ON PRO-ZERO HOMOMORPHISMS AND SEQUENCES IN LOCAL (CO-)HOMOLOGY

PETER SCHENZEL

ABSTRACT. Let $\underline{x}$ denote a system of elements of a commutative ring R. For an R-module M we investigate when $\underline{x}$ is M-pro-regular resp. M-weakly pro-regular as generalizations of M-regular sequences. This is done in terms of Čech co-homology resp. homology, defined by $H^i(\check{C}_{\underline{x}} \otimes_R \cdot)$ resp. by $H_i(\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, \cdot)) \cong H_i(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, \cdot))$, where $\check{C}_{\underline{x}}$ denotes the Čech complex and $\mathcal{L}_{\underline{x}}$ is a bounded free resolution of it as constructed in [17] resp. [16]. The property of $\underline{x}$ being M-pro-regular resp. M-weakly pro-regular follows by the vanishing of certain Čech co-homology resp. homology modules, which is related to completions. This extends previously work by Greenlees and May (see [5]) and Lipman et al. (see [1]). This contributes to a further understanding of Čech (co-)homology in the non-Noetherian case. As a technical tool we use one of Emmanouil's results (see [4]) about the inverse limits and its derived functor. As an application we prove a global variant of the results with an application to prisms in the sense of Bhatt and Scholze (see [3]).

1. INTRODUCTION

Let R denote a commutative ring with $\underline{x} = x_1, \ldots, x_r$ a system of elements. For an R-module M we study generalizations of a M-regular sequence called M-pro-regular sequence and M-weakly pro-regular sequence. To this end we denote by $\check{C}_{\underline{x}}$ the Čech complex with respect to $\underline{x}$ (see e.g. [17, 6.1]). It is a bounded complex of flat R-modules. For an R-module M we write $\check{C}_{\underline{x}}(M) = \check{C}_{\underline{x}} \otimes_R M$. We call $\check{H}^i_{\underline{x}}(M) = H^i(\check{C}_{\underline{x}}(M)), i \in \mathbb{Z}$, the Čech cohomology of M. Dually we look at the complex $\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$ in the derived category. There is a free resolution of $\check{C}_{\underline{x}}$ by a bounded complex $\mathcal{L}_{\underline{x}}$ and $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)$ is a representative of $\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$ (see [17] and [16]). We define $\check{H}^{\underline{x}}_i(M) = H_i(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)) \cong H_i(\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)), i \in \mathbb{Z}$, as the Čech homology of M. For the case of R a Noetherian ring let $\mathfrak{a} = \underline{x}R$ then it follows that $\check{H}^i_{\underline{x}}(M) \cong H^i_{\mathfrak{a}}(M)$, the i-th local cohomology of M with support in $\mathfrak{a}$. At first this was established by Grothendieck (see [6] and [7]). Dually, for Noetherian rings R we have $\check{H}^{\underline{x}}_i(M) \cong \Lambda^{\mathfrak{a}}_i(M)$, where $\Lambda^{\mathfrak{a}}_i(\cdot)$ denotes the left derived functors of the completion $\Lambda^{\mathfrak{a}}(\cdot)$. Contributions were done by Matlis (see [9]), Simon (see [18]), Greenlees and May (see [5]) and others.

Starting with Greenlees and May (see [5]) and Lipman et al. (see [1]) there were extensions to non-Noetherian rings with sequences $\underline{x}$ that are called pro-regular resp. weakly pro-regular (see below for the definitions). In particular, when $\underline{x}$ is weakly pro-regular the isomorphisms $\check{H}^i_{\underline{x}}(M) \cong H^i_{\mathfrak{a}}(M)$ and $\check{H}^{\underline{x}}_i(M) \cong \Lambda^{\mathfrak{a}}_i(M)$ hold for any $i \in \mathbb{Z}$ and any R-module M and more generally for any complex $X \in D(R)$ (see [11], [12] , [14] and [17] for more details).

In the situation of $\underline{x}$ an R-regular sequence there is a corresponding property of $\underline{x}$ being an M-regular sequence (see e.g. [10]). This is a challenge for the study of the relative version that

Date: April 23, 2024.
2020 *Mathematics Subject Classification.* Primary: 13Dxx; Secondary: 13B35, 13C11 .
Key words and phrases. Čech homology and cohomology, pro-zero inverse systems, weakly pro-regular sequences, completion, prisms.

$\underline{x}$ is weakly M-regular for modules instead of $M = R$. Namely, $\underline{x}$ is called an M-weakly pro-regular sequence (see also [17, 7.3.1]) provided the inverse system $\{H_i(\underline{x}^{(n)}; M)\}_{n \geq 1}$ is pro-zero for $i = 1, \ldots, r$, i.e. for each n there is an integer $m \geq n$ such that the natural map $H_i(\underline{x}^{(m)}; M) \to H_i(\underline{x}^{(n)}; M)$ is zero. Here $\underline{x}^{(n)} = x_1^n, \ldots, x_r^n$ and $H_i(\underline{x}^{(n)}; M)$ denotes the Koszul homology. An R-weakly pro-regular sequence is called weakly pro-regular. For a first description of M-weakly pro-regular sequences see [15, Theorem 4.2]. Let $\widehat{M}^{\underline{x}} = \Lambda^{\underline{x}}(M)$ denote the $\underline{x}$-adic completion of M.

Theorem 1.1. *For an R-module M and a sequence $\underline{x} = x_1, \ldots, x_r$ the following is equivalent:*

(i) *$\underline{x}$ is M-weakly pro-regular.*
(ii) *$\check{C}_{\underline{x}}(\operatorname{Hom}_R(M, I))$ is a right resolution of $\operatorname{Hom}_R(\Lambda^{\underline{x}}(M), I)$ for any injective R-module I.*
(iii) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M \otimes_R F)$ is a left resolution of $\Lambda^{\underline{x}}(M \otimes_R F)$ for any free R-module F.*
(iv) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, X)$ is a left resolution of $\Lambda^{\underline{x}}(X)$ for $X = M, M[T]$.*
(v) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M[T])$ is a left resolution of $\Lambda^{\underline{x}}(M[T])$.*

Note that the equivalence of (i), (iii) and (iv) in the particular case of $M = R$ was shown by Positselski (see [12, Theorem 3.6]), that is in the case when $\underline{x}$ is R-weakly pro-regular (or weakly pro-regular for short). Then the complexes $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, X)$ and $\mathrm{L}\Lambda^{\underline{x}}(X)$ are isomorphic in the derived category for all $X \in D(R)$ (see [11] generalizing the case of bounded complexes shown in [16]). For the proof of 1.1 and the notion of left/right resolution see the comments after 3.6.

The notion of a weakly pro-regular sequence $\underline{x} = x_1, \ldots, x_r$ is defined in terms of the Koszul homology of the whole sequence $\underline{x}$. An M-regular sequence is defined by the vanishing of $\underline{x}_{i-1}M :_M x_i / \underline{x}_{i-1}M$ for $i = 1, \ldots, r$, where $\underline{x}_{i-1} = x_1, \ldots, x_{i-1}$. As a generalization of that Greenlees and May (see [5]) resp. Lipman et al. (see [1]) invented the notion of an M-pro-regular sequence. Note that both of the definitions are equivalent (see [15, Proposition 2.2]). A sequence $\underline{x}$ is called M-pro-regular if the inverse system $\{\underline{x}_{i-1}^{(n)}M :_M x_i^n / \underline{x}_{i-1}^{(n)}M)\}_{n \geq 1}$ with multiplication by x_i^n is pro-zero for $i = 1, \ldots, r$. Note that if $\underline{x}$ is M-regular it is also M-weakly pro-regular since $\underline{x}_{i-1}^{(n)}M :_M x_i^n / \underline{x}_{i-1}^{(n)}M = 0$ (see [10, 16.1]). A characterization of pro-regular sequences in terms of Čech cohomology is known (see [15, Theorem 3.2] and 4.4). Here there is a description in the terms of Čech homology. See 4.5 for the following:

Theorem 1.2. *Let $\underline{x} = x_1, \ldots, x_r$ denote a sequence of elements of R. For an R-module M the following conditions are equivalent:*

(i) *The sequence $\underline{x}$ is M-pro-regular.*
(ii) *$\check{H}_0^{x_i}(\Lambda^{\underline{x}_{i-1}}(M \otimes_R F)) \cong \Lambda^{\underline{x}_i}(M \otimes_R F)$ and $\check{H}_1^{x_i}(\Lambda^{\underline{x}_{i-1}}(M \otimes_R F)) = 0$ for $i = 1, \ldots, r$ and any free R-module F.*
(iii) *$\check{H}_0^{\underline{x}_i}(X) \cong \Lambda^{\underline{x}_i}(X)$ and $\check{H}_1^{\underline{x}_i}(X) = 0$ for $i = 1, \ldots, r$ and $X = M, M[T]$.*
(iv) *$\Lambda^{\underline{x}_{i-1}}(M[T])$ is of bounded x_i-torsion for $i = 1, \ldots, r$.*

In the final section we apply the previous results to a global situation. To this end we consider a pair $(\mathcal{I}, x)$ consisting of an effective Cartier divisor $\mathcal{I} \subseteq R$ and an element $x \in R$ (see 5.1 for the definitions). We call it pro-regular whenever the inverse system $\{H_1(x^n; R/\mathcal{I}^n)\}_{n \geq 1}$ is pro-zero. Then our investigations (see 5.5) yield the following:

Corollary 1.3. *With the previous notation the following conditions are equivalent:*

(i) *$R/\mathcal{I}$ is of bounded x-torsion.*
(ii) *$(\mathcal{I}, x)$ is pro-regular.*
(iii) *$\check{H}_0^x(\Lambda^{\mathcal{I}}(F)) \cong \Lambda^{(x,\mathcal{I})}(F))$ and $\check{H}_1^x(\Lambda^{\mathcal{I}}(F)) = 0$ for any free R-module F.*

(iv) $\Lambda^{\mathcal{I}}(R)$ *and* $\Lambda^{\mathcal{I}}(R[T])$ *are of bounded* x*-torsion.*

As shown in [15] this has applications to prisms in the sense of Bhatt and Scholze (see [3]). The equivalent conditions in 1.3 are improvements of the results shown in [15, Corollary 5.7].

In the paper we start with recollections about inverse limits. In particular we include a different proof of one of Emmanouil's results (see [4]) about inverse systems needed in the paper. In the third section we prove additional statements about weakly pro-regular sequences, extending those known before. In section 4 we study pro-regular sequences, continuing the results shown in [15]. Moreover, we prove a necessary and sufficient condition for the isomorphism $\Lambda^x(\Lambda^{\mathcal{I}}(M)) \cong \Lambda^{(x,\mathcal{I})}(M)$ for an ideal $\mathcal{I} \subset R$ and an element $x \in R$ generalizing a result by Greenlees and May (see [5, Lemma 1.6]). Finally in section 5 we study when a pair $(\mathcal{I}, x)$ consisting of an effective Cartier divisor $\mathcal{I}$ and an element $x \in R$ is pro-regular. Finally we apply these results to prisms in the sense of [3] generalizing partial results of [15].

In the terminology we follow that of [17]. In our approach we prefer to work in the category of modules instead of the derived category. For that reason we use a bounded free resolution of the Čech complex (see 3.1).

2. Recollections about inverse limits

Notation 2.1. (A) Let R denote a commutative ring. Let $\{M_n\}_{n\geq 0}$ be an inverse system of R-modules with $\phi_{n,m} : M_m \to M_n$ for all $m \geq n$. Then there is an exact sequence

$$0 \to \varprojlim M_n \to \prod_{n\geq 0} M_n \xrightarrow{\Phi} \prod_{n\geq 0} M_n \to \varprojlim{}^1 M_n \to 0,$$

where Φ denotes the transition map and $\varprojlim{}^1 M_n$ is the first left derived functor of the inverse limit (see e.g. [20, 3.5] or [17, 1.2.2]).
(B) Let M denote an R-module. Let T be a variable over R. In the following we use $M[|T|]$, the formal power series R-module over M. That is, the R-module $M[|T|]$ consists of all formal series $\sum_{i\geq 0} x_i T^i$ with $x_i \in M$ for all $i \geq 0$. Correspondingly, the R-module $M[T]$ consists of all polynomials over M. Therefore, $\sum_{i\geq 0} x_i T^i \in M[T]$ if only finitely many x_i are non-zero. Whence there is an injection $0 \to M[T] \to M[|T|]$ of R-modules.
(C) The inverse system $\{M_n\}_{n\geq 0}$ is called pro-zero if for each n there is an integer $m \geq n$ such that the homomorphism $\phi_{n,m} : M_m \to M_n$ is zero. If $\{M_n\}_{n\geq 0}$ is pro-zero, then it is well known that $\varprojlim M_n = \varprojlim^1 M_n = 0$ since Φ is an isomorphism (see e.g. [17, 1.2.4]).
(D) Let $\{M_n\}_{n\geq 0}$ be an inverse system. Then clearly $\operatorname{Im}\phi_{n,m'} \subseteq \operatorname{Im}\phi_{n,m} \subseteq M_n$ for all $m' \geq m \geq n$. We say that $\{M_n\}_{n\geq 0}$ satisfies the *Mittag-Leffler condition* if for each n the sequence of submodules $\{\operatorname{Im}\phi_{n,m} | m \geq n\}$ stabilizes. For instance, this holds if the maps $\phi_{n,m}$ are surjective or $\{M_n\}_{n\geq 0}$ is an inverse system of Artinian R-modules. It is well-known that $\varprojlim^1 M_n = 0$ if $\{M_n\}_{n\geq 0}$ satisfies the Mittag-Leffler condition (see e.g. [17, 1.2.3]).

For more details about inverse systems we refer to Jensen's exposition in [8] and to [4]. It is remarkable that the vanishing in 2.1 (C) does not imply that $\{M_n\}_{n\geq 0}$ is pro-zero. To this end see the example [17, 1.2.5] or the following generalization:

Example 2.2. Let $(R, \mathfrak{m})$ denote a complete local Noetherian ring with $x \in R$ a non-unit. We consider the direct system $\{R_n\}_{n\geq 0}$ with $R_n = R$ and $\psi_{n,n+1} : R_n \to R_{n+1}$ the multiplication by x. Then $\varinjlim R_n \cong R_x$ and there is a short exact sequence

$$0 \to \oplus_{n\geq 0} R_n \to \oplus_{n\geq 0} R_n \to R_x \to 0.$$

Now we apply $\operatorname{Hom}_R(\cdot, R)$ and obtain the inverse system $\{M_n\}_{n\geq 0}$ with $M_n = \operatorname{Hom}_R(R_n, R)$ and with the multiplication $M_{n+1} \xrightarrow{x} M_n$. By applying $\operatorname{Hom}_R(\cdot, R)$ to the previous short exact sequence it yields the exact sequence

$$0 \to \operatorname{Hom}_R(R_x, R) \to \prod_{n\geq 0} M_n \to \prod_{n\geq 0} M_n \to \operatorname{Ext}^1_R(R_x, R) \to 0.$$

Since R is also xR-complete $\varprojlim M_n = \operatorname{Hom}_R(R_x, R) = 0$ and $\varprojlim^1 M_n = \operatorname{Ext}^1_R(R_x, R) = 0$ (see [17, 3.1.10]) while the inverse system $\{M_n\}_{n\geq 0}$ is neither pro-zero nor satisfies the Mittag-Leffler condition.

In the following we shall discuss necessary and sufficient conditions for an inverse system to be pro-zero. This extends known results. We need a technical construction.

Remark 2.3. An R-module M induces a short exact sequence

$$0 \to M[T] \xrightarrow{T} M[T] \to M \to 0,$$

where T denote the shift operator defined by $\sum_{n\geq 0}^k x_n T^n \mapsto \sum_{n\geq 0}^k x_n T^{n+1}$. The inverse system $\{M_n\}_{n\geq 0}$ induces a short exact sequence of inverse systems

$$0 \to \{M_n[T]\}_{n\geq 0} \xrightarrow{T} \{M_n[T]\}_{n\geq 0} \to \{M_n\}_{n\geq 0} \to 0,$$

induced by the shift operator. Then we have the six-term long exact sequence associated to the inverse limit

$$0 \to \varprojlim M_n[T] \to \varprojlim M_n[T] \to \varprojlim M_n \to \varprojlim{}^1 M_n[T] \to \varprojlim{}^1 M_n[T] \to \varprojlim{}^1 M_n \to 0$$

(see e.g. [17, 1.2.2]) .

By the Example 2.2 it follows that the vanishing of $\varprojlim^1 M_n$ is necessary but not sufficient for the Mittag-Leffler condition of the inverse system $\{M_n\}_{n\geq 0}$. A characterization of the Mittag-Leffler condition was shown by Emmanouil (see [4]). For our purposes we recall part of Emmanouil's result (see [4, Corollary 6]). In our argument we use a certain exact sequence (see the proof of 2.4) and modify an idea of [19, tag 0CQA] as new ingredients.

Lemma 2.4. *Let $\{M_n\}_{n\geq 0}$ denote an inverse system of R-modules. Then the following conditions are equivalent:*

(i) *$\{M_n\}_{n\geq 0}$ satisfies the Mittag-Leffler condition.*
(ii) *$\{M_n[T]\}_{n\geq 0}$ satisfies the Mittag-Leffler condition.*
(iii) *$\varprojlim^1 M_n = 0$ and $\varprojlim^1 M_n[T] = 0$.*
(iv) *$\varprojlim^1 M_n[T] = 0$.*

Proof. (i) $\Longrightarrow$ (ii): This follows since the inverse system $\{M_n[T]\}_{n\geq 0}$ satisfies the Mittag-Leffler condition too.
(ii) $\Longrightarrow$ (iv): This holds trivially.
(iii) $\Longleftrightarrow$ (iv): This is a consequence of the six-term exact sequence in 2.3.
(iii) $\Longrightarrow$ (i): The injections $0 \to M_n[T] \to M_n[|T|]$ induce a short exact sequence of inverse systems

$$0 \to \{M_n[T]\}_{n\geq 0} \to \{M_n[|T|]\}_{n\geq 0} \to \{M_n[|T|]/M_n[T]\}_{n\geq 0} \to 0.$$

By passing to the inverse limit it provides an exact sequence

$$0 \to \varprojlim M_n[T] \to \varprojlim M_n[|T|] \to \varprojlim M_n[|T|]/M_n[T] \to \varprojlim{}^1 M_n[T].$$

Now suppose that $\{M_n\}_{n\geq 0}$ does not satisfy the Mittag-Leffler condition. Then there is an integer m such that the sequence of submodules $\{\operatorname{Im}\phi_{m,k}|k \geq m\}$ of M_m does not stabilize. Whence there is an infinite sequence $m = m_0 < m_1 < \ldots < m_i < \ldots$ and elements $x_i \in M_{m_i}$ such that $\phi_{m,m_i}(x_i) \in M_m \setminus \phi_{m,m_i+1}(M_{m_i+1})$. Now we define $F = (f_n)_{n\geq 0} \in \prod_{n\geq 0} M_n[|T|]$ with $f_n = \sum_{i\geq n} z_{n,i}T^i$ where we put

$$z_{n,i} = \begin{cases} \phi_{n,m_i}(x_i) & if\ m_i \geq n \\ 0 & else. \end{cases}$$

As easily seen $f_n - \phi_{n,n+1}(f_{n+1}) \in M_n[T]$ and F defines an element $F' \in \varprojlim M_n[|T|]/M_n[T]$. Suppose F' has a preimage $G = (g_n)_{n\geq 0} \in \varprojlim M_n[|T|]$ with $g_n = \sum_{i\geq 0} y_{n,i}T^i$ and $y_{n,i} \in M_n$ for all $i \geq 0$. We have that $y_{n,i} = \phi_{n,n+k}(y_{n+k,i})$ for all $k, i \geq 0$ and therefore $y_{n,i} \in \phi_{n,n+k}(M_{n+k})$. That is, $y_{m,i} \in \phi_{m,m_i+1}(M_{m_i+1})$ and $y_{m,i} \neq \phi_{m,m_i}(x_i)$ since $\phi_{m,m_i}(x_i) \in M_m \setminus \phi_{m,m_i+1}(M_{m_i+1})$. Therefore

$$f_m - g_m = \sum_{i\geq 0}(\phi_{m,m_i}(x_i) - y_{m,i})T^i \notin M_m[T]$$

and G can not be a preimage of F', a contradiction to the vanishing of $\varprojlim^1 M_n[T]$. □

As a consequence of 2.4 a characterization of pro-zero inverse systems follows. The vanishing $\varprojlim M_n = \varprojlim^1 M_n = 0$ is not sufficient for $\{M_n\}_{n\geq 1}$ being pro-zero (see 2.2). As shown next it follows by the vanishing $\varprojlim M_n[T] = \varprojlim^1 M_n[T] = 0$ (see 2.5). For the proof we modify Weibel's argument (see the proof [20, 3.5.7]). For an R-module M and a set S we define $M^{(S)} = \oplus_{s\in S} M_s$ with $M_s = M$. Then it is clear that conditions (iii) and (iv) hold also for the inverse system $\{(M_n)^{(S)}\}_{n\geq 0}$ when they hold for $\{M_n\}_{n\geq 0}$.

Corollary 2.5. *Let $\{M_n\}_{n\geq 0}$ denote an inverse system of R-modules. Then the following conditions are equivalent:*

(i) *$\{M_n\}_{n\geq 0}$ is pro-zero.*
(ii) *$\{M_n[T]\}_{n\geq 0}$ is pro-zero.*
(iii) *$\varprojlim M_n = \varprojlim^1 M_n = 0$ and $\varprojlim M_n[T] = \varprojlim^1 M_n[T] = 0$.*
(iv) *$\varprojlim M_n[T] = \varprojlim^1 M_n[T] = 0$.*

Proof. (i) $\Longrightarrow$ (ii): Because $\{M_n\}_{n\geq 0}$ is pro-zero this holds also for the induced inverse system $\{M_n[T]\}_{n\geq 0}$ as easily seen.
(ii) $\Longrightarrow$ (iv): This is obviously true because $\{M_n[T]\}_{n\geq 0}$ is pro-zero.
(iii) $\Longleftrightarrow$ (iv): This is a consequence of the six-term exact sequence in 2.3.
(iii) $\Longrightarrow$ (i): By view of 2.4 the inverse system $\{M_n\}_{n\geq 1}$ satisfies the Mittag-Leffler condition. We define $N_n = \operatorname{Im}\phi_{n,m}$ where $m = m(n)$ is choosen such that $\{\operatorname{Im}\phi_{n,k}\}_{k\geq n}$ becomes stable. Then $\{N_n\}_{n\geq 1}$ becomes an inverse system with surjective maps. Because the inverse system $\{M_n/N_n\}_{n\geq 1}$ is pro-zero the exact sequence $0 \to N_n \to M_n \to M_n/N_n \to 0$ implies $\varprojlim N_n = \varprojlim M_n = 0$ and therefore $N_n = 0$. □

3. Weakly pro-regular sequences

We start with a few recalls of results and definitions of [17] and [16]. As above R denotes a commutative ring.

Notation 3.1. (A) For a system of elements $\underline{x} = x_1, \ldots, x_r$ of R let $\check{C}_{\underline{x}}$ denote the Čech complex

$$\check{C}_{\underline{x}} := \check{C}_{x_1} \otimes_R \cdots \otimes_R \check{C}_{x_r},$$

where $\check{C}_{x_i} : 0 \to R \to R_{x_i} \to 0$ (see e.g. [7] or [17, 6.1]). In the following we look at the complex $\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$ for an R-module M in the derived category. By virtue of [5] there is a finite free resolution of $\check{C}_{\underline{x}}$. We follow here the one $\mathcal{L}_{\underline{x}}$ as given in [16]. Whence $\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)$ is a representative of $\mathrm{R}\operatorname{Hom}_R(\check{C}_{\underline{x}}, M)$. Define the Čech homology $\check{H}_i^{\underline{x}}(M) = H^{-i}(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M))$ and the Čech cohomology $\check{H}^i_{\underline{x}}(M) = H^i(\mathcal{L}_{\underline{x}} \otimes_R M)$ for all $i \in \mathbb{Z}$ (see [17] and [16] for more details).
(B) Let $\underline{U} = U_1, \ldots, U_r$ denote a sequence of r variables over R. For an R-module M we denote, as above, by $M[|\underline{U}|]$ the module of formal power series in the variables $\underline{U}$. Clearly $M[|\underline{U}|] = \varprojlim M[\underline{U}]/\underline{U}^{(n)}M[\underline{U}]$, where $\underline{U}^{(n)} = U_1^n, \ldots, U_r^n$ and $M[\underline{U}]$ is the polynomial module over M. For the sequence $\underline{x} = x_1, \ldots, x_r$ we define the sequence $\underline{x} - \underline{U} = x_1 - U_1, \ldots, x_r - U_r$. As one of the main results of the paper [17, Section 8] the following isomorphisms are shown

$$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M) \cong K_\bullet(\underline{x} - \underline{U}; M[|\underline{U}|]) \cong \varprojlim K_\bullet(\underline{x} - \underline{U}; M[\underline{U}]/\underline{U}^{(n)}M[\underline{U}]),$$

where $K_\bullet(\underline{x} - \underline{U}; \cdot)$ denotes the Koszul complex with respect to the sequence $\underline{x} - \underline{U}$. Moreover there are isomorphisms

$$\mathcal{L}_{\underline{x}} \otimes_R M \cong K^\bullet(\underline{x} - \underline{U}; M[\underline{U}^{-1}]) \cong \varinjlim K^\bullet(\underline{x} - \underline{U}; M[\underline{U}]/\underline{U}^{(n)}M[\underline{U}]),$$

where $M[\underline{U}^{-1}]$ denotes the module of inverse polynomials and $K^\bullet(\underline{x} - \underline{U}; \cdot)$ is the Koszul co-complex (see [16, 4.1] for all of the details).

In the following there is technical result for the computation of $\check{H}_i^{\underline{x}}(M)$ and $\check{H}^i_{\underline{x}}(M)$ resp.

Lemma 3.2. *We fix the notation of 3.1. Furthermore let $\underline{x}^{(n)} = x_1^n, \ldots, x_r^n$ and let $H_i(\underline{x}^{(n)}; M)$ denote the Koszul homology and $H^i(\underline{x}^{(n)}; M)$ the Koszul cohomology.*

(a) *There are isomorphisms $\check{H}^i_{\underline{x}}(M) \cong \varinjlim H^i(\underline{x}^{(n)}; M)$ and short exact sequences*

$$0 \to \varprojlim{}^1 H_{i+1}(\underline{x}^{(n)}; M) \to \check{H}_i^{\underline{x}}(M) \to \varprojlim H_i(\underline{x}^{(n)}; M) \to 0,$$

for all $i \in \mathbb{Z}$.

(b) *For $i > 0$ we have $\check{H}_i^{\underline{x}}(M) = 0$ if and only if $\varprojlim{}^1 H_{i+1}(\underline{x}^{(n)}; M) = \varprojlim H_i(\underline{x}^{(n)}; M) = 0$ and $\check{H}_0^{\underline{x}}(M) \cong \Lambda^{\underline{x}}(M)$ if and only if $\varprojlim{}^1 H_1(\underline{x}^{(n)}; M) = 0$.*

Proof. For the proof of (a) we refer to [17, 6.1.4, 8.1.7] or [16, 5.6]. Then (b) is a consequence of the exact sequences in (a). □

Next we shall give a further characterization for an element $x \in R$ such that an R-module M is of bounded x-torsion.

Definition 3.3. (A) Let M denote an R-module and $x \in R$ an element. Then M is called of bounded x-torsion if the family of increasing submodules $\{0 :_M x^n\}_{n \geq 0}$ stabilizes, that is

$$0 :_M x^n = 0 :_M x^{n+1} \text{ for all } n \gg 0.$$

Note that this is equivalent to the fact that the inverse system $\{0 :_M x^n\}_{n \geq 0}$ with the multiplication map $0 :_M x^m \xrightarrow{x^{m-n}} 0 :_M x^n, m \geq n$, being pro-zero.
(B) It is obvious that M is of bounded x-torsion if and only if the inverse system of Koszul homology modules $\{H_1(x^n; M)\}_{n \geq 0}$ with the multiplication map $H_1(x^m; M) \xrightarrow{x^{m-n}} H_1(x^n; M)$ is pro-zero. With this in mind Lipman (see [2]) introduced the generalization of a weakly pro-regular sequence for a ring R. For a generalization to an R-module M see [17, 7.3.1]. That is, a sequence $\underline{x} = x_1, \ldots, x_r$ is called M-weakly pro-regular, if for $i > 0$ the inverse system

$\{H_i(\underline{x}^{(n)};M)\}_{n\geq 0}$ is pro-zero, where $H_i(\underline{x}^m;M)\to H_i(\underline{x}^n;M), m\geq n$, denotes the natural map induced by the Koszul complexes. A first systematic study of R-weakly pro-regular sequences has been done in [14].

For a characterization of M-weakly pro-regular sequences see [16]. In fact, this is an extension of R-weakly pro-regular sequences shown in [11] which extended the results of [14] to unbounded complexes. Here we shall prove another characterization of M-weakly pro-regular sequences. It is a slight extension of Potsitselski's result see [12, Section 3]) to the case of an R-module M. As above, for an R-module M and a set S we define $M^{(S)}=\oplus_{s\in S}M_s$ with $M_s=M$. Note that $M[T]\cong M^{(\mathbb{N})}$. Moreover, $\Lambda^{\underline{x}}(M)=\widehat{M}^{\underline{x}}=\varprojlim M/\underline{x}^{(n)}M$ denotes the $\underline{x}R$-adic completion of an R-module M.

Theorem 3.4. *Let $\underline{x}=x_1,\ldots,x_r$ denote a sequence of elements of R. For an R-module M the following conditions are equivalent:*

(i) *$\underline{x}$ is M-weakly pro-regular.*
(ii) *For any set S it holds $\check{H}_i^{\underline{x}}(M^{(S)})=0$ for all $i>0$ and $\check{H}_0^{\underline{x}}(M^{(S)})=\Lambda^{\underline{x}}(M^{(S)})$.*
(iii) *$\check{H}_i^{\underline{x}}(M[T])=\check{H}_i^{\underline{x}}(M)=0$ for all $i>0$ and $\check{H}_0^{\underline{x}}(M[T])=\Lambda^{\underline{x}}(M[T])$ and $\check{H}_0^{\underline{x}}(M)=\Lambda^{\underline{x}}(M)$.*
(iv) *$\check{H}_i^{\underline{x}}(M[T])=0$ for all $i>0$ and $\check{H}_0^{\underline{x}}(M[T])=\Lambda^{\underline{x}}(M[T])$.*

Proof. (i) $\Longrightarrow$ (ii): It is clear that for $i>0$ the inverse system $\{H_i(\underline{x}^{(n)};M^{(S)})\}_{n\geq 0}$ is pro-zero too. Then $\varprojlim H_i(\underline{x}^{(n)};M^{(S)})=\varprojlim{}^1 H_i(\underline{x}^{(n)};M^{(S)})=0$ for $i>0$ and (ii) is a consequence of 3.2.
(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv): These hold obviously.
(iv) $\Longrightarrow$ (i): By view of 3.2 the assumptions imply that

$$\varprojlim H_i(\underline{x}^{(n)};M[T])=\varprojlim{}^1 H_i(\underline{x}^{(n)};M[T])=0 \text{ for } i>0.$$

By 2.5 this completes the proof because of $H_i(\underline{x}^{(n)};M[T])\cong H_i(\underline{x}^{(n)};M)[T]$. □

In the following example we show that it is not sufficient to assume S to be finite in 3.4 for the characterization of weakly pro-regular sequences (see also [15, Example 3.3]).

Example 3.5. Let $R=\Bbbk[|x|]$ denote the formal power series ring in the variable x over the field $\Bbbk$. Then define $A=\prod_{n\geq 1}R/x^nR$. By the component wise operations A becomes a commutative ring. The natural map $R\to A, r\to(r+x^nR)_{n\geq 1}$, is a ring homomorphism with $x\mapsto \mathbf{x}:=(x+x^nR)_{n\geq 1}$. As a direct product of xR-complete modules A is an xR-complete R-module (see [17, 2.2.7]). Since R is a Noetherian ring x is R-weakly pro-regular and $\check{H}_i^x(A)\cong H_i(\operatorname{Hom}_R(\mathcal{L}_x,A))=0$ for $i>0$ and $\check{H}_0^x(A)\cong H_0(\operatorname{Hom}_R(\mathcal{L}_x,A))\cong A$. Moreover, by the change of rings there is an isomorphism $\operatorname{Hom}_R(\mathcal{L}_x,A)\cong\operatorname{Hom}_A(\mathcal{L}_{\mathbf{x}},A)$. That is, $\check{H}_i^{\mathbf{x}}(A)=0$ for $i>0$ and $\check{H}_0^{\mathbf{x}}(A)\cong A$. Now note that A is not of bounded $\mathbf{x}$-torsion as easily seen. It follows that the equivalent conditions in 3.4 do not hold for A and $A[T]$. To be more precise, recall $H_1(x^n;A)=\prod_{i\geq 1}(x^{i-n}R/x^iR)$ with $x^{i-n}R=R$ for $i\leq n$, that is

$$H_1(x^n;A)=(\underbrace{R/xR,\ldots,R/x^nR}_{i\leq n},\underbrace{xR/x^{n+1},\ldots,x^{i-n}R/x^iR,\ldots}_{i>n}).$$

Therefore $H_1(x^m;A)$ does not stabilize under the multiplication by x^{m-n} in $H_1(x^n;A)$. Note that the i-component of the image of $H_1(x^m;A)$ under the multiplication by x^{m-n} in $H_1(x^n;A)$ is zero for $i\leq m-n<m$ and non-zero for $i=m-n+1$. Whence $\{H_1(x^n;A)\}_{n\geq 1}$ does not satisfy the Mittag-Leffler condition. By view of 2.4 we have $\varprojlim{}^1 H_1(x^n;A[T])\neq 0$ and $\Lambda_0^x(A[T])\cong\check{H}_0^x(A[T])\twoheadrightarrow\Lambda^x(A[T])$ is not an isomorphism (see 3.2 (a)).

As an application we have another characterization that an R-module M is of bounded x-torsion for an element $x \in R$. Note that (iii) in 3.6 is the analogue to 3.4 (iv).

Corollary 3.6. *For an element $x \in R$ and an R-module M the following conditions are equivalent:*

(i) *M is of bounded x-torsion.*
(ii) *$\check{H}^x_1(M[T]) = \check{H}^x_1(M) = 0$ and $\check{H}^x_0(M[T]) \cong \Lambda^x(M[T])$ and $\check{H}^x_0(M) \cong \Lambda^x(M)$.*
(iii) *$\varprojlim 0 :_{M[T]} x^n = \varprojlim^1 0 :_{M[T]} x^n = 0$.*

Proof. The equivalence of the first two conditions is a particular case of 3.4. The equivalence of the first and third condition is a particular case of 2.5. □

Moreover, the proof of Theorem 1.1 follows by 3.4 and [16, Proposition 5.3]. To this end note that $\check{H}^{\underline{x}}_i(M) = H_i(\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M))$. For an R-module X we call a complex $X_{\cdot} : \ldots \to X_1 \to X_0 \to 0$ a left resolution whenever $X_{\cdot} \xrightarrow{\sim} M$. A co-complex $Y^{\cdot} : 0 \to Y^0 \to Y^1 \to \ldots$ is called a right resolution of X provided $X \xrightarrow{\sim} Y^{\cdot}$.

With the previous results we have the following slight generalization of Potsitselski's result (see [12, Theorem 3.6]). Note that $\underline{x}$ is R-weakly pro-regular if it is $R[T]$-weakly pro-regular as easily seen.

Corollary 3.7. *For a sequence $\underline{x} = x_1, \ldots, x_r$ of a ring R the following conditions are equivalent:*

(i) *$\underline{x}$ is R-weakly pro-regular.*
(ii) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, M)$ is a left resolution of $\Lambda^{\underline{x}}(M)$ for any free R-module M.*
(iii) *$\operatorname{Hom}_R(\mathcal{L}_{\underline{x}}, R[T])$ is a left resolution of $\Lambda^{\underline{x}}(R[T])$.*

Remark 3.8. While the property of R-regular and M-regular sequences are quite "symmetric" this is not the case for the notion of weakly pro-regularity. Let $\underline{x}$ denote a sequence of elements of R. If it is R-weakly pro-regular it follows that $\check{H}^{\underline{x}}_0(M) \cong \Lambda^{\underline{x}}_0(M)$ for any R-module M (see e.g. [17, Chapter 7]). Let $\underline{x}$ be M-weakly pro-regular, then $\check{H}^{\underline{x}}_0(M) \cong \Lambda^{\underline{x}}(M)$ as shown in 3.4. Note that the homomorphism $\Lambda^{\underline{x}}_0(M) \to \Lambda^{\underline{x}}(M)$ is onto (see [17, 2.5.1]) but in general not an isomorphism (see e.g. Example 3.5).

4. Pro-regular sequences

Before we shall investigate pro-regular sequences we need technical results about pro-zero inverse systems. To this end let M denote an R-module with $\{M_n\}_{n \geq 1}$ a decreasing sequence of submodules of M, i.e. $M_{n+1} \subseteq M_n$ for $n \geq 1$. Then $\mathcal{M} = \{M/M_n\}_{n \geq 1}$ forms an inverse system with surjective maps $M/M_{n+1} \to M/M_n$. Moreover, let $\Lambda(\mathcal{M}) = \varprojlim M/M_n$. For a sequence of elements $\underline{x} = x_1, \ldots, x_r \in R$ we consider the induced filtration $\{(\underline{x}^{(n)}M, M_n)\}_{n \geq 1}$, where $\underline{x}^{(n)} = x_1^n, \ldots, x_r^n$. We write $\Lambda(\mathcal{M}/\underline{x}\mathcal{M}) := \varprojlim M/(\underline{x}^{(n)}M, M_n)$ for the inverse limit of the induced filtration. Then there is a natural homomorphism $\Lambda^{\underline{x}}(\Lambda(\mathcal{M})) \to \Lambda(\mathcal{M}/\underline{x}\mathcal{M})$. In the following we will discuss when it is an isomorphism.

Lemma 4.1. *With the previous notation there is a short exact sequence*

$$0 \to \varprojlim{}_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) \to \Lambda^{\underline{x}}(\Lambda(\mathcal{M})) \to \Lambda(\mathcal{M}/\underline{x}\mathcal{M}) \to 0.$$

Therefore $\Lambda^{\underline{x}}(\Lambda(\mathcal{M})) \cong \Lambda(\mathcal{M}/\underline{x}\mathcal{M})$ if and only if $\varprojlim_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) = 0$.

Proof. Let m, n denote positive integers. We investigate the inverse system of Koszul complexes $\{K_\bullet(\underline{x}^{(n)}; M/M_m)\}_{m \geq 1}$. For its inverse limit there are isomorphisms

$$\varprojlim{}_m K_\bullet(\underline{x}^{(n)}, M/M_m) \cong \operatorname{Hom}_R(K^\bullet(\underline{x}^{(n)}), \Lambda(\mathcal{M})) \cong K_\bullet(\underline{x}^{(n)}; \Lambda(\mathcal{M})).$$

The inverse system $\{K_\bullet(\underline{x}^{(n)}; M/M_m)\}_{m\geq 1}$ is degree-wise surjective. Whence for its 0-th homology there is a short exact sequence

$$0 \to \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) \to H_0(\underline{x}^{(n)}; \Lambda(\mathcal{M})) \to \varprojlim{}_m H_0(\underline{x}^{(n)}; M/M_m) \to 0$$

(see [17, 1.2.8]). It forms an exact sequence of inverse systems on n. By passing to the inverse limit it provides the short exact sequence of the statement since $\varprojlim{}^1_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) = 0$ because of the underlying bi-countable indexed system (see the spectral sequence in [13]). Whence the statement follows. $\square$

The previous result is an extension of [5, Lemma 1.6] to the case of a sequence of elements and a more general filtration. Namely, it was shown by Greenlees and May that the vanishing of $\varprojlim{}_n \varprojlim{}^1_m H_1(x^n; M/\mathcal{I}^m M)$ implies the isomorphism $\Lambda^x(\Lambda^{\mathcal{I}}(M)) \cong \Lambda^{(x,\mathcal{I})}(M)$. By 4.1 the vanishing is also necessary for the isomorphism.

For any set S we define also $\Lambda(\mathcal{M}^{(S)}) = \varprojlim M^{(S)}/M_n^{(S)} \cong \varprojlim((M/M_n)^{(S)})$. For an element $x \in R$ we put - as before -

$$\Lambda((\mathcal{M}/x\mathcal{M})^{(S)}) = \varprojlim M^{(S)}/(xM, M_n)^{(S)} \cong \varprojlim((M/(xM, M_n)^{(S)})).$$

Moreover, we study when the inverse system $\{M_n :_M x^n/M_n\}_{n\geq 1}$ with the multiplication by x is pro-zero. That is, when for each $n \geq 1$ there is an $m \geq n$ such that the multiplication map

$$M_m :_M x^m/M_m \xrightarrow{x^{m-n}} M_n :_M x^n/M_n$$

is zero. This is equivalent to the inverse system $\{H_1(x^n; M/M_n)\}_{n\geq 1}$ being pro-zero, where $H_1(x^n; M/M_n)$ denotes the Koszul homology of M/M_n with respect to the element x^n. In other words, for each integer $n \geq 1$ there is an $m \geq n$ such that $M_m :_M x^m \subseteq M_n :_M x^{m-n}$. Note that, if $M_n =: N$ for all $n \geq 1$, then $\{H_1(x^n; M/N)\}_{n\geq 1}$ is pro-zero if and only if M/N is of bounded x-torsion. With this in mind we shall continue with an extension of 3.6.

Theorem 4.2. *With the previous notation the following conditions are equivalent:*

(i) *The inverse system $\{H_1(x^n; M/M_n)\}_{n\geq 1}$ is pro-zero.*
(ii) *$\check{H}_1^x(\Lambda(\mathcal{M}^{(S)})) = 0$ and $\check{H}_0^x(\Lambda(\mathcal{M}^{(S)})) \cong \Lambda((\mathcal{M}/x\mathcal{M})^{(S)})$ for any set S.*
(iii) *Condition (ii) holds for S a set of a single element and $S = \mathbb{N}$.*
(iv) *$\varprojlim H_1(x^n; Y_n) = \varprojlim^1 H_1(x^n; Y_n) = 0$ for both $Y_n = M/M_n$ and $Y_n = M/M_n[T]$.*
(v) *$\varprojlim H_1(x^n; M/M_n[T]) = \varprojlim^1 H_1(x^n; M/M_n[T]) = 0$.*

Proof. (i) $\Longrightarrow$ (ii): We put $X = M^{(S)}$ and $X_n = (M_n)^{(S)}$. Then it follows that $\{H_1(x^n; X/X_n)\}_{n\geq 1}$ is pro-zero too since the Koszul homology commutes with direct sums, therefore

$$\varprojlim H_1(x^n; X/X_n) = \varprojlim{}^1 H_1(x^n; X/X_n) = 0.$$

Furthermore there are isomorphisms

$$\varprojlim_m H_1(x^n; X/X_m) \cong \varprojlim_m \operatorname{Hom}_R(R/x^n R, X/X_m) \cong H_1(x^n; \Lambda(X))$$

for all $n \geq 1$. We have the bi-indexed system $\{H_1(x^n; X/X_m)\}_{n\geq 1, m\geq 1}$ and the diagonal system $\{H_1(x^n; X/X_n)\}_{n\geq 1}$ cofinal in it. There are the isomorphisms and the vanishing

$$\varprojlim_n H_1(x^n; \Lambda(X)) \cong \varprojlim{}_n \varprojlim{}_m H_1(x^n; X/X_m) \cong \varprojlim_{n,m} H_1(x^n; X/X_m) = 0.$$

By virtue of Roos' spectral sequence (see [13] or [20, 5.8.7]) there is a short exact sequence

$$(\#) \quad 0 \to \varprojlim{}^1_n \varprojlim{}_m H_1(x^n; X/X_m) \to \varprojlim{}^1_{n,m} H_1(x^n; X/X_m) \to \varprojlim{}_n \varprojlim{}^1_m H_1(x^n; X/X_m) \to 0$$

and a similar one with m, n reversed. This implies the vanishing $\varprojlim{}^1 H_1(x^n; \Lambda(X)) = 0$ and also $\varprojlim_n \varprojlim{}^1_m H_1(x^n; X/X_m) = 0$. By view of 3.2 and 4.1 this proves the claim.
(ii) $\Longrightarrow$ (iii): This holds trivially.
(iii) $\Longrightarrow$ (iv): By 3.2 the assumption implies that $\varprojlim H_1(x^n; \Lambda(\mathcal{X})) = \varprojlim^1 H_1(x^n; \Lambda(\mathcal{X})) = 0$ for $\mathcal{X} = \mathcal{M}$ and $\mathcal{M}[T]$. Put $X/X_m = \mathcal{M}_m$. Because $\Lambda(\mathcal{X}) \cong \varprojlim_m X/X_m$ and since the inverse limit commutes (as above) with the first Koszul homology it follows that

$$(\star) \qquad \varprojlim_n \varprojlim_m H_1(x^n; X/X_m) = \varprojlim_n{}^1 \varprojlim_m H_1(x^n; X/X_m) = 0.$$

The first vanishing implies that $\varprojlim H_1(x^n; M/M_n) = 0$. In order to continue note that the isomorphism of the assumption $\check{H}^x_0(\Lambda(\mathcal{X})) \cong \Lambda(\mathcal{X}/x\mathcal{X})$ factors through

$$\check{H}^x_0(\Lambda(\mathcal{X})) \xrightarrow{\beta} \Lambda^x(\Lambda(\mathcal{X})) \xrightarrow{\gamma} \Lambda(\mathcal{X}/x\mathcal{X})$$

surjections β (see 3.2) and γ (see 4.1). Whence $\Lambda^x(\Lambda(\mathcal{X})) \to \Lambda(\mathcal{X}/x\mathcal{X})$ is an isomorphism and $\varprojlim_n \varprojlim{}^1_m H_1(\underline{x}^{(n)}; M/M_m) = 0$ (see 4.1). Therefore

$$\varprojlim{}^1_n \varprojlim{}_m H_1(x^n; X/X_m) = \varprojlim{}_n \varprojlim{}^1_m H_1(x^n; X/X_m) = 0.$$

By Roos' exact sequence above (see (#)) $\varprojlim^1 H_1(x^n; M/M_n) = 0$, as required.
(iv) $\Longrightarrow$ (v): This is obvious.
(v) $\Longrightarrow$ (i): The Koszul homology commutes with direct sums. Therefore the implication follows by virtue of 2.5. $\square$

The implication (i) $\Longrightarrow$ (ii) in 4.2 is a generalization of [5, Proposition 1.7]. Furthermore, a certain generalization of bounded torsion to the study of sequences was invented by Greenlees and May (see [5]) and Lipman et al. (see [1]), namely:

Definition 4.3. (A) Let $\underline{x} = x_1, \ldots, x_r$ denote a sequence of elments of R. For an R-module M it is called M-pro-regular if the inverse systems with the multiplication map by x_i^n

$$\{(x_1^n, \ldots, x_{i-1}^n)M :_M x_i^n / (x_1^n, \ldots, x_{i-1}^n)M\}_{n \geq 1}, \quad i = 1, \ldots, r,$$

are pro-zero. This is equivalent to saying that the inverse systems $\{H_1(x_i^{(n)}; M/\underline{x}_{i-1}^{(n)}M)\}_{n \geq 1}$ are pro-zero for $i = 1, \ldots, r$. For a sequence of elements $\underline{x} = x_1, \ldots, x_r$ we specify the subsystems $\underline{x}_i = x_1, \ldots, x_i$ for $i = 0, \ldots, r-1$.
(B) The notion of pro-zero is equivalent to say that for $i = 1, \ldots, r$ and any positive integer n there is an integer $m \geq n$ such that

$$(x_1^m, \ldots, x_{i-1}^m)M :_M x_i^m \subseteq (x_1^n, \ldots, x_{i-1}^n)M :_M x_i^{m-n}.$$

Note that an element $x \in R$ is M-pro-regular if and only if M is of bounded x-torsion.

For a discussion of the notions of pro-regularity of Greenlees and May (see [5]) resp. Lipman (see [1]) we refer to [15]. Moreover, it follows that an M-pro-regular sequence is also M-weakly pro-regular (see e.g. [15, Theorem 2.4]), while the converse does not hold (see [2]). For a homological characterization of M-pro-regular sequences in terms of injective modules we refer to [15, Theorem 2.1]. Here we add a slight extension of [15, Theorem 2.1].

Theorem 4.4. *Let $\underline{x} = x_1, \ldots, x_r$ denote an ordered sequence of elements of R. Let M denote an R-module. Then the following conditions are equivalent.*

(i) *The sequence $\underline{x}$ is M-pro-regular.*
(ii) *The sequence $\underline{x}$ is $(M \otimes_R F)$-pro-regular for any flat R-module F.*

(iii) $\check{H}^1_{x_i}(\Gamma_{\underline{x}_{i-1}}(\operatorname{Hom}_R(M,I)) = 0$ *for* $i = 1,\ldots,k$ *and any injective* R*-module* I.
(iv) $\check{H}^1_{\underline{x}_i}(\operatorname{Hom}_R(M,I)) = 0$ *for* $i = 1,\ldots,k$ *and any injective* R*-module* I.

Proof. For the equivalence of the first three conditions we refer to [15, Theorem 2.1]. For the proof of (iii) $\Longleftrightarrow$ (iv) we put $X = \operatorname{Hom}_R(M,I)$ and recall the following short exact sequence

$$0 \to \check{H}^1_{x_i}(\check{H}^0_{\underline{x}_{i-1}}(X)) \to \check{H}^1_{\underline{x}_i}(X) \to \check{H}^0_{x_i}(\check{H}^1_{\underline{x}_{i-1}}(X)) \to 0$$

for $i = 1,\ldots,r$, (see [17, 6.1.11] or [16, 8.1 (b)]). Then note that $\Gamma_{\underline{x}_{i-1}}(X) \cong \check{H}^0_{\underline{x}_{i-1}}(X)$. If (iv) holds the claim in (iii) follows easily. For the converse we have $\check{H}^1_{\underline{x}_i}(X) \cong \check{H}^1_{x_i}(\check{H}^0_{\underline{x}_{i-1}}(X)) = 0$ for $i = 1,\ldots,r$ and inductively the vanishing of $\check{H}^1_{\underline{x}_i}(X)$ for $i = 1,\ldots,r$, recall that $\check{H}^1_{\underline{x}_{i-1}}(X) = 0$ by the inductive step. This proves (iii). $\square$

Recall that 4.4 provides a characterization of M-pro-regular sequences in terms of Čech cohomology. In the following we shall prove a characterization in terms of Čech homology. This depends upon the results of pro-zero inverse systems as shown above.

Theorem 4.5. *Let* $\underline{x} = x_1,\ldots,x_r$ *denote a sequence of elements of* R*. For an* R*-module* M *the following conditions are equivalent:*

(i) *The sequence* $\underline{x}$ *is* M*-pro-regular.*
(ii) $\check{H}^{x_i}_0(\Lambda^{\underline{x}_{i-1}}(M^{(S)})) \cong \Lambda^{\underline{x}_i}(M^{(S)})$ *and* $\check{H}^{x_i}_1(\Lambda^{\underline{x}_{i-1}}(M^{(S)})) = 0$ *for* $i = 1,\ldots,r$ *and any set* S.
(iii) $\check{H}^{x_i}_0(\Lambda^{\underline{x}_{i-1}}(X)) \cong \Lambda^{\underline{x}_i}(X)$ *and* $\check{H}^{x_i}_1(\Lambda^{\underline{x}_{i-1}}(X)) = 0$ *for* $i = 1,\ldots,r$ *and* $X = M, M[T]$.
(iv) $\check{H}^{\underline{x}_i}_0(X) \cong \Lambda^{\underline{x}_i}(X)$ *and* $\check{H}^{\underline{x}_i}_1(X) = 0$ *for* $i = 1,\ldots,r$ *and* $X = M, M[T]$.
(v) $\Lambda^{\underline{x}_{i-1}}(X)$ *is of bounded* x_i*-torsion for* $i = 1,\ldots,r$ *and* $X = M, M[T]$.

Proof. First note that $\underline{x}$ is $M^{(S)}$-pro-regular for any set S. It turns out since $R/\underline{x}_i^{(n)}R$ is finitely generated and $\operatorname{Hom}_R(R/\underline{x}_i^{(n)}R,\cdot)$ commutes with direct sums. Because of

$$\underline{x}_{i-1}^{(n)}M^{(S)} :_{M^{(S)}} x_i^n/\underline{x}_{i-1}^{(n)}M^{(S)} \cong H_1(x_i^n; H_0(\underline{x}_{i-1}^{(n)}; M^{(S)}))$$

for all $n \geq 0$ and $i = 1,\ldots,r$, it follows that the corresponding inverse systems are isomorphic and pro-zero. Note that $H_0(\underline{x}_{i-1}^{(n)}; M^{(S)}) \cong M^{(S)}/\underline{x}_{i-1}^{(n)}M^{(S)}$. Moreover the condition and Theorem 4.2 proves the equivalence of the first three statements.
(iii) $\Longleftrightarrow$(iv) : By view of [16, 8.1] there are short exact sequences

$$(\dagger) \qquad 0 \to \check{H}^{x_i}_0(\check{H}^{\underline{x}_{i-1}}_j(X)) \to \check{H}^{\underline{x}_i}_j(X) \to \check{H}^{x_i}_1(\check{H}^{\underline{x}_{i-1}}_{j-1}(X)) \to 0$$

for $i = 1,\ldots,r$ and $j = 0,1$. Then the equivalence is easily seen by the exact sequences. More precisely, (iii) $\Longrightarrow$ (iv) follows by increasing induction on i starting at $i = 1$. The converse follows similarly.
(v) $\Longrightarrow$ (iii): The assumption in (v) implies the vanishing

$$\varprojlim H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = \varprojlim{}^1 H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = 0.$$

By virtue of 3.2 it follows that $\check{H}^{\underline{x}_i}_1(\Lambda^{\underline{x}_{i-1}}(X)) = 0$ and $\check{H}^{\underline{x}_i}_0(\Lambda^{\underline{x}_{i-1}}(X)) \cong \varprojlim H_0(x_i^n; \Lambda^{\underline{x}_{i-1}}(X))$. Now we have $\varprojlim H_0(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) \cong \varprojlim_n \varprojlim_m X/(x_i^n, \underline{x}_{i-1}^{(m)})X \cong \Lambda^{\underline{x}_i}(X)$, which proves the claim in (iii).
(iii) $\Longrightarrow$(v): The statement yields $\varprojlim H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = \varprojlim{}^1 H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) = 0$. For a fixed n and $j = 0,1$ we have the short exact sequences

$$0 \to \varprojlim{}^1_m H_{j+1}(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \to H_j(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) \to \varprojlim{}_m H_j(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \to 0.$$

This follows since the inverse system for $\varprojlim_m K_\bullet(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \cong K_\bullet(x_i^n; \Lambda^{\underline{x}_{i-1}}(X))$ has degree wise surjective maps. For $j = 1$ it yields that

$$0 = \varprojlim_n H_1(x_i^n; \Lambda^{\underline{x}_{i-1}}(X)) \cong \varprojlim_n \varprojlim_m H_1(x_i^n; X/\underline{x}_{i-1}^{(m)}X) \cong \varprojlim_n H_1(x_i^n; X/\underline{x}_{i-1}^{(n)}X).$$

It remains to show the vanishing of $\varprojlim^1_n H_1(x_i^n; X/\underline{x}_{i-1}^{(n)}X)$. First note that the above short exact sequence for $j = 1$ provides that $\varprojlim^1_n \varprojlim_m H_1(x_i^n; X/\underline{x}_{i-1}^{(m)}X) = 0$. The same sequence for $j = 0$ yields that $\varprojlim_n \varprojlim^1_m H_1(x_i^n; X/\underline{x}_{i-1}^{(m)}X) = 0$. Then the above sequence (#) (see proof of 4.2) with m, n reversed proves the vanishing $\varprojlim^1_n H_1(x_i^n; X/\underline{x}_{i-1}^{(n)}X) = 0$. □

5. A global variation

As before, let R denote a commutative ring. For an element $f \in R$ we write $D(f) = \operatorname{Spec} R \setminus V(f)$. Note that $D(f)$ is an open set in the Zariski topology of $\operatorname{Spec} R$. For $f \in R$ there is a natural map $\operatorname{Spec} R_f \to \operatorname{Spec} R$ that induces a homeomorphism between $\operatorname{Spec} R_f$ and $D(f)$. Since $\operatorname{Spec} R = \cup_{f\in R} D(f)$ and since $\operatorname{Spec} R$ is quasi-compact there are finitely many $f_1, \ldots, f_r \in R$ such that $\operatorname{Spec} R = \cup_{i=1}^r D(f_i)$. Then we recall the following definitions (see [15]).

Definition 5.1. (A) A sequence $\underline{f} = f_1, \ldots, f_r$ of elements of R is called a covering sequence if $\operatorname{Spec} R = \cup_{i=1}^r D(f_i)$. This is equivalent to saying that $R = \underline{f}R$. Moreover, if $\underline{f}$ is a covering sequence then the natural map $M \to \oplus_{i=1}^r M_{f_i}$ is injective for any R-module M as easily seen.
(B) An ideal $\mathcal{I} \subset R$ is called an effective Cartier divisor if there is a covering sequence $\underline{f} = f_1, \ldots, f_r$ such that $\mathcal{I}R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r$, for non-zerodivisors $x_i/1$ of R_{f_i} with $x_i \in R$. It follows that $\mathcal{I} \subseteq (x_1, \ldots, x_r)R$.
(C) Let $\mathcal{I}$ denote an effective Cartier divisor and $x \in R$. The pair $(\mathcal{I}, x)$ is called pro-regular if for any integer n there is an integer $m \geq n$ such that $\mathcal{I}^m : x^m \subseteq \mathcal{I}^n : x^{m-n}$. This is consistent with the definition in [5] (see 4.3) and is equivalent to the fact that for each n there is an integer $m \geq n$ such that the multiplication map $\mathcal{I}^m :_R x^m/\mathcal{I}^m \xrightarrow{x^{m-n}} \mathcal{I}^n :_R x^n/\mathcal{I}^n$ is the zero map. Moreover, the pair $(\mathcal{I}, x)$ is pro-regular if and only if the inverse system $\{H_1(x^n; R/\mathcal{I}^n)\}_{n\geq 1}$ is pro-zero.

For the following we need a technical result about Cartier divisors and their relation to pro-regularity.

Lemma 5.2. *Let $\mathcal{I} \subseteq R$ be an effective Cartier divisor with the covering sequence $\underline{f} = f_1, \ldots, f_r$ such that $\mathcal{I}R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r$, for non-zerodivisors $x_i/1$ of R_{f_i}. For an element $x \in R$ the following conditions are equivalent:*
- (i) *$R/\mathcal{I}$ is of bounded x-torsion.*
- (ii) *$R_{f_i}/x_i R_{f_i}$ is of bounded $x/1$-torsion for $i = 1, \ldots, r$.*
- (iii) *$x_i/1, x/1$ is pro-regular in R_{f_i} for $i = 1, \ldots, r$ in the sense of 4.3.*
- (iv) *$(\mathcal{I}, x)$ is pro-regular in the sense of 5.1.*

Proof. (i) $\iff$ (ii): For each pair of integers $m \geq n \geq 1$ we have the following commutative diagram where the horizontal maps are injections

$$\begin{array}{ccc} \mathcal{I} :_R x^m/\mathcal{I} & \to & \oplus_{j=1}^r (x_i R_{f_i} :_{R_{f_i}} x^m/1)/x_i R_{f_i} \\ \downarrow^{x^{m-n}} & & \downarrow^{\oplus(x^{m-n}/1)} \\ \mathcal{I} :_R x^n/\mathcal{I} & \to & \oplus_{j=1}^r (x_i R_{f_i} :_{R_{f_i}} x^n/1)/x_i R_{f_i} \end{array}$$

which proves the equivalence.
(ii) $\iff$ (iii): Note that $x_i/1, x/1$ is pro-regular if and and only if $R_{f_i}/x_i^k R_{f_i}$ is of bounded $x/1$-torsion for all $k \geq 1$. The equivalence follows easily: First note that $x_i/1$ is R_{f_i}-regular. Then use induction on the short exact sequence

$$0 \to x_i^k R_{f_i}/x_i^{k+1} R_{f_i} \to R_{f_i}/x_i^{k+1} R_{f_i} \to R_{f_i}/x_i^k R_{f_i} \to 0$$

and recall that $x_i^k R_{f_i}/x_i^{k+1} R_{f_i} \cong R_{f_i}/x_i R_{f_i}$.
(iii) $\iff$ (iv): The equivalence comes out by the following modification of the above commutative diagram

$$\begin{array}{ccc} \mathcal{I}^m :_R x^m/\mathcal{I}^m & \to & \oplus_{j=1}^r (x_i^m R_{f_i} :_{R_{f_i}} x^m/1)/x_i^m R_{f_i} \\ \downarrow^{x^{m-n}} & & \downarrow^{\oplus (x^{m-n}/1)} \\ \mathcal{I}^n :_R x^n/\mathcal{I}^n & \to & \oplus_{j=1}^r (x_i^n R_{f_i} :_{R_{f_i}} x^n/1)/x_i^n R_{f_i}. \end{array}$$

Recall that the horizontal maps are injective (see also [15]). $\square$

Next we apply the previous investigations to the case when the pair $(\mathcal{I}, x)$ is pro-regular in the sense of 5.1.

Lemma 5.3. *Let $\mathcal{I} \subsetneq R$ be an effective Cartier divisor with the covering sequence $\underline{f} = f_1, \ldots, f_r$ such that $\mathcal{I} R_{f_i} = x_i R_{f_i}, i = 1, \ldots, r$, for non-zerodivisors $x_i/1$ of R_{f_i}. For an element $x \in R$ the following conditions are equivalent:*

(i) *$R/\mathcal{I}$ is of bounded x-torsion.*
(ii) *$\check{H}_1^x((R/\mathcal{I})[T]) = 0$ and $\check{H}_0^x((R/\mathcal{I})[T]) \cong \Lambda^x((R/\mathcal{I})[T])$.*
(iii) *$\check{H}_1^x(\Lambda^{\mathcal{I}}(X)) = 0$ and $\check{H}_0^x(\Lambda^{\mathcal{I}}(X)) \cong \Lambda^{(x,\mathcal{I})}(X)$ for $X = R, R[T]$.*
(iv) *$\Lambda^{\mathcal{I}}(R)$ and $\Lambda^{\mathcal{I}}(R[T])$ are of bounded x-torsion.*

Proof. First note that by 5.2 $\{H_1(x^n; R/\mathcal{I})\}_{n\geq 1}$ is pro-zero if and only if $\{H_1(x^k; R/\mathcal{I}^k)\}_{k\geq 1}$ is pro-zero. Then the equivalence of (i) and (ii) follows by 3.4. Moreover, by 4.2 the pro-zero property of the second inverse system above implies the equivalence to (iii). Finally the equivalence of (iii) and (iv) is a consequence of 4.5 and 4.1 since $\varprojlim_n \varprojlim_m^1 H_1(x^n; R/\mathcal{I}^m) = 0$. $\square$

In the following we shall give a comment of the previous investigations to the recent work of Bhatt and Scholze (see [3]) completing the results of [15]. To this end let $p \in \mathbb{N}$ denote a prime number and let $\mathbb{Z}_p := \mathbb{Z}_{\mathfrak{p}}$ the localization at the prime ideal $(p) = \mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$. In the following let R be a $\mathbb{Z}_p$-algebra.

Definition 5.4. (see [3, Definition 1.1]) A prism is a pair $(R, \mathcal{I})$ consisting of a δ-ring R (see [3, Remark 1.2]) and a Cartier divisor $\mathcal{I}$ on R satisfying the following two conditions.

(a) The ring R is $(p, \mathcal{I})$-adic complete.
(b) $p \in \mathcal{I} + \phi_R(\mathcal{I})R$, where ϕ_R is the lift of the Frobenius on R induced by its δ-structure (see [3, Remark 1.2]).

With the previous definition there is the following application of our results.

Corollary 5.5. *Let $(R, \mathcal{I})$ denote a prism. Then the following conditions are equivalent:*

(i) *$\mathcal{I}$ is of bounded p-torsion.*
(ii) *The pair $(\mathcal{I}, p)$ is pro-regular in the sense of 5.1.*
(iii) *$\check{H}_x^1(\operatorname{Hom}_R(R/\mathcal{I}, I)) = 0$ for any injective R-module I.*
(iv) *$\check{H}_0^{pR}(\Lambda^{\mathcal{I}}(R^{(S)}) \cong \Lambda^{(pR,\mathcal{I})}(R^{(S)}))$ and $\check{H}_1^{pR}(\Lambda^{\mathcal{I}}(R^{(S)}) = 0$ for any set S.*
(v) *$\Lambda^{\mathcal{I}}(R^{(S)})$ and $\Lambda^{\mathcal{I}}(R^{(S)})$ are of bounded p-torsion for any set S..*

(vi) $\Lambda^{\mathcal{I}}(R)$ *and* $\Lambda^{\mathcal{I}}(R[T])$ *are of bounded p-torsion.*

Proof. This is a consequence of 5.2, 5.3 and 4.4. $\square$

Note that 5.5 is an essential improvement of [15, Corollary 4.5], where it was shown that (i) implies the equivalent conditions (ii) and (iii).

Acknowledgement. Many thanks to the reviewer for the careful reading of the manuscript and the suggestions for improving the text and correcting references.

References

[1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, *Local homology and cohomology on schemes.*, Ann. Sci. Éc. Norm. Supér. (4), 30 (1997), pp. 1–39. 1, 2, 10
[2] ———, *Local homology and cohomology of schemes, corrections*. https://www.math.purdue.edu/~lipman/papers/homologyfix 2000. 6, 10
[3] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology*, Ann. Math. (2), 196 (2022), pp. 1135–1275. 1, 3, 13
[4] I. Emmanouil, *Mittag-Leffler condition and the vanishing of* $\varprojlim^1$, Topology, 35 (1996), pp. 267–271. 1, 3, 4
[5] J. Greenlees and J. May, *Derived functors of I-adic completion and local homology.*, J. Algebra, 149 (1992), pp. 438–453. 1, 2, 3, 6, 9, 10, 12
[6] A. Grothendieck, *Séminaire de géométrie algébrique par Alexander Grothendieck 1962. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux.* Fasc. I. Exposés I à VIII; Fasc. II. Exposés IX à XIII. 3ieme édition, corrigée. Bures-Sur-Yvette (Essonne): Institut des Hautes Études Scientifiques., 1962. 1
[7] ———, *Local cohomology. A seminar given by A. Grothendieck, Harvard University, Fall 1961. Notes by R. Hartshorne.* Lecture Notes in Mathematics. 41. Berlin-Heidelberg-New York: Springer-Verlag. 106 p., 1967. 1, 6
[8] C. Jensen, *Les foncteurs dérivés de lim et leurs applications en théorie des modules.* Lecture Notes in Mathematics. 254. Berlin-Heidelberg-New York: Springer-Verlag. V, 103 p., 1972. 3
[9] E. Matlis, *The higher properties of R-sequences*, J. Algebra, 50 (1978), pp. 77–112. 1
[10] H. Matsumura, *Commutative ring theory. Transl. from the Japanese by M. Reid.*, Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, XIII, 320 p., 1986. 1, 2
[11] M. Porta, L. Shaul, and A. Yekutieli, *On the homology of completion and torsion*, Algebr. Represent. Theory, 17 (2014), pp. 31–67. 1, 2, 7
[12] L. Positselski, *Remarks on derived complete modules and complexes*, Math. Nachr., 296 (2023), pp. 811–839. 1, 2, 7, 8
[13] J.-E. Roos, *On the derived functors of* $\varprojlim$*. Applications*, C. R. Acad. Sci., Paris, 252 (1961), pp. 3702–3704. 9
[14] P. Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand., 92 (2003), pp. 161–180. 1, 7
[15] ———, *About proregular sequences and an application to prisms*, Commun. Algebra, 49 (2021), pp. 4687–4698. 2, 3, 7, 10, 11, 12, 13, 14
[16] ———, *Čech (co-) complexes as Koszul complexes and applications*, Vietnam J. Math., 49 (2021), pp. 1227–1256. 1, 2, 5, 6, 7, 8, 11
[17] P. Schenzel and A.-M. Simon, *Completion, Čech and local homology and cohomology. Interactions between them.*, Cham: Springer, 2018. 1, 2, 3, 4, 5, 6, 7, 8, 9, 11
[18] A.-M. Simon, *Some homological properties of complete modules*, Math. Proc. Camb. Philos. Soc., 108 (1990), pp. 231–246. 1
[19] T. Stacks project authors, *The stacks project*. https://stacks.math.columbia.edu, 2022. 4
[20] C. A. Weibel, *An introduction to homological algebra*, vol. 38 of Camb. Stud. Adv. Math., Cambridge: Cambridge University Press, 1994. 3, 5, 9

Martin-Luther-Universität Halle-Wittenberg, Institut für Informatik, D — 06 099 Halle (Saale), Germany
Email address: schenzel@informatik.uni-halle.de