k - x) = \operatorname*{CT}_{\lambda} \frac{\lambda^{-a_1a_2\cdots a_k+x} - \lambda^x}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})} = \operatorname*{CT}_{\lambda} \frac{\lambda^x(\lambda^{-a_1a_2\cdots a_k} - 1)}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.$$

The remainder of the argument is analogous to that in Theorem 1.1 and is left to the reader. □

Similar to Corollaries 2.3 and 2.4, we can obtain the following three corollaries. We omit the proofs.

Corollary 4.1 ([12])**.** *Let $A = \{a_1, a_2\}$ with $\gcd(A) = 1$. Let $1 \le x \le a_1 + a_2 - 1$. Then*

$$p_A(a_1a_2 - x) = 1.$$

Corollary 4.2 ([13])**.** *Let $A = \{a_1, a_2, a_3\}$, where a_1, a_2, a_3 are pairwise relatively prime positive integers. Let $1 \le x \le a_1 + a_2 + a_3 - 1$. Then*

$$p_A(a_1a_2a_3 - x) = \frac{a_1a_2a_3 + a_1 + a_2 + a_3}{2} - x.$$

Corollary 4.3. *Following the notation in Theorem 1.2. If $A = \{a_1, a_2, a_3, a_4\}$, then*

$$p_A(a_1a_2a_3a_4 - x) = \frac{1}{12}\Big(3(n-x)(a_1+a_2+a_3+a_4) + 2(n-x)^2 + 2nx + (a_1+a_2+a_3+a_4)^2 + a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4\Big),$$

where $n = a_1a_2a_3a_4 - x$.

If $A = \{a_1, a_2, a_3, a_4, a_5\}$, then

$$p_A(a_1a_2a_3a_4a_5 - x) = \frac{1}{24}\Bigg((n-x)(n^2+x^2) + (2n^2 - 2nx + 2x^2)\Big(\sum_{i=1}^{5} a_i\Big) + (n-x)\Big(\sum_{i=1}^{5} a_i^2\Big) + \sum_{i=1}^{5} a_i^2\Big(\sum_{j=1}^{5} a_j - a_i\Big) + 3(n-x)\sum_{1\le i<j\le 5} a_ia_j + 3\sum_{1\le i<j\le 5} \frac{a_1a_2a_3a_4a_5}{a_ia_j}\Bigg),$$

where $n = a_1a_2a_3a_4a_5 - x$.

5 The Proof of Theorem 1.3

Proof of Theorem 1.3. By Equation (1), we have

$$p_A(a_1a_2\cdots a_k - x) = \operatorname*{CT}_{\lambda} \frac{\lambda^{-a_1a_2\cdots a_k+x}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.$$

Since $a_1 + a_2 + \cdots + a_k \le x \le a_1a_2\cdots a_k$, we have

$$\operatorname*{CT}_{\lambda} \frac{\lambda^{x}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})} = 0.$$

We have

$$p_A(a_1a_2\cdots a_k - x) = \operatorname*{CT}_{\lambda} \frac{\lambda^{-a_1a_2\cdots a_k+x} - \lambda^x}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})} = \operatorname*{CT}_{\lambda} \frac{\lambda^{x}(\lambda^{-a_1a_2\cdots a_k} - 1)}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.$$

Let

$$G(\lambda) = \frac{\lambda^{x-1}(\lambda^{-a_1a_2\cdots a_k} - 1)}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.$$

Then

$$p_A(a_1a_2\cdots a_k - x) = \operatorname*{CT}_{\lambda} \lambda G(\lambda) = \operatorname*{Res}_{\lambda=0} G(\lambda) = -\sum_{\xi} \operatorname*{Res}_{\lambda=\xi} G(\lambda) - \operatorname*{Res}_{\lambda=\infty} G(\lambda),$$

where ξ ranges over all nonzero poles of $G(\lambda)$. Similar to the proof of Theorem 1.1, we have

$$p_A(a_1a_2\cdots a_k - x) = -\operatorname*{Res}_{\lambda=1} G(\lambda) - \operatorname*{Res}_{\lambda=\infty} G(\lambda).$$

By $a_1 + a_2 + \cdots + a_k \le x \le a_1a_2\cdots a_k$, we obtain

$$\operatorname*{Res}_{\lambda=\infty} \frac{\lambda^{x-1-a_1a_2\cdots a_k}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})} = 0. \tag{4}$$

The proof of Equation (4) is similar to Lemma 2.2. Let

$$G_1(\lambda) = \frac{-\lambda^{x-1}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})}.$$

Then

$$\begin{aligned} p_A(a_1a_2\cdots a_k - x) &= -\operatorname*{Res}_{\lambda=1} G(\lambda) - \operatorname*{Res}_{\lambda=\infty} G_1(\lambda) = -\operatorname*{Res}_{s=0} G(e^s)\cdot e^s - \operatorname*{Res}_{\lambda=0} G_1(\lambda^{-1})\cdot \frac{-1}{\lambda^2} \\ &= -\operatorname*{CT}_{s} G(e^s)\cdot e^s \cdot s + \operatorname*{CT}_{\lambda} G_1(\lambda^{-1}) \cdot \frac{1}{\lambda} \\ &= -\operatorname*{CT}_{s} G(e^s)\cdot e^s \cdot s + \operatorname*{CT}_{\lambda} \frac{(-1)^{k+1}\lambda^{a_1+a_2+\cdots+a_k-x}}{(1-\lambda^{a_1})(1-\lambda^{a_2})\cdots(1-\lambda^{a_k})} \\ &= -\operatorname*{CT}_{s} G(e^s)\cdot e^s \cdot s + (-1)^{k+1} p_A(x - a_1 - a_2 - \cdots - a_k). \end{aligned}$$

The remainder of the argument is analogous to Equation (3) and is left to the reader. □

Corollary 5.1. *Let $A = \{a_1, a_2, a_3\}$, where a_1, a_2, a_3 are pairwise relatively prime positive integers. Let $a_1 + a_2 + a_3 \le x \le a_1a_2a_3$. Then*

$$p_A(a_1a_2a_3 - x) - p_A(x - a_1 - a_2 - a_3) = \frac{a_1a_2a_3 + a_1 + a_2 + a_3}{2} - x.$$

When $x = a_1 + a_2 + a_3$ (and $x = a_1 + a_2 + a_3 + 1$) in Corollary 5.1, we obtain the results of Brown, Chou, and Shiue [5] as follows:

$$p_A(a_1a_2a_3 - a_1 - a_2 - a_3) = \frac{a_1a_2a_3 - a_1 - a_2 - a_3}{2} + 1,$$

and

$$p_A(a_1a_2a_3 - a_1 - a_2 - a_3 - 1) = p_A(1) + \frac{a_1a_2a_3 - a_1 - a_2 - a_3}{2} - 1.$$

Note: $p_A(1) = 0$ when $a_1, a_2, a_3 \ge 2$.

Acknowledgements: This work is partially supported by the National Natural Science Foundation of China [12071311].

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