STABILITY ESTIMATES FOR THE INVERSE SOURCE PROBLEM WITH PASSIVE MEASUREMENTS*

KRISTOFFER LINDER-STEINLEIN†, MIRZA KARAMEHMEDOVIĆ‡, AND FAOUZI TRIKI, §

Abstract.
We consider the multi-frequency inverse source problem in the presence of a non-homogeneous medium using passive measurements. Precisely, we derive stability estimates for determining the source from the knowledge of only the imaginary part of the radiated field on the boundary for multiple frequencies. The proof combines a spectral decomposition with a quantification of the unique continuation of the resolvent as a holomorphic function of the frequency. The obtained results show that the inverse problem is well posed when the frequency band is larger than the spatial frequency of the source.

Key words. Passive-imaging, Stability estimate, Helmholtz equation, Inverse source problem.

AMS subject classifications. 34A55

1. Introduction.

This paper is concerned with the stability estimate for determining a one-dimensional source in the presence of a medium using passive measurements. That is, reconstruction of the source having access to only the imaginary part of the resulting wave-field at the domain's boundary. This is a type of analysis of the time-reversal experiment [11]. Time reversal has been well studied and is one of the most commonly used reconstruction methods in direct imaging for inverse source problems. It was first proposed for energy focusing in physics. Here we apply the Helmholtz-Kirchhoff identity to study time reversal [2, 11].

Multiple results already exist in the cases where the entire wave-field at the boundary is obtainable [5, 8, 4, 13, 9, 6, 1]. We here prove for the one-dimensional Helmholtz equation that it is possible to complete the missing real part of the resulting wave-field using the measurable quantities, and to recover the source. These results can be seen in Theorem 1.1 for a large frequency band and in Theorem 1.2 for a short frequency band.

The remaining part of this section introduces the mathematical model and spaces from which the sources and media originate. This is followed by a statement regarding the available data and the considered inverse problem with passive measurements. In section 2 the proof of the first main result is provided. It is based on a spectral decomposition of the source in a specific orthonormal basis of eigenfunctions.

In the last section, the second main stability estimate for the inverse problem is established using a quantification of the unique continuation for the resolvent of the Helmholtz operator as a holomorphic function of the frequency and making use of the results of the previous section 2.

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†Department of Applied Mathematics, Technical University of Denmark, Kgs. Lyngby, Denmark (krlin@dtu.dk, https://orbit.dtu.dk/en/persons/kristoffer-linder-steinlein-2).
‡Department of Applied Mathematics, Technical University of Denmark, Kgs. Lyngby, Denmark (mika@dtu.dk, https://orbit.dtu.dk/en/persons/mirza-karamehmedovic).
§Laboratoire Jean Kuntzmann, Université Grenoble-Alpes, Grenoble, France (Faouzi.Triki@univ-grenoble-alpes.fr , https://membres-ljk.imag.fr/Faouzi.Triki/contact.html).

1.1. Mathematical model. We here focus on the one dimensional Helmholtz equation, which can be expressed as:

$$\phi''(x,k) + k^2(1+q(x))\phi(x,k) = f(x), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $1+q$ and f are real-valued functions called respectively the refractive index and the source; the resulting solution ϕ to (1.1) is the field generated by the source f in the presence of the medium q both supported in the interval $[0,1]$. The coefficient k is any positive number and referred to as the frequency. We furthermore impose the Sommerfeld radiation condition on the field:

$$\begin{aligned} \phi'(0,k) + ik\phi(0,k) &= 0, \\ \phi'(1,k) - ik\phi(1,k) &= 0. \end{aligned} \tag{1.2}$$

The medium and source functions are assumed to belong to the spaces of real-valued functions

$$\mathcal{M}(q_0, M, m, n_0) := \{q \in C_0^{m+1}([0,1]) : \ \|q - q_0\|_{C^{m+1}([0,1])} \leq M, n_0 \leq 1 + q\},$$

where $m \geq 1$, $M > 0$, $q_0 \in C_0^{m+1}([0,1])$ and $1 + q_0 \geq n_0$ for some $n_0 \in (0,1)$ and

$$\mathcal{F}(L) := \{f \in H^1(0,1) : \ \|f\|_{H^1(0,1)} \leq L, \ \operatorname{supp} f \subset (0,1)\},$$

for $L > 0$.

1.2. Passive measurements model and main results. The data assumed accessible in this work is passive measurements. We consider conventional full apparatus-based passive imaging, that is, the sensors completely surround the domain imaged. The described setup is an application of the analysis of the time-reversal experiment [11]. Time reversal has been well studied and is one of the most commonly used reconstruction methods in direct imaging for inverse source problems. It was first proposed for energy focusing in physics. Here we apply the Helmholtz-Kirchhoff identity to study time reversal [2, 11].

The analysis is based on an integral representation of the solution of the Helmholtz equation (1.1), given by

$$\phi(x,k) = \int_0^1 G(x,z,k) f(z) dz, \tag{1.3}$$

where G denotes the Green function of the Helmholtz equation (1.1), satisfying

$$\begin{cases} G''(x,z,k) + k^2(1+q(x))G(x,z,k) = \delta_z(x), \qquad x \in \mathbb{R}, \\ G'(0,z,k) + ikG(0,z,k) = 0, \\ G'(1,z,k) - ikG(1,z,k) = 0. \end{cases} \tag{1.4}$$

The time-reversal imaging functional in the multidimensional case given by [2]

$$\mathcal{I}(x,k) = \int_\Gamma \overline{G(x,z,k)} \phi(z,k) ds(z).$$

where Γ is a closed curve where the measurements will be taken. We here define the integral along the end-points of an interval as the Lebesgue integral w.r.t. the signed measure $\gamma(E) = \mathbb{1}_{b\in E} - \mathbb{1}_{a\in E}$ leading to

$$\mathcal{I}(x,k) = \int_{\{0,1\}} \overline{G(x,z,k)} \phi(z,k) d\gamma(z).$$

Based on [2, Corollary 2.1] or [11, Theorem 2.2] the above imaging functional is equivalent to

$$\mathcal{I}(x,k) = -\frac{1}{k}\int_0^1 \Im G(x,y,k) f(y)dy,$$

where the one-dimensional version of the Helmholtz-Kirchhoff identity is used reading

$$k\int_{\{0,1\}} \overline{G(x,z,k)} G(y,z,k) d\gamma(z) = -\Im G(x,y,k).$$

Therefore

$$\mathcal{I}(x,k) = -\frac{1}{k}\Im\phi(x,k),\ x\in\{0,1\}.$$

In [11] it is proven that the Helmholtz-Kirchhoff identity is closely related to the cross-correlation. The Helmholtz-Kirchhoff identity not only holds for the Green's functions but for the total field as well, and it is shown in [10], in higher dimensions, the cross-correlation matrix is given by the imaginary near field, however, in one dimension this amounts to the cross-correlation being given by the imaginary field at the boundary.

Let K be a fixed positive constant, and let $I = (0,K)$ be the index set for the wave-numbers used for multifrequency measurements. We will only work with measurements at the boundary $x\in\{0,1\}$, and the inverse problem can be stated as:

From measurements $\Im\phi(0,k)$ and $\Im\phi(1,k),\ k\in I$, reconstruct the source f.

This is solved in two steps, first reconstructing the source f using the system satisfied by $\Re\phi$ on a large band of frequencies, and second employing unique continuation techniques for holomorphic functions to recover the required boundary data from a small band of frequencies. We give here the main results of the paper.

Theorem 1.1. *Let $f\in\mathcal{F}$. Then there exist constants $c_0>0$ and $C_0>0$ that only depend on $\mathcal{M}$ such that*

$$\|f\|_{H^{-1}} \le C_0(\mu_f+1)\left(\|\Im\phi(0,\cdot)\|_{L^\infty(0,c_0\mu_f)} + \|\Im\phi(1,\cdot)\|_{L^\infty(0,c_0\mu_f)}\right), \tag{1.5}$$

where $\mu_f = \|f\|_{L^2}/\|f\|_{H^{-1}}$.

Theorem 1.2. *Assume that*

$$\varepsilon := C_0(\mu_f+1)\left(\|\Im\phi(0,\cdot)\|_{L^\infty(0,K)} + |\Im\phi(1,\cdot)\|_{L^\infty(0,K)}\right) < 1.$$

Then there exist constants $C = C(\mathcal{M},L)>0$ and $n_h = n_h(\mathcal{M})\in\mathbb{N}^*$ such that the inequality

$$\|f\|_{H^{-1}} \le C^{1-\eta(c_0\mu_f,K)}\varepsilon^{\eta(c_0\mu_f,K)}, \tag{1.6}$$

holds, where $\mu_f = \|f\|_{L^2}/\|f\|_{H^{-1}}$, and

$$\eta(s,K) := \frac{2}{\pi}\arctan\left(\frac{(e^K-1)^{n_h}}{\left((e^s-1)^{2n_h} - (e^K-1)^{2n_h}\right)^{\frac{1}{2}}}\right), \quad \textit{for } s>K.$$

Remark 1.3. i) The number μ_f represents the frequency of the source and characterizes its spatial oscillations [3, 16].

ii) The stability estimate (1.5) shows that the inverse problem with passive measurements is well posed if the frequency band is large enough and covers the spatial frequency of the source. The Lipschitz constant grows linearly with respect to the frequency of the source which is in agreement with the known physical resolution in source imaging.

iii) The stability estimate (1.6) indicates that the inverse problem with passive measurements becomes ill-posed if the frequency band shrinks to zero. Notice that when $K = c_0\mu_f$ we recover the stability estimate (1.5) from (1.6).

iv) The integer n_h is related to $h = \frac{2\pi}{n_h}$ which is the width of the complex strip S around the real axis (see Proposition 3.1) where the system (1.1) is free from scattering resonances. It can be shown that $\eta(c_0\mu_f, K)$ decreases exponentially to zero when h tends to zero. This shows that recovering the source from a small band of frequencies becomes ill-posed when the imaginary part of the resonances are closer to the real axis (trapped modes).

2. Proof of Theorem 1.1.

Let $w = \Re(\phi)$. We deduce from (1.1) and (1.2) that w satisfies the following system

$$
\begin{cases} w''(x,k) + k^2(1+q(x))w(x,k) = f(x), & x \in \mathbb{R}, \\ w'(0,k) = k\Im\phi(0,k), \\ w'(1,k) = -k\Im\phi(1,k). \end{cases} \tag{2.1}
$$

Based on the passive measurements the data at hand is equivalent to the Neumann boundary data in the system (2.1). Recall that this later may not have a unique solution for all $k \in I$.

The strategy next is to recover $f(x)$, $x \in (0,1)$ from $\Im(\phi(x,k))$, $k \in I$, $x \in \{0,1\}$ using the system (2.1). The approach is based on a spectral decomposition of the source in an orthonormal basis formed by the eigenfunctions of the system. Taking the frequency within the set of the associated real eigenvalues we are able to determine the coefficient of the expansion of the source in terms of passive boundary measurements. This method has been applied previously in multifrequency inverse source problems with full data [1, 13].

2.1. Spectral decomposition.

Since $q \in \mathcal{M}$, the weighted space $L^2_q(0,1)$ endowed with the inner product

$$
\langle \varphi, \psi \rangle_{L^2_q} = \int_0^1 (1+q(x))\varphi(x)\psi(x)dx,
$$

is well defined, and its norm $\|\cdot\|_{L^2_q}$ is equivalent to the classical L^2 norm.

Let $\{\mu_j, \phi_j\}_{j=1}^\infty$ be all the pairs solutions to the Neumann spectral problem:

$$
\begin{cases} -\phi_j''(x) = \mu_j^2(1+q(x))\phi_j(x) & x \in (0,1), \\ \phi_j' = 0 & x \in \{0,1\}, \\ \|\phi_j\|_{L^2_q} = 1. \end{cases} \tag{2.2}
$$

Since $q \in \mathcal{M}$, we have $\mu_1 = 0$, $(\mu_j)_{j\in\mathbb{N}^*}$ is an increasing real sequence, $\phi_1 = \|1+q\|_{L^1}^{-1/2}$, and

$(\phi_j)_{j\in\mathbb{N}^*}$ is an orthonormal basis of $L^2_q(0,1)$ [14].

Therefore

$$f(x)(1+q(x))^{-1} = \sum_{j=1}^{\infty} f_j\phi_j(x), \quad x \in (0,1),$$

where $f_j = \langle f(1+q), \phi_j\rangle_{L^2_q} = \int_0^1 f(x)\phi_j dx$.

Multiplying the Helmholtz equation (2.1) taken at $k = \mu_j$ by ϕ_j, and integrating by parts, lead to

$$f_j = \mu_j \left(\phi_j(0)\mathfrak{I}(\phi)(0,\mu_j) - \phi_j(1)\mathfrak{I}(\phi)(1,\mu_j)\right). \tag{2.3}$$

Proposition 2.1. *Let (μ_j, ϕ_j) be a pair of eigenelements of the Neumann spectral problem (2.2). Then there exists a constant $C = C(\mathcal{M}) > 0$ such that*

$$|\phi_j(0)| + |\phi_j(1)| \le C(\mu_j + 1), \quad \forall j \in \mathbb{N}^*. \tag{2.4}$$

Proof. We have

$$\phi_j(0) = \phi_j(x) - \int_0^x \phi_j'(t)dt, \quad \forall x \in (0,1).$$

Hence

$$|\phi_j(0)| \le |\phi_j(x)| + \|\phi_j'\|_{L^2}.$$

Integrating both sides over $(0,1)$, yields

$$|\phi_j(0)| \le \|(1+q)^{-\frac{1}{2}}\|_{L^2} + \|\phi_j'\|_{L^2}.$$

On the other hand, we deduce from (2.2) that $\|\phi_j'\|_{L^2} = \mu_j$. By taking $C = \max(1, \|(1+q)^{-\frac{1}{2}}\|_{L^2})$, we then obtain the final estimate for $\phi_j(0)$. The estimate for $\phi_j(1)$ can be derived by following the same steps of the previous proof for $\phi_j(0)$. ∎

Combining the identity (2.3) with the estimate (2.4), we get

$$|f_j| \le C(1+\mu_j)\left(|\mathfrak{I}(\phi)(0,\mu_j)| + |\mathfrak{I}(\phi)(1,\mu_j)|\right), \quad \forall j \in \mathbb{N}^*. \tag{2.5}$$

For $\psi \in H^s(0,1)$ with $s \in \mathbb{R}$, we further define the norm

$$\|\psi\|_s := \left(\sum_{j=1}^{\infty}(1+\mu_j^2)^s\psi_j^2\right)^{\frac{1}{2}},$$

where $\psi_j = \langle \psi, \phi_j\rangle_{L^2_q}$. For $\mu \in (0,+\infty)$, we then have

$$\|f(1+q)^{-1}\|_{-1}^2 = \sum_{j=1}^{\infty}(1+\mu_j^2)^{-1}f_j^2 \le \sum_{\mu_j\le\mu}(1+\mu_j^2)^{-1}f_j^2 + \frac{1}{\mu^2}\sum_{\mu_j>\mu} f_j^2.$$

Consequently

$$(1-\frac{\tilde{\mu}_f^2}{\mu^2})\|f(1+q)\|_{-1}^2 \le \sum_{\mu_j\le\mu}(1+\mu_j^2)^{-1}f_j^2,$$

where

$$\tilde{\mu}_f = \frac{\|f(1+q)^{-1}\|_0}{\|f(1+q)^{-1}\|_{-1}}.$$

Taking $\mu = \sqrt{2}\tilde{\mu}_f$, we obtain

$$\|f(1+q)^{-1}\|_{-1}^2 \le 2 \sum_{\mu_j \le \sqrt{2}\tilde{\mu}_f} (1+\mu_j^2)^{-1} f_j^2. \tag{2.6}$$

On the other hand we deduce from estimate (2.3)

$$|f_j|^2 \le 4C(1+\mu_j^2)\left(|\Im(\phi)(0,\mu_j)|^2 + |\Im(\phi)(1,\mu_j)|^2\right), \quad \forall j \in \mathbb{N}^*. \tag{2.7}$$

Combining inequalities (2.6) and (2.7) yields

$$\|f(1+q)^{-1}\|_{-1}^2 \le 8C\left(\sum_{\mu_j \le \sqrt{2}\tilde{\mu}_f} 1\right) \sup_{k\in(0,\sqrt{2}\tilde{\mu}_f)} \left(|\Im(\phi)(0,k)|^2 + |\Im(\phi)(1,k)|^2\right). \tag{2.8}$$

Proposition 2.2. *Let μ_j be an eigenvalue of the Neumann spectral problem* (2.2)*. Then there exist constants $c_i = c_i(\mathcal{M}) > 0$, $i = 1, 2$, such that*

$$c_1(j-1) \le \mu_j \le c_2(j-1), \quad \forall j \in \mathbb{N}^*. \tag{2.9}$$

Proof. Using the Min-max principle we have [12]

$$\mu_j = \min_{E_j\subset H^1(0,1),\, \dim(E_j)=j}\ \max_{\phi\in E_j\setminus\{0\}} \frac{\|\phi'\|_{L^2}^2}{\|\phi\|_{L^2_q}^2}.$$

Since

$$\min_{x\in[0,1]}(1+q(x))\|\phi\|_{L^2} \le \|\phi\|_{L^2_q} \le \max_{x\in[0,1]}(1+q(x))\|\phi\|_{L^2},$$

we obtain

$$\left(\max_{x\in[0,1]}(1+q(x))\right)^{-1}\pi(j-1) \le \mu_j \le \left(\min_{x\in[0,1]}(1+q(x))\right)^{-1}\pi(j-1).$$

Consequently there exist constants $c_i > 0$, $i = 1, 2$, that depend only on $\mathcal{M}$ such that the inequalities (2.9) hold. ∎

We deduce from estimates (2.9) that

$$\sum_{\mu_j \le \sqrt{2}\tilde{\mu}_f} 1 \le c_3\tilde{\mu}_f + 1,$$

for some constant $c_3 > 0$ that depend only on $\mathcal{M}$. Then it follows from inequality (2.8) that

$$\|f(1+q)^{-1}\|_{-1}^2 \le 8C(c_3\tilde{\mu}_f + 1) \sup_{k\in(0,\sqrt{2}\tilde{\mu}_f)} \left(|\Im(\phi)(0,k)|^2 + |\Im(\phi)(1,k)|^2\right). \tag{2.10}$$

Since $q \in \mathcal{M}$, one can easily show that for $s \in \mathbb{R}$, the norm $\|\cdot\|_{H^s}$ is equivalent to $\|\cdot\|_s$. Hence there exists a constant $c_4 > 0$ that only depend on $\mathcal{M}$ such that $\tilde{\mu}_f \le c_4\mu_f$. Finally the main result of the Theorem follows directly from the fact that $q \in \mathcal{M}$ and the last inequality (2.10).

3. Proof of Theorem 1.2. In this section, we adopt the method used in [13] to the problem considered here. The significant differences hinge on the fact that no Black-Box operator theory is used, and the region where the resolvent is holomorphic is given by [7][Proposition 2.3], which follows from Gel'fand-Levitan techniques that convert the Helmholtz equation into a Schrödinger equation. Indeed in the obtained Schrödinger equation, the refractive index and the frequency are separated, which allows a better understanding of the behavior of the solutions as functions of the frequency. For the sake of completeness, [7][Proposition 2.3] is restated:

Proposition 3.1. *There exists a constant* $h = h(\mathcal{M}) > 0$ *such that the strip*

$$S = \{k \in \mathbb{C};\ -h \le \Im(k) \le h\},$$

is free from resonances of the system (1.1)*.*

We deduce from Proposition 3.1 that the Green function defined in (1.4) has no poles in the strip S. Therefore $\phi(x, \cdot)$ which satisfies the integral representation (1.3) is holomorphic and bounded in S. Let

$$M_f = \max_{k\in S} \left(|\phi(0, \cdot)| + |\phi(1, \cdot)|\right). \tag{3.1}$$

Notice that $M_f > 0$ depends only on $\mathcal{M}$ and L. We also remark that $\overline{\phi(\cdot, k)}$ is a solution to the system (1.1) with radiation conditions (1.2) when substituting k by $-k$. We deduce from the uniqueness of the system that $\overline{\phi(\cdot, k)} = \phi(\cdot, -k)$. Therefore for fixed $x \in \mathbb{R}$, $k \to \Im(\phi(x, k)) = \frac{1}{2i}\left(\phi(\cdot, k) - \phi(x, -k)\right)$, is also a holomorphic function in the strip S.

Consequently

$$F(z) := \Im(\phi(0, z)) + i\Im(\phi(1, z)), \tag{3.2}$$

is holomorphic in S. In addition it satisfies $F(-z) = F(z)$ for $z \in \mathbb{R}$, and $|F(z)| \le 2M_f,\ \forall z \in S$.

Next, we aim to estimate $F(z)$ within the strip S in terms of its values on a segment $F(z), z \in (-K, K)$.

Without loss of generality we can assume that $h = \frac{\pi}{2n_h}$, where $n_h \in \mathbb{N}^*$. Let $S_+ = \{k \in \mathbb{C}; \Re(k) > 0,\ |\Im(k)| < h\}$, be half a strip, and $w(k, K)$ be the harmonic measure of the complex open domain $S_+ \setminus [0, K] \times \{0\}$. It is the unique solution to the system:

$$\begin{cases} \Delta w(k, K) &= 0 \quad k \in S_+ \setminus [0, K] \times \{0\}, \\ w(k, K) &= 0 \quad k \in \partial S_+, \\ w(k, K) &= 1 \quad k \in (0, K] \times \{0\}. \end{cases} \tag{3.3}$$

We infer from the maximum principle that $w_0(k, K) \in [0, 1]$ for all $k \in S_+$. Moreover the holomorphic unique continuation of the functions F using the Two constants Theorem ([15, Chap. III, Section 2.1], [17]), gives:

Lemma 3.2. *Let* F *be defined by* (3.2)*, let* M_f *be given by* (3.1) *and let* w *be the same as in* (3.3)*. Then, we have*

$$|F(k)| \le (2M_f)^{1-w_0(k,K)} \|F\|_{L^\infty(0,K)}^{w_0(k,K)}, \qquad \forall k \in (K, +\infty).$$

The following estimate is needed [7, Proposition 5.1].

PROPOSITION 3.3. *Let $w_0(k,K)$ be the harmonic measure of $S_+ \setminus (0,K] \times \{0\}$. Then*

$$w_0(k,K) \geq \frac{2}{\pi} \arctan\Big(\frac{(e^K-1)^{n_h}}{\big((e^k-1)^{2n_h}-(e^K-1)^{2n_h}\big)^{\frac{1}{2}}}\Big), \tag{3.4}$$

for all $k \geq K$.

Without loss of generality we can assume that $2M_f \geq 1$. Since $\|F\|_{L^\infty(0,c_0\mu_f)} < 1$, we deduce from Lemma 3.2 that

$$\|F\|_{L^\infty(0,c_0\mu_f)} \leq (2M_f)^{1-w_0(c_0\mu_f,K)} \|F\|_{L^\infty(0,K)}^{w_0(c_0\mu_f,K)}.$$

Applying now the derived lower bound in (3.4) on the last inequality, we get

$$\|F\|_{L^\infty(0,c_0\mu_f)} \leq (2M_f)^{1-\eta(c_0\mu_f,K)} \|F\|_{L^\infty(0,K)}^{\eta(c_0\mu_f,K)}.$$

Combining this inequality with the estimate in Lemma 3.2 achieves the proof of the theorem.

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