# A topological sphere theorem for submanifolds of the hyperbolic space

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**Abstract**

We identify as topological spheres those complete submanifolds lying with any codimension in hyperbolic space whose Ricci curvature satisfies a lower bound contingent solely upon the length of the mean curvature vector of the immersion.

There are numerous papers that characterize the topology of compact submanifolds in space forms of nonnegative sectional curvature under pinching assumptions that encompass both intrinsic and extrinsic data. The former are given in terms of some metric curvature, while the latter incorporates concepts derived from the second fundamental form of the submanifold, quite often emphasizing its norm. Most of these papers are [2], [3], [8], [11], [12], [15], [16], [17], [18] and [19].

In rather stark contrast, the situation diverges significantly when the ambient space form features negative sectional curvature, a scenario addressed by this paper. The papers we have been able to find pertaining to this case, namely [5], [9] and [13], do not offer results related to the one presented here.

Let $f\colon M^n \to \mathbb{H}_c^{n+m}$, $n \geq 4$, be an isometric immersion with codimension $m$ of a complete $n$-dimensional Riemannian manifold into the hyperbolic space of constant sectional curvature $c < 0$. Let $\mathrm{Ric}_M$ stand for the (not normalized) Ricci curvature of $M^n$ and denote the norm of the (normalized) mean curvature vector field $\mathcal{H}$ by $H$.

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**Theorem 1.** *Let $f\colon M^n \to \mathbb{H}_c^{n+m}$, $n \geq 4$, be an isometric immersion of a complete Riemannian manifold. If at any point it holds that*

$$Ric_M \geq (n-4)c + (n-2)H^2 \tag{$*$}$$

*then $M^n$ is homeomorphic to a sphere $\mathbb{S}^n$.*

A well-known result due to Hamilton [6] gives that for dimension $n = 3$ the submanifold is diffeomorphic to a spherical space form.

When $M^n$ possesses the topological structure of a sphere, a conjecture for the weaker bound $(n-2)(c+H^2)$ under the assumption $c+H^2 \geq 0$ have been put forth by Xu and Gu [16]. It proposes that the submanifold should not merely be topologically equivalent but diffeomorphic to a sphere. In our case this holds true for dimensions $n = 5, 6, 12, 56, 61$ as in these cases it has been established that the differentiable structure is unique; see Corollary 1.15 in [14].

There are plenty of compact submanifolds in the hyperbolic space that satisfy $(*)$ strictly at any point. In fact, this is the case for any totally umbilical $n$-dimensional submanifold, being the inequality strict at any point only if $H > \sqrt{-3c}$. Notice that it will persist in its strict form after subjecting a totally umbilical submanifold to a sufficiently small smooth deformation.

Finally, notice that the theorem extends its applicability to compact submanifolds within both Euclidean space and the round sphere, as these are umbilical submanifolds of the hyperbolic space. However, in such cases the assumption regarding the Ricci curvature is more restrictive compared to that stipulated in [3].

# 1 The pinching condition

The following result for complete simply connected spaces forms of sectional curvature $c > 0$ has been proved by Lawson and Simons [7], and then by Xin [15] for $c = 0$ when strict inequality holds in $(*)$ at any point. Elworthy and Rosenberg [4, p. 71] observed that the result still holds by only requiring the bound to be strict at some point of the submanifold. In case $c < 0$ the result was proved by Fu and Xu [5] if strict inequality holds at any point. But the observation by Elworthy and Rosenberg also applies to this case.

**Theorem 2.** *Let $f\colon M^n \to \mathbb{Q}_c^{n+m}$, $n \geq 4$ be an isometric immersion of a compact manifold and $p$ an integer such that $1 \leq p \leq n-1$. Assume that at any point $x \in M^n$ and for* any *orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $T_xM$ the second fundamental form $\alpha_f\colon TM \times TM \to N_fM$ satisfies*

$$\Theta_p=\sum_{i=1}^{p}\sum_{j=p+1}^{n}\big(2\|\alpha_f(e_i,e_j)\|^2-\langle\alpha_f(e_i,e_i),\alpha_f(e_j,e_j)\rangle\big)\leq p(n-p\,\mathrm{sign}(c))c. \quad (\#)$$

*If the inequality (#) is strict at some point $x \in M^n$ and for any orthonormal basis of $T_xM$, then there are no stable $p$-currents and the homology groups satisfy that $H_p(M^n;\mathbb{Z}) = H_{n-p}(M^n;\mathbb{Z}) = 0$.*

In the sequel we will make use of the following lemma.

**Lemma 3.** *Let $f\colon M^n \to \mathbb{H}_c^{n+m}$, $n \geq 3$, be an isometric immersion that satisfies at $x \in M^n$ the inequality $(*)$. Then for the traceless part of the second fundamental form $\phi = \alpha_f - \langle\,,\,\rangle\mathcal{H}$ at $x$ we have $\frac{1}{n}\|\phi\|^2 \leq H^2 + 3c$. Hence $H \geq \sqrt{-3c}$ and if $H = \sqrt{-3c}$ then $f$ is totally umbilical at $x$.*

*Proof.* From $(*)$ the scalar curvature satisfies $\tau \geq n(n-4)c + n(n-2)H^2$. On the other hand, the Gauss equation gives that

$$\tau = n(n-1)c + n^2H^2 - S, \quad (1)$$

where $S$ is the norm of the second fundamental form. Then $S \leq 3nc+2nH^2$. Since $\|\phi\|^2 = S - nH^2$ we have the desired inequality. □

Recall that a vector in the normal space $\eta \in N_fM(x)$ at $x \in M^n$ is named a *Dupin principal normal* of $f\colon M^n \to \mathbb{H}_c^{n+m}$ at $x \in M^n$ if the associated tangent vector subspace

$$E_\eta(x) = \{X \in T_xM \colon \alpha_f(X,Y) = \langle X,Y\rangle\eta \text{ for all } Y \in T_xM\}$$

is at least two dimensional. The dimension of $E_\eta(x)$ is the *multiplicity* of $\eta$.

The proof of the following result is inspired by computations given by Xu and Gu in [16] and more recently by us in [2].

**Proposition 4.** *Let $f\colon M^n \to \mathbb{H}_c^{n+m}$, $n \geq 4$, be an isometric immersion satisfying the inequality $(*)$ at $x \in M^n$. Then at $x \in M^n$ the following assertions hold:*

*(i) The inequality (#) is satisfied for any integer $2 \leq p \leq n/2$ and for any orthonormal basis of $T_xM$. Moreover, if the inequality $(*)$ is strict or if $p < n/2$ then also (#) is strict for any orthonormal basis of $T_xM$.*

*(ii) Assume that equality holds in (#) for a certain integer $2 \leq p \leq n/2$ and an orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $T_xM$. Then $n = 2p$ and*

$$Ric_M(X) = (n-4)c + (n-2)H^2 \ \text{ for any unit } \ X \in T_xM.$$

*Moreover, we have:*

- *(a) If $n \geq 6$ then either $f$ is totally umbilical with $H = \sqrt{-3c}$ or we have that $H > \sqrt{-3c}$ and there are distinct Dupin principal normals $\eta_1$ and $\eta_2$ such that $E_{\eta_1} = span\{e_1, \ldots, e_p\}$ and $E_{\eta_2} = span\{e_{p+1}, \ldots, e_n\}$.*
- *(b) If $n = 4$ there are normal vectors $\eta_j$, $j = 1, 2$, such that*

$$\pi_{V_j} \circ A_\xi|_{V_j} = \langle \xi, \eta_j \rangle I \ \text{ for any } \xi \in N_f(x) \tag{2}$$

*where $V_1 = span\{e_1, e_2\}$, $V_2 = span\{e_3, e_e\}$ and $\pi_{V_j} \colon T_xM \to V_j$ are the projections.*

*Proof.* Recall that the Gauss equation of $f$ yields that the Ricci curvature for any unit vector $X \in T_xM$ is given by

$$\mathrm{Ric}_M(X) = (n-1)c + \sum_{\alpha=1}^{m} (\mathrm{tr} A_\alpha)\langle A_\alpha X, X\rangle - \sum_{\alpha=1}^{m} \|A_\alpha X\|^2, \tag{3}$$

where the $A_\alpha$, $1 \leq \alpha \leq m$, stand for the shape operators of $f$ associated to an orthonormal basis $\{\xi_\alpha\}_{1\leq\alpha\leq m}$ of the normal vector space $N_fM(x)$.

From now on, the basis $\{\xi_\alpha\}_{1\leq\alpha\leq m}$ is taken such that $\mathcal{H}(x) = H(x)\xi_1$ when $H(x) \neq 0$. For a given orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of $T_xM$, we denote for simplicity $\alpha_{ij} = \alpha_f(e_i, e_j)$, $1 \leq i, j \leq n$. Then, we have

$$\begin{aligned}
\Theta_p = {} & 2\sum_{i=1}^{p}\sum_{j=p+1}^{n} \|\alpha_{ij}\|^2 - n\sum_{i=1}^{p} \langle \alpha_{ii}, \mathcal{H}\rangle + \sum_{i,j=1}^{p} \langle \alpha_{ii}, \alpha_{jj}\rangle \\
= {} & 2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha} \langle A_\alpha e_i, e_j\rangle^2 - nH\sum_{i=1}^{p} \langle A_1 e_i, e_i\rangle + \sum_{\alpha} \Big(\sum_{i=1}^{p} \langle A_\alpha e_i, e_i\rangle\Big)^2 \\
\leq {} & 2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha} \langle A_\alpha e_i, e_j\rangle^2 - nH\sum_{i=1}^{p} \langle A_1 e_i, e_i\rangle + p\sum_{\alpha}\sum_{i=1}^{p} \langle A_\alpha e_i, e_i\rangle^2, 
\end{aligned} \tag{4}$$

where the inequality part was obtained using the Cauchy-Schwarz inequality

$$\Big(\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle\Big)^2 \le p\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle^2. \tag{5}$$

Since $p \ge 2$ by assumption, then

$$\begin{aligned}
2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha}\langle A_\alpha e_i, e_j\rangle^2 + p\sum_{i=1}^{p}\sum_{\alpha}\langle A_\alpha e_i, e_i\rangle^2 \\
\le p\sum_{i=1}^{p}\sum_{j=p+1}^{n}\sum_{\alpha}\langle A_\alpha e_i, e_j\rangle^2 + p\sum_{i=1}^{p}\sum_{\alpha}\langle A_\alpha e_i, e_i\rangle^2 \\
\le p\sum_{i=1}^{p}\sum_{\alpha}\|A_\alpha e_i\|^2
\end{aligned} \tag{6}$$

and thus (4) implies that

$$\Theta_p \le p\sum_{i=1}^{p}\sum_{\alpha}\|A_\alpha e_i\|^2 - nH\sum_{i=1}^{p}\langle A_1 e_i, e_i\rangle.$$

Setting $\varphi = A_1 - HI$ and using (3), we obtain

$$\begin{aligned}
\Theta_p &\le p\sum_{i=1}^{p}((n-1)c - \mathrm{Ric}_M(e_i)) + (p-1)nH\sum_{i=1}^{p}\langle A_1 e_i, e_i\rangle \\
&= p\sum_{i=1}^{p}\big((n-1)(c+H^2) - \mathrm{Ric}_M(e_i)\big) - p(n-p)H^2 \\
&\quad + (p-1)nH\sum_{i=1}^{p}\langle \varphi e_i, e_i\rangle.
\end{aligned} \tag{7}$$

Then

$$\Theta_p \le p^2\big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x)\big) - p(n-p)H^2 + (p-1)nH\sum_{i=1}^{p}\langle \varphi e_i, e_i\rangle \tag{8}$$

where

$$\mathrm{Ric}_M^{\min}(x) = \min\left\{\mathrm{Ric}_M(X)\colon X \in T_xM, \|X\| = 1\right\}.$$

We have that

$$(n-1)(c+H^2) \geq \mathrm{Ric}_M^{\min}(x) \tag{9}$$

and that equality holds if $f$ is totally umbilical at $x \in M^n$. In fact, it follows from (1) that

$$S \leq n(n-1)c + n^2H^2 - n\mathrm{Ric}_M^{\min}(x). \tag{10}$$

Therefore,

$$(n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x) \geq \frac{1}{n}(S - nH^2) = \frac{1}{n}\|\phi\|^2,$$

and (9) follows.

From (9) and having that $p \leq n/2$, then

$$p^2\big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x)\big) \leq p(n-p)\big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x)\big). \tag{11}$$

Therefore, from (8) it follows the estimate

$$\Theta_p \leq p(n-p)\big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x) - H^2\big) + (p-1)nH\sum_{i=1}^{p}\langle \varphi e_i, e_i\rangle. \tag{12}$$

Next, we obtain a second estimate of

$$\Theta_p = \sum_\alpha \Big(2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\langle A_\alpha e_i, e_j\rangle^2 - \sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle\Big).$$

Decomposing

$$\begin{aligned}
&\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle \\
=&\frac{n-p}{n}\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle + \frac{p}{n}\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
\Theta_p = \sum_\alpha \Big(&2\sum_{i=1}^{p}\sum_{j=p+1}^{n}\langle A_\alpha e_i, e_j\rangle^2 - \frac{n-p}{n}\mathrm{tr}\, A_\alpha \sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle \\
&+ \frac{n-p}{n}\big(\sum_{i=1}^{p}\langle A_\alpha e_i, e_i\rangle\big)^2 - \frac{p}{n}\mathrm{tr}\, A_\alpha \sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle + \frac{p}{n}\big(\sum_{j=p+1}^{n}\langle A_\alpha e_j, e_j\rangle\big)^2\Big).
\end{aligned}$$

Since $p(n-p)/n \geq 1$, we obtain using the Cauchy-Schwarz inequality that

$$\begin{aligned}\Theta_p \leq & \sum_\alpha \Big(2\sum_{i=1}^{p}\sum_{j=p+1}^{n} \langle A_\alpha e_i, e_j\rangle^2 - \frac{n-p}{n}\mathrm{tr}\, A_\alpha \sum_{i=1}^{p} \langle A_\alpha e_i, e_i\rangle \\ & + \ \frac{p(n-p)}{n}\sum_{i=1}^{p} \langle A_\alpha e_i, e_i\rangle^2 - \frac{p}{n}\mathrm{tr}\, A_\alpha \sum_{j=p+1}^{n} \langle A_\alpha e_j, e_j\rangle \\ & + \ \frac{p(n-p)}{n}\sum_{j=p+1}^{n} \langle A_\alpha e_j, e_j\rangle^2\Big) \\ \leq & \ \frac{p(n-p)}{n}S - (n-p)H\sum_{i=1}^{p} \langle A_1 e_i, e_i\rangle - pH\sum_{j=p+1}^{n} \langle A_1 e_j, e_j\rangle.\end{aligned}$$

It follows that

$$\Theta_p \leq \frac{p(n-p)}{n}S - 2p(n-p)H^2 - (n-2p)H\sum_{i=1}^{p} \langle \varphi e_i, e_i\rangle.$$

Then we have from (10) that

$$\Theta_p \leq p(n-p)\big((n-1)(c+H^2) - \mathrm{Ric}_M^{\min}(x) - H^2\big) - (n-2p)H\sum_{i=1}^{p} \langle \varphi e_i, e_i\rangle. \quad (13)$$

By computing $(n-2p)\times(12)+n(p-1)\times(13)$, we obtain

$$\Theta_p \leq p(n-p)\left((n-1)c + (n-2)H^2 - \mathrm{Ric}_M^{\min}(x)\right). \quad (14)$$

It follows from (14) using $(*)$ that

$$\Theta_p - p(n+p)c \leq 2p(n-2p)c, \quad (15)$$

and the inequality (#) has been proved. Clearly, if the inequality $(*)$ is strict or if $p < n/2$ then (#) is strict, and this completes the proof of part $(i)$.

We prove part $(ii)$. From part $(i)$ we have that $n = 2p$ and that all the inequalities that range from (4) to (8) as well as the ones from (11) to (15) become equalities.

From (5) we obtain

$$\langle A_\alpha e_i, e_i\rangle = \rho_\alpha \ \text{ for all } 1 \leq i \leq p, \ \ 1 \leq \alpha \leq m. \quad (16)$$

We have that (6) gives

$$(p-2)\langle A_\alpha e_i, e_j\rangle = 0 \text{ for all } 1 \le i \le p,\ p+1 \le j \le n,\ 1 \le \alpha \le m, \tag{17}$$

and

$$\langle A_\alpha e_i, e_{i'}\rangle = 0 \text{ for all } 1 \le i \ne i' \le p,\ 1 \le \alpha \le m. \tag{18}$$

From (7) and (8) we have $\mathrm{Ric}_M(e_i) = \mathrm{Ric}_M^{\min}(x)$. Then (14) and (15) give

$$\mathrm{Ric}_M(e_i) = \mathrm{Ric}_M^{\min}(x) = (n-4)c + (n-2)H^2 \text{ for all } 1 \le i \le p. \tag{19}$$

Since $n = 2p$ then equality also holds in (#) for the reordered orthonormal basis $\{e_{p+1}, \dots, e_n, e_1, \dots, e_p\}$ of $T_xM$. Therefore, we also have that

$$\langle A_\alpha e_j, e_j\rangle = \mu_\alpha \text{ for all } p+1 \le j \le n,\ 1 \le \alpha \le m, \tag{20}$$

$$\langle A_\alpha e_j, e_{j'}\rangle = 0 \text{ for all } p+1 \le j \ne j' \le n,\ 1 \le \alpha \le m, \tag{21}$$

and

$$\mathrm{Ric}_M(e_j) = \mathrm{Ric}_M^{\min}(x) = (n-4)c + (n-2)H^2 \text{ for all } p+1 \le j \le n. \tag{22}$$

Hence, we obtain from (19) and (22) that

$$\mathrm{Ric}_M(X) = (n-4)c + (n-2)H^2 \text{ for any unit } X \in T_xM. \tag{23}$$

In particular, it follows at $x$ from (16), (17), (18), (20) and (21) that the vectors $\eta_1 = \sum_\alpha \rho_\alpha \xi_\alpha$ and $\eta_2 = \sum_\alpha \mu_\alpha \xi_\alpha$ are Dupin principal normals with $E_{\eta_1} = \mathrm{span}\{e_1, \dots, e_p\}$ and $E_{\eta_2} = \mathrm{span}\{e_{p+1}, \dots, e_n\}$. If $\eta_1 = \eta_2$, then $f$ at $x$ is totally umbilical and equality holds in (9). This combined with (23) yields $H = \sqrt{-3c}$ at $x$. If otherwise, then Lemma 3 gives at $x$ that $H > \sqrt{-3c}$ and that $\eta_1$ and $\eta_2$ are distinct Dupin principal normals, and this concludes the proof of part $(a)$.

Finally, if $n = 4$ then for any $\xi \in N_f(x)$ we have (2) where $\eta_1 = \sum_\alpha \rho_\alpha \xi_\alpha$ and $\eta_2 = \sum_\alpha \mu_\alpha \xi_\alpha$, and part $(ii)$ has also been proved. ☐

# 2 The proof of Theorem 1

For the proof of Theorem 1 we initially establish a topological result.

**Lemma 5.** *Let $f\colon M^n \to \mathbb{H}_c^{n+m}$, $n \geq 4$, be an isometric immersion of a compact manifold satisfying*

$$Ric_M > \frac{1}{n+2}\big((n^2-n-4)c + n(n-1)H^2\big). \tag{24}$$

*Then $\pi_1(M^n) = 0$ and $H_{n-1}(M^n, \mathbb{Z}) = 0$.*

*Proof.* From (1) and (24) it follows that

$$\|\phi\|^2 \leq \frac{2n}{n+2}\big((n+1)c + (n-1)H^2\big). \tag{25}$$

Let $\{e_i\}_{1\leq i\leq n}$ and $\{\xi_\alpha\}_{1\leq\alpha\leq m}$ be orthonormal tangent and normal bases at $x \in M^n$. Using (3) we obtain that

$$\begin{aligned}
&\sum_{j=2}^{n}\big(2\|\alpha_{1j}\|^2 - \langle\alpha_{11}, \alpha_{jj}\rangle\big)\\
&\quad = 2\sum_\alpha\sum_{j=2}^{n}\langle A_\alpha e_1, e_j\rangle^2 - \sum_\alpha\langle A_\alpha e_1, e_1\rangle\sum_{j=2}^{n}\langle A_\alpha e_j, e_j\rangle\\
&\quad = \sum_\alpha\sum_{j=2}^{n}\langle A_\alpha e_1, e_j\rangle^2 - \sum_\alpha \operatorname{tr} A_\alpha\langle A_\alpha e_1, e_1\rangle + \sum_\alpha\|A_\alpha e_1\|^2\\
&\quad = \sum_{j=2}^{n}\|\phi(e_1, e_j)\|^2 + (n-1)c - \mathrm{Ric}_M(e_1).
\end{aligned}$$

Then this together with (24) and (25) give

$$\sum_{j=2}^{n}\big(2\|\alpha_{1j}\|^2 - \langle\alpha_{11}, \alpha_{jj}\rangle\big) \leq \frac{1}{2}\|\phi\|^2 + (n-1)c - \mathrm{Ric}_M(e_1) < (n+1)c.$$

Hence, by Theorem 2 there are no stable 1-currents on $M^n$ and therefore $H_1(M^n, \mathbb{Z}) = H_{n-1}(M^n, \mathbb{Z}) = 0$. Since in each nontrivial free homotopy class there is a length minimizing curve, we conclude that $\pi_1(M^n) = 0$. $\square$

*Proof of Theorem 1:* We have from Lemma 3 that $H \geq \sqrt{-3c}$. It then follows from $(*)$ that $\mathrm{Ric}_M \geq -2(n-1)c$ and hence $M^n$ is compact by the classical Bonnet-Myers theorem. Moreover, since

$$\mathrm{Ric}_M \geq (n-4)c + (n-2)H^2 > \frac{1}{n+2}\big((n^2-n-4)c + n(n-1)H^2\big)$$

then Lemma 5 yields that $M^n$ is simply connected and $H_{n-1}(M^n, \mathbb{Z}) = 0$.

According to part $(i)$ of Proposition 4 the inequality (#) is satisfied at any point of $M^n$ for any $2 \leq p \leq n/2$ and for any orthonormal tangent basis at that point. We argue that the homology groups satisfy

$$H_p(M^n; \mathbb{Z}) = 0 = H_{n-p}(M^n; \mathbb{Z}) \text{ for all } 2 \leq p \leq n/2. \tag{26}$$

Suppose to the contrary that (26) does not hold. Consider the nonempty set

$$P = \{2 \leq p \leq n/2 : H_p(M^n; \mathbb{Z}) \neq 0 \text{ or } H_{n-p}(M^n; \mathbb{Z}) \neq 0\}$$

and denote $k = \max P$. Hence $H_k(M^n; \mathbb{Z}) \neq 0$ or $H_{n-k}(M^n; \mathbb{Z}) \neq 0$. By Theorem 2 at any point $x \in M^n$ there is an orthonormal tangent basis such that equality holds in (#) for $p = k$. Moreover, we have from part $(ii)$ of Proposition 4 that (23) holds. In this situation it is well-known that $M^n$ is an Einstein manifold and, in particular, it follows that $H$ is a positive constant.

We need to differentiate between two cases based on the dimension of the submanifold.

*Case* $n \geq 6$. Part $(ii)$ of Proposition 4 yields $k = n/2$. We argue that $H > \sqrt{-3c}$. If we have otherwise, then the submanifold is totally umbilical by part $(ii)$ of Proposition 4. Hence, we have from the Gauss equation that $M^n$ has constant sectional curvature $-2c$. But then $M^n$ would be isometric to a round sphere, and this contradicts our assumption that $H_k(M^n; \mathbb{Z}) \neq 0$.

Since $H > \sqrt{-3c}$, according to part $(ii)$ of Proposition 4 there are smooth Dupin principal normal vector fields $\eta_1$ and $\eta_2$ of multiplicity $k$ and corresponding smooth distributions $E_1$ and $E_2$. Let $\{X_\ell\}_{1 \leq \ell \leq n}$ be a smooth local orthonormal frame satisfying that $E_1 = \operatorname{span}\{X_1, \ldots, X_k\}$ and $E_2 = \operatorname{span}\{X_{k+1}, \ldots, X_n\}$. Then $\alpha_f(X_i, X_i) = \eta_1$ if $1 \leq i \leq k$ and $\alpha_f(X_j, X_j) = \eta_2$ if $k+1 \leq j \leq n$.

If follows from the Gauss equation that

$$\mathrm{Ric}_M(X) = (n-1)c\|X\|^2 + n\langle \mathcal{H}, \alpha_f(X, X)\rangle - \mathit{III}(X) \text{ for any } X \in \mathfrak{X}(M),$$

where $\mathit{III}(X) = \sum_{\ell=1}^n \|\alpha_f(X, X_\ell)\|^2$ is the so called third fundamental form of $f$. Since $\mathcal{H} = (\eta_1 + \eta_2)/2$, then

$$4H^2 = \|\eta_1\|^2 + \|\eta_2\|^2 + 2\langle \eta_1, \eta_2\rangle. \tag{27}$$

Moreover, we have for $1 \leq i \leq k$ that

$$III(X_i) = \sum_{\ell=1}^{n} \|\alpha(X_\ell, X_i)\|^2 = \|\eta_1\|^2$$

and

$$\begin{aligned}(n-4)c + (n-2)H^2 = \mathrm{Ric}_M(X_i) &= (n-1)c + n\langle \mathcal{H}, \alpha(X_i, X_i)\rangle - III(X_i) \\ &= (n-1)c + k\langle \eta_1 + \eta_2, \eta_1\rangle - \|\eta_1\|^2.\end{aligned}$$

Thus

$$(n-4)c + (n-2)H^2 = (n-1)c + (k-1)\|\eta_1\|^2 + k\langle \eta_1, \eta_2\rangle. \tag{28}$$

Arguing similarly for $k+1 \leq j \leq n$, we obtain

$$(n-4)c + (n-2)H^2 = (n-1)c + (k-1)\|\eta_2\|^2 + k\langle \eta_1, \eta_2\rangle. \tag{29}$$

It follows from (28) and (29) that $\|\eta_1\| = \|\eta_2\|$, and hence (27) becomes

$$2H^2 = \|\eta_1\|^2 + \langle \eta_1, \eta_2\rangle. \tag{30}$$

Combining (28) with (30) gives

$$\langle \eta_1, \eta_2\rangle = -3c. \tag{31}$$

Then, we conclude from (30) that

$$\|\eta_1\|^2 = \|\eta_2\|^2 = 2H^2 + 3c > 0. \tag{32}$$

The Codazzi equation for $f$ is easily seen to yield

$$\langle \nabla_X Y, Z\rangle(\eta_i - \eta_j) = \langle X, Y\rangle \nabla^\perp_Z \eta_i \text{ if } i \neq j \tag{33}$$

for any $X, Y \in E_i, Z \in E_j$. Using (31) and (32) then (33) gives

$$2\langle \nabla_X Y, Z\rangle(H^2 + 3c) = \langle X, Y\rangle \langle \nabla^\perp_Z \eta_i, \eta_i\rangle = 0$$

for all $X, Y \in E_i$ and $Z \in E_j$, $i \neq j$, that is, the distributions $E_1$ and $E_2$ are totally geodesic. Being simply connected, then de Rham theorem gives that $M^n$ is a Riemannian product $M_1^k \times M_2^k$ such that $TM_j^k = E_j$, $j = 1, 2$. It follows from the Gauss equation that the manifolds $M_1^k$ and $M_2^k$ have both constant sectional curvature $2H^2 + 4c$. But then the Ricci curvature of $M^n = M_1^k \times M_2^k$ is $(n-2)(H^2 + 2c)$, which is in contradiction with (23).

*Case* $n = 4$. We have that $k = 2$ and $H_2(M^4; \mathbb{Z}) \neq 0$. Since $\mathrm{Ric}_M = 2H^2$ then $\tau = 8H^2$ and hence (1) gives $S = 12c + 8H^2$ or, equivalently, that $\|\phi\|^2 = 12c + 4H^2$. It then follows from Proposition 16 in [8] that the Bochner operator $\mathcal{B}^{[2]} \colon \Omega^2(M^4) \to \Omega^2(M^4)$, a certain symmetric endomorphism of the bundle of 2-forms $\Omega^2(M^4)$, satisfies for any $\omega \in \Omega^2(M^4)$ the inequality

$$\langle \mathcal{B}^{[2]}\omega, \omega\rangle \geq \big((4(c + H^2) - \|\phi\|^2\big)\|\omega\|^2 = -8c\|\omega\|^2.$$

Hence $\mathcal{B}^{[2]}$ is positive definite.

We claim that the second Betti number $\beta_2(M^4)$ of the manifold vanishes. If otherwise, then there would exist a nonzero harmonic 2-form $\omega \in \Omega^2(M^4)$. By the Bochner-Weitzenböck formula the Laplacian of $\omega$ is given by

$$0 = \Delta\omega = \nabla^*\nabla\omega + \mathcal{B}^{[2]}\omega,$$

where $\nabla^*\nabla$ is the rough Laplacian. From this we obtain

$$\|\nabla\omega\|^2 + \langle \mathcal{B}^{[2]}\omega, \omega\rangle + \frac{1}{2}\,\Delta\|\omega\|^2 = 0.$$

Then the maximum principle and the fact that $\mathcal{B}^{[2]}$ is positive definite imply that $\omega = 0$, which proves the claim.

From the claim, we have that $H_2(M^4; \mathbb{Z})$ is a nontrivial torsion group and Poincaré duality gives that the torsion of $H^2(M^4; \mathbb{Z})$ is isomorphic to $H_2(M^4; \mathbb{Z})$. On the other hand, the universal coefficient theorem of cohomology yields that the torsion subgroups of $H^2(M^4; \mathbb{Z})$ and $H_1(M^4; \mathbb{Z})$ are isomorphic (cf. [10, p. 244 Corollary 4]). Since $M^4$ is simply connected, we have that $H_1(M^n; \mathbb{Z}) = 0$ and thus $H^2(M^4; \mathbb{Z})$ is torsion free. This is a contradiction and completes the proof that (26) holds.

Hence $M^n$ is a simply connected homology sphere and it follows from the Hurewicz isomorphism theorem that $M^n$ is a homotopy sphere. Finally, the resolution of the generalized Poincaré conjecture gives that $M^n$ is homeomorphic to $\mathbb{S}^n$. $\square$

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