Perfect Matching in Product Graphs and in their Random Subgraphs

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Abstract

For $t \in \mathbb{N}$ and every $i \in [t]$, let H_i be a d_i-regular connected graph, with $1 < |V(H_i)| \leq C$ for some integer $C \geq 2$. Let $G = \square_{i=1}^{t} H_i$ be the Cartesian product of $H_1, \ldots, H_t$. We show that if $t \geq 5C \log_2 C$ then G contains a (nearly-)perfect matching.

Then, considering the random graph process on G, we generalise the result of Bollobás on the binary hypercube Q^t, showing that with high probability, the hitting times for minimum degree one, connectivity, and the existence of a (nearly-)perfect matching in the G-random-process are the same. We develop several tools which may be of independent interest in a more general setting of the typical existence of a perfect matching under percolation.

1 Introduction

1.1 Background and main results

Given two graphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, the *Cartesian product* $H = H_1 \square H_2$ is the graph whose vertex set is $V_1 \times V_2$, and for $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$, we have that $\{(u_1, u_2), (v_1, v_2)\} \in E(H)$ either if $u_1 = v_1$ and $\{u_2, v_2\} \in E(H_2)$ or if $u_2 = v_2$ and $\{u_1, v_1\} \in E(H_1)$. More generally, given t graphs, $H_i, \ldots, H_t$, their Cartesian product $G = \square_{i=1}^{t} H_i$ is the graph with the vertex set

$$V \coloneqq \{v = (v_1, \ldots, v_t) \colon v_i \in V(H_i) \text{ for all } i \in [t]\},$$

and the edge set

$$\left\{ uv \colon \begin{array}{l} \text{there is some } i \in [t] \text{ such that } u_j = v_j \\ \text{for all } i \neq j \text{ and } \{u_i, v_i\} \in E\left(H_i\right) \end{array} \right\}.$$

We call $H_1, H_2, \ldots, H_t$ the *base graphs of G*. Cartesian product graphs arise naturally in many contexts and have received much attention in combinatorics, probability, and computer science. Many classical graphs, which have been extensively studied, are in fact Cartesian product graphs: the t-dimensional torus is the Cartesian product of t copies of the cycle C_k, the t-dimensional grid is the Cartesian product of t copies of the path P_k, and the binary t-dimensional hypercube Q^t is the Cartesian product of t copies of a single edge K_2. We refer the reader to [16] for a systematic coverage of Cartesian product graphs, and related product structures on graphs. Throughout the paper, whenever we write product graphs we consider the Cartesian product as defined above.

In this paper, we study *perfect matchings* in product graphs. One can convince oneself that if one of the base graphs has a perfect matching, then the product graph G has a perfect matching as well. Indeed, Kotzig [19] showed something even stronger: let $G = \square_{i=1}^{t} H_i$ be such that each H_i is regular, then if at least one of the base graphs has a 1-factorisation (recall that a

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1-factorisation is a decomposition of the edges of the graph into disjoint perfect matchings), or at least two base graphs have perfect matchings, then G has a 1-factorisation; and furthermore these sufficient conditions are not necessary. However, when none of the base graphs have a perfect matching, it is not clear whether their product graph will have a perfect matching. Indeed, not every connected regular graph has a perfect matching (or nearly-perfect matching, that is, a matching covering all but one vertex when the order of the graph is odd). However, note that given two graphs H_1 and H_2, the proportion of vertices covered by a largest matching is at least as large as this proportion in any of the base graphs. Thus, taking the product of many graphs, one can hope to improve this proportion. Indeed, our first result gives a simple sufficient condition for the existence of a perfect matching in product graphs in this spirit.

Theorem 1. *Let $C > 1$ be a constant, and let $t \geq 5C \log_2 C$ be an integer. For every $i \in [t]$, let H_i be a d_i-regular connected graph with $1 < |V(H_i)| \leq C$. Let $G = \square_{i=1}^{t} H_i$ and suppose that $|V(G)|$ is even. Then G has a perfect matching.*

Observe that since each H_i is a d_i-regular connected graph, $G = \square_{i=1}^{t} H_i$ is a connected d-regular graph with $d = \sum_{i=1}^{t} d_i$, and in particular $d = \Theta_C(\log |V(G)|)$. Furthermore, we note that we have not tried to optimise the constants in the above. Theorem 1 shows that if the dimension of the product graph is sufficiently large, with respect to the maximum size of the base graphs, then the product graph has a perfect matching — regardless of whether the base graphs themselves contain any perfect matchings. Furthermore, from the same proof it follows that if G has an odd number of vertices, then it has a nearly-perfect matching.

As mentioned above, a well-studied product graph is the binary hypercube, Q^t, in particular in terms of *bond percolation* on it. Given a graph G and probability p, we form the percolated random subgraph $G_p \subseteq G$ by including every edge of G independently with probability p (note that $G(n,p)$ is then $(K_n)_p$). The study of Q^t_p has been initiated by Sapoženko [22] and by Burtin [8], who showed that the sharp threshold for connectivity is $p^* = \frac{1}{2}$: when $p < \frac{1}{2}$, **whp**[1] Q^t_p is disconnected, whereas for $p > \frac{1}{2}$, **whp** Q^t_p is connected. Erdős and Spencer [14] conjectured that Q^t_p undergoes a phase transition with respect to its component structure, that is, the typical emergence of a giant component (a connected component containing a linear fraction of the vertices) around $p = \frac{1}{t}$, similar to that of $G(n,p)$ around $p = \frac{1}{n}$. This conjecture was confirmed by Ajtai, Komlós, and Szemerédi [1], with subsequent work by Bollobás, Kohayakawa, and Łuczak [6]. We refer the reader to [20] for a modern short proof of this result.

In recent years, there has been an effort to generalise these results to a wider family of product graphs. Lichev [21] gave sufficient conditions, in terms of the base graphs, for the typical emergence of a giant component in bond percolation on high-dimensional product graphs. Diskin, Erde, Kang, and Krivelevich [11] improved upon this, giving sufficient and tight conditions for the typical emergence of a giant component. Furthermore, they showed that assuming the base graphs are regular, one can give a rather precise description, similar to that in $G(n,p)$, both of the typical component structure [10], and of the asymptotic combinatorial properties of the giant component [12].

Returning to the dense regime, that is, when p is constant, studying connectivity and the existence of a perfect matching in Q^t_p, Bollobás [4] obtained a *hitting time* result for the *random graph process* on Q^t. Given a graph Γ, the random graph process on Γ is defined as a random sequence of nested graphs $\Gamma(0) \subseteq \ldots \subseteq \Gamma(|E(\Gamma)|)$ together with an ordering σ on $E(\Gamma)$, chosen uniformly at random from among all $|E(\Gamma)|!$ such orderings. We set $\Gamma(0)$ to be the empty graph on $V(\Gamma)$. Given $\Gamma(i)$, with $0 \leq i < |E(\Gamma)|$, we form $\Gamma(i+1)$ by adding the $(i+1)$-th edge, according to the ordering σ, to $\Gamma(i)$. The hitting time of a monotone increasing, non-empty graph property $\mathcal{P}$, is a random variable equal to the index τ for which $\Gamma(\tau) \in \mathcal{P}$, but $\Gamma(\tau - 1) \notin \mathcal{P}$. Note that having minimum degree one, connectivity, and the existence of a perfect matching are all monotone increasing properties. Furthermore, observe that for a graph

[1] With high probability, that is, with probability tending to one as t tends to infinity.

Γ to be connected or contain a perfect matching, the minimum degree of Γ has to be at least one. A classical result of Erdős and Rényi [13], and of Bollobás and Thomason [7], is that in the random process on K_n, **whp** the hitting time for minimum degree one, connectivity, and the existence of a perfect matching are the same. In 1990, Bollobás [4] showed that the same phenomenon holds in the random graph process on Q^t.

Theorem 1.1. *Consider the random graph process on Q^t. Let τ_1 be the hitting time of minimum degree one, let τ_2 be the hitting time for connectivity, and let τ_3 be the hitting time for the existence of a perfect matching. Then,* ***whp****, $\tau_1 = \tau_2 = \tau_3$.*

Subsequent work by Joos determined the threshold for connectivity for Cartesian powers of graphs [17, 18], that is, $\prod_{i=1}^{t} H_i$ where all the base graphs H_i are the same.

Our main result generalises the result of Bollobás [4] to a wider family of product graphs.

Theorem 2. *Let $C > 1$ be an integer. For every $i \in [t]$, let H_i be a d_i-regular connected graph, with $1 < |V(H_i)| \le C$. Let $G = \square_{i=1}^{t} H_i$, and suppose that $|V(G)|$ is even. Consider the random graph process on G. Let τ_1 be the hitting time of minimum degree one, let τ_2 be the hitting time for connectivity, and let τ_3 be the hitting time for the existence of a perfect matching. Then,* ***whp****, $\tau_1 = \tau_2 = \tau_3$.*

In fact, we prove the following description of G_p, from which standard results (see, for example, [5] and [4, Lemma 2]), allow one to derive Theorem 2.

Theorem 3. *Let $C > 1$ be an integer. For every $i \in [t]$, let H_i be a d_i-regular connected graph with $1 < |V(H_i)| \le C$. Let $G = \square_{i=1}^{t} H_i$, and suppose that $n := |V(G)|$ is even. Let $d := d(G) = \sum_{i=1}^{t} d_i$. Let $\epsilon \ge 0$ be a sufficiently small constant, and let p be such that $(1-p)^d \le n^{-(1-\epsilon)}$. Then,* ***whp****, the following holds in G_p.*

(a) There exists a unique giant component, spanning all but $o(n)$ of the vertices. All the other components of G_p, if there are any, are isolated vertices. Furthermore, every two isolated vertices in G_p are at distance at least two in G.

(b) The giant component of G_p has a (nearly-)perfect matching.

A few comments are in place. Note that the threshold probability p^*, at which the expected number of isolated vertices is zero, is $p^* = 1 - \left(\frac{1}{n}\right)^{1/d}$. Since $2^d \le n \le C^d$, one can observe that p^* is bounded away from zero and one. Since $d = \Theta_C(\log n)$, note that if p is such that $(1-p)^d \le n^{-(1-\epsilon)}$ for small enough constant $\epsilon \ge 0$, then $p = \Theta_C(1)$, and in particular is bounded away from zero and one as well.

Furthermore, let us remark that Theorem 3(a) implies that **whp** adding any edge to G_p, which is not contained in the giant component, must connect an isolated vertex to the giant component, and indeed this suffices to show that the hitting times of minimum degree one and connectivity are **whp** the same. Theorem 3(b) then shows that **whp** the only obstacle to a perfect matching in G_p itself is the existence of isolated vertices. Moreover, the proof of Theorem 3(a) does not rely on the product structure of the graph, and only uses the assumptions that the graph is d-regular, has optimal edge-expansion for sets of size polynomial in d, and has some mild edge-expansion for larger sets.

Let us briefly remark that since the hypercube is bipartite, in order to find a perfect matching it suffices to apply Hall's theorem. In our setting, however, the graph G is not necessarily bipartite, which means that in order to show the typical existence of a perfect matching, one needs to consider the Tutte-Berge formula. This requires a much more delicate treatment, which is detailed in the proof outline in the subsequent section, and also requires us to develop tools which are valid for more general graphs (see Lemmas 3.2 and 3.7) and could be of independent interest when treating perfect matchings in the setting of percolation.

The paper is structured as follows. In Section 1.2 we give an outline of the proofs. In Section 2 we prove Theorem 3(a), and in Section 3 we prove Theorem 3(b). Finally, in Section 4 we conclude with a short discussion of the results and possible avenues for future research.

1.2 Proofs outline

Notation and definitions. Throughout the paper we let $C > 1$ be an integer, $(H_i)_{i=1}^t$ be a sequence of d_i-regular and connected graphs with $1 < |V(H_i)| \le C$, and let $G = \square_{i=1}^t H_i$ be their product. We let $G = (V, E)$. We call t the dimension of G, and given $u = (u_1, \ldots, u_t) \in V$ we call u_i the i-th coordinate of u. Furthermore, we denote by $n := |V|$ the order of G, and by $d \coloneqq d(G)$ the degree of G, noting once again that $d = \sum_{i=1}^t d_i = \Theta_C(\log n)$.

Given a graph Γ and subsets $S_1, S_2 \subseteq V(\Gamma)$ with $S_1 \cap S_2 = \varnothing$, we denote by $E_\Gamma(S_1)$ the set of edges in Γ whose both endpoints are in S_1, and by $E_\Gamma(S_1, S_2)$ the edges in Γ with one endpoint in S_1 and the other endpoint in S_2. We set $e_\Gamma(S_1) \coloneqq |E_\Gamma(S_1)|$ and $e_\Gamma(S_1, S_2) \coloneqq |E_\Gamma(S_1, S_2)|$. If the choice of Γ is clear, we may omit the subscript. Moreover, given $S \subseteq V(\Gamma)$, we define S^C to be $V(\Gamma) \setminus S$.

Finally, throughout the paper, we let $\epsilon \ge 0$ be a sufficiently small constant, and let p be such that $(1-p)^d \le n^{-(1-\epsilon)}$, recalling that this implies that $p = \Theta_C(1)$.

External results. We make extensive use of the following generalisations of Harper's inequality to regular high-dimensional product graphs.

Theorem 1.2 (Theorems 1 and 2 in [12]). *For every $S \subseteq V$ with $1 \le |S| \le n$,*

$$e(S, S^C) \ge \frac{|S|(d - \log_C |S|)}{C-1} \quad \textit{and}$$
$$e(S, S^C) \ge |S|(d - (C-1)\log_2 |S|).$$

We also utilise the following bound on the number of trees on k vertices in a d-regular graph, that are rooted at a fixed vertex.

Lemma 1.3 (Lemma 2 in [3]). *Let Γ be a d-regular graph, let k be a positive integer and let $v \in V(\Gamma)$. Denote by $t_k(v)$ the number of trees on k vertices rooted at v in Γ. Then $t_k(v) \le (ed)^{k-1}$.*

Proof outline of Theorem 3(a). The proof follows from a double-exposure argument, similar in spirit to the classical argument of Ajtai, Komlós, and Szemerédi [1]. Let $p_2 \coloneqq \frac{1}{d^2}$ and let p_1 be such that $(1-p_1)(1-p_2) = 1-p$. Note that $G_{p_1} \cup G_{p_2} \sim G_p$, and $p_1 \approx p - \frac{1}{d^2}$. We first show, using a first-moment calculation, that **whp** every two vertices u, v which are isolated in G_{p_1} are at distance at least two in G (Lemma 2.1). Then, utilising Theorem 1.2 and Lemma 1.3, we show that **whp** there are no components in G_{p_1} whose order is in $\left[2, d^{20}\right]$ (Lemma 2.2). We then turn to show that typically components whose order is at least d^{20} merge after sprinkling with p_2 — this is fairly standard, and follows the same approach as in [1]. Noting that any vertex which was an isolated vertex in G_{p_1}, is **whp** either an isolated vertex after sprinkling with p_2 or merged into a component which was of size at least d^{20} in G_{p_1}, we obtain that **whp** G_p has a unique large connected component, and all the other vertices are isolated vertices. Finally, from Markov's inequality it follows that the total volume of isolated vertices in G_p is at most $n^{\epsilon/2} = o(n)$.

Proof outline of Theorem 3(b). Recall that we seek to show that the giant component of G_p has a (nearly-)perfect matching. We know from Theorem 3(a) that **whp** every vertex of G which is not in the giant component is an isolated vertex in G_p. Thus, it suffices to show that **whp** G_p has a perfect matching if there are typically no isolated vertices in G_p. To that end, we will utilise the Tutte-Berge formula [2], which implies that the number of unmatched vertices in a maximum matching in Γ is equal to

$$\max_{U\subseteq V(\Gamma)} \left(\mathrm{odd}\left(\Gamma[V(\Gamma)\setminus U]\right)-|U|\right),$$

where $\mathrm{odd}\left(\Gamma[V(\Gamma)\setminus U]\right)$ is the number of connected components with an odd number of vertices in $\Gamma[V(\Gamma)\setminus U]$. In particular, a graph Γ has a perfect matching if and only if for every subset $U\subseteq V(\Gamma)$, the subgraph $\Gamma[V(\Gamma)\setminus U]$ has at most $|U|$ connected components with an odd number of vertices. Note here that if Γ has isolated vertices, then choosing $U=\varnothing$ witnesses that there is no perfect matching in Γ.

We thus define Tutte-like *obstructions* — we say that $U\subseteq V$ is an obstruction (for G_p) if $|U|\geq 1$, and the number of components of size different than two in $G_p[V\setminus U]$ is at least $|U|+1$. We say that the size of the obstruction is $|U|$. Note that if G_p has no such obstructions, then the only possible obstructions to a perfect matching are isolated vertices, and **whp** the giant component has a perfect matching. Furthermore, observe that this definition does not capture all the properties coming from the Tutte-Berge formula – indeed, we treat components with an even number of vertices, except two, the same as components with an odd number of vertices – but this crude, yet much-simplified, outlook will suffice for our needs.

We denote the components in $G_p[V\setminus U]$ of size different from two by $K_1,\dots,K_\ell$, where $\ell\geq |U|+1$ if U is an obstruction, and let $K=\bigcup_{i=1}^{\ell} V(K_i)$ and $k_i:=|K_i|$ for every $i\in[\ell]$. We denote the components of size two in $G_p[V\setminus U]$ by $W_1,\dots,W_{w/2}$, such that their total volume is w.

We say that U is a *minimal obstruction*, if it is an obstruction with the smallest size of U. Note that if a graph has an obstruction, then it must also have a minimal obstruction. We will use the convention that $u:=|U|$ and $k:=|K|$. Note that $u+k+w=n$.

We will require a finer description of the components in K. Let us partition the vertices in K into three sets, according to the size of the component in $G_p[V\setminus U]$ they belong to. Let ℓ_1 be the number of components of size one in $G_p[V\setminus U]$, let ℓ_2 be the number of components whose size is in $[3,d^2]$ in $G_p[V\setminus U]$, and let $\ell_3=\ell-\ell_1-\ell_2$ be the number of components in $G_p[V\setminus U]$ that contain more than d^2 vertices. We further denote the set of vertices in components of $G_p[V\setminus U]$ whose size is one by V_1, those whose size is in $[3,d^2]$ by S, and those whose size is larger than d^2 by B. We write $s:=|S|$ and $b:=|B|$, and note that $\ell_1=|V_1|$.

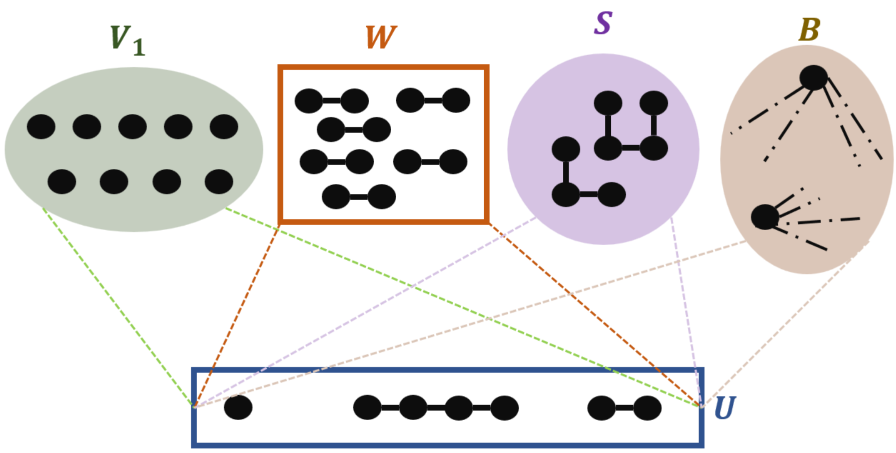

Figure 1: Illustration of an obstruction, with the sets U, V_1, W, S and B. Note that the only edges in G_p, which are not induced by U or by components K_i in K or W_i in W, are in $E(U,V_1\cup W\cup S\cup B)$.

Note that an obstruction is uniquely determined by the choice of U, and that U is not necessarily connected, and thus the number of choices could be $\binom{n}{u}$. As is often the case, a key ingredient will be to efficiently enumerate the number of possible obstructions, and $\binom{n}{u}$ will often be an inefficient bound. We thus prove two claims, which could be of independent interest in arguing about perfect matchings under percolation for general graphs. First, in Lemma 3.2, we show that a minimal obstruction of size u is determined by the choice of $W \cup S \cup B$. We stress that this result holds for any graph G, without requirements on its degree or structure. As these sets contain connected components, by Lemma 1.3 there are at most (roughly) $n^{w/2+\ell_2+\ell_3}(ed)^{s+w+b}$ choices for a minimal obstruction of size u (see Lemma 3.3 for a precise statement). Since $d = \Theta_C(\log n)$, for 'large' u this will be a much more efficient bound than $\binom{n}{u}$. As it turns out, for certain types of obstructions, neither of these bounds will be efficient enough, and we will require a different approach. Utilising the isoperimetric properties of G (Theorem 1.2), we show in Lemma 3.7 that **whp** if U is an obstruction of order polynomial in n (that is, n^a for some constant $0 < a < 1$), then there is a set M such that $U \subseteq M$, $|M| \leq 10u$, and there are relatively few connected components in M — that is, U must reside in a relatively well-connected set of order proportional to $|U|$. This allows us to more efficiently bound the number of choices of such obstructions, using Lemma 1.3.

With these definitions and tools at hand, we can now describe how to proceed in showing that typically obstructions do not exist. Using Theorem 1.2 one can deduce a lower bound of the number of edges in $E(K_i, K_i^C)$. Note that for U to be an obstruction, $E_{G_p}(K_i, K_i^C) \subseteq E(K_i, U)$ for all $i \in [\ell]$. Since G is d-regular, there are at most du edges touching U. So in order to have an obstruction in G_p, many of the edges in $\bigcup_{i\in\ell} E(K_i, K_i^C)$ cannot be present in G_p. Utilising the enumeration given by Lemma 3.7 and this probabilistic bound, we show in Lemma 3.8 that **whp** there are no obstructions with $1 \leq u \leq \frac{n}{d^{C^3/p}}$. Then, using the enumeration given by Lemma 3.2 and the aforementioned probabilistic bound, we show that **whp** there are no obstructions with b not 'too large'. For larger values of b, the aforementioned probabilistic bound no longer suffices. Thus, the final ingredient, similar in spirit to the approach of Bollobás in [4], will be Lemma 3.5, which allows to efficiently bound the number of choices for sets B that have 'bad expansion', that is, the number of edges in $E(B, B^C)$ is small. We note that this is the only place where we use the structure of G as a product graph. Indeed, sets B with bad expansion leave a large 'fingerprint' on some coordinates of the product, in particular, if B intersects non-trivially with a projection $G(I)$ of G onto some coordinates $I \subseteq [t]$, then $G(I)$ spans many edges of $E(B, B^C)$.

Finally, let us briefly comment on the difference between the proof here and the proof for the hypercube, given in [4]. Since the hypercube is a bipartite graph, it suffices there to apply Hall's theorem, and in particular, one does not need to consider the entirety of the giant component, but only sets of size at most $\frac{n}{2}$ which expand badly. Hence, Harper's isoperimetric inequality for the hypercube, together with a fingerprint argument given by Bollobás [4], allows one to show that **whp** there are no obstructions (bar isolated vertices) to a perfect matching. Here, since G is not necessarily bipartite, one needs to consider Tutte-type obstructions, and in particular sets encompassing the entirety of the giant component. This, in turn, requires a much more delicate treatment, and in particular, to consider the structure of obstructions, and the more efficient enumerations given by Lemmas 3.2 and 3.7.

2 Connectivity

We begin by showing that isolated vertices in G_p are typically at distance at least two in G.

Lemma 2.1. ***Whp****, every two isolated vertices in G_p are at distance at least two.*

Proof. Fix an edge $\{u, v\} \in E$. Since G is d-regular, the total number of edges meeting u and v is $2d - 1$. Thus, the probability that u and v are isolated vertices in G_p is at most

$$(1-p)^{2d-1} \leq (1-p)^{-1} n^{-2(1-\epsilon)} \leq n^{-2(1-2\epsilon)},$$

where we used that $(1-p)^d \le n^{-(1-\epsilon)}$. There are $\frac{dn}{2}$ edges to consider. Thus, by the union bound, the probability that two isolated vertices in G_p are at distance one in G is at most $\frac{dn}{2}n^{-2(1-2\epsilon)} = o(1)$. □

Let us further show the following 'gap' statement, which is an almost-immediate corollary of Theorem 1.2 and Lemma 1.3. We note that d^{20} in the following lemma is chosen rather arbitrarily, and a much tighter bound can be obtained.

Lemma 2.2. ***Whp****, there are no connected components K in G_p with $|V(K)| =: k \in [2, d^{20}]$.*

Proof. Fix $k \in [2, d^{20}]$. Let us estimate the probability that there is a connected component K of order k in G_p. By Theorem 1.2, we have that $e(K, K^C) \ge k(d-(C-1)\log_2 k) \ge \frac{9kd}{10}$, where we used the fact that $k \le d^{20}$. Let $\mathcal{T}_k$ be the set of trees of order k, and let $\mathcal{A}_k$ be the event that there exists a connected component of order k in G_p. Thus, by the union bound and by Lemma 1.3,

$$\mathbb{P}\left[\mathcal{A}_k\right] \le \sum_{T \in \mathcal{T}_k} (1-p)^{e(T,T^C)} \le n(ed)^{k-1}(1-p)^{\frac{9kd}{10}} \le n^{1-(1-\epsilon)\frac{9k}{10}}(ed)^{k-1} \le n^{1-(1-2\epsilon)\frac{9k}{10}}.$$

By the union bound over the less than d^{20} possible values of k, we have that the probability of an event violating the statement of the lemma is at most

$$\sum_{k=2}^{d^{20}} n^{1-(1-2\epsilon)\frac{9k}{10}} \le 2n^{1-(1-2\epsilon)\frac{18}{10}} \le n^{-7/10} = o(1).$$

□

We are now ready to prove the key result of this section, that is, Theorem 3(a), whose proof will utilise the classical double-exposure argument of [1].

Proposition 2.3. ***Whp*** *there is a unique giant component in G_p whose order is $n - o(n)$, and all the other components in G_p are isolated vertices. Furthermore, every two isolated vertices are at distance at least two from each other.*

Proof. Let $p_2 = \frac{1}{d^2}$, and let p_1 be such that $(1-p_1)(1-p_2) = 1-p$. Note there exists $\epsilon' > 0$ such that $(1-p_1)^d = n^{-(1-\epsilon')}$, and thus we may apply Lemmas 2.1 and 2.2 on G_{p_1}. Furthermore, observe that G_p has the same distribution as $G_{p_1} \cup G_{p_2}$.

We begin by percolating with probability p_1. Note that by Lemma 2.2, **whp** in G_{p_1} there are only isolated vertices and components of order at least d^{20}. We begin by showing that after sprinkling with p_2, typically all the components of order at least d^{20} merge. Let W be the set of vertices in components of order at least d^{20} in G_{p_1}. Let $A \sqcup B = W$ be a partition of W which respects the components of G_{p_1}. We may assume that $|A| \le |B|$ and let $a := |A|$.

Let $A' := (A \cup N_G(A)) \setminus B$, where $N_G(X)$ is the external neighbourhood of X in G, and let $B' := (B \cup N_G(B)) \setminus A'$. By Lemma 2.1, **whp** every two isolated vertices in G_{p_1} are at distance at least two from each other in G. We continue assuming this holds deterministically. Thus, every isolated vertex in G_p is in the neighbourhood of A or B, and we have that $V = A' \sqcup B'$, that is, $B' = V \setminus A'$. By Theorem 1.2, we have that $e(A', B') \ge \frac{a(d-\log_C a)}{C-1} \ge \frac{a}{C-1}$, since $a \le \frac{n}{2}$. We can thus extend these edges to $\frac{a}{C-1}$ paths of length at most 3 between A and B. Very naively, we can trim these to a set of $\frac{a}{Cd^2}$ edge-disjoint paths of length at most 3 between A and B.

We now sprinkle with probability p_2. The probability none of these paths are in G_{p_2} is thus at most $(1-p_2^3)^{\frac{a}{Cd^2}} \le \exp\left\{-\frac{a}{d^9}\right\}$. Thus, by the union bound, the probability that there is a component of order at least d^{20} in G_{p_1} which does not merge in $G_{p_1} \cup G_{p_2}$ is at most

$$\sum_{a=d^{20}}^{\frac{n}{2d^{20}}} \binom{n/d^{20}}{a/d^{20}} \exp\left\{-\frac{a}{d^9}\right\} \leq \sum_{a=d^{20}}^{n} (en)^{a/d^{20}} \exp\left\{-\frac{a}{d^9}\right\} = o(1),$$

where we used the fact that $\ln n \leq \ln(C^d) \leq d^2$, and that $a \geq d^{20}$. Thus, all the components of order at least d^{20} in G_{p_1} merge after sprinkling with probability p_2.

Note that by Lemma 2.1, **whp** every two isolated vertices in G_{p_1} are not connected by an edge of G. Hence, adding any edge touching an isolated vertex connects it to a component whose order is at least d^{20} in G_{p_1}, and these components all merge **whp**. Hence, **whp**, there exists a unique connected component in G_{p_1} whose order is at least d^{20}, and all the other components are isolated vertices, whose distance in G is at least two. Finally, let Y be the random variable counting the number of isolated vertices in G_p. Then, $\mathbb{E}[Y] = n(1-p)^d \leq n^{\epsilon}$. Thus, by Markov's inequality, **whp** $Y \leq n^{\epsilon/2} = o(n)$, completing the proof. $\square$

3 Perfect matching

We begin with the proof of Theorem 1. Throughout the section, we assume divisibility by two whenever is necessary, noting that minor modifications will allow us to argue for the existence of nearly-perfect matching otherwise. Let us first show that, by Theorem 1.2, for 'high enough' dimension of G every edge-cut of G has at least d edges.

Lemma 3.1. *Suppose that $t \geq 5C\log_2 C$. Then, we have that G is d-edge-connected.*

Proof. It suffices to show that $e(S, S^C) \geq d$ for all $S \subseteq V$ with $1 \leq |S| \leq n-1$, and in fact by symmetry, for S of size at most $\frac{n}{2}$. By Theorem 1.2,

$$e(S, S^C) \geq |S|(d - (C-1)\log_2 |S|). \tag{1}$$

For all $|S|$ such that $d \geq C\log_2|S| + C/\ln 2$, (1) is increasing with $|S|$. Since $d \geq t \geq 5C\log_2 C$, we have that

$$d - C\log_2 d \geq \frac{1.1C\log_2 C}{\ln 2} > \frac{C}{\ln 2},$$

where we further used the fact that $C \geq 2$. Hence, $|S|(d-(C-1)\log_2|S|)$ is increasing with $|S|$ for $|S| \leq d$, and for all S with $|S| \leq d$ we have that $e(S, S^C) \geq d$, as required. For $d \leq |S| \leq 2^{\frac{d-1}{C-1}}$ we have $d - (C-1)\log_2|S| \geq 1$, and thus $e(S, S^C)$ is at least d, as required. Thus, suppose that $2^{\frac{d-1}{C-1}} \leq |S| \leq \frac{n}{2}$. Note that $|S| \leq n/2 \leq C^t/2 \leq C^d/2$, which implies that $d - \log_C|S| \geq \log_C 2$. Hence, by Theorem 1.2,

$$e(S, S^C) \geq \frac{|S|(d - \log_C|S|)}{C-1} \geq \frac{|S|\log_C 2}{C-1} \geq \frac{2^{\frac{d-1}{C-1}}\log_C 2}{C-1} \geq \frac{2^{\frac{d-1}{C-1}}}{2C\ln C}. \tag{2}$$

We claim that for our choice of t, (2) is at least d. Note that the $\frac{2^{\frac{d-1}{C-1}}}{2C\ln C} - d$ is increasing as d is increasing, and indeed

$$\frac{2^{\frac{5C\log_2 C-1}{C-1}}}{2C\ln C} - 5C\log_2 C > \frac{C^3}{2\ln C} - 5C\log_2 C \geq 0,$$

for $C \geq 2$. $\square$

The proof of Theorem 1 then follows immediately from Lemma 3.1, since every d-regular, $(d-1)$-edge-connected graph has a perfect matching (see [2, Theorem 7, Chapter 18]).

3.1 Threshold for a perfect matching

Recall the definitions of an obstruction and the sets U, V_1, W, S and B given in Section 1.2, and that p satisfies $(1-p)^d \le n^{-(1-\epsilon)}$ for a sufficiently small constant $\epsilon \ge 0$. We begin by collecting several lemmas which we will utilise to show that **whp** there are no obstructions with $u \ge 1$.

3.2 Typical properties of obstructions

We begin with the following Lemma, which will allow us to bound the number of possible obstructions more efficiently.

Lemma 3.2. *There are at most two minimal obstructions, U and U', with $|U| = |U'| = u \ge 2$ and the same choice of $W \cup S \cup B$.*

Proof. Fix a minimal obstruction U with $u = |U| \ge 2$. We have that $V \setminus (U \cup W \cup S \cup B) = V_1$.

Suppose first that there are two sets, $\varnothing \neq A_1 \subsetneq U$ and $\varnothing \neq A_2 \subseteq V_1$, such that $X := (U \setminus A_1) \cup A_2$ is a minimal obstruction as well. By minimality, we have that $|A_1| = |A_2|$. Since X is also an obstruction and has the same set of components of size strictly larger than one, every $v \in V_1 \setminus A_2$ has that $N(v) \subseteq U \setminus A_1$. But then, we have that $U \setminus A_1$ is also an obstruction, where $1 \le |U \setminus A_1| < u$ — contradicting the minimality of U. Indeed, there are $\ell_1 - |A_2|$ components of size 1 in $G_p[V \setminus (U \setminus A_1)]$, and $\ell_2 + \ell_3$ components of size at least three in $G_p[S \cup B]$, thus at least $\ell_1 + \ell_2 + \ell_3 - |A_2| = \ell - |A_2| \ge u + 1 - |A_2| = |X| + 1$ components in total.

Now, suppose towards contradiction that there are three minimal obstructions U, U', U'' with the same choice of $W \cup S \cup B$. Then, by the above, we have $\varnothing \neq U', U'' \subseteq V_1$ with $U' \neq U''$, such that $u = |U'| = |U''|$. But then, note that there must be some $v \in U'' \subseteq V_1$, with $v \notin U'$. Since $N(v) \subseteq U$, $N(v) \subseteq U'$, and $U' \cap U = \varnothing$, we have that v is an isolated vertex. But then $U'' \setminus \{v\}$ is also an obstruction of size $u - 1 \ge 1$ — contradiction. □

We will often seek to enumerate the number of minimal obstructions, having fixed w, ℓ_2, s, ℓ_3, and b.

Lemma 3.3. *Given u, w, ℓ_2, s, ℓ_3, and b, the number of minimal obstructions is at most*

$$2n^{w/2+\ell_2+2\ell_3}(ed)^{w/2+2s+b}.$$

Proof. Recall that W is a set of $w/2$ edges. We thus have at most $\binom{nd/2}{w/2} \le (nd)^{w/2}$ ways to choose W. As for S, it has ℓ_2 components. Let us denote the sizes of these components by $s_1, \ldots, s_{\ell_2}$, where we have that $3 \le s_i \le d^2$ for every $i \in [\ell_2]$. Thus there are at most $d^{2\ell_2}$ ways to choose $s_1, \ldots, s_{\ell_2}$. We then have at most $\binom{n}{\ell_2} \le n^{\ell_2}$ ways to choose roots for some spanning trees of these components. Thus by Lemma 1.3, the number of ways to choose S is at most $d^{2\ell_2} n^{\ell_2} \prod_{i=1}^{\ell_2} (ed)^{s_i - 1} \le n^{\ell_2}(ed)^{2s}$. Finally, B has ℓ_3 components, and here we use the crude bound of at most n^{ℓ_3} ways to choose their sizes. Then, similarly to the above, by Lemma 1.3 there are at most $n^{2\ell_3}(ed)^b$ ways to choose B.

By Lemma 3.2, given $W \cup S \cup B$ and u there are at most 2 minimal obstructions. Thus, given u, w, ℓ_2, s, ℓ_3, and b, the number of minimal obstructions is at most $2n^{w/2+\ell_2+2\ell_e}(ed)^{w/2+2s+b}$. □

Noting that there are at most du edges touching U in G, let us bound from below the number of edges leaving the components K_i for $i \in [\ell]$. To that end, for $m \in [n/2]$, let

$$f(m) = \max\left\{m(d - (C-1)\log_2 m), \frac{m(d - \log_C m)}{C-1}\right\}, \tag{3}$$

where for $m \in [n/2+1, n]$ we set $f(m) = f(n-m)$.

Lemma 3.4. *Suppose $U \subseteq V$ is an obstruction. Then*

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) \geq d(\ell-1) + f(k-\ell+1),$$

where f is defined according to (3)*.*

Proof. By Theorem 1.2, we have that $\sum_{i=1}^{\ell} e(K_i, K_i^C) \geq \sum_{i=1}^{\ell} f(k_i)$. We have that

$$\sum_{i=1}^{\ell} f(k_i) = \sum_{i=1}^{\ell} \max\left\{k_i(d-(C-1)\log_2 k_i), \frac{k_i(d-\log_C k_i)}{C-1}\right\}.$$

We claim that the function f is concave, and thus the minimum of the above sum is obtained with $k_i = 1$ for all $i \in [\ell-1]$, and $k_\ell = k-(\ell-1)$. Indeed, note that $g_1(x) = x(d-(C-1)\log_2 x)$ and $g_2(x) = \frac{x(d-\log_C x)}{C-1}$ are concave. While the maximum of two concave functions is not necessarily concave, observe that there is some minimal integer m_0 such that for all $m \geq m_0$, $g_2(m) \geq g_1(m)$, and for all $m' < m_0$, $g_2(m') \leq g_1(m)$. It thus suffices to verify that that the discrete second derivative at m_0 is decreasing. Indeed, the function $g(x) = g_2(x) - g_1(x-1)$ is decreasing at $x = m_0$, and thus $f(x)$ is concave. $\square$

We will further make use of the following estimate on the number of sets whose size is not too small, yet having a small edge-boundary.

Lemma 3.5. *Let $a \geq \frac{n}{d^{\ln^2 d}}$. Then, the number of sets $A \subseteq V$ of size a with $e(A, A^C) < a\ln^2 d$ is at most* $\exp\left\{\frac{2a}{\ln d}\right\}$.

Proof. Let $\mathcal{F}$ be the family of $A \subseteq V$ satisfying the conditions of the lemma.

For $i \in [t]$ and any $A \subseteq V$, let $E_i(A, A^C) \subseteq E(A, A^C)$ be the set of edges in $E(A, A^C)$ corresponding to a change in the i-th coordinate, and let $e_{i,A} := |E_i(A, A^C)|$. Moreover, given $I \subseteq [t]$, let $e_{I,A} = \sum_{i \in I} e_{i,A}$. We say that A is *bad* with respect to a set of coordinates I, if $e_{I,A} < a\ln^2 d \cdot \frac{|I|}{t}$. Let $\mathcal{A}_I$ be the family of sets A which are bad with respect to some $I \subseteq [t]$. Note that for every fixed $m \in \mathbb{N}$, if $A \in \mathcal{F}$, then there is some I with $|I| = m$ such that $e_{I,A} < a\ln^2 d \cdot \frac{|I|}{t}$. Thus,

$$|\mathcal{F}| \leq \sum_{\substack{I \subseteq [t] \\ |I|=m}} |\mathcal{A}_I| \leq \binom{t}{m} \max_{\substack{I \subseteq [t] \\ |I|=m}} |\mathcal{A}_I|.$$

We now set $m = \log_C\left(\ln^5 d\right)$, and turn to estimate $|\mathcal{A}_I|$ for any $I \subseteq [t]$ with $|I| = m$.

For such an I and $v \in V$, let $G(I,v) := \square_{i \in [t]\setminus I}\{v_i\}\square_{i \in I} H_i \subseteq G$. Observe that $2^{|I|} \leq |V(G(I,v))| \leq C^{|I|}$, and that for every $v \neq u \in V$, $V(G(I,v))$ and $V(G(I,u))$ are either disjoint or identical. Thus, fixing I with $|I| = m$, there are at most $\frac{n}{2^m}$ different subgraphs $G(I,v)$, and their union is V. We say that A *intersects non-trivially* with $G(I,v)$ if $V(G(I,v)) \cap A \neq \varnothing$ and $V(G(I,v)) \setminus A \neq \varnothing$. By Theorem 1.2 and Lemma 3.1, we have that if A intersects non-trivially with $G(I,v)$, then $G(I,v)$ spans at least $|I| = m$ edges of $E(A, A^C)$. Thus, if $A \in \mathcal{A}_I$, we have that A intersects non-trivially at most $\frac{a\ln^2 d}{t}$ such subgraphs. Indeed, otherwise, there would be no I with $|I| = m$ and $e_{I,A} < a\ln^2 d \cdot \frac{m}{t}$. Therefore, a set $A \in \mathcal{A}_I$ contains at least

$$\frac{a - C^m \frac{a\ln^2 d}{t}}{C^m}$$

such subgraphs, and at most $C^m\frac{a\ln^2 d}{t}$ other vertices. Recalling $a \geq \frac{n}{d^{\ln^2 d}}$ we thus obtain that

$$\begin{aligned}|\mathcal{A}_I| &\leq \binom{\frac{n}{2^m}}{\frac{a}{C^m}}\binom{n}{C^m\frac{a\ln^2 d}{t}} \leq \left(\frac{enC^m}{a}\right)^{\frac{a}{C^m}}\left(\frac{ent}{C^m a\ln^2 d}\right)^{C^m\frac{a\ln^2 d}{t}} \\ &\leq \left(ed^{\ln^2 d}C^m\right)^{\frac{a}{C^m}}\left(\frac{etd^{\ln^2 d}}{C^m\ln^2 d}\right)^{C^m\frac{a\ln^2 d}{t}} \leq \left(ed^{\ln^2 d}\ln^5 d\right)^{\frac{a}{\ln^5 d}}\left(\frac{ed^{\ln^2 d+1}}{\ln^7 d}\right)^{\frac{a\ln^7 d}{t}} \\ &\leq \exp\left\{\frac{a}{\ln d}\right\}.\end{aligned}$$

Altogether, we obtain that $|\mathcal{F}| \leq \binom{t}{\log_C(\ln^5 d)}\exp\left\{\frac{a}{\ln d}\right\} \leq \exp\left\{\frac{2a}{\ln d}\right\}$, as required. $\square$

We finish this section by showing that given an obstruction with $u \geq 2^{\frac{d}{10C}}$, **whp** U lies in a set which is not 'too disconnected', and whose size is proportional to that of U. First, we require the following fairly simple claim.

Claim 3.6. ***Whp*** *there are no obstructions such that one (or more) of the following holds,*

(a) $\ell_1 \geq 5u$,

(b) $s \geq 6u$,

(c) $\ell_3 \geq \frac{6u}{d^2}$ *and* $u \geq d^2$.

Proof. We first claim that in any of the above cases, we would have that

$$\sum_{i\in[\ell]} e(K_i, K_i^C) \geq 5du.$$

Indeed, if $\ell_1 = |V_1| \geq 5u$, since the graph G is d-regular, $\sum_{i\in[\ell]} e(K_i, K_i^C) \geq d\ell_1 \geq 5du$. Similarly, since by Theorem 1.2, every component K_i in $G_p[S]$ satisfies $e(K_i, K_i^C) \geq |K_i|(d - 2(C-1)\log_2 d)$, if $s \geq 6u$ then $\sum_{i\in[\ell]} e(K_i, K_i^C) \geq (1-o(1))ds \geq 5du$. Finally, if $\ell_3 \geq \frac{6u}{d^2}$, then, by Theorem 1.2 and by convexity arguments similar to Lemma 3.4, we have that

$$\begin{aligned}\sum_{i\in[\ell]} e(K_i, K_i^C) &> (\ell_3 - 1)d^2(d - 2(C-1)\log_2 d) \\ &\geq 6du - d^3 + 2(C-1)\log_2 d \geq 5du,\end{aligned}$$

where we used the assumption that $u \geq d^2$.

Fix u and recall that U has at most du edges touching it. Note that any edge between two components K_i, K_j for $i,j \in [\ell]$ would contradict the fact that we consider an obstruction. Hence, the probability of an obstruction with $\sum_{i\in[\ell]} e(K_i, K_i^C) \geq 5du$ is at most

$$\binom{n}{u}(1-p)^{5du-du} \leq n^u n^{-(1-\epsilon)4u} \leq n^{-\frac{5u}{2}},$$

where we used $(1-p)^d \leq n^{-(1-\epsilon)}$, and recalling that choosing U determines the obstruction. A union bound over all possible values of u, and the at most n^2 choices for s and ℓ_3 yields that the probability of an event violating the statement of the claim is at most

$$3n^2\sum_{u=1}^{n} n^{-\frac{5u}{2}} \leq 3n^2 \cdot 2n^{-\frac{5}{2}} = o(1).$$

$\square$

We are now ready to describe for some range of u a set M containing U with relatively few components.

Lemma 3.7. ***Whp*** *for every obstruction with* $u \geq 2^{\frac{d}{10C}}$*, there is a* $M \subseteq V$ *satisfying the following,*

1. $U \subseteq M$*; and,*
2. $|M| \leq 10u$*; and,*
3. $G[M]$ *contains at most* $u/2^{\frac{d}{(100C)^3}}$ *components.*

Proof. Suppose that U is an obstruction with $u = |U|$, $|V_1| = \ell_1$ and $s = |S|$. By Claim 3.6, we have that **whp** there is no obstruction with $\ell_1 \geq 4u$, as well as no obstruction with $s \geq 5u$, and no obstruction with $\ell_3 \geq 10u/d^2$. We may thus assume that $\ell_1 < 4u, s < 5u$ and $\ell_3 < 10u/d^2$. In particular, $\ell_1 + \ell_2 \geq u + 1 - 10u/d^2$.

For every component K_i in $G_p[S \cup V_1]$, we have that $E_{G_p}(K_i, K_i^C) \subseteq E(K_i, U)$. Now, by Theorem 1.2,

$$\sum_{K_i \in G_p[S \cup V_i]} e(K_i, K_i^C) \geq d\ell_1 + (1 + o(1))ds \geq d(\ell_1 + \ell_2) \geq d(u + 1 - 10u/d^2),$$

where in the second inequality we used that $s \geq 3\ell_2$. Thus, the probability that there is such an obstruction with $e(U, V_1 \cup S) \leq \frac{du}{100C^2}$ is at most

$$\begin{aligned}\binom{n}{u}(1-p)^{d(u+1-10u/d^2-\frac{u}{100C^2})} &\leq \left(\frac{en}{u}\right)^u n^{-\left(1-\frac{1}{95C^2}\right)u} \leq \exp\left\{\frac{u}{95C^2}\ln n + u - u\ln u\right\} \\ &\leq \exp\left\{u\left(\frac{\ln n}{95C^2} - \frac{\ln 2 \cdot d}{10C}\right)\right\} = o\left(n^{-3}\right),\end{aligned}$$

where we used $n \leq C^d$, $C \geq 2$ and $u \geq 2^{\frac{d}{10C}}$. Hence, by the union bound over the at most n^3 possible choices for u, s and ℓ_3, we have that **whp** any obstruction U with $u \geq 2^{\frac{d}{10C}}$ has $\ell_1 < 4u$, $s < 5u$ and there are at least $\frac{du}{100C^2}$ edges in $G[U \cup V_1 \cup S]$.

Let $M = U \cup V_1 \cup S$. We have that $U \subseteq M$ and $|M| = u + \ell_1 + s \leq 10u$, by our assumptions $\ell_1 < 4u$ and $s < 5u$. Theorem 1.2 implies that for any set $M \subseteq V(G)$

$$e_G(M) \leq |M|d - |M|(d - (C-1)\log_2 |M|) = |M|(C-1)\log_2 |M|.$$

If $G[M]$ had at least $u/2^{\frac{d}{(100C)^3}}$ components, then by standard convexity arguments, we would have that

$$\begin{aligned}e_G(M) &\leq |M|(C-1)\log_2\left(\frac{|M|}{u/2^{\frac{d}{(100C)^3}}}\right) \\ &\leq 10u(C-1)\log_2\left(10 \cdot 2^{\frac{d}{(100C)^3}}\right) \leq \frac{du}{(100C)^2} < \frac{du}{100C^2},\end{aligned}$$

a contradiction. □

3.3 Typically no obstructions

We are now ready to show that **whp** there are no obstructions with $u \geq 1$ in G_p. We consider several cases separately: when $|U|$ is small; when $s = b = 0$; when $s \neq 0$ and $b = 0$; when $0 \neq b \leq \frac{n}{2}$; and, when $b > \frac{n}{2}$. We show that for each of these cases, **whp** there are no obstructions, thus completing the proof of Theorem 3(b).

Obstructions with small U. We denote by $\mathcal{B}_u$ the event there exists an obstruction in G_p with $|U| = u$.

Lemma 3.8. *We have that*

$$\bigcup_{1\le u\le \frac{n}{d^{C^3/p}}} \mathbb{P}\left(\mathcal{B}_u\right) = o(1).$$

Proof. Suppose there exists a construction with $|U| = u$ and $u \in \left[\frac{n}{d^{C^3/p}}\right]$. By Lemma 3.4, we have that

$$\begin{aligned}\sum_{j=1}^{j=w/2} e(W_j, W_j^C) + \sum_{i=1}^{\ell} e(K_i, K_i^C) &\ge w(d-1) + d(\ell-1) + f(k-\ell+1)\\ &\ge w(d-1) + du + f(n-2u-w) \ge du + f(2u),\end{aligned}$$

where we used that $\ell \ge u+1, k = n-u-w$ and the definition of $f(x)$.

We continue by considering two ranges separately. First, suppose that $u \le 2^{\frac{d}{10C}}$. We then have that $(C-1)\log_2 u \le (C-1)\frac{d}{10C} < \frac{d}{10}$ and thus $f(2u) \ge 2u(d-(C-1)\log_2 u) > \frac{3du}{2}$. As there are at most du edges touching U, and noting that any edge between any two different components in $G_p[V\setminus U]$ would rule out the existence of such an obstruction, we have that

$$\mathbb{P}\left(\mathcal{B}_u\right) \le \binom{n}{u}(1-p)^{3du/2} \le n^{-(1-\epsilon)\frac{3u}{2}+u} < n^{-u/3}.$$

Therefore, by the union bound,

$$\bigcup_{1\le u\le 2^{\frac{d}{10C}}} \mathbb{P}\left(\mathcal{B}_u\right) \le \sum_{u=1}^{2^{\frac{d}{10C}}} \frac{1}{n^{u/3}} = o(1).$$

We now turn to $2^{\frac{d}{10C}} \le u \le \frac{n}{d^{C^3/p}}$. By Lemma 3.7, we may assume that there exists M such that $|M| \le 10u$, $U \subseteq M$ and there are at most $u/2^{\frac{d}{(100C)^3}} =: r$ components in $G[M]$. We have at most n^r ways to choose the sizes of these components. Thus, by Lemma 1.3, there are at most $\binom{10u}{u}n^{2r}(ed)^{10u}$ ways to choose U. Since $u \le \frac{n}{d^{C^3/p}}$, we have that

$$f(2u) \ge \frac{2u(d-\log_C u)}{C} \ge \frac{2uC^2\log_C d}{p}.$$

Thus, the probability of such an obstruction is at most

$$\begin{aligned}\binom{10u}{u}n^{2r}(ed)^{10u}(1-p)^{\frac{2uC^2\log_C d}{p}} &\le (10e)^u n^{2r}(ed)^{10u}\exp\left\{-2uC^2\log_C d\right\}\\ &\le \exp\left\{u\left(11+\ln 10+\frac{2\ln n}{2^{d/(100C)^3}}+10\ln d-2C^2\log_C d\right)\right\}\\ &\le \exp\left\{u\left(15+\frac{2\ln n}{2^{d/(100C)^3}}-\left(10-\frac{2C^2}{\ln C}\right)\ln d\right)\right\}\\ &= \exp\left\{-\frac{u\ln d}{2}\right\} \le d^{-u/2},\end{aligned}$$

using $\frac{2C^2}{\ln C} \ge 11$ for $C \ge 2$ and $\frac{2\ln n}{2^{d/(100C)^3}} = o(\ln d)$. A union bound over the less than n possible values of u completes the proof. $\square$

We now turn to consider obstructions with large U. We consider several cases separately. The proofs in Lemmas 3.9-3.14 follow quite similar calculations, however, each such calculation requires a slightly different approach. As these differences are telling of the different challenges in showing that such typical obstructions do not appear, we left the calculations explicit.

Obstructions with $s = b = 0$.

Lemma 3.9. ***Whp**, there are no obstructions with $s = b = 0$.*

Proof. Fix u and w, and let $\mathcal{B}_{u,w}$ be the event that there is a minimal obstruction with $u = |U|, w = |W|$ and $s = b = 0$. By Lemma 3.3, there are at most $2n^{w/2}(ed)^{w/2}$ such minimal obstructions. In particular, $\ell = |V_1| = n - u - w$.

We have that

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) + \sum_{j=1}^{w/2} e(W_j, W_j^C) \geq d\ell + w(d-1).$$

As there are at most du edges touching U in G, we have by the union bound

$$\begin{aligned}\mathbb{P}\left(\bigcup_{\substack{u\in[1,n/2-1]\\ w\in[0,n-2u-1]}} \mathcal{B}_{u,w}\right) &\leq \sum_{u=1}^{n/2-1}\sum_{w=0}^{n-2u-1} 2n^{w/2}(ed)^{w/2}(1-p)^{d(\ell+w-u)-Cw} \\ &\leq \sum_{u=1}^{n/2-1}\sum_{w=0}^{n-3} 2n^{-(1-\epsilon)(\ell-u)-w/3}.\end{aligned}$$

Recall that since u and w are fixed, $\ell - u$ is fixed as well and ranges from 2 to $n-2$. Furthermore, note that the sum over w is a geometric sum and is at most twice its value when $w = 0$. Thus, we have that

$$\mathbb{P}\left(\bigcup_{\substack{u\in[1,n/2-1]\\ w\in[0,n-2u-1]}} \mathcal{B}_{u,w}\right) \leq 4\sum_{j=2}^{n-2} n^{-(1-\epsilon)j} = o(1),$$

as required. $\square$

Obstructions with $s \neq 0, b = 0$.

Lemma 3.10. ***Whp**, there are no obstructions with $s \neq 0$ and $b = 0$.*

Proof. Fix u, w and s, and let $\mathcal{B}_{u,w,s}$ be the event that there is a minimal obstruction with $u = |U|$, $w = |W|$, $0 \neq s = |S|$ and $b = 0$. We then have that

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) + \sum_{j=1}^{j=w/2} e(W_j, W_j^C) \geq d\ell_1 + ds - 2Cs\log_2 d + w(d-1),$$

since the components in S are of size at most d^2, and thus by Theorem 1.2, any K_i of size at most d^2 has that $e(K_i, K_i^C) \geq |K_i|(d - 2C\log_2 d)$.

By Lemma 3.3 there are at most $2n^{w/2+\ell_2}(ed)^{w/2+2s}$ choices for such a minimal obstruction with $u = |U|$, $w = |W|$, $s = |S|$ and $b = 0$. As before, we obtain that

$$\begin{aligned}\mathbb{P}\left(\bigcup_{w\in[0,n-u-s-\ell_1]} \mathcal{B}_{u,w,s}\right) &\leq \sum_{w=0}^{n-u-s-\ell_1} 2n^{w/2+\ell_2}(ed)^{w/2+2s}(1-p)^{d(\ell_1+s+w-u)-2Cs\log_2 d-w} \\ &\leq 4n^{\ell_2}(ed)^{2s}(1-p)^{d(\ell_1+s-u)-2Cs\log_2 d}.\end{aligned}$$

Now, $\ell_1+\ell_2 \geq u+1$, and thus $\ell_1-u \geq 1-\ell_2$. Hence,

$$\begin{aligned}
n^{\ell_2}(ed)^{2s}(1-p)^{d(\ell_1+s-u)-2C\log_2 ds} &\leq n^{\ell_2}(ed)^{2s}(1-p)^{d(s-\ell_2+1)-2Cs\log_2 d}\\
&\leq n^{-(1-\epsilon)(s-2\ell_2+1)}(ed)^{2s}(1-p)^{-2Cs\log_2 d}\\
&\leq \exp\left\{-(1-\epsilon)(s-2\ell_2+1)\ln n + 5C\left(1+\ln\left(\frac{1}{1-p}\right)\right)s\ln d\right\}.
\end{aligned}$$

Note that the above decreases as s increases, and therefore, using $s \geq 3\ell_2$ and $\ell_2 \geq \frac{s}{d^2}$, we have that

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{\substack{s\in[3,n-u-\ell_1]\\ w\in[0,n-u-s-\ell_1]}} \mathcal{B}_{u,w,s}\right) &\leq 4\sum_{s=3}^{n-u-\ell_1} \exp\left\{-(1-\epsilon)(\ell_2+1)\ln n + 15C\left(1+\ln\left(\frac{1}{1-p}\right)\right)\ell_2\ln d\right\}\\
&\leq 4d^2\sum_{\ell_2=1}^{n}\exp\left\{-(1-2\epsilon)(\ell_2+1)\ln n\right\}\\
&\leq 5d^2\exp\left\{-(1-2\epsilon)2\ln n\right\} = o(1/n),
\end{aligned}$$

where we used the fact that p is bounded away from one. Therefore, by the union bound over the at most n values of u, we have that **whp** there is no obstruction violating the statement of the Lemma. $\square$

Obstructions with $0 \neq b \leq \frac{n}{2}$. We first treat this case under the assumption that the edge boundary of B in G is not too small.

Lemma 3.11. ***Whp*** *there are no obstructions with $0 \neq b$ and $e(B, B^C) \geq b\ln^2 d$.*

Proof. Fix u, w, s, ℓ_2, b, and ℓ_3, and let $\mathcal{B}_{u,w,s,b}$ be the event that there is a minimal obstruction with $u=|U|$, $w=|W|$, $s=|S|$ and $0 \neq b=|B|$. Similarly to before, we then have that

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) + \sum_{j=1}^{w/2} e(W_j, W_j^C) \geq \ell_1 d + (1+o(1))sd + e(B,B^C) + wd - w.$$

Since we assume that $e(B,B^C) \geq b\ln^2 d$, we obtain that

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) + \sum_{j=1}^{w/2} e(W_j, W_j^C) \geq \ell_1 d + (1+o(1))sd + b\ln^2 d + wd - w.$$

Now, by Lemma 3.3, there are at most $2n^{w/2+\ell_2+2\ell_3}(ed)^{w/2+2s+b} = 2n^{\ell-\ell_1+\ell_3+w/2}(ed)^{2s+b+w/2}$ such minimal obstructions with $u=|U|, w=|W|, s=|S|$ and $b=|B|$. As before, we obtain that

$$\begin{aligned}
\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right) &\leq 2n^{\ell-\ell_1+\ell_3}(ed)^{2s+b}(1-p)^{d(\ell_1+(1+o(1))s-u)+b\ln^2 d}\cdot n^{w/2}(ed)^{w/2}(1-p)^{dw-w}\\
&\leq 2n^{\ell-\ell_1+\ell_3}(ed)^{2s+b}(1-p)^{d(\ell_1+(1+o(1))s-u)+b\ln^2 d}\cdot n^{-w/3}\\
&\leq 2n^{\ell-\ell_1+\ell_3}(ed)^{2s+b}(1-p)^{d(\ell_1+(1+o(1))s-u)+b\ln^2 d}\\
&\leq 2\exp\left\{(\ell-\ell_1+\ell_3)\ln n + (2s+b)\ln(ed) - pb\ln^2 d\right\} n^{-(1-\epsilon)(\ell_1+(1+o(1))s-u)}\\
&\leq 2\exp\left\{(\ell-\ell_1)\ln n + 2s(\ln d+1) - \frac{p}{2}b\ln^2 d - (1-\epsilon)(\ell_1+(1+o(1))s-u)\ln n\right\}\\
&\leq 2\exp\left\{(\ell+\ell_3-(2-\epsilon)\ell_1+(1-\epsilon)u)\ln n - s(1-2\epsilon)\ln n - \frac{p}{2}b\ln^2 d\right\}.
\end{aligned}$$

Recall that $\ell = \ell_1 + \ell_2 + \ell_3$ and $\ell_2 \le \frac{s}{3}$ and $\ell_3 \le \frac{b}{d^2}$. Thus,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right) \le 2\exp&\left\{(\ell+\ell_3-(2-\epsilon)(\ell-\ell_2-\ell_3)+(1-\epsilon)u)\ln n - s(1-2\epsilon)\ln n - \frac{p}{2}b\ln^2 d\right\}\\
\le& 2\exp\left\{\left((1-\epsilon)\left(u-\ell\right)+\frac{2s}{3}+\frac{3b}{d^2}\right)\ln n - s(1-2\epsilon)\ln n-\frac{p}{2}b\ln^2 d\right\}\\
\le& 2\exp\left\{(1-\epsilon)(u-\ell)\ln n + O\left(\frac{b}{d}\right) - s\left(1-2\epsilon-\frac{2}{3}\right)\ln n - \frac{p}{2}b\ln^2 d\right\}\\
\le& 2\exp\left\{O\left(\frac{b}{d}\right)-\frac{p}{2}b\ln^2 d\right\},
\end{aligned}$$

where in the last step we used that $u < \ell$. Since $b \ge d^2$, by the union bound over the at most n^6 values for u, w, s, ℓ_2, b and ℓ_3, we have that the probability of an obstruction violating the statement of the claim is at most

$$n^6 \exp\left\{-\Omega\left(d^2\ln^2 d\right)\right\} = o(1).$$

□

We now turn to obstructions where B has a small edge-boundary.

Lemma 3.12. ***Whp*** *there are no obstructions with $0 \neq b \le \frac{n}{2}$ and $e(B, B^C) \le b\ln^2 d$.*

Proof. Note that every set $B \subseteq V$ with $e(B,B^C) \le b\ln^2 d$ is of size at least $\frac{n}{d^{\ln^2 d}}$ since by Theorem 1.2, for $b < \frac{n}{d^{\ln^2 d}}$,

$$e(B,B^C) \ge \frac{b(d-\log_C b)}{C} \ge \frac{b\ln^2 d \log_C d}{C} > b\ln^2 d$$

Fix u, w, s, ℓ_2, b, and ℓ_3, and let $\mathcal{B}_{u,w,s,b}$ be the event that there is a minimal obstruction with $u = |U|$, $w = |W|$, $s = |S|$ and $0 \neq b = |B| \le \frac{n}{2}$. Similarly to before, we then have that

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) + \sum_{j=1}^{w/2} e(W_j, W_j^C) \ge \ell_1 d + (1+o(1))sd + f(b) + wd - w.$$

By the proof of Lemma 3.3, there are at most $n^{\ell_2+w/2}(ed)^{2s+w/2}$ ways to choose $W \cup S$. Furthermore, since we assume $e(B,B^C) \le b\ln^2 d$ and $b \ge \frac{n}{d^{\ln^2 d}}$, by Lemma 3.5 there are at most $\exp\left\{\frac{2b}{\ln d}\right\}$ choices for B. Therefore, by Lemma 3.2, there are at most $2n^{\ell_2+w/2}(ed)^{2s+w/2}\exp\left\{\frac{2b}{\ln d}\right\}$ such minimal obstructions. We obtain that

$$\begin{aligned}
\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right) &\le 2n^{\ell_2}(ed)^{2s}\exp\left\{\frac{2b}{\ln d}\right\}(1-p)^{d(\ell_1+(1+o(1))s-u)+f(b)}\cdot n^{w/2}(ed)^{w/2}(1-p)^{dw-w}\\
&\le 2n^{\ell_2}(ed)^{2s}\exp\left\{\frac{2b}{\ln d}\right\}(1-p)^{d(\ell_1+(1+o(1))s-u)+f(b)}\\
&\le 2\exp\left\{\ell_2\ln n + 2s\ln(ed)+\frac{2b}{\ln d}-pf(b)\right\}n^{-(1-\epsilon)(\ell_1+(1+o(1))s-u)}\\
&\le 2\exp\left\{\ell_2\ln n + 2s(\ln d+1)+\frac{2b}{\ln d}-pf(b)-(1-\epsilon)(\ell_1+(1+o(1))s-u)\ln n\right\}\\
&\le 2\exp\left\{(\ell_2-(1-\epsilon)(\ell_1-u))\ln n - s(1-2\epsilon)\ln n + \frac{2b}{\ln d}-pf(b)\right\}.
\end{aligned}$$

Recall that $\ell = \ell_1 + \ell_2 + \ell_3$ and $\ell_2 \le \frac{s}{3}$ and $\ell_3 \le \frac{b}{d^2}$. Thus,

$$\begin{aligned}\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right) &\le 2\exp\left\{(\ell_2 - (1-\epsilon)(\ell - \ell_2 - \ell_3 - u))\ln n - s(1-2\epsilon)\ln n + \frac{2b}{\ln d} - pf(b)\right\}\\ &\le 2\exp\left\{\left(\epsilon\frac{s}{3} + (1-\epsilon)\frac{b}{d^2} - (1-\epsilon)(\ell-u)\right)\ln n - s(1-2\epsilon)\ln n + \frac{2b}{\ln d} - pf(b)\right\}\\ &\le 2\exp\left\{\frac{3b}{\ln d} - pf(b)\right\},\end{aligned}$$

where in the last step we used that $u < \ell$. Since we assume that $b \le \frac{n}{2}$ and $n \le C^d$, we have that $\log_C b \le \log_C(C^d/2) \le d - \frac{1}{C}$. Hence, $f(b) \ge \frac{b}{C^2}$. Therefore,

$$\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right) \le 2\exp\left\{b\left(\frac{3}{\ln d} - \frac{p}{C^2}\right)\right\} \le \exp\left\{-\frac{d^2}{2C^2}\right\}.$$

union bound over the at most n^6 values for u, w, s, ℓ_2, n and ℓ_3 completes the proof. □

Obstructions with $b > \frac{n}{2}$. Note that here, by Lemma 3.8, we may assume that $u \ge \frac{n}{d^{C^3/p}}$, and thus $|B^C| \ge \frac{n}{d^{C^3/p}}$. We begin by assuming that B^C has a large edge-boundary.

Lemma 3.13. ***Whp*** *there are no obstructions with $b > \frac{n}{2}$ and $e(B^C, B) \ge |B^C| \ln^2 d$.*

Proof. Fix u, ℓ_1, w, s and b. We then have that

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) + \sum_{j=1}^{w/2} e(W_j, W_j^C) \ge \ell_1 d + (1+o(1))sd + e(B, B^C) + wd - w.$$

By Claim 3.6, we may assume that $\ell_3 < \frac{6u}{d^2}$, and since $u + 1 \le \ell_1 + \ell_2 + \ell_3$, we have that

$$\ell_1 d + (1+o(1))sd + e(B, B^C) + wd - w \ge du - \frac{6u}{d} + |B^C|\ln^2 d \ge du + \frac{u\ln^2 d}{2}.$$

We have at most $\binom{n}{u}$ ways to choose the obstruction. Thus, the probability of having an obstruction with such u, ℓ_1, w, s, b is at most

$$\begin{aligned}\binom{n}{u}(1-p)^{u\ln^2 d/2} &\le \left(\frac{en}{u}\right)^u \exp\left\{-\frac{pu\ln^2 d}{2}\right\}\\ &\le \exp\left\{u\left(1 + \ln n - \ln u - \frac{p\ln^2 d}{2}\right)\right\}\\ &\le \exp\left\{u\left(1 + \frac{C^3}{p}\ln d - \frac{p\ln^2 d}{2}\right)\right\} \le \exp\left\{-\frac{pu\ln^2 d}{3}\right\},\end{aligned}$$

where in the penultimate inequality we used $u \ge \frac{n}{d^{C^3/p}}$. Union bound over the at most n^5 choices of u, ℓ_1, w, s and b completes the proof. □

We now turn the case where B^C has a small edge-boundary.

Lemma 3.14. ***Whp*** *there are no obstructions with $b > \frac{n}{2}$ and $e(B^C, B) \le |B^C| \ln^2 d$.*

Proof. We follow a similar argument to that in Lemma 3.12. Fix $u, w, s,$ and b, and let $\mathcal{B}_{u,w,s,b}$ be the event that there is a minimal obstruction with $u = |U|$, $w = |W|$, $s = |S|$ and $\frac{n}{2} \neq b = |B|$. Similarly to before, we then have that

$$\sum_{i=1}^{\ell} e(K_i, K_i^C) + \sum_{j=1}^{w/2} e(W_j, W_j^C) \ge \ell_1 d + (1+o(1))sd + e(B, B^C) + wd - w.$$

By Theorem 1.2 $e(B,B^C)\ge f(b)=f(n-b)$ and we obtain

$$\sum_{i=1}^{\ell} e(K_i,K_i^C)+\sum_{j=1}^{w/2} e(W_j,W_j^C)\ge \ell_1 d+(1+o(1))sd+f(n-b)+wd-w.$$

By the proof of Lemma 3.3, there are at most $n^{\ell_2+w/2}(ed)^{2s+w/2}$ ways to choose $W\cup S$. Furthermore, since we assume $e(B,B^C)\le |B^C|\ln^2 d$ and $|B^C|\ge u\ge \frac{n}{d^{\ln^2 d}}$, by Lemma 3.5 there are at most $\exp\left\{\frac{2|B^C|}{\ln d}\right\}=\exp\left\{\frac{2(n-b)}{\ln d}\right\}$ choices for B^C, and hence for B. Therefore, by Lemma 3.2, there are at most $2n^{\ell_2+w/2}(ed)^{2s+w/2}\exp\left\{\frac{2(n-b)}{\ln d}\right\}$ such minimal obstructions. We thus obtain that

$$\begin{aligned}
\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right)&\le 2n^{\ell_2}(ed)^{2s}\exp\left\{\frac{2(n-b)}{\ln d}\right\}(1-p)^{d(\ell_1+(1+o(1))s-u)+f(n-b)}\cdot n^{w/2}(ed)^{w/2}(1-p)^{dw-w}\\
&\le 2n^{\ell_2}(ed)^{2s}\exp\left\{\frac{2(n-b)}{\ln d}\right\}(1-p)^{d(\ell_1+(1+o(1))s-u)+f(n-b)}\\
&\le 2\exp\left\{(\ell_2-(1-\epsilon)(\ell_1-u))\ln n-s(1-2\epsilon)\ln n+\frac{2(n-b)}{\ln d}-pf(n-b)\right\}.
\end{aligned}$$

Recall that $\ell=\ell_1+\ell_2+\ell_3$ and $\ell_2\le\frac{s}{3}$ and by Claim 3.6 $\ell_3\le\frac{6u}{d^2}$. Thus,

$$\begin{aligned}
\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right)&\le 2\exp\left\{(\ell_2-(1-\epsilon)(\ell-\ell_2-\ell_3-u))\ln n-s(1-2\epsilon)\ln n+\frac{2(n-b)}{\ln d}-pf(n-b)\right\}\\
&\le 2\exp\left\{\left(\frac{6u}{d^2}+\epsilon\frac{s}{3}-s(1-2\epsilon)-(1-\epsilon)(\ell-u)\right)\ln n+\frac{2(n-b)}{\ln d}-pf(n-b)\right\}\\
&\le 2\exp\left\{\frac{3(n-b)}{\ln d}-pf(n-b)\right\},
\end{aligned}$$

where in the last step we used that $u<\ell$. Since $b\ge\frac{n}{2}$, we have that $\log_C(n-b)\le\log_C(C^d/2)\le d-\frac{1}{C}$. Hence, $f(n-b)\ge\frac{n-b}{C^2}$. Moreover, $n-b\ge u$. Therefore,

$$\mathbb{P}\left(\mathcal{B}_{u,w,s,b}\right)\le 2\exp\left\{u\left(\frac{3}{\ln d}-\frac{1}{C^2}\right)\right\}\le\exp\left\{-\frac{u}{2C^2}\right\}.$$

A union bound over the at most n^6 values for u,w,s,ℓ_2,b and ℓ_3 completes the proof. □

4 Discussion

We have extended the classical result concerning hitting times of minimum degree one, connectivity, and the existence of a perfect matching to random subgraphs of regular Cartesian product graphs. In particular, this includes a simplified self-contained version of the connectivity result for bond percolation on the hypercube. Let us mention that, independently, Collares, Doolittle, and Erde use a similar approach – that is, sprinkling with probabilities p_1 and p_2 – to show a connectivity result for bond percolation on the permutahedron [9]. There, however, similarly to the approach of [1, 6, 20], one utilises that in G_{p_1}, large components are relatively well-spread, that is, typically every vertex in G is quite close (in G) to a large component of G_{p_1}. In this paper, we neither require nor utilise such a 'density' statement, and instead use the fact that the isolated vertices are 'sparsely spread'.

We note that Lemma 3.2 does not require anything from the host graph G, and Lemma 3.7 only utilises some of the isoperimetric profile of G, yet does not utilise the product structure of G. Hence, these two results could be of independent interest, in particular in questions concerning the typical existence of a perfect matching under percolation in a more general setting. Still, in this proof, the bound on the number of sets with 'bad expansion' from Lemma 3.5 exploits the coordinate structure present in product graphs.

Many other random graph models are known to have typically the same hitting times for minimum degree one, connectivity, and the existence of a perfect matching (see, for example, [15] and the references therein). It is thus natural to ask what are the minimal requirements on G for this phenomenon to hold. As a step towards this, we propose the following question, considering regular graphs with high-degree.

Question 4.1. *Let G be a d-regular graph on n vertices, with $d=\omega(1)$ and n divisible by two. What minimal requirements are needed on G, such that in the graph process on G, the hitting times for minimum degree one, connectivity, and the existence of a perfect matching are the same?*

Utilising the Tutte-Berge formula we aimed for a (nearly-)perfect matching, that is, a matching missing none or only one vertex (in the case when n is odd) of the graph. A natural extension is to look for the threshold to have a matching that covers all but a small fraction of the vertices.

Question 4.2. *Let $G=\square_{i=1}^t H_i$ for H_i connected, regular and of bounded size. What is the threshold p^* such that for all $p \geq p^*$, **whp**, the giant component of G_p contains a (nearly-)perfect matching?*

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