

# Quantum master equation for many-body systems: Derivation based on the Lieb-Robinson bound

Koki Shiraishi,<sup>1,\*</sup> Masaya Nakagawa,<sup>1</sup> Takashi Mori,<sup>2</sup> and Masahito Ueda<sup>1,3,4</sup>

<sup>1</sup>*Department of Physics, The University of Tokyo,  
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan.*

<sup>2</sup>*Department of Physics, Keio University, 3-14-1 Hiyoshi, Yokohama, Kanagawa 223-8522, Japan.*

<sup>3</sup>*RIKEN Center for Emergent Matter Science (CEMS), Wako, Saitama 351-0198, Japan.*

<sup>4</sup>*Institute for Physics of Intelligence, The University of Tokyo,  
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan.*

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The local Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) quantum master equation is a powerful tool for the study of open quantum many-body systems. However, its microscopic derivation applicable to many-body systems is available only in limited cases of weak internal couplings, and it has yet to be fully understood under what microscopic conditions the local GKSL equation is valid. We derive the local GKSL equation on the basis of the Lieb-Robinson bound, which provides an upper bound of the propagation of information in quantum many-body systems. We numerically test the validity of the derived local GKSL equation for a one-dimensional tight-binding fermion chain.

## I. INTRODUCTION

Most quantum systems are inevitably influenced by surrounding environments, i.e., they are open quantum systems [1]. In recent years, open quantum many-body systems have intensively been studied both theoretically and experimentally due to advances in experimental techniques [2, 3]. The standard method to describe an open quantum system is based on the quantum master equations (QMEs) [4–10], such as the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation [9, 10] and the Redfield equation [8]. In fact, the QMEs can correctly reproduce many experimental results about open quantum many-body systems [11–16]. The QMEs have also been used to analyze transport phenomena in nonequilibrium steady states of quantum systems in contact with reservoirs [17–24].

Among the QMEs, the local GKSL equation is widely applied to open many-body systems [2, 19, 25]. The GKSL equation is given by (with  $\hbar = 1$ )

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_{\mu} \left( L_{\mu}\rho L_{\mu}^{\dagger} - \frac{1}{2}\{L_{\mu}^{\dagger}L_{\mu}, \rho\} \right), \quad (1)$$

where  $\rho$  is the density matrix of a system,  $H$  is the Hamiltonian of the system, and  $L_{\mu}$ 's are the Lindblad operators. Equation (1) is called the local GKSL equation if each Lindblad operator in Eq. (1) acts only on a spatially local subsystem. The local GKSL equation reflects the local nature of the effect of dissipation due to the environment and consequently guarantees the physically desirable properties. For example, the locality of the Lindblad operators guarantees the existence of the Lieb-Robinson bound in open quantum many-body systems [26], thermalization of bulk-dissipated systems in the weak cou-

pling regime [27], and local conservation laws in the bulk of boundary-dissipated many-body systems [28].

However, since the local GKSL equation is usually given phenomenologically without any microscopic derivation [19, 29], it is not clear how properties such as the temperature of the environment should be incorporated in the Lindblad operators. It is also unclear when the local GKSL equation is valid. The validity of the local GKSL equation is debated in terms of thermodynamics [30, 31], conservation laws [28], dynamics [32] and phase transitions [33]. Unfortunately, the microscopic derivations of the local GKSL equations known so far are limited to the case of quantum many-body systems consisting of multiple sites where the intersite couplings are weak enough to be treated perturbatively [34–36]. Therefore, a microscopic derivation of the local GKSL equation applicable to generic many-body systems is highly desired.

In this paper, we propose a method to derive the local GKSL equation by using the Lieb-Robinson bound [37, 38], which provides a fundamental limitation on the speed of information propagation in locally interacting quantum many-body systems. The GKSL equation is commonly derived from the Redfield equation by using the Born-Markov approximation and further approximations in the weak-coupling regime, where the system-environment interaction is weak [1]. Here, we use the Lieb-Robinson bound to impose locality on the Redfield equation before applying existing approximations to obtain the GKSL equation such as the rotating-wave (secular) approximation [39–42], time coarse-graining [43–48], and approximation of the sum of the spectral densities by the product of the square roots of the spectral densities [49–51]. On the basis of this microscopic derivation, we find that the Lindblad operators of the local GKSL equation should have a support of size  $\zeta_0\tau_B$ , where  $\zeta_0$  is the propagation velocity of the system and  $\tau_B$  is the relaxation time of the environment (their precise definitions are given later). We also numerically demonstrate

\* shiraishi@cat.phys.s.u-tokyo.ac.jp

that the local GKSL equation can correctly describe the steady states of open many-body systems when the relaxation of the environment is sufficiently fast.

Our microscopic derivation clarifies how the locality of the Lindblad operators is related to the time scales of the system, the environment, and the interaction between them. Moreover, our derivation reduces the computational cost of time evolution since the Lindblad operators can be calculated only from diagonalization of the Hamiltonian of a local subsystem. The use of local GKSL equations based on the microscopic derivation is expected to deepen understanding of nonequilibrium phenomena in open quantum many-body systems.

The rest of this paper is organized as follows. In Sec. II, we introduce the Redfield equation and two types of the GKSL equations, namely, the Davies equation [39] and the universal Lindblad equation [50]. In Sec. III, we derive the local GKSL equation on the basis of the Lieb-Robinson bound. In Sec. IV, we numerically evaluate the error of our estimates about the equilibrium and nonequilibrium steady states described by the derived local GKSL equation for a quadratic fermionic system and show that the error becomes small when the parameter regime is consistent with our derivation based on the Lieb-Robinson bound. In Sec. V, we discuss the numerical cost for the analysis of open quantum many-body systems on the basis of the derived local GKSL equation. Finally, we conclude this paper in Sec. VI.

In Appendix A, we give the derivation of the universal Lindblad equation in the frequency domain. In Appendix B, we show that the largest eigenvalue of the dissipator of the QME in the quadratic open fermionic system can be efficiently calculated. In Appendix C, we provide additional numerical results.

## II. QUANTUM MASTER EQUATIONS

In this section, we introduce three QMEs, namely, the Redfield equation, the Davies equation, and the universal Lindblad equation (ULE). We summarize in Table I the conditions under which these and two related QMEs are valid, including the local GKSL equations in the next section.

Let us consider a quantum system  $S$ , which interacts with a bath  $B$  and is represented by a finite-dimensional Hilbert space. Since we consider a many-body system, the dimension of the Hilbert space grows exponentially with increasing the system size. The Hamiltonian of the total system is given by  $H_{\text{tot}} = H_S \otimes I_B + I_S \otimes H_B + H_{SB}$ , where  $H_S$ ,  $H_B$ , and  $H_{SB}$  are the Hamiltonians of the system, the bath, and the system-bath interaction, respectively, and  $I_{S(B)}$  is the identity operator on the system (bath). The interaction Hamiltonian is represented by

$$H_{SB} = \sum_{\mu} A_{\mu} \otimes B_{\mu}, \quad (2)$$

where  $A_{\mu}$ 's and  $B_{\mu}$ 's are operators that act on the system and the bath, respectively. The time evolution of the

total system is described by the von-Neumann equation

$$\frac{d}{dt} \rho_{\text{tot}}(t) = -i[H_{\text{tot}}, \rho_{\text{tot}}(t)], \quad (3)$$

where  $\rho_{\text{tot}}(t)$  is the density matrix of the total system at time  $t$ .

### II.1. Redfield equation

We first introduce the Redfield equation, from which the GKSL equation can be derived. We assume that the interaction between the system and the bath is weak and that the state  $\rho_B$  of the bath is the Gibbs state  $\rho_B = e^{-\beta H_B} / \text{Tr}[e^{-\beta H_B}]$  with inverse temperature  $\beta$  throughout time evolution. The correlation functions of operators  $B_{\mu}$  of the bath are denoted by  $C_{\mu\nu}(t) = \text{tr}(B_{\mu}^{\dagger}(t) B_{\nu} \rho_B)$ , where  $B_{\mu}(t) = e^{iH_B t} B_{\mu} e^{-iH_B t}$ .

We introduce the time scale  $\tau_B$  of the bath and the time scale  $\tau_{SB}$  of the system-bath interaction. The precise definitions of the time scales are given below. If the system-bath interaction is so weak and the relaxation of the bath is so fast that there is a separation of the two time scales such that  $\tau_B \ll \tau_{SB}$ , then the Born-Markov approximation [1, 8] is justified and the Redfield equation

$$\begin{aligned} \frac{d}{dt} \rho(t) = & -i[H_S, \rho(t)] \\ & + \sum_{\mu, \nu} \int_0^{\infty} ds [C_{\mu\nu}(s) (A_{\nu}(-s) \rho(t) A_{\mu}^{\dagger} \\ & - A_{\mu}^{\dagger} A_{\nu}(-s) \rho(t)) + \text{H.c.}] \end{aligned} \quad (4)$$

can be derived from Eq. (3) [8]. Here,  $\rho(t) = \text{tr}_B[\rho_{\text{tot}}(t)]$  and we assume the condition  $\text{tr}_B[H_{SB}, \rho(0) \otimes \rho_B] = 0$  for the initial state  $\rho(0)$  of the system. Let  $E_n$  and  $|E_n\rangle$  be an eigenvalue of  $H_S$  and the corresponding eigenstate. By decomposing  $A_{\mu}(t)$  into the sum of  $A_{\mu}(\omega)$ 's given by

$$A_{\mu}(\omega) = \sum_{E_n - E_m = \omega} |E_m\rangle \langle E_m| A_{\mu} |E_n\rangle \langle E_n|, \quad (5)$$

QME	Conditions
Redfield [1, 8]	$\tau_B \ll \tau_{SB}$
Davies [1, 39]	$\tau_B \ll \tau_{SB}, \tau_S \ll \tau_{SB}$
ULE [50, 51]	$\tau_B \ll \tau_{SB}$
Local Davies	$\tau_B \ll \tau_{SB}, \tau_B \ll R/\zeta_0^a, \tau_{S,\Omega} \ll \tau_{SB}^b$
Local ULE	$\tau_B \ll \tau_{SB}, \tau_B \ll R/\zeta_0^a$

<sup>a</sup> Discussed in Sec. III.2. Numerically shown in Sec. IV.

<sup>b</sup> Discussed in Sec. III.3. Numerically shown in Sec. IV.4.

TABLE I. Conditions required for various QMEs to be valid, where  $\tau_B$ ,  $\tau_{SB}$ ,  $\tau_S$ , and  $\tau_{S,\Omega}$  are the relaxation time of the bath, the time scale of the system-bath interaction, the time scale of the system, and the time scale of the local subsystem  $\Omega$ , respectively,  $R$  is the radius of  $\Omega$ , and  $\zeta_0$  is the propagation velocity of the system. The detailed definitions are given in the text.

we can rewrite the Redfield equation (4) as

$$\begin{aligned} \frac{d}{dt}\rho(t) = & -i[H_S + H_{LS}, \rho(t)] \\ & + \sum_{\mu, \nu} \sum_{\omega, \omega'} \left[ \frac{\gamma_{\mu\nu}(\omega) + \gamma_{\mu\nu}(\omega')}{2} + i(\eta_{\mu\nu}(\omega) - \eta_{\mu\nu}(\omega')) \right] \\ & \times \left( A_\nu(\omega)\rho(t)A_\mu^\dagger(\omega') - \frac{1}{2}\{A_\mu^\dagger(\omega')A_\nu(\omega), \rho(t)\} \right), \end{aligned} \quad (6)$$

where the Lamb-shift Hamiltonian  $H_{LS}$  is defined as

$$\begin{aligned} H_{LS} = & \sum_{\mu, \nu} \sum_{\omega, \omega'} \left( \frac{\eta_{\mu\nu}(\omega) + \eta_{\mu\nu}(\omega')}{2} \right. \\ & \left. + i \frac{\gamma_{\mu\nu}(\omega) - \gamma_{\mu\nu}(\omega')}{4} \right) A_\mu^\dagger(\omega') A_\nu(\omega). \end{aligned} \quad (7)$$

Here, the spectral density function  $\gamma_{\mu\nu}(\omega)$  and the principal density function  $\eta_{\mu\nu}(\omega)$  are the Hermitian matrices satisfying

$$\Gamma_{\mu\nu}(\omega) := \int_0^\infty ds e^{i\omega s} C_{\mu\nu}(s) = \frac{1}{2}\gamma_{\mu\nu}(\omega) + i\eta_{\mu\nu}(\omega). \quad (8)$$

The spectral density function  $\gamma_{\mu\nu}(\omega)$  is given by

$$\gamma_{\mu\nu}(\omega) = \int_{-\infty}^\infty ds e^{i\omega s} C_{\mu\nu}(s). \quad (9)$$

The time scale  $\tau_B$  of the bath is defined by the relaxation time of the correlation functions as

$$\tau_B = \max_{\mu, \nu} \frac{\int_0^\infty t |C_{\mu\nu}(t)| dt}{\int_0^\infty |C_{\mu\nu}(t)| dt}, \quad (10)$$

where  $|C_{\mu\nu}|$  is the absolute value of  $C_{\mu\nu}$ . The time scale  $\tau_{SB}$  of the system-bath interaction is defined as the inverse rate of the time evolution due to the system-bath interaction. Let  $\mathcal{D}[\rho(t)]$  denote the second term in Eq. (4) as

$$\begin{aligned} \mathcal{D}[\rho(t)] = & \sum_{\mu, \nu} \int_0^\infty ds [C_{\mu\nu}(s)(A_\nu(-s)\rho(t)A_\mu^\dagger \\ & - A_\mu^\dagger A_\nu(-s)\rho(t)) + \text{H.c.}], \end{aligned} \quad (11)$$

which represents the time evolution caused by the system-bath interaction. The rate of the evolution caused by  $\mathcal{D}[\rho(t)]$  is bounded from above by the operator norm of  $\mathcal{D}$ :

$$\|\mathcal{D}[\rho(t)]\|_1 \leq \|\mathcal{D}\| \|\rho(t)\|_1 \leq \|\mathcal{D}\| =: \tau_{SB}^{-1}, \quad (12)$$

where  $\|\cdot\|$  is the operator norm induced by the trace norm  $\|\cdot\|_1$ . We note that  $\tau_{SB}^{-1}$  only gives an upper bound on the rate of time evolution caused by the system-bath interaction, but does not necessarily characterize the rate of time evolution itself. Therefore,  $\tau_B \ll \tau_{SB}$  is a sufficient condition for deriving the Redfield equation, but not a necessary condition.

Equation (6) is not in the GKSL form and does not possess the complete positivity because the Hermitian matrix  $\gamma_{\nu\omega, \mu\omega'} := (\gamma_{\mu\nu}(\omega) + \gamma_{\mu\nu}(\omega'))/2 + i(\eta_{\mu\nu}(\omega) - \eta_{\mu\nu}(\omega'))$  is not necessarily positive semidefinite. To recover the complete positivity, we need further approximations to the Redfield equation to derive the GKSL equations. Such derivations have been studied in Refs. [34, 35, 39–53]. In the following, we introduce two types of the microscopically derived GKSL equations that we use in this paper.

## II.2. Davies equation

The Davies equation [39] is the well-known GKSL equation derived by applying the rotating-wave approximation to the Redfield equation. To derive the Davies equation, we assume that the gaps between energy-level spacings are so large that the time scale  $\tau_S$  of the system defined by a typical value of  $|\omega - \omega'|^{-1}$  in Eq. (6) is much smaller than  $\tau_{SB}$ . A typical value is, for example, defined as the maximum value of  $|\omega - \omega'|^{-1}$  or the maximum value of  $|\omega - \omega'|^{-1}$  over  $\omega$  and  $\omega'$  satisfying that  $\gamma_{\mu\nu}(\omega)$  or  $\gamma_{\mu\nu}(\omega')$  is sufficiently large. Here, we do not fix the definition of  $\tau_S$  and only assume  $\tau_S \ll \tau_{SB}$ . Then, we can neglect the rapidly oscillating terms in Eq. (6) where  $\omega \neq \omega'$  and obtain the Davies equation

$$\begin{aligned} \frac{d}{dt}\rho(t) = & -i[H_S + H_{LS}, \rho(t)] + \sum_{\omega} \sum_{\mu, \nu} \gamma_{\mu\nu}(\omega) \\ & \times \left( A_\nu(\omega)\rho(t)A_\mu^\dagger(\omega) - \frac{1}{2}\{A_\mu^\dagger(\omega)A_\nu(\omega), \rho(t)\} \right), \end{aligned} \quad (13)$$

where  $H_{LS}$  is the Lamb-shift Hamiltonian given by

$$H_{LS} := \sum_{\omega} \sum_{\mu, \nu} \eta_{\mu\nu}(\omega) A_\mu^\dagger(\omega) A_\nu(\omega). \quad (14)$$

Since the Lindblad operators of the Davies equation cause quantum jumps between energy eigenstates, the Davies equation (13) leads to a global change of the state and therefore does not have desired locality in many-body systems. Moreover, the rotating-wave approximation often fails for many-body systems [34, 54] because the energy levels of many-body systems become exponentially small with increasing the system size (i.e. the condition  $\tau_S \ll \tau_{SB}$  is violated).

## II.3. Universal Lindblad equation

The universal Lindblad equation (ULE) [50] can be derived under the same assumptions as those made to derive the Redfield equation and it can describe nonequilibrium steady states and dynamics with a small error of  $\mathcal{O}(\tau_B/\tau_{SB})$  [50, 55]. See Refs. [50, 51] for the detailed derivations. We also give a derivation of the ULE in Appendix A for the sake of self-containedness of this paper.

Let us define the Lindblad operators  $L_\lambda$  and the Lamb-shift Hamiltonian  $H_{\text{LS}}$  as

$$L_\lambda := \sum_\mu \sum_\omega \gamma_{\lambda\mu}^{1/2}(\omega) A_\mu(\omega), \quad (15)$$

$$H_{\text{LS}} := \sum_{\mu,\nu} \sum_{\omega,\omega'} \left[ \eta_{\mu\nu} \left( \frac{\omega + \omega'}{2} \right) + i \frac{\gamma_{\mu\nu}(\omega) - \gamma_{\mu\nu}(\omega')}{4} \right] A_\mu^\dagger(\omega) A_\nu(\omega'), \quad (16)$$

where  $\gamma_{\mu\nu}^{1/2}(\omega)$  is defined to satisfy  $\gamma_{\mu\nu}(\omega) = \sum_\lambda \gamma_{\mu\lambda}^{1/2}(\omega) \gamma_{\lambda\nu}^{1/2}(\omega)$ . Then, we can write the ULE in the following form:

$$\begin{aligned} \frac{d}{dt} \rho = & -i[H_S + H_{\text{LS}}, \rho] \\ & + \sum_{\lambda=1}^k \left( L_\lambda \rho L_\lambda^\dagger - \frac{1}{2} \{L_\lambda^\dagger L_\lambda, \rho\} \right). \end{aligned} \quad (17)$$

The existence of  $\gamma_{\mu\nu}^{1/2}(\omega)$  is ensured by the positive semidefiniteness of the matrix  $\gamma_{\mu\nu}(\omega)$  at any fixed  $\omega$ .

The Lindblad operators (15) of the ULE can also be written as

$$L_\lambda = \sum_\mu \int_{-\infty}^{\infty} ds g_{\lambda\mu}(-s) A_\mu(s), \quad (18)$$

where  $g_{\lambda\mu}(s)$  is defined as the Fourier transformation of  $\gamma_{\lambda\mu}^{1/2}(\omega)$ :

$$g_{\lambda\mu}(s) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\lambda\mu}^{1/2}(\omega) e^{i\omega s} d\omega. \quad (19)$$

### III. LOCAL GKSL EQUATION BASED ON THE LIEB-ROBINSON BOUND

In this section, we use the Lieb-Robinson bound to derive the local GKSL equations, which are useful for the analysis of open many-body systems. We first introduce the Lieb-Robinson bound in Sec. III.1, and use it to impose the locality on the Redfield equation in Sec. III.2. We use this local Redfield equation to derive the local GKSL equations, which include the local Davies equation in Sec. III.3 and the local universal Lindblad equation in Sec. III.4.

#### III.1. Lieb-Robinson bound

In the following, we consider a many-body system on a lattice with Hamiltonian  $H_S = \sum_{X \subseteq \Lambda} h_X$ , where  $\Lambda$  denotes the set of sites and  $h_X$  is an operator that acts nontrivially only on a local region  $X \subset \Lambda$ . A distance  $\text{dist}(p, q)$  between lattice sites  $p$  and  $q$  is defined by the number of sites in the shortest pass from site  $p$  to site  $q$ .

We also define the distance between sets  $X$  and  $Y$  of sites by  $\text{dist}(X, Y) := \min_{p \in X, q \in Y} \text{dist}(p, q)$ . We assume the Hamiltonian  $H_S$  to be strictly local, that is,  $h_X = 0$  holds for  $X$  whose radius exceeds a certain constant value.

We follow Ref. [38] to introduce the Lieb-Robinson bound in the form of Lemma 1.

**Lem. 1** *Let  $H = \sum_Y h_Y$  be a local Hamiltonian, and  $O_X$  be any operator acting on the sites belonging to a region  $X$ . Suppose that we take a set of sites  $\Omega \subseteq \Lambda$  which satisfies  $l = \text{dist}(X, \Lambda \setminus \Omega)$ . Then,*

$$\left\| (U_t^H)^\dagger O_X U_t^H - (U_t^{H_\Omega})^\dagger O_X U_t^{H_\Omega} \right\| \leq |X| \|O_X\| \frac{(2\zeta_0 |t|)^l}{l!}, \quad (20)$$

where  $H_\Omega := \sum_{Y \subseteq \Omega} h_Y$  is a Hamiltonian of the subsystem  $\Omega$ ,  $U_t^H = \exp(-itH)$  is a unitary time evolution operator, and  $\zeta_0 := \max_{p \in \Lambda} \sum_{Z \ni p} |Z| \|h_Z\| = \mathcal{O}(L^0)$  is the propagation velocity of the system which does not depend on the system size  $L$ . The norm  $\|\cdot\|$  is the operator norm and  $|X|$  denotes the number of sites in  $X$ .

This lemma states that in many-body systems the time evolution of a local operator  $O_X$  acting on a local region  $X$  during time  $T$  can be approximated by an operator acting on a set of sites within a distance of  $\zeta_0 T$  from  $X$ .

#### III.2. Localizing the dissipators in the Redfield equation

Using the Lieb-Robinson bound, we can approximate the Redfield equation (5) so that its locality is apparent. Focusing on the integrand on the right-hand side in Eq. (4), we find that the correlation function  $C_{\mu\nu}(s)$  decays with the relaxation time  $\tau_B$  of  $C_{\mu\nu}(s)$ . Thus, the range of the integral can be well approximated by  $[0, T]$ , where  $T \sim \tau_B$ . Therefore, it is legitimate to replace the upper bound of the integral with  $T$ . We assume that  $A_\mu$  acts on a local region  $X_\mu$  and that the correlation function  $C_{\mu\nu}(s)$  decays sufficiently fast as the distance between the two regions,  $\text{dist}(X_\mu, X_\nu)$ , increases.

Here, we can use the Lieb-Robinson bound in Lem. 1 and approximate  $A_\mu(s)$  in Eq. (4) by

$$A_\mu^{\text{loc}}(s) := \exp(isH_{\Omega_\mu}) A_\mu \exp(-isH_{\Omega_\mu}), \quad (21)$$

where  $\Omega_\mu$  is the subsystem which is chosen to satisfy  $\text{dist}(X_\mu, \Lambda \setminus \Omega_\mu) \gtrsim \zeta_0 \tau_B$  (see Fig. 1). For example, if  $X_\mu$  consists of a single site,  $\Omega_\mu$  is constituted from sites whose distance from the single site is less than  $R \gtrsim \zeta_0 \tau_B$ . Then, we obtain the local Redfield equation:

$$\begin{aligned} \frac{d}{dt} \rho(t) = & -i[H_S, \rho(t)] + \sum_{\mu,\nu} \int_0^\infty ds [C_{\mu\nu}(s) \\ & \times (A_\nu^{\text{loc}}(-s) \rho(t) A_\mu - A_\mu A_\nu^{\text{loc}}(-s) \rho(t)) + \text{H.c.}], \end{aligned} \quad (22)$$

where all the operators  $A_\nu^{\text{loc}}$  and  $A_\mu$  in the dissipator act on local subsystems. When all  $A_\mu$ 's act on the same region  $X$ , the second term in Eq. (22) is local, acting

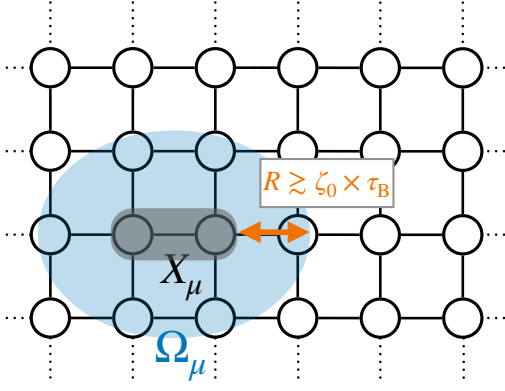


FIG. 1. Subsystem  $\Omega_\mu$  on which a local operator  $A_\mu^{\text{loc}}$  acts in the dissipator. Circles represent the sites of the system. The inner shaded region represents  $X_\mu$ , the set of sites on which  $A_\mu$  acts. The outer shaded region represents the subsystem  $\Omega_\mu$  on which the system-bath coupling  $A_\mu \otimes B_\mu$  exerts during the relaxation time  $\tau_B$  of the bath. This subsystem is chosen to be composed of the sites within the distance shorter than  $R \gtrsim \zeta_0 \times \tau_B$  from every site connected to the corresponding bath.

only on  $\Omega$  which satisfies  $\text{dist}(X, \Lambda \setminus \Omega) \gtrsim \zeta_0 \tau_B$ . When  $A_\mu$  acts on a spatially distant region  $X_\mu$ , the second term in Eq. (22) is also local, if the correlation function  $C_{\mu\nu}(s)$  of the bath decays sufficiently fast as the spatial distance

$\text{dist}(X_\mu, X_\nu)$  increases.

### III.3. Local Davies equation

By making the rotating-wave approximation in Eq. (22), the local Davies equation can be derived. The local Davies equation has been used in the literature of quantum thermodynamics [30, 32, 56] because it can describe the relaxation to the Gibbs state at the temperature of a bath.

An operator  $A_\mu$  can be decomposed into the sum of its frequency components  $A_\mu^{\text{loc}}(\omega)$  over  $\omega$ ,

$$A_\mu = \sum_{\omega} A_\mu^{\text{loc}}(\omega), \quad (23)$$

where  $A_\mu^{\text{loc}}(\omega)$  is defined as

$$A_\mu^{\text{loc}}(\omega) := \sum_{E_n^{\text{loc}} - E_m^{\text{loc}} = \omega} |E_m^{\text{loc}}\rangle \langle E_m^{\text{loc}}| A_\mu |E_n^{\text{loc}}\rangle \langle E_n^{\text{loc}}|, \quad (24)$$

with  $E_n^{\text{loc}}$  and  $|E_n^{\text{loc}}\rangle$  being an eigenvalue and the corresponding eigenstate of  $H_{\Omega_\mu}$ . We note that the sum on the right-hand side runs over all pairs of energy levels in  $H_{\Omega_\mu}$  whose spacing is equal to  $\omega$ .

By substituting Eq. (23) into Eq. (22), we obtain

$$\frac{d}{dt}\rho(t) = -i[H_S, \rho(t)] + \sum_{\mu, \nu} \sum_{\omega, \omega'} \left[ \left( \frac{1}{2} \gamma_{\mu\nu}(\omega) + i\eta_{\mu\nu}(\omega) \right) (A_\nu^{\text{loc}}(\omega)\rho(t)A_\mu^{\text{loc}\dagger}(\omega') - \rho(t)A_\mu^{\text{loc}\dagger}(\omega')A_\nu^{\text{loc}}(\omega)) + \text{H.c.} \right]. \quad (25)$$

We define the time scale  $\tau_{S, \Omega_\mu}$  of the subsystem by a typical value of  $|\omega - \omega'|^{-1}$ , where  $\omega$  and  $\omega'$  are the energy-level spacings of  $H_{\Omega_\mu}$ . By assuming that the time scale  $\tau_{S, \Omega_\mu}$  is sufficiently shorter than the time scale  $\tau_{SB}$  of the system-bath interaction (see Eq. (12) for its definition), we can ignore the terms with  $\omega \neq \omega'$  in Eq. (25), and obtain the local Davies equation as

$$\begin{aligned} \frac{d}{dt}\rho(t) = & -i[H_S + H_{\text{LS}}, \rho(t)] \\ & + \sum_{\mu, \nu} \sum_{\omega} \gamma_{\mu\nu}(\omega) \left( A_\nu^{\text{loc}}(\omega)\rho(t)A_\mu^{\text{loc}\dagger}(\omega) \right. \\ & \left. - \frac{1}{2} \left\{ A_\mu^{\text{loc}\dagger}(\omega)A_\nu^{\text{loc}}(\omega), \rho(t) \right\} \right), \end{aligned} \quad (26)$$

where the Lamb-shift Hamiltonian  $H_{\text{LS}}$  is defined as

$$H_{\text{LS}} := \sum_{\mu, \nu} \sum_{\omega} \eta_{\mu\nu}(\omega) A_\mu^{\text{loc}\dagger}(\omega) A_\nu^{\text{loc}}(\omega). \quad (27)$$

The Lindblad operators  $A_\mu^{\text{loc}}(\omega)$  in Eq. (26) induce the transitions between energy eigenstates of the Hamiltonian  $H_{\Omega_\mu}$  of the local subsystem and satisfy the detailed balance condition for  $H_{\Omega_\mu}$ , provided that the bath is at thermal equilibrium.

In order for the rotating-wave approximation for the Hamiltonian of the local subsystem  $H_{\Omega_\mu}$  to be valid, the energy-level spacings of  $H_{\Omega_\mu}$  must be sufficiently large so that the time scale  $\tau_{S, \Omega_\mu}$  of the subsystem defined by a typical value of  $|\omega - \omega'|^{-1}$  must be much smaller than the time scale  $\tau_{SB}$  of the system-bath interaction, where  $\omega$  and  $\omega'$  are the energy-level spacings of  $H_{\Omega_\mu}$ . Since the energy-level spacings become smaller as the size of the subsystem becomes larger, we cannot take the subsystem  $\Omega_\mu$  too large. In contrast, the replacement of  $A_\mu(t)$  by  $A_\mu^{\text{loc}}(t)$  becomes a better approximation as we take a larger subsystem. Therefore, we expect a trade-off relationship between the error arising from localization of the Redfield equation and the error arising from the rotating-wave approximation. We numerically show in Sec. IV that such a trade-off relationship indeed exists.

### III.4. Local universal Lindblad equation

In a manner similar to the derivation of the ULE in Sec. II.3, starting from Eq. (22), we can obtain the local ULE where  $A_\mu(\omega)$ 's in Eq. (15) are replaced by  $A_\mu^{\text{loc}}(\omega)$ . The Lindblad operators and the Lamb-shift Hamiltonian

of the local ULE can be represented in terms of  $A_\mu^{\text{loc}}(\omega)$  as

$$L_\lambda = \sum_\mu \sum_\omega \gamma_{\lambda\mu}^{1/2}(\omega) A_\mu^{\text{loc}}(\omega), \quad (28)$$

$$H_{\text{LS}} = \sum_{\mu,\nu} \sum_{\omega,\omega'} \left[ \eta_{\mu\nu} \left( \frac{\omega + \omega'}{2} \right) + i \frac{\gamma_{\mu\nu}(\omega) - \gamma_{\mu\nu}(\omega')}{4} \right] A_\mu^{\text{loc}\dagger}(\omega) A_\nu^{\text{loc}}(\omega'). \quad (29)$$

These Lindblad operators and the Lamb-shift Hamiltonian can be obtained from the diagonalization of  $H_{\Omega_\mu}$ , while it is necessary to diagonalize the full Hamiltonian  $H_S$  to calculate the Lindblad operators (15) and the Lamb-shift Hamiltonian (16) of the original ULE. The local ULE derived here is yet another efficient approach to avoiding the diagonalization of the full Hamiltonian in numerical calculations.

The Lindblad operator can also be written as

$$L_\lambda = \sum_\mu \int_{-\infty}^{\infty} ds g_{\lambda\mu}(-s) A_\mu^{\text{loc}}(s). \quad (30)$$

The relaxation time of  $g_{\mu\mu}(t)$  is also of the order of  $\tau_B$ . Because we have chosen the subsystem  $\Omega_\mu$  for  $A_\mu$  so that  $\text{dist}(X_\mu, \Lambda \setminus \Omega_\mu) \gtrsim \zeta_0 \tau_B$ , where  $X_\mu$  is the support of  $A_\mu$ ,  $A_\mu(s)$  in Eq. (18) can be replaced by  $A_\mu^{\text{loc}}(s)$  by using the Lieb-Robinson bound. We find that Eq. (30) reduces to Eq. (18) in the limit of  $\tau_B \rightarrow 0$ . Since the error of the ULE relative to the Redfield equation is  $\mathcal{O}(\tau_B/\tau_{\text{SB}})$ , the error of the local ULE from the Redfield equation vanishes in the limit of  $\tau_B \rightarrow 0$ .

#### IV. NUMERICAL TEST OF THE LOCAL GKSL EQUATIONS

In this section, we numerically test the validity of the local Davies equation and the local ULE derived in Sec. III. Here, we show that the distance between the generator of the local GKSL equation and that of the Redfield equation becomes small if  $\tau_B \ll R/\zeta_0$ , which is consistent with the condition shown in Table I. This result guarantees that the dynamics in a sufficiently short time is described by the local GKSL equation with a small error. However, it does not guarantee the correctness of the steady state because the small error of the generator can accumulate to grow exponentially in a sufficiently long time compared with  $\tau_{\text{SB}}$ . Even if the generators of time evolution change only slightly, the steady state can change significantly. Therefore, we investigate the errors in steady states obtained from the local GKSL equations and confirm that the steady states can also be accurately described by the local GKSL equations.

##### IV.1. Model

We consider spinless fermions on a one-dimensional lattice with  $L$  sites. The Hamiltonian of the system is given

by

$$H_S = \omega_0 \sum_{j=1}^L a_j^\dagger a_j - J \sum_{j=1}^{L-1} (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) \quad (31)$$

$$=: \sum_{i,j=1}^L h_{ij} a_i^\dagger a_j, \quad (32)$$

where  $\omega_0$  is an on-site energy and  $J > 0$  is the hopping amplitude. Here,  $a_j$  and  $a_j^\dagger$  represent the annihilation and creation operators at site  $j$ , and they satisfy the anticommutation relations  $\{a_i, a_j^\dagger\} = \delta_{i,j}$ . The propagation velocity of the system is given by  $\zeta_0 = 4J$  (see Lem. 1 in Sec. III.1 for the definition of the propagation velocity).

We couple  $N$  sites from each edge of the one-dimensional lattice to identical baths constituted of free fermions (see Fig. 2). The state  $\rho_{B,j}$  of the bath connected to site  $j$  is assumed to be the Gibbs state at inverse temperature  $\beta_j$  and the average number of fermions therein with wave number  $\mathbf{k}$  is given by the Fermi-Dirac distribution  $f_\beta(\omega(\mathbf{k})) = 1/(1 + e^{\beta_j \omega(\mathbf{k})})$ , where  $\omega(\mathbf{k})$  is the dispersion relation of fermions in the bath measured from the chemical potential. Here, the chemical potentials at the baths are set to be equal. The temperature  $\beta_j$  is set to be  $\beta_j = \beta_l$  ( $\beta_r$ ) if site  $j$  is close to the left (right) edge. In this section, we consider the two settings: an equilibrium setting where  $\beta_l = \beta_r$ , and a nonequilibrium setting where there is a temperature difference,  $\beta_l \neq \beta_r$  (see Fig. 2). The Hamiltonian  $H_{B,j}$  of the bath at each site and the system-bath interaction Hamiltonian  $H_{\text{SB},j}$  at each site are given as

$$H_{B,j} = \sum_{\mathbf{k}} \omega(\mathbf{k}) c_{\mathbf{k}}^{(j)\dagger} c_{\mathbf{k}}^{(j)}, \quad (33)$$

$$H_{\text{SB},j} = \frac{J_{\text{int}}}{\sqrt{V}} \sum_{\mathbf{k}} (a_j^\dagger c_{\mathbf{k}}^{(j)} + c_{\mathbf{k}}^{(j)\dagger} a_j), \quad (34)$$

where  $V$  is the volume of the baths, and  $c_{\mathbf{k}}^{(j)}$  and  $c_{\mathbf{k}}^{(j)\dagger}$  are the annihilation and creation operators of a fermion of the bath with wave number  $\mathbf{k}$ , respectively. The total Hamiltonian is written as

$$H_{\text{tot}} = H_S + \sum_{j=1}^N (H_{\text{SB},j} + H_{B,j}) + \sum_{j=L-N+1}^L (H_{\text{SB},j} + H_{B,j}). \quad (35)$$

To introduce the time scale  $\tau_B$  of the bath in this model, we assume that the density of states  $D(\omega)$  of the baths in the limit  $V \rightarrow \infty$  is well-approximated by the Cauchy-Lorentz distribution as

$$D(\omega) := \frac{1}{V} \sum_{\mathbf{k}} \delta(\omega - \omega(\mathbf{k})) \rightarrow \frac{2\tau_B^{-1}}{\tau_B^{-2} + (\omega - \omega_0)^2}. \quad (36)$$

Here, we assume that the density of states peaks at  $\omega_0$  for simplicity. The interaction Hamiltonian at site  $j$  can be rewritten as  $H_{\text{SB},j} = A_{1,j} \otimes B_{1,j} + A_{2,j} \otimes B_{2,j}$ , where

$$A_{1,j} := a_j, A_{2,j} := a_j^\dagger, \quad B_{1,j} := \frac{J_{\text{int}}}{\sqrt{V}} \sum_{\mathbf{k}} c_{\mathbf{k}}^{(j)\dagger}, B_{2,j} := \frac{J_{\text{int}}}{\sqrt{V}} \sum_{\mathbf{k}} c_{\mathbf{k}}^{(j)}. \quad (37)$$

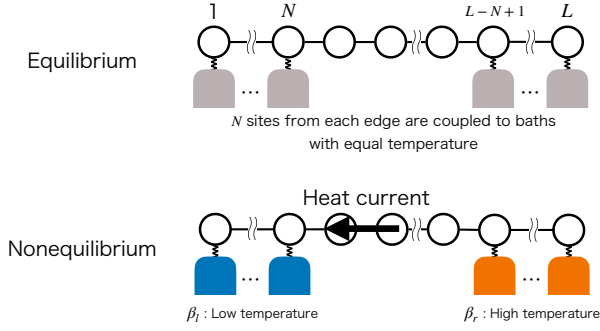


FIG. 2. Schematic illustration of the model for numerical demonstration of the validity of the local GKSL equations. A spinless fermion chain is connected to baths so that an equilibrium (top,  $\beta_l = \beta_r$ ) or a nonequilibrium (bottom,  $\beta_l \neq \beta_r$ ) steady state is achieved. The round square under each site represents a bath connected to it.

Since the state of each bath is assumed to be given by the Gibbs state, the correlation functions  $C_{\mu\nu}^{(j)}(t) = \text{tr}[B_{\mu,j}^\dagger(t)B_{\nu,j}\rho_{B,j}]$  in the limit of  $V \rightarrow \infty$  are written as

$$\begin{aligned} C_{11}^{(j)}(t) &= J_{\text{int}}^2 \int_{-\infty}^{\infty} d\omega e^{-i\omega t} (1 - f_{\beta_j}(\omega)) D(\omega), \\ C_{22}^{(j)}(t) &= J_{\text{int}}^2 \int_{-\infty}^{\infty} d\omega e^{-i\omega t} f_{\beta_j}(\omega) D(\omega), \\ C_{12}^{(j)}(t) &= C_{21}^{(j)}(t) = 0, \end{aligned} \quad (38)$$

and the spectral density functions are given as

$$\begin{aligned} \gamma_{11}^{(j)}(\omega) &= J_{\text{int}}^2 (1 - f_{\beta_j}(\omega)) D(\omega), \\ \gamma_{22}^{(j)}(\omega) &= J_{\text{int}}^2 f_{\beta_j}(\omega) D(\omega), \\ \gamma_{12}^{(j)}(\omega) &= \gamma_{21}^{(j)}(\omega) = 0. \end{aligned} \quad (39)$$

In the infinite-temperature limit  $\beta_j = 0$ , the correlation functions are given by

$$C_{11}^{(j)}(t) = C_{22}^{(j)}(t) = J_{\text{int}}^2 e^{-t/\tau_B}. \quad (40)$$

Thus, at high temperature,  $\tau_B$  characterizes the time scale of decay of the correlation functions of the bath.

Here we derive the local GKSL equation for this model by using the method in Sec. III. To obtain the local Lindblad operators that describe the dissipation due to the bath at site  $j$ , we make the subsystem  $\Omega_R^{(j)}$  composed of sites whose distance from site  $j$  is shorter than  $R$  (see Fig. 1). We call this  $R$  the radius of the subsystem  $\Omega_R^{(j)}$ . The Hamiltonian  $H_{\text{loc},j}$  of the subsystem is then given by

$$\begin{aligned} H_{\text{loc},j} &= \omega_0 \sum_{i \in \Omega_R^{(j)}} a_i^\dagger a_i \\ &\quad - J \sum_{\{i,i+1\} \subset \Omega_R^{(j)}} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) \\ &=: \sum_{i,l \in \Omega_R^{(j)}} h_{il}^{(\text{loc},j)} a_i^\dagger a_l. \end{aligned} \quad (41)$$

The matrix  $h_{il}^{(\text{loc},j)}$  can be diagonalized by an orthogonal matrix  $O$  as

$$\sum_{i,l} O_{mi} h_{il}^{(\text{loc},j)} O_{nl} = \omega_m^{(\text{loc},j)} \delta_{nm}, \quad (42)$$

and the subsystem Hamiltonian can be written as

$$H_{\text{loc},j} = \sum_{m=1}^{|\Omega_R^{(j)}|} \omega_m^{(\text{loc},j)} d_m^\dagger d_m, \quad (43)$$

where  $d_m = \sum_{i \in \Omega_R^{(j)}} O_{mi}^* a_i$  is the annihilation operator of the energy eigenmode of the subsystem Hamiltonian. Therefore,  $A_{1,j}$  and  $A_{2,j}$  can be decomposed as (see Eq. (23))

$$A_{1,j} = \sum_{m=1}^{|\Omega_R^{(j)}|} A_{1,j}^{(\text{loc})}(\omega_m^{(\text{loc},j)}) = \sum_{m=1}^{|\Omega_R^{(j)}|} O_{mj} d_m, \quad (44)$$

$$A_{2,j} = \sum_{m=1}^{|\Omega_R^{(j)}|} A_{1,j}^{(\text{loc})}(-\omega_m^{(\text{loc},j)}) = \sum_{m=1}^{|\Omega_R^{(j)}|} O_{mj}^* d_m^\dagger. \quad (45)$$

We can use Eqs. (44) and (45) to derive the local GKSL equation in a manner similar to the derivation in the previous section.

The obtained local GKSL equations and the Redfield equation can be written in the form of

$$\begin{aligned} \frac{d\rho}{dt} &= -i[H_S, \rho] + \sum_{m,n} (M_{mn} [w_m \rho, w_n] + \text{H.c.}) \\ &=: -i[H_S, \rho] + \mathcal{D}[\rho], \end{aligned} \quad (46)$$

where  $M = (M_{mn})$  is a Hermitian matrix and

$$w_{2j-1} := a_j + a_j^\dagger, \quad w_{2j} := i(a_j - a_j^\dagger). \quad (47)$$

Here, we neglect the small Lamb-shift Hamiltonian for the sake of simplicity. For such QMEs, the steady-state expectation value of an observable which is written in the quadratic form of the annihilation and creation operators can be efficiently calculated [57–60]. In addition, the modulus of the largest eigenvalue  $\lambda_{\text{max}}$  of  $\mathcal{D}$  is also efficiently calculated from the matrix  $M$  in Eq. (46) as  $|\lambda_{\text{max}}| = 2|\text{tr} M|$  as long as the dissipator has a single zero eigenvalue and the real part of all the other eigenvalues of the dissipator are negative (see Appendix B for a proof). This condition is satisfied in all the following numerical calculations.

The norm of the dissipator  $\mathcal{D}^{(\text{Redfield})}$  in the Redfield equation cannot be computed efficiently. Therefore, although the norm  $\|\mathcal{D}^{(\text{Redfield})}\|$  and the modulus  $|\lambda_{\text{max}}|$  of the largest eigenvalue do not generally coincide, here we expect the norm and the maximum eigenvalue to take similar values and characterize the time scale  $\tau_{\text{SB}}$  by  $|\lambda_{\text{max}}|$  as

$$\tau_{\text{SB}}^{-1} := \|\mathcal{D}^{(\text{Redfield})}\| \simeq |\lambda_{\text{max}}| = 2|\text{tr} M^{(\text{Redfield})}|. \quad (48)$$



This time scale  $\tau_{\text{SB}}$  can be adjusted by changing the coupling  $J_{\text{int}}$  between the system and the bath. In the following, we discuss the validity of the local GKSL equations by changing the ratios of the three time scales,  $\tau_{\text{B}}$ ,  $\tau_{\text{SB}}$ , and  $R\zeta_0^{-1} = R/4J$ . In the numerical calculation, the unit of time is set to be the inverse hopping rate  $J^{-1}$ .

#### IV.2. Error in the generator of the local GKSL equation

To confirm the validity of the local GKSL equations derived in Sec. III, we numerically calculate the distance between the generator of the time evolution of the Redfield equation and that of the derived local GKSL equation. We fix  $\tau_{\text{SB}}$  and show that the distance becomes small under the condition  $\tau_{\text{B}} < R/\zeta_0$  by decreasing the value of  $\tau_{\text{B}} (\ll \tau_{\text{SB}})$ . We define the distance between the generators of the two QMEs in terms of the ratio of the Hilbert-Schmidt norm of the difference of the matrix  $M$  in Eq. (46) to that of  $M$  of the Redfield equation as

$$\frac{\sqrt{\text{tr}[|M(\text{locGKSL}) - M(\text{Redfield})|^2]}}{\sqrt{\text{tr}[|M(\text{Redfield})|^2]}}, \quad (49)$$

where  $|M|^2 = M^\dagger M$ . This distance represents the error in the generator of the local GKSL equation.

The distance (49) depends on the radius  $R$  of the subsystem chosen in the derivation of the local GKSL equation. For local GKSL equations with a different choice of the radius of the subsystem, the distances are plotted in Fig. 3 against the relaxation time  $\tau_{\text{B}}$  of the baths. As local GKSL equations, we consider the local Davies equation (see Sec. III.3) and the local ULE (see Sec. III.4).

The distance becomes small for both the local Davies equation and the local ULE when the relaxation time  $\tau_{\text{B}}$  is so short that the radius  $R$  is larger than  $\zeta_0\tau_{\text{B}}$ , where  $\zeta_0 = 4J$  is the propagation velocity of the system. This result is consistent with the condition used in the derivation of the local GKSL equation in Sec. III, where the radius  $R$  of the subsystem should be chosen to be larger than  $\zeta_0\tau_{\text{B}}$  according to the Lieb-Robinson bound. By comparing the two local GKSL equations, we find that the error is smaller for the local ULE when  $\tau_{\text{B}}$  is short, which indicates that the local ULE is a better approximation in terms of accuracy. For the local Davies equation, the distance is nonzero in the limit of  $\tau_{\text{B}} \rightarrow 0$  (Fig. 3-(a,b)) while it vanishes for the local ULE (Fig. 3-(c,d)) as mentioned in Sec. III.4. This is because the error due to the rotating-wave approximation, which is determined by the time scales  $\tau_{\text{S},\Omega}$  and  $\tau_{\text{SB}}$ , remains nonzero in the limit of  $\tau_{\text{B}} \rightarrow 0$ . In addition, the distance for the local Davies equation does not decrease monotonically with increasing the radius, even though the distance for the local ULE decreases as the radius increases. The origin of this behavior will be discussed in more detail in Sec. IV.4.

As shown in Appendix C, the behavior of the distance is quantitatively different depending on whether the ra-

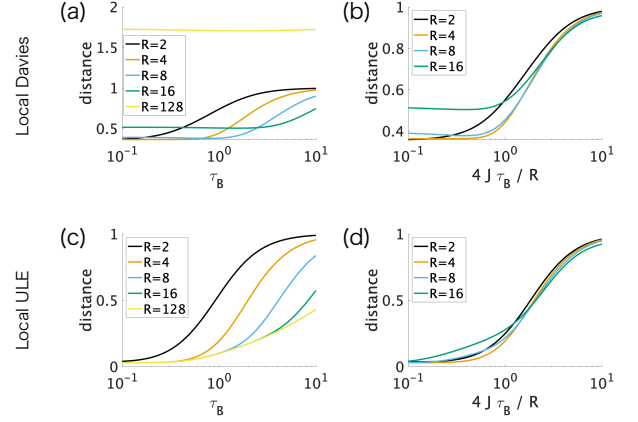


FIG. 3. Distance between the generator of each local GKSL equation and that of the Redfield equation plotted against the relaxation time  $\tau_{\text{B}}$  of the baths. Here  $R$  represents the radius of the local subsystem upon which the Lindblad operators act. The parameters used are  $L = 128$ ,  $N = 16$ ,  $\tau_{\text{SB}} = 100$ ,  $J = 1$ ,  $\omega_0 = 0$ ,  $\beta_l = 0.5$  and  $\beta_r = 0.1$ . (a) Distance for the local Davies equation in the nonequilibrium setting. The distance varies nonmonotonically with increasing the radius. (b) The same quantity as in (a) with only the horizontal axis rescaled. The distance for the local Davies equation decreases significantly below  $\tau_{\text{B}} = R/4J$ . (c) Distance for the ULE in the nonequilibrium setting. The distance decreases with increasing the radius. (d) The same quantity as in (c) with only the horizontal axis rescaled. The distance for the local ULE decreases significantly below  $\tau_{\text{B}} = R/4J$ .

dius  $R$  is even or odd. We emphasize that the difference is noticeable for large  $\tau_{\text{B}}$ , where the localization of the Redfield equation is not justified in the first place. Therefore, the difference does not affect the validity of the local GKSL equation. In this section, we focus on only even  $R$ .

#### IV.3. Error in the steady state of the local GKSL equation

We numerically calculate the error in the steady state described by the local GKSL equation from the steady state described by the Redfield equation. We again fix  $\tau_{\text{SB}}$  and show that the error in the steady state also becomes small as long as the condition  $\tau_{\text{B}} < R/\zeta_0$  is satisfied. The accuracy of the GKSL equation cannot be evaluated by the errors of the generators alone. This is because even if the error in the generator is small, the error can accumulate over time and the error in the steady state can become large. Nevertheless, we argue that the time coarse-grained dynamics or steady state can accurately be described by the GKSL equation if the error in the generator is small. To demonstrate this, we evaluate the error in the steady states of the local GKSL equations.

For the steady state of each QME, we follow Refs. [57–60] to compute the  $2L \times 2L$ -matrix  $W = (W_{mn})$  defined



by

$$W_{mn} = \text{tr}(w_m w_n \rho_{\text{steady}}), \quad (50)$$

where  $\rho_{\text{steady}}$  is the density matrix of the steady state. The error in the steady state of the local GKSL equation is evaluated by the normalized maximal difference between the  $W$  matrix of the steady state of the local GKSL equation and that of the Redfield equation:

$$\Delta = \frac{\max_{mn} |W_{mn}^{\text{locGKSL}} - W_{mn}^{\text{Redfield}}|}{\max_{m \neq n} |W_{mn}^{\text{Redfield}}|}, \quad (51)$$

where we normalize the error by the maximum off-diagonal element of the  $W$  matrix of the Redfield equation. We note that the diagonal elements of  $W$  are always unity and do not reflect the state of the system.

The errors for each local GKSL equation are evaluated for both an equilibrium steady state and a nonequilibrium steady state. The results for the local Davies equation and the local ULE are shown in Figs. 4 and 5, respectively. The errors in the steady states for both of the local GKSL equations become small when  $\tau_B < R/4J$  in a similar way as the distance between the generators. This is consistent with the condition for applying the approximation to the Redfield equation using the Lieb-Robinson bound described in Sec. III.

For the local Davies equation (Fig. 4), the error in the steady states are reduced to about 10% for  $R \geq 4$  and  $\tau_B \ll R/\zeta_0$ , even though the generators produce errors larger than 40% (see Fig. 3 (b)). In the equilibrium case, the Gibbs state is the exact steady state of the Davies equation and the steady state described by the local Davies equation approaches the Gibbs state as  $R$  increases despite the large error in the generators. The error in the nonequilibrium steady state of the local Davies equation is also as small as that in the equilibrium steady state even though the nonequilibrium steady state of the Davies equation and that of the Redfield equation do not coincide. The error is nonzero in the limit of  $\tau_B \rightarrow 0$  since the generator of the local Davies equation has nonzero error as mentioned in Sec. IV.2.

For the local ULE (Fig. 5), the error is smaller than that of the local Davies equation in both equilibrium and nonequilibrium cases. In contrast to the local Davies equation, the error in the steady state tends to be larger in the nonequilibrium case than the equilibrium case (Fig. 5 (c)). For  $R = 2$  and 4, the steady state of the local ULE has nonzero errors even in the limit of  $\tau_B \rightarrow 0$ , whereas the steady states for the local ULEs of  $R \geq 8$  and the ULE have negligible errors in the same limit. This result can be attributed to the fact that small errors in the generators of the dynamics may accumulate in the long time and result in a nonzero error of the steady state. In fact, we have numerically confirmed that the distance of the generator vanishes in the limit of  $\tau_B \rightarrow 0$ . There may be such errors even for  $R > 4$  although they are too small to be seen and buried in numerical errors.

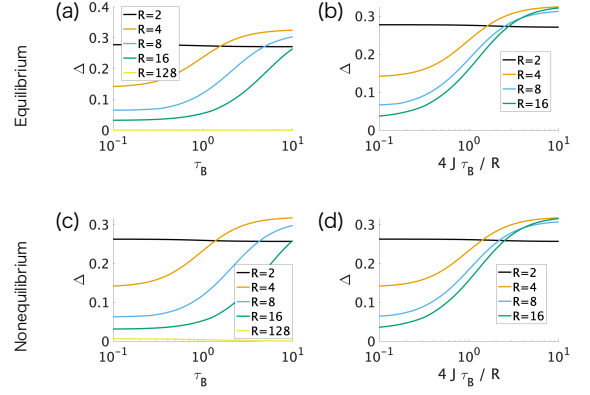


FIG. 4. Error of the steady state described by the local Davies equation where Lindblad operators act on local subsystems with radius  $R$ . The parameters are set to be  $L = 128$ ,  $N = 16$ ,  $\tau_{\text{SB}} = 100$ ,  $J = 1$  and  $\omega_0 = 0$ . (a) Case of an equilibrium steady state, where the inverse temperatures of the baths are set to be equal:  $\beta_l = \beta_r = 0.1$ . It can be seen that the error between the Redfield equation and the Davies equation ( $R = 128$ ) vanishes because both lead to the same steady state (the Gibbs state). (b) The same quantity as in (a) with only the horizontal axis rescaled. We see that the error significantly decreases around the region  $4J\tau_B/R < 1$  where the radius  $R$  of the subsystem is larger than  $\zeta_0\tau_B$  ( $\zeta_0 = 4J$  is the propagation velocity) except for the case of  $R = 2$ . (c) Case of a nonequilibrium steady state, where the inverse temperatures of the baths at the ends are  $\beta_l = 0.5$  for the left and  $\beta_r = 0.1$  for the right. The behavior of the steady-state error is qualitatively the same for the equilibrium and nonequilibrium cases except that the error for the Davies equation ( $R = 128$ ) is nonzero since the steady state is no longer the Gibbs state. (d) The same quantity as in (c) with only the horizontal axis rescaled. As in (b), we also see that the error significantly decreases around the region  $4J\tau_B/R < 1$  where the radius  $R$  of the subsystem is larger than  $\zeta_0\tau_B$  except for the case of  $R = 2$ .

#### IV.4. Trade-off between two errors in the steady state of the local Davies equation

We numerically show a trade-off relationship between two errors caused by the rotating-wave approximation and the localization using the Lieb-Robinson bound. We fix  $\tau_{\text{SB}}$  and  $\tau_B$  and change the radius  $R$  of the subsystem to investigate how the errors in the generator and the steady state change as the time scale  $\tau_{\text{S},\Omega}$  of the subsystem is changed. As shown in Fig. 6, both the error in the generator and the error in the steady state decrease with increasing  $R$  because the error due to localization becomes smaller. However, the errors begin to increase at certain values of  $R$  because the error caused by the rotating-wave approximation becomes larger. The error in the generator and that in the steady state behave differently. While the error in the generator increases rapidly after  $R = 16$ , the error in the steady state first decrease and then gradually increases from  $R = 32$ . This result shows that when one uses the local Davies equation, the size of the subsystem should be taken appropri-

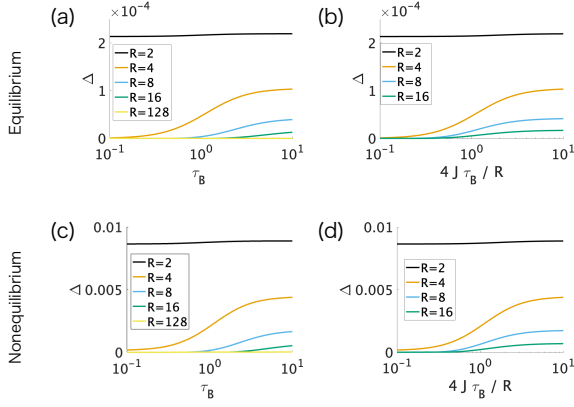


FIG. 5. Error of the steady state described by the local ULE where Lindblad operators act on local subsystems with the radius  $R$ . The parameters are set to be  $L = 128$ ,  $N = 16$ ,  $\tau_{SB} = 100$ ,  $J = 1$  and  $\omega_0 = 0$ . (a) Case of an equilibrium steady state, where the inverse temperatures of the baths are set to be equal as  $\beta_l = \beta_r = 0.1$ . There is a nonzero error in the steady state of the  $R = 2$  local ULE even in the limit of  $\tau_B \rightarrow 0$ , where the local ULE reduces to the ULE. (b) The same quantity as in (a) with only the horizontal axis rescaled. We see that the error becomes small in the region  $4J\tau_B/R < 1$ . (c) Case of a nonequilibrium steady state, where the inverse temperatures of the baths are  $\beta_l = 0.5$  and  $\beta_r = 0.1$ . Errors are larger than those in the equilibrium case. (d) The same quantity as in (c) with only the horizontal axis rescaled. We see that the local ULE can describe the steady state with a sufficiently small error in the region  $4J\tau_B/R < 1$ .

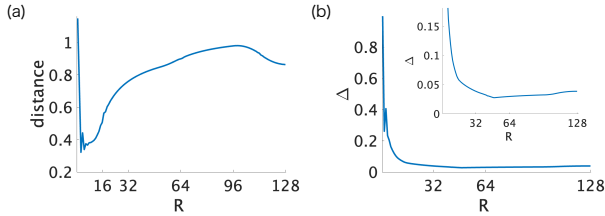


FIG. 6. Errors in the generator (a) and the steady state (b) of the local Davies equation plotted against the radius  $R$  of the local subsystem. The parameters are set to be  $L = 128$ ,  $N = 16$ ,  $\tau_{SB} = 10$ ,  $\tau_B = 1$ ,  $\beta_l = 0.5$ ,  $\beta_r = 0.1$ ,  $J = 1$  and  $\omega_0 = 0$ . As  $R$  increase, both errors initially decrease and then begin to increase at certain values of  $R$ . For (b), this can be clearly seen in the inset.

ately depending on whether one focuses on the dynamics or the steady state.

## V. DISCUSSION

We discuss the advantages of the local GKSL equations from the viewpoint of the numerical analysis of open quantum many-body systems compared with the existing GKSL equations. To numerically simulate the dynamics of an open quantum many-body system, for

practical purposes, it is desirable that the QME satisfies the following three conditions: (i) The QME preserves the positive semidefiniteness of the density matrix, i.e., it takes a GKSL form as Eq. (1). As a consequence, the QME can be efficiently solved by the Monte Carlo method [1, 2]. (ii) The number of Lindblad operators of the QME is small enough to be numerically tractable, since we have to calculate the probabilities of quantum jumps caused by the Lindblad operators at each iteration of the Monte Carlo method [2]. For example, in cases where the number of Lindblad operators grows exponentially with respect to the system size, it is necessary to compute the probabilities associated with an exponential number of quantum jumps at each iteration of the Monte Carlo method. This exponential growth requires a large numerical cost for a large system size. (iii) The numerical cost for the computation of the Lindblad operators in Eqs. (13) and (15) is not so high. For example, it is practically impossible to compute Lindblad operators of the Davies equation and the ULE in cases where the diagonalization of the Hamiltonian is not feasible, such as in many-body systems [50].

We first discuss whether or not each of these three conditions holds in the existing microscopically derived GKSL equations introduced in Sec. II. All GKSL equations satisfy condition (i). However, the Davies equation does not satisfy conditions (ii) and (iii). The Lindblad operators in Eq. (13) represent the transitions between energy levels. Therefore, there are as many Lindblad operators as possible energy transitions due to interactions with the bath, the number of which is exponentially large in many-body systems. In addition, to obtain Lindblad operators, we need to diagonalize the Hamiltonian  $H_S$  of the many-body system of interest, find the eigenvalue  $E_n$  and its eigenstate  $|E_n\rangle$ , and calculate  $A_\mu(\omega)$ . On the other hand, the ULE satisfies the condition (ii) because the number of Lindblad operators is not more than the number of interaction terms in Eq. (2). However, it is still necessary to diagonalize the Hamiltonian of the entire many-body system to calculate the Lindblad operators in Eq. (15) as in the Davies equation. Therefore, the ULE does not satisfy condition (iii).

In contrast, the local GKSL equations satisfy all these three conditions (i)-(iii). Owing to the locality, the number of Lindblad operators representing the effect from a bath is at most limited to the dimension of the space of those operators that act on the local subsystem, and the cost of computing the Lindblad operators is as small as the cost of diagonalizing the operators acting on the local subspace. The computational cost required for computing the dynamics of the system by the Monte Carlo method is given by  $M \times D^2$ , where  $D$  is the dimension of the Hilbert space of the system and  $M$  is the number of sampled quantum trajectories [2]. Here,  $M$  is smaller than  $D$  for most cases [2]. In the case of the GKSL equation which requires the diagonalization of the Hamiltonian of the entire system, the computational cost of diagonalization is of the order of  $D^3$ . Thus, the total computational cost for computing the dynamics with

such a GKSL equation is also of the order of  $D^3$ . On the other hand, in the case of the local GKSL equations based on the Lieb-Robinson bound, diagonalization of the Hamiltonian of the local subsystem is required to compute the Lindblad operators, and the computational cost stays roughly the same with respect to the system size since the dimension of the local subsystem is independent from the system size. Therefore, the total computational cost for computing the dynamics with the local GKSL equations is of the order of  $M \times D^2$ . Thus, the use of the local GKSL equations reduces the computational cost in the numerical analysis of open quantum many-body systems.

## VI. CONCLUSION

We have proposed a method of microscopic derivation of the local GKSL equation by combining the localization of the Redfield equation on the basis of the Lieb-Robinson bound and the existing derivations of the GKSL equation [34, 35, 39–53]. While other derivations of the local GKSL equation [34–36, 53] treat the couplings between sites perturbatively, the causal structure given by the Lieb-Robinson bound makes the derivation valid for more general cases. This derivation shows that the locality of the Lindblad operators is determined by the relaxation time  $\tau_B$  of the bath and the propagation velocity  $\zeta_0$  of the system. In this paper, we have introduced the local Davies equation and the local ULE; the former is derived from the local Redfield equation with the rotating-wave approximation, and the latter is derived from the local Redfield equation with the derivation of the ULE. The different local GKSL equations can be used depending on the purpose of applications. The local Davies equation can be used for thermodynamic settings, such as the case where dissipation due to a bath satisfies the detailed-balance condition for each subsystem. The local ULE can be used to accurately describe a steady state as well as the dynamics.

By the numerical calculations for the one-dimensional tight-binding fermion chain, we have shown that the er-

rors in the generators and in the steady state of the local GKSL equations become small when localization of the Redfield equation is performed so as to be compatible with the Lieb-Robinson bound, i.e., the radius of the local subsystem on which Lindblad operators act is taken to be larger than  $\zeta_0\tau_B$ . These numerical results support the validity of our derivation. However, since small errors in the generators may accumulate over a long time, a small error in the generators does not necessarily guarantee a small error in the steady state. While we have numerically shown that the errors in the steady state are indeed small for some cases, it is still an open problem how the behavior of errors in a steady state can be understood from the microscopic derivation.

The local GKSL equation is a QME that can efficiently analyze open quantum many-body systems. However, the local GKSL equation previously discussed in literature [19, 29] has been given phenomenologically and does not reflect details of baths such as relaxation times. Our microscopic derivation allows us to efficiently analyze open quantum many-body systems in such a manner as to accommodate detailed properties of surrounding baths.

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## Appendix A: Derivation of the universal Lindblad equation in the frequency domain

For the sake of self-containedness, we provide a derivation of the universal Lindblad equation (ULE) [50]. While the original derivation in Ref. [50] is performed in the time domain, here we present another derivation in the frequency domain. In the derivation of the ULE, we only assume that the time scale  $\tau_{SB}$  of the time evolution caused by the bath is much larger than the relaxation time  $\tau_B$  of the bath i.e.,  $\tau_{SB} \gg \tau_B$ , — this condition is assumed to derive the Redfield equation. Therefore, the ULE can describe the dynamics of an open system within the error of the same order as that of the Redfield equation [50].

Let us begin with Eq. (6). If the condition  $\tau_{SB} \gg \tau_B$  is satisfied, there always exists an intermediate time scale  $\Delta t$  such that  $\tau_B \ll \Delta t \ll \tau_{SB}$ . To derive the ULE, we modify the terms in Eq. (6) with  $|\omega - \omega'| > \omega_c = \Delta t^{-1}$  so that the dynamics of the obtained GKSL equation does not deviate from that of the Redfield equation in the coarse-grained time scale larger than  $\Delta t$ . The idea of the derivation presented here is fundamentally the same as that in Ref. [51], with a slightly different definition of the Lamb-shift Hamiltonian.

Expanding the Lamb-shift term in Eq. (6), we obtain

$$\begin{aligned}
\frac{d}{dt}\rho = & -i[H_S, \rho] + \sum_{\mu, \nu} \sum_{\omega, \omega'} \left[ \frac{\gamma_{\mu\nu}(\omega) + \gamma_{\mu\nu}(\omega')}{2} + i(\eta_{\mu\nu}(\omega) - \eta_{\mu\nu}(\omega')) \right] \left( A_\nu(\omega) \rho A_\mu^\dagger(\omega') - \frac{1}{2} \{A_\mu^\dagger(\omega') A_\nu(\omega), \rho\} \right) \\
& + i \sum_{\mu, \nu} \sum_{\omega, \omega'} \frac{1}{2} \left( \eta_{\mu\nu}(\omega') + i \frac{\gamma_{\mu\nu}(\omega) - \gamma_{\mu\nu}(\omega')}{2} \right) (A_\mu^\dagger(\omega') A_\nu(\omega) \rho - A_\nu(\omega) \rho A_\mu^\dagger(\omega')) \\
& + i \sum_{\mu, \nu} \sum_{\omega, \omega'} \frac{1}{2} \left( \eta_{\mu\nu}(\omega) + i \frac{\gamma_{\mu\nu}(\omega) - \gamma_{\mu\nu}(\omega')}{2} \right) (A_\nu(\omega) \rho A_\mu^\dagger(\omega') - \rho A_\mu^\dagger(\omega') A_\nu(\omega)).
\end{aligned} \tag{A1}$$

Here, we approximate the coefficients composed of  $\gamma(\omega)$  and  $\eta(\omega)$  as follows:

$$\frac{\gamma_{\mu\nu}(\omega) + \gamma_{\mu\nu}(\omega')}{2} \simeq \sum_{\lambda} \gamma_{\mu\lambda}^{1/2}(\omega) \gamma_{\lambda\nu}^{1/2}(\omega'), \tag{A2}$$

$$\eta_{\mu\nu}(\omega), \eta_{\mu\nu}(\omega') \simeq \eta_{\mu\nu} \left( \frac{\omega + \omega'}{2} \right). \tag{A3}$$

The square root  $\gamma_{\mu\nu}^{1/2}$  of the spectral density function is defined as

$$\gamma_{\mu\nu}(\omega) = \sum_{\lambda} \gamma_{\mu\lambda}^{1/2}(\omega) \gamma_{\lambda\nu}^{1/2}(\omega). \tag{A4}$$

Since we approximate the spectral density function using its square root, this approach to deriving the ULE is called the “ $\sqrt{\text{SD}}$ -approach” in Ref. [51]. This approximation is justified in the following two situations. The first situation is that the condition  $|\omega - \omega'| \lesssim \Delta t^{-1} \ll \tau_B^{-1}$  holds so that  $\gamma_{\mu\nu}(\omega) \simeq \gamma_{\mu\nu}(\omega')$  and  $\eta_{\mu\nu}(\omega) \simeq \eta_{\mu\nu}(\omega')$ . In fact,

$$\begin{aligned}
\gamma_{\mu\nu}(\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} C_{\mu\nu}(t) \\
&\simeq \int_{-\sqrt{\tau_B \Delta t}}^{\sqrt{\tau_B \Delta t}} dt e^{i\omega t} C_{\mu\nu}(t) \quad (C(t) \sim 0 \text{ for } t > \sqrt{\tau_B \Delta t} \gg \tau_B) \\
&= \int_{-\sqrt{\tau_B \Delta t}}^{\sqrt{\tau_B \Delta t}} dt e^{i\omega' t} e^{i(\omega - \omega')t} C_{\mu\nu}(t) \\
&\simeq \int_{-\sqrt{\tau_B \Delta t}}^{\sqrt{\tau_B \Delta t}} dt e^{i\omega' t} C_{\mu\nu}(t) \quad (e^{i(\omega - \omega')t} \simeq 1 \text{ because } |(\omega - \omega')t| \ll 1 \text{ at } |t| \ll \Delta t.) \\
&\simeq \int_{-\infty}^{\infty} dt e^{i\omega' t} C_{\mu\nu}(t) = \gamma_{\mu\nu}(\omega').
\end{aligned} \tag{A5}$$

Since  $\gamma_{\mu\nu}(\omega) \simeq \gamma_{\mu\nu}(\omega')$  holds,  $\eta_{\mu\nu}(\omega) \simeq \eta_{\mu\nu}(\omega')$  also holds for  $|\omega - \omega'| \lesssim \Delta t^{-1} \ll \tau_B^{-1}$  and the approximation in Eq. (A3) is also justified. The second situation is that  $|\omega - \omega'| \gtrsim \Delta t^{-1}$  holds so that the effect of such terms is canceled out after the coarse graining of time in the scale of  $\Delta t$ . In this sense, the dynamics described by the obtained GKSL equation does not deviate from that described by the Redfield equation up to the coarse graining of time.

With this approximation, Eq. (A1) can be transformed as

$$\begin{aligned}
\frac{d}{dt}\rho = & -i[H_S, \rho] + \sum_{\lambda} \sum_{\mu, \nu} \sum_{\omega, \omega'} \gamma_{\mu\lambda}^{1/2}(\omega) \gamma_{\lambda\nu}^{1/2}(\omega') \left( A_\nu(\omega') \rho A_\mu^\dagger(\omega) - \frac{1}{2} \{A_\mu^\dagger(\omega) A_\nu(\omega'), \rho\} \right) \\
& + i \sum_{\mu, \nu} \sum_{\omega, \omega'} \left( \eta_{\mu\nu} \left( \frac{\omega + \omega'}{2} \right) + i \frac{\gamma_{\mu\nu}(\omega) - \gamma_{\mu\nu}(\omega')}{4} \right) (A_\mu^\dagger(\omega) A_\nu(\omega') \rho - \rho A_\mu^\dagger(\omega) A_\nu(\omega')).
\end{aligned} \tag{A6}$$

By defining the Lindblad operators  $L_\lambda$  and the Lamb-shift Hamiltonian  $H_{\text{LS}}$  as

$$L_\lambda = \sum_{\nu} \sum_{\omega} \gamma_{\lambda\nu}^{1/2}(\omega) A_\nu(\omega), \tag{A7}$$

$$H_{\text{LS}} = \sum_{\mu, \nu} \sum_{\omega, \omega'} \left( \eta_{\mu\nu} \left( \frac{\omega + \omega'}{2} \right) + i \frac{\gamma_{\mu\nu}(\omega) - \gamma_{\mu\nu}(\omega')}{4} \right) A_\mu^\dagger(\omega) A_\nu(\omega'), \tag{A8}$$

Eq. (A6) is written in the following GKSL form:

$$\frac{d}{dt}\rho = -i[H_S + H_{LS}, \rho] + \sum_{\lambda=1}^k \left( L_{\lambda}\rho L_{\lambda}^{\dagger} - \frac{1}{2}\{L_{\lambda}^{\dagger}L_{\lambda}, \rho\} \right). \quad (\text{A9})$$

According to the derivation in Ref. [50], the Lindblad operators of the ULE are given by

$$L_{\lambda} = \sum_{\nu} \int_{-\infty}^{\infty} ds g_{\lambda\nu}(-s) A_{\nu}(s), \quad (\text{A10})$$

where  $g_{\mu\nu}(s)$  is defined as the Fourier transformation of  $\gamma_{\mu\nu}^{1/2}(\omega)$ :

$$g_{\lambda\nu}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\lambda\nu}^{1/2}(\omega) e^{-i\omega s} d\omega. \quad (\text{A11})$$

The Lindblad operators in Eq. (A10) are equivalent to those in Eq. (A7) since

$$\begin{aligned} L_{\lambda} &= \int_{-\infty}^{\infty} ds \sum_{\nu} g_{\lambda\nu}(-s) A_{\nu}(s) \\ &= \sum_{m,n} \int_{-\infty}^{\infty} ds \sum_{\nu} g_{\lambda\nu}(-s) e^{i(E_n - E_m)s} \langle E_m | A_{\nu} | E_n \rangle | E_m \rangle \langle E_n | \\ &= \sum_{\omega} \int_{-\infty}^{\infty} ds \sum_{\nu} g_{\lambda\nu}(-s) e^{i\omega s} A_{\nu}(\omega) \\ &= \sum_{\omega} \sum_{\nu} \gamma_{\lambda\nu}^{1/2}(\omega) A_{\nu}(\omega). \end{aligned} \quad (\text{A12})$$

By writing the Lindblad operators of the ULE as in Eq. (A10), we can also derive the local ULE by replacing  $A_{\mu}(s)$  with  $A_{\mu}^{\text{loc}}(s)$  in the Lindblad operators (A10). In this sense the local ULE can also be considered as the approximation of the original ULE. Therefore, the local ULE becomes more accurate for a larger subsystem  $\Omega_{\mu}$  in contrast to the local Davies equation. However, the computational costs of calculating the Lindblad operators and the Lamb-shift Hamiltonian increase as the size of the subsystem  $\Omega_{\mu}$  increases.

## Appendix B: Efficient calculation of the largest eigenvalue of a dissipator

In this appendix, we show that the largest eigenvalue of a dissipator of the quadratic open fermionic system can efficiently be calculated. In the quadratic open fermionic system, the QME can be represented as in Eq. (46). The problem of finding the steady state of this system is then reduced to diagonalizing the matrix  $A$  [57, 59], whose nonzero components are given by

$$\begin{aligned} A_{2m-1, 2n-1} &= -2iH_{m,n} - M_{m,n} + M_{n,m}, & A_{2m, 2n} &= -2iH_{m,n} + M_{m,n} - M_{n,m}, \\ A_{2m-1, 2n} &= iM_{n,m} + iM_{m,n}, & A_{2m, 2n-1} &= -iM_{m,n} - iM_{n,m}. \end{aligned} \quad (\text{B1})$$

Here,  $H = (H_{m,n})$  is an antisymmetric Hermitian matrix which satisfies  $H_S = \sum_{m,n} H_{m,n} w_m w_n$ . Since  $A$  is antisymmetric, eigenvalues of  $A$  are expressed as  $\beta_j, -\beta_j$ , ( $j = 1, \dots, 2L$ ), where  $\text{Re } \beta_j \geq 0$ .

Let  $\mathcal{L}[\rho]$  be given by the right-hand side of Eq. (46). We assume that the real part of each eigenvalue of  $\mathcal{L}$  is negative except for a single zero eigenvalue, which corresponds to a steady state. Here, we focus on the dynamics in subspace  $\mathcal{K}^+$  composed of an even number of fermionic operators  $w_m$  since we are interested in the expectation values of the product of the even number of  $w_m$ 's such as  $W_{mn} = \langle w_m w_n \rangle$ . For the precise definition of  $\mathcal{K}^+$ , see Refs. [57, 59]. Let  $\mathcal{P}_+$  be a projector onto  $\mathcal{K}^+$ . The dynamics in subspace  $\mathcal{K}^+$  is described by  $\mathcal{L}_+ := \mathcal{P}_+ \mathcal{L} \mathcal{P}_+$ . Note that the generator  $\mathcal{L}$  is block diagonalized as  $\mathcal{L} = \mathcal{P}_+ \mathcal{L} \mathcal{P}_+ + (1 - \mathcal{P}_+) \mathcal{L} (1 - \mathcal{P}_+)$  [57, 59].

Assuming that the real part of the nonzero eigenvalues of  $\mathcal{L}$  is negative and that  $\mathcal{L}$  has a unique zero eigenvalue,  $\mathcal{L}$  is expressed as

$$\mathcal{L}_+ = -2 \sum_i \beta_i b'_i b_i, \quad (\text{B2})$$

where  $b'_i$  and  $b_i$  play a similar role as the creation and annihilation operators of normal modes and satisfy the anticommutation relations  $\{b'_i, b_j\} = \delta_{ij}$ ,  $\{b_i, b_j\} = \{b'_i, b'_j\} = 0$ . The steady state  $\rho_{\text{steady}}$  of  $\mathcal{L}$  is given by the “vacuum

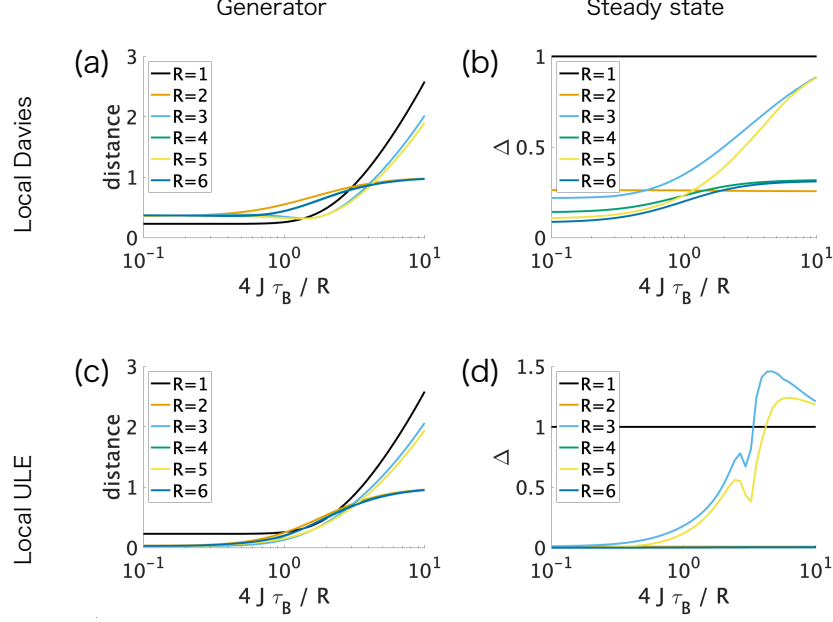


FIG. 7. (a, c) Distance between generators of the local GKSL equations and that of the Redfield equation. (b, d) Errors in the steady states of the local Davies equation and the local ULE. The parameters are set to be  $L = 128$ ,  $N = 16$ ,  $\tau_{SB} = 100$ ,  $J = 1$  and  $\omega_0 = 0$ . The inverse temperatures of the baths are set to be  $\beta_r = 0.5$  and  $\beta_l = 0.1$ . When the relaxation time  $\tau_B$  is small, both the distance and the error become small regardless of whether  $R$  is even or odd except for the case of  $R = 1$ . When  $\tau_B$  is large, the error tends to be large for odd  $R$ . (a) Distance between the generator of the local Davies equation and that of the Redfield equation. (b) Error in the steady state of the local Davies equation. (c) Distance between the generator of the local ULE and that of the Redfield equation. (d) Error in the steady state of the local ULE. The errors for even  $R$  are negligible compared with those for odd  $R$ . The error is larger around  $\tau_B = 1$  when  $R$  is odd while the distances of the generators are comparable regardless of whether  $R$  is even or odd as seen in (c).

stat" satisfying  $b_i \rho_{\text{steady}} = 0$  for all  $i$ . See Ref. [57] for the details. Therefore, the eigenvalues  $\lambda_\nu$  of  $\mathcal{L}_+$  are given by all the possible binary linear combinations of  $\beta_j$ 's:

$$\lambda_\nu = -2 \sum_j v_{\nu,j} \beta_j, \quad v_{\nu,j} \in \{0, 1\}. \quad (\text{B3})$$

To evaluate  $\tau_{SB}$ , we need to have the eigenvalues of the dissipator  $\mathcal{D}$  defined in Eq. (11). They are obtained by setting  $H_{m,n} = 0$  in Eq. (B1). For the GKSL equation,  $M_{m,n}$  is a Hermitian matrix. Therefore, when  $H_{m,n} = 0$ , the matrix  $A$  corresponding to  $\mathcal{D}$  is also a Hermitian matrix and all the eigenvalues of  $A$  are real. Since  $\text{Re } \beta_j \geq 0$ , the eigenvalue  $\lambda_{\max}$  with the largest absolute value of  $\mathcal{D}_+ := \mathcal{P}_+ \mathcal{D} \mathcal{P}_+$  is given by

$$\lambda_{\max} = -2 \sum_j \beta_j, \quad (\text{B4})$$

if the real part of each nonzero eigenvalue of  $\mathcal{D}$  is negative. Thus, the eigenvalue with the largest absolute value of a dissipator of QMEs can be calculated efficiently by using the sum rule shown in Ref. [57]:

$$2 \sum_j \beta_j = 2 \text{tr } M. \quad (\text{B5})$$

### Appendix C: Additional numerical results

Here we discuss the errors in the generators and the steady state when the radius  $R$  of the subsystem is given by an odd integer. Compared with the case of even  $R$ , the distance between generators for odd  $R$  becomes larger in the region where  $\tau_B$  is large for both the local Davies equation and the local ULE (see Figs. 7 (a) and (c)). This makes the error in the steady state for odd  $R$  larger in the region where  $\tau_B$  is large (see Figs. 7 (b) and (d)). If  $\tau_B$  is sufficiently

small, the behavior of the errors does not significantly depend on whether  $R$  is even or odd. Since the local GKSL equation is valid when  $\tau_B$  is smaller than  $R/\zeta_0$  (where  $\zeta_0 = 4J$ ), the even-odd dependence on  $R$  in the error for large  $\tau_B$  is not relevant to the validity of the local GKSL equation.

The error of the local GKSL equation with  $R = 1$  is always equal to one because the steady state of the local GKSL equation with  $R = 1$  is a product state and the off-diagonal components of  $W$  vanish when  $\omega_0 = 0$ . Since the diagonal components of  $W$  are always one, the error  $\Delta$  is given by

$$\Delta := \frac{\max_{kl} |W_{kl}^{\text{locGKSL}} - W_{kl}^{\text{Redfield}}|}{\max_{k \neq l} W_{kl}^{\text{Redfield}}} = \frac{\max_{k \neq l} |W_{kl}^{\text{Redfield}}|}{\max_{k \neq l} W_{kl}^{\text{Redfield}}} = 1. \quad (\text{C1})$$

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