Volumes of components of Lelong upper level sets II

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Abstract

Let X be a compact Kähler manifold of dimension n, and let T be a closed positive $(1,1)$-current in a nef cohomology class on X. We establish an optimal upper bound for the volume of components of Lelong upper level sets of T in terms of cohomology classes of non-pluripolar self-products of T.

Keywords: relative non-pluripolar product, density current, Lelong number.

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1 Introduction

The aim of this paper is to investigate the singularities of closed positive currents on compact Kähler manifolds. Let X be a compact Kähler manifold of dimension n, and let T be a closed positive $(1,1)$-current on X. We are interested in understanding the set of points where T has a strictly positive Lelong numbers. By the celebrated upper semi-continuity of Lelong numbers by Siu [12], we know that this set is a countable union of proper analytic subsets on X. Our goal is to estimate the size of this upper level set. We note that by [13], for every closed positive current R on X, there always exists a closed positive $(1,1)$-current whose Lelong numbers coincide with those of R. Thus, to study questions about Lelong numbers in this paper, it suffices to consider only currents of bi-degree $(1,1)$. To delve into details, let us first introduce some necessary notions.

Let ω be a fixed smooth Kähler form on X. We equip X with the Riemannian metric induced by ω. Let S be a closed positive (p,p)-current for some $0 \le p \le n$. We define the mass $\|S\|$ of S to be equal to $\int_X S \wedge \omega^{n-p}$. For an analytic set V of pure dimension l in X, we recall that

$$\mathrm{vol}(V) = \frac{1}{l!}\int_{\mathrm{Reg}V} \omega^l,$$

where Reg$V$ is the regular locus of V. Given a closed positive (p,p)-current R, we denote by $\{R\} \in H^{p,p}(X,\mathbb{R})$ the cohomology class of R. We say $\alpha \in H^{p,p}(X,\mathbb{R})$ is *pseudoeffective* if $\alpha = \{R\}$ for some closed positive (p,p)-current R. Let $\alpha, \beta \in H^{p,p}(X,\mathbb{R})$. We say $\alpha \ge \beta$ if $\alpha - \beta$ is a pseudoeffective class.

For every $x \in X$, let $\nu(T,x)$ denote the Lelong number of T at x. For every irreducible analytic subset V in X, we recall that the generic Lelong number $\nu(T,V)$ of T along V is defined as $\inf_{x\in V}\{\nu(T,x)\}$. The Lelong number is a notion measuring the singularities of T. We refer to [4] for the basics of Lelong numbers. For every constant $c>0$, we denote $E_c(T) := \{x \in X | \nu(T,x) \geq c\}$ and $E_+ := \{x \in X|\nu(T,x) > 0\}$. By [12], $E_c(T)$ is a proper analytic subset in X, and $E_+(T) = \cup_{m\in\mathbb{N}^*} E_{1/m}(T)$ is a countable union of analytic sets.

Let W be an irreducible analytic subset of dimension m in X. We denote by $E_+^W(T) := \{x \in W | \nu(T,x) > \nu(T,W)\}$ the Lelong upper level set of T on W, which is also a countable union of proper analytic subsets in W. Let $V \subset E_+^W(T)$ be an irreducible analytic set. We say that V is *maximal* if there is no irreducible analytic subset V' of $E_+^W(T)$ such that V is a proper subset of V'. We call V *a component of the Lelong upper level set of T along W*, and let $\mathscr{V}_{T,W}$ be the set of such components V. Observe that $\mathscr{V}_{T,W}$ has at most countably many elements. For $0 \leq l \leq m$, we denote by $\mathscr{V}_{l,T,W}$ the set of $V \in \mathscr{V}_{T,W}$ such that $\dim V = l$.

Write $T = dd^c u$ locally, where u is a plurisubharmonic (psh in short) function. We define $T|_{\mathrm{Reg}W}$ to be $dd^c(u|_{\mathrm{Reg}W})$ if $u \not\equiv -\infty$ on Reg$W$, and $T|_{\mathrm{Reg}W} := 0$ otherwise. One sees that this definition is independent of the choice of u. Thus, $T|_{\mathrm{Reg}W}$ is a current on Reg$W$. Here is our main result in the paper.

Theorem 1.1. *Let α be a nef $(1,1)$-class and let W be an irreducible analytic subset of dimension m in X. Let T be a closed positive current in α such that $\nu(T,W) = 0$. Let $1 \leq m' \leq m$ be an integer. Then, we have*

$$\sum_{V\in\mathscr{V}_{m-m',T,W}} \nu(T,V)^{m'} \operatorname{vol}(V) \leq \frac{1}{(m-m')!}\int_{\mathrm{Reg}W} \big(\alpha^{m'} - \langle (T|_{\mathrm{Reg}W})^{m'}\rangle\big) \wedge \omega^{m-m'}, \quad (1.1)$$

where in the integral, we identify α with a smooth closed form in α.

We have some comments on (1.1). To see why the term

$$I := \int_{\mathrm{Reg}W} \big(\alpha^{m'} - \langle (T|_{\mathrm{Reg}W})^{m'}\rangle\big) \wedge \omega^{m-m'}$$

is non-negative, one can consider the case where W is smooth. Then, by a monotonicity of non-pluripolar products (see Theorem 3.3 below), the cohomology class $(\alpha|_W)^{m'} - \{\langle (T|_W)^{m'}\rangle\}$ is pseudoeffective. Hence the integral in the right-hand side of (1.1) is non-negative. In the general case where W is singular, one can use either a desingularisation of W or interpret I as the mass of some non-pluripolar product relative to $[W]$ (the current of integration along W); see Lemma 3.2 below. We underline however that in order to prove Theorem 1.1, it is not possible to use desingularisation of W to reduce to the case where W is smooth. The reason is that in the process of desingularisation, one has to blow up submanifolds of W which in general could be some of components of the Lelong upper level sets of T on W.

In [8], a less precise upper bound of volume of components of the Lelong upper level set was given in terms of the volume of W and the mass of T; see also Theorem 3.7 for a more general statement. The estimate (1.1) is indeed optimal. To see this, consider $n = m = m' = 1$ and T is a sum of Dirac masses. In this case, (1.1) becomes an equality.

If we consider $W = X$, then the generic Lelong number of T along W is zero. Thus, by Theorem 1.1, we have the following result.

Corollary 1.2. *Let α be a nef $(1,1)$-class, and T be a closed positive current in α. For $0 \le l \le n$, let $\mathscr{V}_{l,T}$ be the set of $V \in \mathscr{V}_{T,X}$ such that $\dim V = l$. Let $1 \le m' \le n$ be an integer. Then, we have*

$$\sum_{V \in \mathscr{V}_{n-m',T}} \nu(T,V)^{m'} \operatorname{vol}(V) \le \frac{1}{(n-m')!} \int_X \left(\alpha^{m'} - \{\langle T^{m'} \rangle\}\right) \wedge \omega^{n-m'}.$$

Corollary 1.2 generalizes [5, Corollary 7.6] by Demailly in which it was assumed additionally that the components of the upper Lelong level set of T are only of dimension 0 (hence the cohomology class of T is necessarily nef, see [5, Lemma 6.3]). The feature of Corollary 1.2 is that it holds for any current in a nef class.

This paper refines and substitutes [16]. The proof of Theorem 1.1 requires both the theory of density currents in [7] and relative non-pluripolar products in [17] (see also [1, 2]). One of the keys is Theorem 3.6 below following from a general comparison of Lelong numbers for density currents (see Corollary 2.6).

The organization of this paper is as follows. In Section 2, we recall basic properties of density currents from [7]. In Section 3, we discuss the connection between the non-pluripolar product and density currents, and prove Theorem 1.1.

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2 Density currents

We first recall some basic properties of density currents introduced by Dinh-Sibony in [7].

Let X be a complex Kähler manifold of dimension n, and V a smooth complex submanifold of X of dimension l. Let T be a closed positive (p,p)-current on X, where $0 \le p \le n$. Denote by $\pi : E \to V$ the normal bundle of V in X and $\overline{E} := \mathbb{P}(E \oplus \mathbb{C})$ the projective compactification of E. By abuse of notation, we also use π to denote the natural projection from $\overline{E}$ to V.

Let U be an open subset of X with $U \cap V \neq \varnothing$. Let τ be a smooth diffeomorphism from U to an open neighborhood of $V \cap U$ in E which is identity on $V \cap U$ such that the restriction of its differential $d\tau$ to $E|_{V \cap U}$ is identity. Such a map is called *an admissible map*. Note that in [7], to define an admissible map, it is required furthermore that $d\tau$ is $\mathbb{C}$-linear at every point of V. This difference doesn't affect what follows. When U is a small enough tubular neighborhood of V, there always exists an admissible map τ by [7, Lemma 4.2]. In general, τ is not holomorphic. When U is a small enough local chart, we can choose a holomorphic admissible map by using suitable holomorphic coordinates on U. For $\lambda \in \mathbb{C}^*$, let $A_\lambda : E \to E$ be the multiplication by λ on fibers of E. Here is the first fundamental result for density currents.

Theorem 2.1. *([7, Theorem 4.6]) Let τ be an admissible map defined on a tubular neighborhood of V. Then, the family $(A_\lambda)_*\tau_*T$ is of mass uniformly bounded in λ on compact subsets in E, and if S is a limit current of the last family as $\lambda \to \infty$, then S is a current on E which can be extended trivially through $\overline{E}\backslash E$ to be a closed positive current on $\overline{E}$ such that the cohomology class $\{S\}$ of S in $\overline{E}$ is independent of the choice of S, and $\{S\}|_V = \{T\}|_V$, and $\|S\| \leq C\|T\|$ for some constant C independent of S and T, where $\{S\}|_V$ denotes the pull-back of $\{S\}$ under the natural inclusion map from V to $\overline{E}$.*

We call S *a tangent current to T along V*, and its cohomology class is called *the total tangent class of T along V* and is denoted by $\kappa^V(T)$. Tangent currents are not unique in general. By [7, Theorem 4.6] again, if

$$S = \lim_{k\to\infty}(A_{\lambda_k})_*\tau_*T$$

for some sequence $(\lambda_k)_k$ converging to ∞, then for every open subset U' of X and every admissible map $\tau' : U' \to E$, we also have

$$S = \lim_{k\to\infty}(A_{\lambda_k})_*\tau'_*T.$$

This is equivalent to saying that tangent currents are independent of the choice of the admissible map τ.

Definition 2.2. *([7, Definition 3.1]) Let F be a complex manifold and $\pi_F : F \to V$ a holomorphic submersion. Let S be a positive current of bi-degree (p,p) on F.* The h-dimension *of S with respect to π_F is the biggest integer q such that $S \wedge \pi_F^*\theta^q \neq 0$ for some Hermitian metric θ on V.*

By a bi-degree reason, the h-dimension of S is in $[\max\{l-p, 0\}, \min\{\dim F - p, l\}]$. We have the following description of currents with minimal h-dimension.

Lemma 2.3. *([7, Lemma 3.4]) Let $\pi_F : F \to V$ be a submersion. Let S be a closed positive current of bi-degree (p,p) on F of h-dimension $(l-p)$ with respect to π_F. Then $S = \pi^*S'$ for some closed positive current S' on V.*

By [7, Lemma 3.8], the h-dimensions of tangent currents to T along V are the same and this number is called *the tangential h-dimension of T along V*.

Let $m \geq 2$ be an integer. Let T_j be a closed positive current of bi-degree (p_j, p_j) for $1 \leq j \leq m$ on X and let $T_1 \otimes \cdots \otimes T_m$ be the tensor current of $T_1, \ldots, T_m$ which is a current on X^m. A *density current* associated to $T_1, \ldots, T_m$ is a tangent current to $\otimes_{j=1}^m T_j$ along the diagonal Δ_m of X^m. Let $\pi_m : E_m \to \Delta$ be the normal bundle of Δ_m in X^m. Denote by $[V]$ the current of integration along V. When $m = 2$ and $T_2 = [V]$, the density currents of T_1 and T_2 are naturally identified with the tangent currents to T_1 along V (see [14, Lemma 2.3]).

The unique cohomology class of density currents associated to $T_1, \ldots, T_m$ is called *the total density class of $T_1, \ldots, T_m$*. We denote the last class by $\kappa(T_1, \ldots, T_m)$. The tangential h-dimension of $T_1 \otimes \cdots \otimes T_m$ along Δ_m is called *the density h-dimension* of $T_1, \ldots T_m$.

Lemma 2.4. *([7, Section 5]) Let T_j be a closed positive current of bi-degree (p_j, p_j) on X for $1 \leq j \leq m$ such that $\sum_{j=1}^m p_j \leq n$. Assume that the density h-dimension of $T_1, \ldots, T_m$ is minimal,* i.e*, equals to $n - \sum_{j=1}^m p_j$. Then the total density class of $T_1, \ldots, T_m$ is equal to $\pi_m^*(\wedge_{j=1}^m \{T_j\})$.*

Let $h_{\overline{E}}$ be the Chern class of the dual of the tautological line bundle of $\overline{E}$. By [7, Page 535], we have

$$\kappa^V(T) = \sum_{j=\max\{0,l-p\}}^{\min\{l,n-p-1\}} \pi^*\big(\kappa_j^V(T)\big) \wedge h_{\overline{E}}^{p-(l-j)}, \tag{2.1}$$

where $\pi : \overline{E} \to V$ is the natural projection and $\kappa_j^V(T) \in H^{2l-2j}(V, \mathbb{R})$. The tangential h-dimension of T along V is exactly equal to the maximal j such that $\kappa_j^V(T) \neq 0$, and it was known that the class $\kappa_j^V(T)$ is pseudoeffective ([7, Lemma 3.15]).

Theorem 2.5. *([7, Proposition 4.13]) Let V' be a submanifold of V and let T be a closed positive current on X. Let T_∞ be a tangent current to T along V. Let s be the tangential h-dimension of T_∞ along V'. Then, the tangential h-dimension of T along V' is at most s, and we have*

$$\kappa_s^{V'}(T) \leq \kappa_s^{V'}(T_\infty).$$

As a consequence, we obtain the following result generalizing the well-known lower bound of Lelong numbers of intersection of $(1,1)$-currents due to Demailly [6, Page 169] in the compact setting. It is probably the first result dealing with comparison of Lelong numbers for intersection of currents of arbitrary bi-degree.

Corollary 2.6. *Let T_j be a closed positive current on X for $1 \leq j \leq m$. Then, for every $x \in X$ and for every density current S associated to $T_1, \ldots, T_m$, we have*

$$\nu(S, x) \geq \nu(T_1, x) \cdots \nu(T_m, x). \tag{2.2}$$

Proof. Let $x \in X$. Let Δ_m be the diagonal of X^m. Let $\pi : E \to \Delta_m$ be the natural projection from the normal bundle of the diagonal Δ_m of X^m in X^m. We identify x with a point in Δ_m via the natural identification Δ_m with X. Put $T := \otimes_{j=1}^m T_j$ and $V' := \{x\}$. By [11, Lemma 2.4], we have $\nu(T, x) \geq \nu(T_1, x) \cdots \nu(T_m, x)$. By [7, Proposition 5.6], we have

$$\kappa_0^{V'}(S) = \nu(S, x)\delta_x, \quad \kappa_0^{V'}(T) = \nu(T, x)\delta_x,$$

where δ_x is the Dirac measure on x (notice here $\dim V' = 0$). This combined with Theorem 2.5 applied to X^m, $T := \otimes_{j=1}^m T_j$, Δ_m the diagonal of X^m and $V' := \{x\}$ implies

$$\nu(S, x) \geq \nu(T, x) \geq \nu(T_1, x) \cdots \nu(T_m, x).$$

Hence, the desired inequality follows. The proof is finished. $\square$

3 Relative non-pluripolar products

We first recall basic facts about the relative non-pluripolar product of currents and discuss its connection with the density current.

Let X be a compact Kähler manifold of dimension n. Let $T_1, \dots, T_m$ be closed positive $(1,1)$-currents on X, and T be a closed positive (p,p)-current. By [17], the T-relative non-pluripolar product $\langle \wedge_{j=1}^m T_j \dot\wedge T \rangle$ is defined in a way similar to that of the usual non-pluripolar products in [1, 2]. For readers' convenience, we explain briefly how to do it.

Write $T_j = dd^c u_j + \theta_j$, where θ_j is a smooth form and u_j is a θ_j-psh function. Put

$$R_k := \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \wedge_{j=1}^m (dd^c \max\{u_j, -k\} + \theta_j) \wedge T$$

for $k \in \mathbb{N}$. By the strong quasi-continuity of bounded psh functions ([17, Theorems 2.4 and 2.9]), we have

$$R_k = \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \wedge_{j=1}^m (dd^c \max\{u_j, -l\} + \theta_j) \wedge T$$

for every $l \geq k \geq 1$. One can check that R_k is positive (see [17, Lemma 3.2]).

As in [2], we have that R_k is of mass bounded uniformly in k and $(R_k)_k$ converges to a closed positive current as $k \to \infty$. This limit is denoted by $\langle \wedge_{j=1}^m T_j \dot\wedge T \rangle$. We refer to [17, Proposition 3.5] for properties of relative non-pluripolar products.

For every closed positive $(1,1)$-current P, we denote by I_P the set of $x \in X$ so that local potentials of P are equal to $-\infty$ at x. Note that I_P is a complete pluripolar set. It is clear from the definition that $\langle \wedge_{j=1}^m T_j \dot\wedge T \rangle$ has no mass on $\cup_{j=1}^m I_{T_j}$. Furthermore, for every locally complete pluripolar set A (*i.e,* A is locally equal to $\{\psi = -\infty\}$ for some psh function ψ), if T has no mass on A, then so does $\langle \wedge_{j=1}^m T_j \dot\wedge T \rangle$. The following is deduced from [17, Proposition 3.5].

Proposition 3.1. *(i) For $R := \langle \wedge_{j=l+1}^m T_j \dot\wedge T \rangle$, we have $\langle \wedge_{j=1}^m T_j \dot\wedge T \rangle = \langle \wedge_{j=1}^l T_j \dot\wedge R \rangle$.*

(ii) For every complete pluripolar set A, we have

$$\mathbf{1}_{X \setminus A} \langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \dot\wedge T \rangle = \big\langle T_1 \wedge T_2 \wedge \cdots \wedge T_m \dot\wedge (\mathbf{1}_{X \setminus A} T) \big\rangle.$$

In particular, the equality

$$\langle \wedge_{j=1}^m T_j \dot\wedge T \rangle = \langle \wedge_{j=1}^m T_j \dot\wedge T' \rangle$$

holds, where $T' := \mathbf{1}_{X \setminus \cup_{j=1}^m I_{T_j}} T$.

Let V be an irreducible analytic set in X. For the case $T = [V]$, we have the following lemma.

Lemma 3.2. *([18, Lemma 2.3]) Let $T_1, \dots, T_m$ be closed positive $(1,1)$-currents on X. Then the following properties hold:*

(i) If V is contained in $\cup_{j=1}^m I_{T_j}$, then $\langle T_1 \wedge \cdots \wedge T_m \dot\wedge [V] \rangle = 0$ and there is $1 \leq j_0 \leq m$ so that $V \subset I_{T_{j_0}}$.

(ii) If V is not contained in $\cup_{j=1}^m I_{T_j}$, then

$$\langle \wedge_{j=1}^m T_j \dot{\wedge} [V] \rangle = i_* \langle T_{1,V} \wedge \cdots \wedge T_{m,V} \rangle,$$

where $i\colon \mathrm{Reg}(V) \to X$ is the natural inclusion, and $T_{j,V} := dd^c(u_j|_{\mathrm{Reg}(V)})$ if $dd^c u_j = T_j$ locally.

Let T, T' be closed positive $(1,1)$-currents in the same cohomology class on X. T' is said to be less singular than T if for every local potentials u of T and u' of T', $u \le u' + O(1)$. Here is a crucial property of relative non-pluripolar products.

Theorem 3.3. *([17, Theorem 1.1]) Let T_j' be closed positive $(1,1)$-current in the cohomology class of T_j on X such that T_j' is less singular than T_j for $1 \le j \le m$. Then we have*

$$\{\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle\} \le \{\langle T_1' \wedge \cdots \wedge T_m' \dot{\wedge} T \rangle\}.$$

Weaker versions of the above result were proved in [2, 3, 19]. Let $\alpha_1, \ldots, \alpha_m$ be big $(1,1)$-classed of X. A current $T_{j,\min} \in \alpha_j$ is said to have minimal singularities if it is less singular than any closed positive current in α_j.

By Theorem 3.3, the class $\{\langle T_{1,\min} \wedge \cdots \wedge T_{m,\min} \dot{\wedge} T \rangle\}$ is a well-defined pseudoeffective class which is independent of the choice of $T_{j,\min}$. We denote the last class by $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \dot{\wedge} T \rangle\}$. When $T \equiv 1$, we simply write $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ for $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \dot{\wedge} T \rangle\}$. In this case, the product $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ was introduced in [2].

Regarding the relation between relative non-pluripolar products and density currents, the following fact was proved in [15, Theorem 3.5], see also [9, 10].

Theorem 3.4. *Let R_∞ be a density current associated to $T_1, \ldots, T_m, T$. Then we have*

$$\pi_{m+1}^* \langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle \le R_\infty, \tag{3.1}$$

where π_{m+1} is the natural projection from the normal bundle of the diagonal Δ of X^{m+1} to Δ, and as usual we identified Δ with X.

We will need the following to estimate the density h-dimension of currents.

Proposition 3.5. *([15, Proposition 3.6]) Let $T_1, \ldots, T_m, T$ be as above. Let A be a Borel subset of X. Assume that for every $J \subset \{1, \ldots, m\}$ and for every density current R_J associated to $(T_j)_{j \in J}, T$ with $m_J := |J| < m$, we have that R_J has no mass on the set*

$$\pi_{m_J+1}^* (\cap_{j \notin J} \{x \in A : \nu(T_j, x) > 0\}).$$

Then for every density current S associated to $T_1, \ldots, T_m, T$, the h-dimension of the current $\mathbf{1}_A S$ is equal to $n - p - m$, where $\mathbf{1}_A$ denotes the characteristic function of A.

For every pseudoeffective (p,p)-class γ on X, we put $\|\gamma\| := \int_X \Theta \wedge \omega^{n-p}$, where Θ is any closed smooth form in γ. This definition is independent of the choice of Θ and is nonnegative because of the pseudoeffectivity of γ.

Theorem 3.6. *Let P and T be closed positive currents of bi-degree $(1,1)$ and (p,p) respectively on X, where $1 \le p \le n-1$. Assume that T has no mass on I_P. Then, the cohomology class*

$$\gamma := \{P\} \wedge \{T\} - \{\langle P \dot{\wedge} T \rangle\}$$

is pseudoeffective and we have

$$\|\gamma\| \ge \sum_V \nu(P,V)\nu(T,V) n_V! \operatorname{vol}(V), \tag{3.2}$$

where the sum is taken over every irreducible subset V of dimension at least $n-p-1$ in X, and $n_V := \dim V$.

Proof. Let $\mathscr{V}$ be the set of irreducible analytic subsets V of dimension at least $n-p-1$ in X such that $\nu(T,V) > 0$ and $\nu(P,V) > 0$. We note that in (3.2), it is enough to consider $V \in \mathscr{V}$. We will see below that $\mathscr{V}$ has at most countable elements.

Observe that if $\nu(P,x) > 0$, then $x \in I_P$. Hence, by hypothesis, the trace measure of T has no mass on the set $\{x \in X : \nu(P,x) > 0\}$. This allows us to apply Proposition 3.5 to P and T to obtain that the density h-dimension of P and T is minimal. Using this and Lemma 2.4 gives

$$\kappa(P,T) = \pi^*(\{P\} \wedge \{T\}), \tag{3.3}$$

where π is the natural projection from the normal bundle of the diagonal Δ of X^2 to Δ.

Let S be a density current associated to P and T. Since the h-dimension of S is minimal, using Lemma 2.3, we get that there exists a current S' on X such that $S = \pi^* S'$ (recall Δ is identified with X). Since the relative non-pluripolar product is dominated by density currents (Theorem 3.4), the current $S' - \langle P \dot{\wedge} T \rangle$ is closed and positive. Moreover, by (3.3), the cohomology class of the last current is equal to γ. It follows that γ is pseudoeffective.

It remains to prove (3.2). Let $V \in \mathscr{V}$. By definition, the generic Lelong number of T along V is positive. Since T is of bi-degree (p,p), the dimension of V must be at most $n-p$. Hence, we have two possibilities: either $\dim V = n-p-1$ or $\dim V = n-p$. The latter case cannot happen. If T has no mass on V, then $\nu(T,V) = 0$, which leads to a contradiction. If T has mass on V, which is contained in I_P (for $\nu(P,V) > 0$), then this contradicts the hypothesis that T has no mass on I_P.

Let $V \in \mathscr{V}$. Since the Lelong numbers are preserved by submersion maps ([11, Proposition 2.3]), by applying Corollary 2.6 to P, T and generic $x \in V$, we obtain

$$\nu(S',V) = \nu(S,V) \ge \nu(P,V)\nu(T,V).$$

This combined with the fact that $\dim V = n-p-1$ implies $S' \ge \nu(P,V)\nu(T,V)\,[V]$. We deduce that

$$\begin{aligned} S' &\ge \langle P \dot{\wedge} T \rangle + \mathbf{1}_{I_P} S' \\ &\ge \langle P \dot{\wedge} T \rangle + \sum_{V \in \mathscr{V}} \nu(P,V)\nu(T,V)\,[V]. \end{aligned}$$

The second inequality comes from Siu's decomposition theorem ([6, 2.18]), and this also shows that $\mathscr{V}$ has at most countable elements. The desired assertion follows and the proof is finished. $\square$

We now prove Theorem 1.1.

Proof of Theorem 1.1. It suffices to consider the case where α is Kähler by using $\alpha + \epsilon\{\omega\}, T+\epsilon\omega$ instead of α, T and letting $\epsilon \to 0$. Hence we assume from now on that α is Kähler. By abuse of notation, we also denote by α a smooth Kähler form in α. By Lemma 3.2, the right hand-hand side of (1.1) can be written as

$$\frac{1}{(m-m')!}\int_{\mathrm{Reg}W}\left(\alpha^{m'} - \langle (T|_{\mathrm{Reg}W})^{m'}\rangle\right)\wedge\omega^{m-m'} = \frac{1}{(m-m')!}\left\|\langle\alpha^{m'}\dot{\wedge}[W]\rangle - \langle T^{m'}\dot{\wedge}[W]\rangle\right\|.$$

Step 1. We first focus on the case that T has analytic singularities. Set $S = \langle T^{m'-1}\dot{\wedge}[W]\rangle$. Since α is Kähler, by the monotonicity of non-pluripolar product (Theorem 3.3) and Proposition 3.1 (i), we get

$$\begin{aligned}\left\|\langle\alpha^{m'}\dot{\wedge}[W]\rangle - \langle T^{m'}\dot{\wedge}[W]\rangle\right\| &\geq \left\|\langle\alpha\wedge T^{m'-1}\dot{\wedge}[W]\rangle - \langle T^{m'}\dot{\wedge}[W]\rangle\right\| \\ &= \|\alpha\wedge S - \langle T\dot{\wedge}S\rangle\|\end{aligned} \tag{3.4}$$

We now show that S has no mass on I_T. For $m' > 1$, this directly follows from the definition of non-pluripolar product. For $m' = 1$, the current S is just $[W]$. Since we assume that T has analytic singularities, the polar locus I_T is an analytic subset and it doesn't contain W. Hence, $[W]$ also has no mass on I_T. Therefore, we can apply Lemma 3.6 to T, S, and get

$$\|\alpha\wedge\{S\} - \{\langle T\dot{\wedge}S\rangle\}\| \geq (m-m')!\sum_{V\in\mathscr{V}_{m-m',T,W}}\nu(T,V)\nu(S,V)\,\mathrm{vol}(V) \tag{3.5}$$

Let $V \in \mathscr{V}_{m-m',T,W}$ and let $\mathrm{Sing}(I_T\cap W)$ be the singular locus of the analytic set $I_T\cap W$. Put $X' := X\backslash\mathrm{Sing}(I_T\cap W)$ which is an open subset in X. Now, we claim that the classical intersection $T^{m'-1}\wedge[W]$ is well-defined on some open neighborhood U of $V\backslash\mathrm{Sing}(I_T\cap W)$ in X', in the sense given in [4, Chapter III]. Since T has analytic singularities, the Lelong number $\nu(T,x)$ is strictly positive if and only if x belongs to I_T. This implies that V is contained in $I_T\cap W$. Let $K_1,\dots,K_s$ be the irreducible components of $I_T\cap W$. Since V is a maximal irreducible proper analytic set such that $\nu(T,V)>0$, V must be one of the irreducible components. By rearranging the index, we may assume $V = K_1$. Let U be an open neighborhood of $V\backslash\mathrm{Sing}(I_T\cap W)$ such that

$$U\cap K_j = \emptyset,\ \text{for } j = 2,\dots,s.$$

Notice that $V\backslash\mathrm{Sing}(I_T\cap W)$is contained in $\mathrm{Reg}(V)$, and is of dimension $m-m'$. Consequently, for $0\leq j'\leq m'-1$,

$$\begin{aligned}\mathscr{H}_{2m-2j'+1}(L(T)|_U\cap W) &= \mathscr{H}_{2m-2j'+1}(I_T\cap W\cap U) \\ &= \mathscr{H}_{2m-2j'+1}(V\backslash\mathrm{Sing}(I_T\cap W)) \\ &= 0,\end{aligned}$$

where $L(T)$ is the set of $x \in X$ such that the local potential of T is unbounded on any neighborhood of x. This allows us to apply [4, Chapter III, Theorem 4.5] and get the well-definedness of $T^{m'-1} \wedge [W]$ on U. By [17, Proposition 3.6], we obtain

$$\langle T^{m'-1} \dot{\wedge} [W] \rangle = \mathbf{1}_{U \setminus I_T} T^{m'-1} \wedge [W]. \tag{3.6}$$

Actually, the equality also holds on $U \cap I_T$. To show this, we need to check that $T^{m'-1} \wedge [W]$ has no mass on $U \cap I_T$. Since $\dim(U \cap I_T \cap W) = m - m'$ and $T^{m'-1} \wedge [W]$ is of bi-dimension $(m - m' + 1, m - m' + 1)$, the current $T^{m'-1} \wedge [W]$ must have no mass on $U \cap I_T \cap W$. Also, by the fact $\mathrm{Supp}(T^{m'-1} \wedge [W]) \subset W$, the current $T^{m'-1} \wedge [W]$ also has no mass on $(U \cap I_T) \setminus W$. Therefore, the equality (3.6) extends to U. This implies that the Lelong number $\nu(S, V)$ equals $\nu(T^{m'-1} \wedge [W], V \setminus \mathrm{Sing}(I_T \cap W))$ (remember that we consider the current $T^{m'-1} \wedge [W]$ on U, and $V \setminus \mathrm{Sing}(I_T \cap W)$ is an analytic subset in U), and we then have

$$\begin{aligned}
\nu(S, V) &= \nu(T^{m'-1} \wedge [W], V \setminus \mathrm{Sing}(I_T \cap W)) \\
&\geq \nu(T, V \setminus \mathrm{Sing}(I_T \cap W))^{m'-1} \nu([W], V \setminus \mathrm{Sing}(I_T \cap W)) \\
&\geq \nu(T, V)^{m'-1}.
\end{aligned} \tag{3.7}$$

By (3.4), (3.5) and (3.7), the desired inequality follows in the case where T has analytic singularities.

Step 2. Now, we remove the assumption that T has analytic singularities. First, we write $T = dd^c u + \theta$, where θ is a closed smooth $(1,1)$- form and $u \in \mathrm{PSH}(X, \theta)$. Demailly's analytic approximation theorem (see [6, Corollary 14.13]) allows us to construct a sequence $u_k^D \in \mathrm{PSH}(X, \theta + \epsilon_k \omega)$, where ϵ_k decreases to 0, such that

(1) $u_k^D \geq u$ and u_k^D converges to u in L^1.

(2) u_k^D has analytic singularities.

(3) $\nu(T_k, x)$ converges to $\nu(T, x)$ uniformly on X, where $T_k = dd^c u_k^D + (\theta + \epsilon_k \omega)$.

By the monotonicity property of non-pluripolar product (Theorem 3.3), we have

$$\begin{aligned}
\left\| \langle \alpha^{m'} \dot{\wedge} [W] \rangle - \langle T^{m'} \dot{\wedge} [W] \rangle \right\| &= \lim_{k \to \infty} \left\| \langle (\alpha + \epsilon_k \omega)^{m'} \dot{\wedge} [W] \rangle - \langle (T + \epsilon_k \omega)^{m'} \dot{\wedge} [W] \rangle \right\| \\
&\geq \limsup_{k \to \infty} \left\| \langle (\alpha + \epsilon_k \omega)^{m'} \dot{\wedge} [W] \rangle - \langle T_k^{m'} \dot{\wedge} [W] \rangle \right\|
\end{aligned} \tag{3.8}$$

For every constant $r > 0$, set $A_r := \{V \in \mathscr{V}_{m-m',T,W} | \nu(T, V) \geq r\}$. Observe that A_r increases to $\mathscr{V}_{m-m',T,W}$ as $r \to 0$. Since $\nu(T_k, x)$ converges to $\nu(T, x)$ uniformly and T_k is less singular than T, for every fixed $r > 0$ we have

$$A_r \subset \mathscr{V}_{m-m',T_k,W}$$

when k is large enough. By Step 1, we therefore have

$$\begin{aligned}
\left\| \langle (\alpha + \epsilon_k \omega)^{m'} \dot{\wedge} [W] \rangle - \langle T_k^{m'} \dot{\wedge} [W] \rangle \right\| &\geq (m - m')! \sum_{V \in \mathscr{V}_{m-m',T_k,W}} \nu(T_k, V)^{m'} \mathrm{vol}(V) \\
&\geq (m - m')! \sum_{V \in A_r} \nu(T_k, V)^{m'} \mathrm{vol}(V).
\end{aligned}$$

Letting $k \to \infty$ and using (3.8) give

$$\begin{aligned}\left\|\langle \alpha^{m'} \dot{\wedge} [W]\rangle - \langle T^{m'} \dot{\wedge} [W]\rangle\right\| &\geq (m-m')! \limsup_{k\to\infty} \sum_{V \in A_r} \nu(T_k, V)^{m'} \operatorname{vol}(V) \\ &= (m-m')! \sum_{V\in A_r} \nu(T,V)^{m'} \operatorname{vol}(V),\end{aligned}$$

for every constant $r > 0$. Letting $r \to 0$, we obtain the desired estimate. $\square$

For the general case where $\nu(T, W) > 0$. We couldn't directly compare the volume of Lelong upper level sets of T on W and the mass of $\{\langle \alpha^{m'} \dot{\wedge} [W]\rangle\} - \{\langle T^{m'} \dot{\wedge} [W]\rangle\}\}$. In this case, we have the following modified inequality which is stronger than [8, Theorem 1.1].

Theorem 3.7. *Let α be a nef $(1,1)$-class. Let W be an irreducible analytic subset in X. Let T be a closed positive current in α. Let $1 \leq m' \leq m$ be an integer. Then we have*

$$\begin{aligned}(m-m')! \sum_{V \in \mathscr{V}_{m-m',T,W}} \big(\nu(T,V) - \nu(T,W)\big)^{m'} \operatorname{vol}(V) \leq \\ \left\|(\alpha + c\{\omega\})^{m'} \wedge \{[W]\} - \{\langle (T + c\omega)^{m'} \dot{\wedge} [W]\rangle\}\right\| \quad (3.9)\end{aligned}$$

where $c = c_1 \cdot \nu(T,W)$ and $c_1 > 0$ is a constant independent of α, T, W. In particular, there is a constant $c_2 > 0$ independent of α, T, W such that

$$\sum_{V \in \mathscr{V}_{m-m',T,W}} \big(\nu(T,V) - \nu(T,W)\big)^{m'} \operatorname{vol}(V) \leq c_2 \operatorname{vol}(W) \|T\|^{m'}. \quad (3.10)$$

Proof. The desired inequality (3.10) follows directly from (3.9). We prove now (3.9). For convenience, set $c_3 := \nu(T, W) > 0$. The regularization theorem of currents introduced in [5] allows us to cut down the Lelong upper level set $\{x \in X | \nu(T, V) \geq c_3\}$ from T. More precisely, by [5, Theorem 1.1], there exists a sequence of almost positive closed $(1,1)$-currents $T_{c_3,k}$ in α such that

(1) $T_{c_3,k} \geq -(c_1 \cdot c_3 + \epsilon_k)\omega$, where $\lim_{k\to\infty} \epsilon_k = 0$ and $c_1 > 0$ is a constant independent of α, T and W.

(2) The global potentials of $T_{c_3,k}$ decreases to the global potential of T.

(3) $\nu(T_{c_3,k}, x) = \max\{\nu(T,x) - c_3, 0\}$.

Set $\widetilde{T}_{c_3,k} = T_{c_3,k} + (c_1 \cdot c_3 + \epsilon_k)\omega$, which is a closed positive $(1,1)$-current. By Theorem 3.3, we have

$$\begin{aligned}&\left\|(\alpha + c_1 \cdot c_3\{\omega\})^{m'} \wedge \{[W]\} - \langle (T + c_1 \cdot c_3 \omega)^{m'} \dot{\wedge} [W]\rangle\right\| \\ &\geq \limsup_{k\to\infty} \left\|\big(\alpha + (c_1 \cdot c_3 + \epsilon_k)\{\omega\}\big)^{m'} \wedge \{[W]\} - \langle \widetilde{T}^{m'}_{c_3,k} \dot{\wedge} [W]\rangle\right\|.\end{aligned}$$

Since $\nu(\widetilde{T}_{c_3,k}, W) = 0$, we can apply Theorem 1.1 to the right-hand side of the above inequality and get

$$\begin{aligned}&\big\|\big(\alpha + (c_1 \cdot c_3 + \epsilon_k)\{\omega\}\big)^{m'} \wedge \{[W]\} - \langle \widetilde{T}^{m'}_{c_3,k} \dot{\wedge} [W]\rangle\big\| \\ &\geq (m-m')! \sum_{V \in \mathscr{V}_{m-m', \tilde{T}_{c_3,k}, W}} \nu(\widetilde{T}_{c_3,k}, V)^{m'} \operatorname{vol}(V),\end{aligned} \tag{3.11}$$

By the above properties of $T_{c_3,k}$, we have

$$\mathscr{V}_{m-m', \tilde{T}_{c_3,k}, W} = \mathscr{V}_{m-m', T, W}.$$

Therefore, the right-hand side of (3.11) is equal to

$$\begin{aligned}&(m-m')! \sum_{V \in \mathscr{V}_{m-m', T, W}} \nu(\widetilde{T}_{c_2,k}, V)^{m'} \operatorname{vol}(V) \\ &= (m-m')! \sum_{V \in \mathscr{V}_{m-m', T, W}} \big(\nu(T, V) - \nu(T, W)\big)^{m'} \operatorname{vol}(V).\end{aligned}$$

This completes the proof. □

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