The coproduct for the affine Yangian and the parabolic induction for non-rectangular W-algebras

Mamoru Ueda*

Department of Mathematical and Statistical Sciences, University of Alberta, 11324 89 Ave NW, Edmonton, AB T6G 2J5, Canada

Abstract

By using the coproduct and evaluation map for the affine Yangian and the Miura map for non-rectangular W-algebras, we construct a homomorphism from the affine Yangian associated with $\widehat{\mathfrak{sl}}(n)$ to the universal enveloping algebra of a non-rectangular W-algebra of type A, which is an affine analogue of the one given in De Sole-Kac-Valeri [6]. As a consequence, we find that the coproduct for the affine Yangian is compatible with the parabolic induction for non-rectangular W-algebras via this homomorphism. We also show that the image of this homomorphism is contained in the affine coset of the W-algebra in the special case that the W-algebra is associated with a nilpotent element of type $(1^{m-n}, 2^n)$.

1 Introduction

The Yangian $Y_\hbar(\mathfrak{g})$ associated with a finite dimensional simple Lie algebra $\mathfrak{g}$ was introduced by Drinfeld ([7], [8]). The Yangian $Y_\hbar(\mathfrak{g})$ is a quantum group which is a deformation of the current algebra $\mathfrak{g}\otimes\mathbb{C}[z]$. The affine Yangian associated with $\widehat{\mathfrak{sl}}(n)$ was first introduced by Guay ([14], [15]). The affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is a 2-parameter Yangian and is the deformation of the universal enveloping algebra of the central extension of $\mathfrak{sl}(n)[u^{\pm 1}, v]$. Guay-Nakajima-Wendlandt [16] gave the coproduct for the Yangian associated with the Kac-Moody Lie algebra of affine type. The coproduct for the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is a homomorphism satisfying the coassociativity:

$$\Delta^n\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)) \to Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\widehat{\otimes} Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)),$$

where $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\widehat{\otimes} Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is the standard degreewise completion of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\otimes Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$.

Recently, the relationships between affine Yangians and W-algebras have been studied. The W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is a vertex algebra associated with a reductive Lie algebra $\mathfrak{g}$, a nilpotent element $f \in \mathfrak{g}$ and a complex number k. In [28], we constructed a surjective homomorphism from the affine Yangian to the universal enveloping algebra of a rectangular W-algebra of type A, which is a W-algebra associated with $\mathfrak{gl}(ln)$ and a nilpotent element of type (l^n). Kodera and the author [20] gave another proof of the construction of this homomorphism by using the coproduct and evaluation map for the affine Yangian. The evaluation map for the affine Yangian is a non-trivial homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the standard degreewise completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(n)$:

$$\mathrm{ev}^n\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)) \to \mathcal{U}(\widehat{\mathfrak{gl}}(n)),$$

where $\mathcal{U}(\widehat{\mathfrak{gl}}(n))$ is the standard degreewise completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(n)$.

*mueda@ualberta.ca

In [29], we gave a homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the universal enveloping algebra of a non-rectangular W-algebra of type A. However, the proof of [29] is so complicated and we cannot understand what is the meaning of the construction way of [29]. In this article, by the similar way to [20], we give the another proof to [29]. We use the homomorphism given in [25] and [27]:

$$\Psi^{m,m+n}\colon Y_{\hbar,\varepsilon+n\hbar}(\widehat{\mathfrak{sl}}(m))\to Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(m+n)).$$

By using $\Psi^{m,m+n}$, we can construct a homomorphism

$$\Delta^l\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(q_{Min}))\to\bigotimes_{1\leq i\leq l}Y_{\hbar,\varepsilon-(q_i-q_{Min})\hbar}(\widehat{\mathfrak{sl}}(q_i)).$$

We also use the Miura map for the W-algebra. The Miura map was first introduced by Kac-Wakimoto [17]. In type A setting, the Miura map is an embedding from a W-algebra of type A to the tensor product of universal affine vertex algebras associated with $\mathfrak{gl}(n)$. Let us fix positive integers and its partition:

$$N=q_1+q_2+\cdots+q_l,\qquad q_1\leq q_2\leq\cdots\leq q_v,\ q_{v+1}\geq q_{v+2}\geq\cdots\geq q_l.$$

We take a nilpotent element f whose column heights of the diagram is $(q_1,\cdots,q_l)$. Then, the Miura map for the W-algebra associated with $\mathfrak{gl}(N)$ and f is an embedding

$$\mathcal{W}^k(\mathfrak{gl}(N),f)\hookrightarrow\bigotimes_{1\leq i\leq l}V^{\kappa_i}(\mathfrak{gl}(q_i)).$$

The Miura map induces the injective map

$$\widetilde{\mu}\colon\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(N),f))\hookrightarrow\widehat{\bigotimes}_{1\leq i\leq l}U(\widehat{\mathfrak{gl}}(q_i)),$$

where $\widehat{\bigotimes}_{1\leq i\leq l}U(\widehat{\mathfrak{gl}}(q_i))$ is the standard degreewise completion of $\bigotimes_{1\leq i\leq l}U(\widehat{\mathfrak{gl}}(q_i))$.

Theorem 1.1. *For the simplicity, we assume that q_l is the minimum value of $\{q_i\}$. There exists a homomorphism*

$$\Phi\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(q_{Min}))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(N),f))$$

uniquely determined by

$$\widetilde{\mu}\circ\Phi=\bigotimes_{1\leq i\leq l}\mathrm{ev}^{q_i}\circ\Delta^l,$$

where $q_{Min}=\min(q_1,q_l)$.

This theorem gave the relationship between the coproduct for the affine Yangian and the parabolic induction for a non-rectangular W-algebra. Let us consider two W-algebras associated with $(\mathfrak{gl}(N_1),f_1)$ and $(\mathfrak{gl}(N_2),f_2)$, where

$$N_1=q_1+\cdots+q_w,\ N_2=q_{w+1}+\cdots+q_l$$

and nilpotent elements f_1 and f_2 are defined in the same way as f. In [13], Genra constructed a homomorphism called the parabolic induction:

$$\Delta_W\colon\mathcal{W}^k(\mathfrak{gl}(N),f)\to\mathcal{W}^{k+N_2}(\mathfrak{gl}(N_1),f_1)\otimes\mathcal{W}^{k+N_1}(\mathfrak{gl}(N_2),f_2).$$

By Theorem 1.1, we obtain two homomorphisms

$$\Phi_1\colon Y_{\hbar,\varepsilon+(\min(q_1,q_w)-q_{Min})\hbar}(\widehat{\mathfrak{sl}}(\min(q_1,q_w)))\to\mathcal{U}(\mathcal{W}^{k+N_2}(\mathfrak{gl}(N_1),f_1)),$$
$$\Phi_2\colon Y_{\hbar,\varepsilon+(\min(q_{w+1},q_l)-q_{Min})\hbar}(\widehat{\mathfrak{sl}}(\min(q_{w+1},q_l)))\to\mathcal{U}(\mathcal{W}^{k+N_1}(\mathfrak{gl}(N_2),f_2)).$$

By the construction way in Theorem 1.1, we obtain the following relations:

$$(\Phi_1 \circ \tau_{-\gamma_{w+1}\hbar} \circ \Psi^{q_l,\min(q_1,q_w)}) \otimes \Phi_2 \circ \Delta = \Delta_W \circ \Phi \text{ if } q_1 \geq q_l,$$
$$.(\Phi_1 \circ \tau_{-\gamma_{w+1}\hbar}) \otimes (\Phi_2 \circ \Psi^{q_1,\min(q_l,q_{w+1})}) \circ \Delta = \Delta_W \circ \Phi \text{ if } q_1 < q_l.$$

where $\tau_{-\gamma_{w+1}\hbar}$ is a shift operator of the affine Yangian.

The motivation of this theorem is the generalization of the AGT conjecture. The AGT conjecture suggests that there exists a representation of the principal W-algebra of type A on the equivariant homology space of the moduli space of $U(r)$-instantons. Schiffmann and Vasserot [23] gave this representation by using an action of the Yangian associated with $\widehat{\mathfrak{gl}}(1)$ on this equivariant homology space. It is conjectured by Crutzig-Diaconescu-Ma [3] that an action of an iterated W-algebra of type A on the equivariant homology space of the affine Laumon space will be given through an action of an affine shifted Yangian constructed in [9]. For the resoulution of the Crutzig-Diaconescu-Ma's conjecture, we need to construct a homomorphism from the shifted affine Yangian to the universal enveloping algebra of a iterated W-algebra. However, the shifted affine Yangian is so complicated that we cannot directly construct this homomorphism.

In finite setting, Brundan-Kleshchev [2] gave a surjective homomorphism from a shifted Yangian, which is a subalgebra of the finite Yangian associated with $\mathfrak{gl}(n)$, to a finite W-algebra ([22]) of type A for its general nilpotent element. A finite W-algebra $\mathcal{W}^{\text{fin}}(\mathfrak{g}, f)$ is an associative algebra associated with a reductive Lie algebra $\mathfrak{g}$ and its nilpotent element f and is a finite analogue of a W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ ([5] and [1]). In [6], De Sole, Kac and Valeri constructed a homomorphism from the finite Yangian of type A to the finite W-algebras of type A by using the Lax operator, which is a restriction of the homomorphism given by Brundan-Kleshchev in [2]. Actually, the homomorphism Φ is the affine analogue of the De Sole-Kac-Valeri's homomorphism (see [26]). Thus, we expect that we can extend Φ to the shifted affine Yangian and solve the Crutzig-Diaconescu-Ma's conjecture.

At last of this article, we give a new interpretation of the homomorphism Φ in the case that $l = 2$. In this case, we have already computed all of the OPEs in [24]. By using the OPEs, we find that there exists a natural embedding from the universal affine vertex algebra associated with $\mathfrak{gl}(q_1 - q_2)$ to the W-algebra. By a direct computation, we find that the image of Φ is contained in the universal enveloping algebra of the coset of the pair of the universal affine vertex algebra and the W-algebra. The coset of the pair of the pair of the universal affine vertex algebra and the W-algebra is connected to Y-algebras and the Gaiotto-Rapcak's triality (see [11] and [4]). We expect that the homomorphism Φ will be helpful for the generalization of the Gaiotto-Rapcak's triality.

2 Affine Yangian

Let us recall the definition of the affine Yangian of type A (Definition 3.2 in [14] and Definition 2.3 in [15]). Hereafter, we sometimes identify $\{0, 1, 2, \cdots, n-1\}$ with $\mathbb{Z}/n\mathbb{Z}$. we also set$\{X, Y\} = XY + YX$ and

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } j = i \pm 1, \\ 0 & \text{otherwise} \end{cases}$$

for $i \in \mathbb{Z}/n\mathbb{Z}$.

Definition 2.1. Suppose that $n \geq 3$. The affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is the associative algebra generated by $X^+_{i,r}, X^-_{i,r}, H_{i,r}$ $(i \in \{0, 1, \cdots, n-1\}, r = 0, 1)$ subject to the following defining relations:

$$[H_{i,r}, H_{j,s}] = 0, \tag{2.2}$$
$$[X^+_{i,0}, X^-_{j,0}] = \delta_{i,j} H_{i,0}, \tag{2.3}$$

$$[X^+_{i,1}, X^-_{j,0}] = \delta_{i,j} H_{i,1} = [X^+_{i,0}, X^-_{j,1}], \tag{2.4}$$
$$[H_{i,0}, X^{\pm}_{j,r}] = \pm a_{i,j} X^{\pm}_{j,r}, \tag{2.5}$$
$$[\tilde{H}_{i,1}, X^{\pm}_{j,0}] = \pm a_{i,j} \left(X^{\pm}_{j,1}\right), \text{ if } (i,j) \neq (0,n-1),(n-1,0), \tag{2.6}$$
$$[\tilde{H}_{0,1}, X^{\pm}_{n-1,0}] = \mp \left(X^{\pm}_{n-1,1} + (\varepsilon + \frac{n}{2}\hbar) X^{\pm}_{n-1,0}\right), \tag{2.7}$$
$$[\tilde{H}_{n-1,1}, X^{\pm}_{0,0}] = \mp \left(X^{\pm}_{0,1} - (\varepsilon + \frac{n}{2}\hbar) X^{\pm}_{0,0}\right), \tag{2.8}$$
$$[X^{\pm}_{i,1}, X^{\pm}_{j,0}] - [X^{\pm}_{i,0}, X^{\pm}_{j,1}] = \pm a_{ij} \frac{\hbar}{2} \{X^{\pm}_{i,0}, X^{\pm}_{j,0}\} \text{ if } (i,j) \neq (0,n-1),(n-1,0), \tag{2.9}$$
$$[X^{\pm}_{0,1}, X^{\pm}_{n-1,0}] - [X^{\pm}_{0,0}, X^{\pm}_{n-1,1}] = \mp \frac{\hbar}{2} \{X^{\pm}_{0,0}, X^{\pm}_{n-1,0}\} + (\varepsilon + \frac{n}{2}\hbar)[X^{\pm}_{0,0}, X^{\pm}_{n-1,0}], \tag{2.10}$$
$$(\mathrm{ad}\, X^{\pm}_{i,0})^{1+|a_{i,j}|}(X^{\pm}_{j,0}) = 0 \ \text{ if } i \neq j, \tag{2.11}$$

where we set $\widetilde{H}_{i,1} = H_{i,1} - \frac{\hbar}{2} H^2_{i,0}$.

By (2.3), (2.4), (2.6) and (2.8), we find that the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is generated by $X^{\pm}_{i,0}$ for $0 \leq i \leq n-1$ and $H_{j,1}$ for $1 \leq j \leq n-1$.

Let us take Chevalley generators of $\widehat{\mathfrak{sl}}(n)$ and denote them by $\{h_i, x^{\pm}_i \mid 0 \leq i \leq n-1\}$. By the defining relations (2.2)-(2.11), we can give a homomorphism from the universal enveloping algebra of $\widehat{\mathfrak{sl}}(n)$ to the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ given by $h_i \mapsto H_{i,0}$ and $x^{\pm}_i \mapsto X^{\pm}_{i,0}$. We denote the image of $x \in Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ via this homomorphism by x.

We take one completion of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$. We set the degree of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ by

$$\deg(H_{i,r}) = 0,\ \deg(X^{\pm}_{i,r}) = \begin{cases} \pm 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

This degree is compatible with the natural degree on the universal enveloping algebra of $\widehat{\mathfrak{sl}}(n)$. We denote the standard degreewise completion of $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ by $\widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$.

3 The coproduct for the affine Yangian

By using the minimalistic presentation of the Yangian, Guay-Nakajima-Wendlandt [16] gave a coproduct for the Yangian associated with a Kac-Moody Lie algebra of the affine type.

Theorem 3.1 (Theorem 5.2 in [16])**.** *There exists an algebra homomorphism*

$$\Delta^{\pm} \colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)) \to Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)) \widehat{\otimes} Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$$

determined by

$$\Delta^{\pm}(X^{\pm}_{j,0}) = X^{\pm}_{j,0} \otimes 1 + 1 \otimes X^{\pm}_{j,0} \ \textit{for } 0 \leq j \leq n-1,$$
$$\Delta^{\pm}(\widetilde{H}_{i,1}) = \widetilde{H}_{i,1} \otimes 1 + 1 \otimes \widetilde{H}_{i,1} + A^{\pm}_i \ \textit{for } 1 \leq i \leq n-1,$$

where $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)) \widehat{\otimes} Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ is the standard degreewise completion of $\otimes^2 Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ and

$$\begin{aligned} A^+_i &= -\hbar(E_{i,i} \otimes E_{i+1,i+1} + E_{i+1,i+1} \otimes E_{i,i}) \\ &\quad + \hbar \sum_{s \geq 0} \sum_{u=1}^{i} (-E_{u,i} t^{-s-1} \otimes E_{i,u} t^{s+1} + E_{i,u} t^{-s} \otimes E_{u,i} t^{s}) \\ &\quad + \hbar \sum_{s \geq 0} \sum_{u=i+1}^{n} (-E_{u,i} t^{-s} \otimes E_{i,u} t^{s} + E_{i,u} t^{-s-1} \otimes E_{u,i} t^{s+1}) \end{aligned}$$

$$
\begin{aligned}
&- \hbar\sum_{s\geq 0}\sum_{u=1}^{i}(-E_{u,i+1}t^{-s-1}\otimes E_{i+1,u}t^{s+1}+E_{i+1,u}t^{-s}\otimes E_{u,i+1}t^{s})\\
&- \hbar\sum_{s\geq 0}\sum_{u=i+1}^{n}(-E_{u,i+1}t^{-s}\otimes E_{i+1,u}t^{s}+E_{i+1,u}t^{-s-1}\otimes E_{u,i+1}t^{s+1}),\\
A_i^- &= -\hbar(E_{i,i}\otimes E_{i+1,i+1}+E_{i+1,i+1}\otimes E_{i,i})\\
&+ \hbar\sum_{s\geq 0}\sum_{u=1}^{i}(-E_{i,u}t^{s+1}\otimes E_{u,i}t^{-s-1}+E_{u,i}t^{s}\otimes E_{i,u}t^{-s})\\
&+ \hbar\sum_{s\geq 0}\sum_{u=i+1}^{n}(-E_{i,u}t^{s}\otimes E_{u,i}t^{-s}+E_{u,i}t^{s+1}\otimes E_{i,u}t^{-s-1})\\
&- \hbar\sum_{s\geq 0}\sum_{u=1}^{i}(-E_{i+1,u}t^{s+1}\otimes E_{u,i+1}t^{-s-1}+E_{u,i+1}t^{s}\otimes E_{i+1,u}t^{-s})\\
&- \hbar\sum_{s\geq 0}\sum_{u=i+1}^{n}(-E_{i+1,u}t^{s}\otimes E_{u,i+1}t^{-s}+E_{u,i+1}t^{s+1}\otimes E_{i+1,u}t^{-s-1}).
\end{aligned}
$$

The homomorphism Δ^n is said to be the coproduct for the affine Yangian since Δ^n satisfies the coassociativity.

4 The evaluation map for the affine Yangian

The evaluation map for the affine Yangian is a non-trivial homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the completion of the universal enveloping algebra of the affinization of $\mathfrak{gl}(n)$. We set a Lie algebra

$$\widehat{\mathfrak{gl}}(n)=\mathfrak{gl}(n)\otimes\mathbb{C}[z^{\pm 1}]\oplus\mathbb{C}\tilde{c}\oplus\mathbb{C}z$$

whose commutator relations are given by

$$
\begin{gathered}
z \text{ and } \tilde{c} \text{ are central elements of } \widehat{\mathfrak{gl}}(n),\\
[E_{i,j}\otimes t^u,E_{p,q}\otimes t^v]=(\delta_{j,p}E_{i,q}-\delta_{i,q}E_{p,j})\otimes t^{u+v}+\delta_{u+v,0}u(\delta_{i,q}\delta_{j,p}\widetilde{c}+\delta_{i,j}\delta_{p,q}z).
\end{gathered}
$$

We take the grading of $U(\widehat{\mathfrak{gl}}(n))/U(\widehat{\mathfrak{gl}}(n))(z-1)$ as $\deg(Xt^s)=s$ and $\deg(\tilde{c})=0$. We denote the degreewise completion of $U(\widehat{\mathfrak{gl}}(n))/U(\widehat{\mathfrak{gl}}(n))(z-1)$ by $\mathcal{U}(\widehat{\mathfrak{gl}}(n))$.

Theorem 4.1 (Theorem 3.8 in [19] and Theorem 4.18 in [18]). *Suppose that $\tilde{c}=\dfrac{\varepsilon}{\hbar}$. For $a\in\mathbb{C}$, there exists an algebra homomorphism*

$$\mathrm{ev}_{\hbar,\varepsilon}^{n,a}\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\to\mathcal{U}(\widehat{\mathfrak{gl}}(n))$$

uniquely determined by

$$\mathrm{ev}_{\hbar,\varepsilon}^{n,a}(X_{i,0}^+)=\begin{cases}E_{n,1}t & \text{if } i=0,\\ E_{i,i+1} & \text{if } 1\leq i\leq n-1,\end{cases}\quad \mathrm{ev}_{\hbar,\varepsilon}^{n,a}(X_{i,0}^-)=\begin{cases}E_{1,n}t^{-1} & \text{if } i=0,\\ E_{i+1,i} & \text{if } 1\leq i\leq n-1,\end{cases}$$

and

$$
\begin{aligned}
\mathrm{ev}_{\hbar,\varepsilon}^{n,a}(H_{i,1})&=(a-\frac{i}{2}\hbar)\,\mathrm{ev}_{\hbar,\varepsilon}^{n}(H_{i,0})-\hbar E_{i,i}E_{i+1,i+1}\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=1}^{i}E_{i,k}t^{-s}E_{k,i}t^{s}+\hbar\sum_{s\geq 0}\sum_{k=i+1}^{n}E_{i,k}t^{-s-1}E_{k,i}t^{s+1}
\end{aligned}
$$

$$-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}E_{i+1,k}t^{-s}E_{k,i+1}t^{s}-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{n}E_{i+1,k}t^{-s-1}E_{k,i+1}t^{s+1}$$

for $i\neq 0$.

We note that the universal enveloping algebra of the universal affine vertex algebra associated with $\mathfrak{gl}(n)$ coincides with the stadard degreewise completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(n)$. Also, the universal affine vertex algebra is a W-algebra associated with a nilpotent element 0. Thus, the evaluation map is the easiest example of a homomorphism from the affine Yangian to the universal enveloping algebra of a W-algebra.

5 A homomorphism from the affine Yangian $Y_{\hbar,\varepsilon+m\hbar}(\widehat{\mathfrak{sl}}(n))$ to the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+m))$

In [25], we gave a homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$. In [27]. we gave another homomorphism from the affine Yangian $Y_{\hbar,\varepsilon+\hbar}(\widehat{\mathfrak{sl}}(n))$ to the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$.

Theorem 5.1 (Theorem 3.1 in [27])**.** *For* $m\geq 3$ *and* $n\geq 1$*, there exists a homomorphism*

$$\Psi^{m,m+n}\colon Y_{\hbar,\varepsilon+n\hbar}(\widehat{\mathfrak{sl}}(m))\to\widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(m+n))$$

determined by

$$\Psi^{m,m+n}(X^+_{i,0})=\begin{cases}E_{m+n,n+1}t & \text{if } i=0,\\ E_{n+i,n+i+1} & \text{if } i\neq 0,\end{cases}\quad \Psi^{m,m+n}(X^-_{i,0})=\begin{cases}E_{n+1,m+n}t^{-1} & \text{if } i=0,\\ E_{n+i+1,n+i} & \text{if } i\neq 0,\end{cases}$$

and

$$\begin{aligned}\Psi^{m,m+n}(H_{i,1})&=H_{i+n,1}+\hbar\sum_{s\geq 0}\sum_{k=1}^{n}E_{k,n+i}t^{-s-1}E_{n+i,k}t^{s+1}\\&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{n}E_{k,n+i+1}t^{-s-1}E_{n+i+1,k}t^{s+1}\end{aligned}$$

for $i\neq 0$.

6 W-algebras of type A

We fix some notations for vertex algebras. For a vertex algebra V, we denote the generating field associated with $v\in V$ by $v(z)=\sum_{n\in\mathbb{Z}}v_{(n)}z^{-n-1}$. We also denote the OPE of V by

$$u(z)v(w)\sim\sum_{s\geq 0}\frac{(u_{(s)}v)(w)}{(z-w)^{s+1}}$$

for all $u,v\in V$. We denote the vacuum vector (resp. the translation operator) by $|0\rangle$ (resp. ∂). We denote the universal affine vertex algebra associated with a finite dimensional Lie algebra $\mathfrak{g}$ and its inner product κ by $V^\kappa(\mathfrak{g})$. By the PBW theorem, we can identify $V^\kappa(\mathfrak{g})$ with $U(t^{-1}\mathfrak{g}[t^{-1}])$. In order to simplify the notation, here after, we denote the generating field $(ut^{-1})(z)$ as $u(z)$. By the definition of $V^\kappa(\mathfrak{g})$, the generating fields $u(z)$ and $v(z)$ satisfy the OPE

$$u(z)v(w)\sim\frac{[u,v](w)}{z-w}+\frac{\kappa(u,v)}{(z-w)^2}\tag{6.1}$$

for all $u, v \in \mathfrak{g}$.

We take a positive integer and its partition:

$$N = \sum_{i=1}^{l} q_i, \qquad q_1 \le q_2 \le \cdots \le q_v,\ q_{v+1} \ge .q_{v+2} \ge \cdots \ge q_l.$$

We also set

$$q_{Max} = \max(q_v, q_{v+1}),\ q_{Min} = \min(q_1, q_l).$$

We also fix an inner product of $\mathfrak{gl}(N)$ determined by

$$(E_{i,j}|E_{p,q}) = k\delta_{i,q}\delta_{p,j} + \delta_{i,j}\delta_{p,q}.$$

For $1 \le i \le N$, we set $1 \le \mathrm{col}(i) \le l$ and $q_{Max} - q_{\mathrm{col}(i)} < \mathrm{row}(i) \le q_{Max}$ satisfying

$$\mathrm{col}(i) = s \text{ if } \sum_{j=1}^{s-1} q_j < i \le \sum_{i=1}^{s} q_j,\ \mathrm{row}(i) = i - \sum_{j=1}^{\mathrm{col}(i)-1} q_j + q_{Max} - q_{\mathrm{col}(i)}.$$

We take a nilpotent element f as

$$f = \sum_{1 \le j \le N} e_{\hat{j},j},$$

where the integer $1 \le \hat{j} \le N$ is determined by

$$\mathrm{col}(\hat{j}) = \mathrm{col}(j) + 1,\ \mathrm{row}(\hat{j}) = \mathrm{row}(j).$$

The W-algebra is a vertex algebra associated with the reductive Lie algebra $\mathfrak{g}$ and a nilpotent element f. We denote the W-algebra associated with $\mathfrak{gl}(N)$ and f by $\mathcal{W}^k(\mathfrak{gl}(N), f)$.

We set the inner product on $\mathfrak{gl}(q_s)$ by

$$\kappa_s(E_{i,j}, E_{p,q}) = \delta_{j,p}\delta_{i,q}\alpha_s + \delta_{i,j}\delta_{p,q},$$

where $\alpha_s = k + N - q_s$. Then, by Corollary 5.2 in [12], the W-algebra $\mathcal{W}^k(\mathfrak{gl}(N), f)$ can be embedded into $\bigotimes_{1 \le s \le l} V^{\kappa_s}(\mathfrak{gl}(q_s))$. This embedding is called the Miura map. We can prove the following theorem in the same way as Theorem 4.6 in [29].

Theorem 6.2. *We set $\gamma_a = \sum\limits_{u=a+1}^{l} \alpha_u$. For positive integers $p, q > q_{Max} - q_{Min}$, the folloing elemets of $\bigotimes_{1 \le s \le l} V^{\kappa_s}(\mathfrak{gl}(q_s))$ are contained in $\mathcal{W}^k(\mathfrak{gl}(N), f)$.*

$$\begin{aligned}
W^{(1)}_{p,q} &= \sum_{1 \le r \le l} E^{(r)}_{p,q}[-1],\\
W^{(2)}_{p,q} &= -\sum_{1 \le r \le l} \gamma_r E^{(r)}_{p,q}[-2] + \sum_{\substack{r_1 < r_2 \\ u > q_{Max} - q_{Min}}} E^{(r_1)}_{u,q}[-1]E^{(r_2)}_{p,u}[-1]\\
&\quad - \sum_{\substack{r_1 \ge r_2 \\ q_{Max} - min(q_{r_1}, q_{r_2}) < u \le q_{Max} - q_{Min}}} E^{(r_1)}_{u,q}[-1]E^{(r_2)}_{p,u}[-1],
\end{aligned}$$

where we denote $E_{i,j}t^{-u} \in U(t^{-1}\mathfrak{gl}(q_s)[t^{-1}]) = V^{\kappa}(\mathfrak{gl}(q_s))$ by $E^{(s)}_{i,j}[-u]$.

Let us recall the definition of a universal enveloping algebra of a vertex algebra in the sense of [10] and [21]. For any vertex algebra V, let $L(V)$ be the Borchards Lie algebra, that is,

$$L(V) = V \otimes \mathbb{C}[t, t^{-1}]/\mathrm{Im}(\partial \otimes \mathrm{id} + \mathrm{id} \otimes \frac{d}{dt}), \tag{6.3}$$

where the commutation relation is given by

$$[ut^a, vt^b] = \sum_{r\geq 0} \binom{a}{r} (u_{(r)}v)t^{a+b-r}$$

for all $u, v \in V$ and $a, b \in \mathbb{Z}$.

Definition 6.4 (Section 6 in [21])**.** We set $\mathcal{U}(V)$ as the quotient algebra of the standard degreewise completion of the universal enveloping algebra of $L(V)$ by the completion of the two-sided ideal generated by

$$(u_{(a)}v)t^b - \sum_{i\geq 0} \binom{a}{i} (-1)^i (ut^{a-i}vt^{b+i} - (-1)^a vt^{a+b-i}ut^i), \tag{6.5}$$

$$|0\rangle t^{-1} - 1. \tag{6.6}$$

We call $\mathcal{U}(V)$ the universal enveloping algebra of V.

Induced by the Miura map μ, we obtain the embedding

$$\widetilde{\mu}\colon \mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(N), f)) \to \widehat{\bigoplus}_{1\leq s\leq l} U(\widehat{\mathfrak{gl}}(q_s)),$$

where $\widehat{\bigotimes}_{1\leq s\leq l} U(\widehat{\mathfrak{gl}}(q_s))$ is the standard degreewise completion of $\bigotimes_{1\leq s\leq l} U(\widehat{\mathfrak{gl}}(q_s))$.

By the definition of $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$, we have

$$\widetilde{\mu}(W^{(1)}_{i,j}t^s) = \sum_{1\leq r\leq l} E^{(r)}_{i,j}t^s, \tag{6.7}$$

$$\begin{aligned}\widetilde{\mu}(W^{(2)}_{i,j}t) &= \sum_{r=1}^{n} \gamma_r E^{(r)}_{i,j} + \sum_{s\in\mathbb{Z}} \sum_{r_1<r_2} \sum_{u>q_{Max}-q_{Min}} e^{(r_1)}_{u,j}t^{-s}e^{(r_2)}_{i,u}t^s\\ &\quad - \sum_{s\geq 0} \sum_{r\geq 0} \sum_{1\leq u\leq q_{Max}-q_r} (E^{(r)}_{u,j}t^{-s-1}E^{(r)}_{i,u}t^{s+1} + E^{(r)}_{i,u}t^{-s}E^{(r)}_{u,j}t^s)\\ &\quad - \sum_{s\in\mathbb{Z}} \sum_{r_1<r_2} \sum_{q_{Max}-\min(q_{r_1},q_{r_2})<u\leq q_{Max}-q_{Min}} E^{(r_1)}_{i,u}t^{-s}E^{(r_2)}_{u,j}t^s.\end{aligned} \tag{6.8}$$

7 A parabolic induction for a W-algebra of type A

Let us recall the parabolic induction for W-algebras of type A (see Theorem 6.1 in [13]). Let us take two positive integers

$$N_1 = q_1 + \cdots + q_w,\ N_2 = q_{w+1} + \cdots + q_l.$$

and $f_1 \in \mathfrak{gl}(N_1)$ and $f_2 \in \mathfrak{gl}(N_2)$ are nilpotent elements defined by the same way as f. Let us denote the Miura map associated with $\mathcal{W}^{k+N_2}(\mathfrak{gl}(N_1), f_1)$ and $\mathcal{W}^{k+N_1}(\mathfrak{gl}(N_2), f_2)$ by μ_1 and μ_2.

Theorem 7.1 (Theorem 6.1 in [13])**.** *There exists a homomorphism*

$$\Delta_W\colon \mathcal{W}^k(\mathfrak{gl}(N), f) \to \mathcal{W}^{k+N_2}(\mathfrak{gl}(N_1), f_1) \otimes \mathcal{W}^{k+N_1}(\mathfrak{gl}(N_2), f_2) \tag{7.2}$$

satisfying that

$$\mu = (\mu_1 \otimes \mu_2) \circ \Delta_W. \tag{7.3}$$

We call this homomorphism the parabolic induction for a W-algebra. By (7.3), we find that

$$\Delta_W(W^{(1)}_{i,j}) = W^{(1)}_{i,j} \otimes 1 + 1 \otimes W^{(1)}_{i,j},$$

$$\Delta_W(W^{(2)}_{i,j})=\begin{cases}W^{(2)}_{i,j}\otimes 1+1\otimes W^{(2)}_{i,j}-\gamma_{w+1}\partial W^{(1)}_{i,j}\otimes 1\\ \quad -\sum_{q_{Max}-\min(q_1,q_w)\leq u\leq q_{Max}-q_{Min}}(W^{(1)}_{u,j})_{(-1)}W^{(1)}_{i,u}\otimes 1 \text{ if } q_1\geq q_l,\\ W^{(2)}_{i,j}\otimes 1+1\otimes W^{(2)}_{i,j}-\gamma_{w+1}\partial W^{(1)}_{i,j}\otimes 1\\ \quad -1\otimes\sum_{q_{Max}-\min(q_{w+1},q_l)\leq u\leq q_{Max}-q_{Min}}(W^{(1)}_{u,j})_{(-1)}W^{(1)}_{i,u} \text{ if } q_1<q_l.\end{cases}$$

We also denote by Δ_W a homomorphism

$$\Delta_W\colon\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(N),f))\to\mathcal{U}(\mathcal{W}^{k+N_2}(\mathfrak{gl}(N_1),f_1))\otimes\mathcal{U}(\mathcal{W}^{k+N_1}(\mathfrak{gl}(N_2),f_2)),$$

which is induced by Δ^W in (7.2).

8 A homomorphism from the affine Yangian to the universal enveloping algebra of $\mathcal{W}^k(\mathfrak{gl}(N),f)$

In this section, we will give the another proof to Theorem 6.1 in [29]. The following theorem is the same as Theorem 6.1 in [29].

Theorem 8.1. *Assume that $q_{Min}\geq 3$ and $\varepsilon=\hbar(k+N-q_{Min})$. Then, there exists an algebra homomorphism*

$$\Phi\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(q_{Min}))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(N),f))$$

determined by

$$\Phi(X^+_{i,0})=\begin{cases}W^{(1)}_{q_{Max},q_{Max}-q_{Min}+1}t&\text{if } i=0,\\ W^{(1)}_{q_{Max}-q_{Min}+i,q_{Max}-q_{Min}+i+1}&\text{if } i\neq 0,\end{cases}$$

$$\Phi(X^-_{i,0})=\begin{cases}W^{(1)}_{q_{Max}-q_{Min}+1,q_{Max}}t^{-1}&\text{if } i=0,\\ W^{(1)}_{q_{Max}-q_{Min}+i+1,q_{Max}-q_{Min}+i}&\text{if } i\neq 0,\end{cases}$$

and

$$\begin{aligned}\Phi(H_{i,1})&=-\hbar(W^{(2)}_{q_{Max}-q_{Min}+i,q_{Max}-q_{Min}+i}t-W^{(2)}_{q_{Max}-q_{Min}+i+1,q_{Max}-q_{Min}+i+1}t)\\&\quad-\frac{i}{2}\hbar(W^{(1)}_{q_{Max}-q_{Min}+i,q_{Max}-q_{Min}+i}-W^{(1)}_{q_{Max}-q_{Min}+i+1,q_{Max}-q_{Min}+i+1})\\&\quad+\hbar W^{(1)}_{q_{Max}-q_{Min}+i,q_{Max}-q_{Min}+i}W^{(1)}_{q_{Max}-q_{Min}+i+1,q_{Max}-q_{Min}+i+1}\\&\quad+\hbar\sum_{s\geq 0}\sum_{u=1}^{i}W^{(1)}_{q_{Max}-q_{Min}+i,q_{Max}-q_{Min}+u}t^{-s}W^{(1)}_{q_{Max}-q_{Min}+u,q_{Max}-q_{Min}+i}t^{s}\\&\quad+\hbar\sum_{s\geq 0}\sum_{u=i+1}^{n}W^{(1)}_{q_{Max}-q_{Min}+i,q_{Max}-q_{Min}+u}t^{-s-1}W^{(1)}_{q_{Max}-q_{Min}+u,q_{Max}-q_{Min}+i}t^{s+1}\\&\quad-\hbar\sum_{s\geq 0}\sum_{u=1}^{i}W^{(1)}_{q_{Max}-q_{Min}+i+1,q_{Max}-q_{Min}+u}t^{-s}W^{(1)}_{q_{Max}-q_{Min}+u,q_{Max}-q_{Min}+i+1}t^{s}\\&\quad-\hbar\sum_{s\geq 0}\sum_{u=i+1}^{n}W^{(1)}_{q_{Max}-q_{Min}+i+1,q_{Max}-q_{Min}+u}t^{-s-1}W^{(1)}_{q_{Max}-q_{Min}+u,q_{Max}-q_{Min}+i+1}t^{s+1}\end{aligned}$$

for $i\neq 0$.

Remark 8.2. We note that the Definition 2.1 is different from the one in [29]. The parameter ε in this article is equal to $-n\hbar-\varepsilon$ in [29]. This is the reason that the assumption of ε is different from the one in [29].

In [29], we gave the proof to Theorem 8.1 by a direct computation. In this article, we prove it by showing the following theorem. Let us take the permutation of $\sigma \in S_l$ satisfying that

$$q_{\sigma(1)} > q_{\sigma(2)} > \cdots > q_{\sigma_l}.$$

Let us set a homomorphism

$$\Delta^l = (\prod_{s=1}^{l-1} \Delta^{q_{\sigma(s)},q_{\sigma(s+1)}} \otimes \mathrm{id}^{l-s-1})\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(q_{Min})) \to \bigotimes_{1\leq i\leq l} Y_{\hbar,\varepsilon-\hbar(q_{\sigma(i)}-q_{Min}}(\widehat{\mathfrak{sl}}(q_{\sigma(i)}))$$

where we set

$$\Delta^{q_{\sigma(s)},q_{\sigma(s+1)}} = \begin{cases} \Delta^+ \text{ if } q_{\sigma(s)} < q_{\sigma(s+1)}, \\ \Delta^- \text{ if } q_{\sigma(s)} > q_{\sigma(s+1)}. \end{cases}$$

We also define a natural homomorphism

$$\Sigma\colon \bigotimes_{1\leq i\leq l} Y_{\hbar,\varepsilon-\hbar(q_{\sigma(i)}-q_{Min})}(\widehat{\mathfrak{sl}}(q_{\sigma(i)})) \to \bigotimes_{1\leq i\leq l} Y_{\hbar,\varepsilon-\hbar(q_i-q_{Min})}(\widehat{\mathfrak{sl}}(q_i))).$$

Theorem 8.3. *We obtain the following relation:*

$$\bigotimes_{1\leq s\leq l} \mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon-(q_s-q_{Min})\hbar} \circ \Sigma \circ \Delta^l = \widetilde{\mu} \circ \Phi.$$

Since $\widetilde{\mu}$ is an injective homomorphism ([12]) and $\mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon+(q_s-q_{Min})\hbar}$, Δ^l and $\widetilde{\mu}$ are homomorphisms, we obtain Theorem 8.3.

The proof of Theorem 8.3. In order to simplify the notation, we only show the case that $q_1 \geq q_2 \geq \cdots \geq q_l$. The other cases can be proven in a similar way.

In this proof, we denote $E_{i,j} \in \mathfrak{gl}(q_s)$ by $e_{q_1-q_s+i,q_1-q_s+j}$. It is enough to show the relations:

$$\bigotimes_{1\leq s\leq l} \mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon-(q_s-q_{Min})\hbar} \circ \Delta^l(X^\pm_{j,0}) = \widetilde{\mu} \circ \Phi(X^\pm_{j,0}) \tag{8.4}$$

for $0 \leq j \leq n-1$ and

$$\bigotimes_{1\leq s\leq l} \mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon-(q_s-q_{Min})\hbar} \circ \Delta^l(H_{i,1}) = \widetilde{\mu} \circ \Phi(H_{i,1}) \tag{8.5}$$

for $1 \leq i \leq n-1$. It is trivial that (8.4) holds. We only show the relation (8.5). For $A \in \widetilde{Y}_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n)), \mathcal{U}(\widehat{\mathfrak{gl}}(q_s))$, we denote $A^{(s)} = 1^{\otimes s-1} \otimes A \otimes 1^{\otimes l-s}$. By the definition of $\Psi^{n,m+n}$ and Δ^n, we obtain

$$\Delta^l(H_{i,1}) = \sum_{1\leq s\leq l} H^{(s)}_{i+q_s-q_l} + B_i + C_i,$$

where we set

$$\begin{aligned} B_i &= -\hbar \sum_{r_1<r_2} E^{(r_1)}_{q_{r_1}-q_l+i,q_{r_1}-q_l+i} E^{(r_2)}_{q_{r_2}-q_l+i+1,q_{r_2}-q_l+i+1} \\ &\quad - \hbar \sum_{r_1<r_2} E^{(r_1)}_{q_{r_1}-q_l+i+1,q_{r_1}-q_l+i+1} E^{(r_2)}_{q_{r_2}-q_l+i,q_{r_2}-q_l+i} \\ &\quad - \hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=1}^{q_{r_2}-q_l+i} E^{(r_1)}_{q_{r_1}-q_{r_2}+u,q_{r_1}-q_l+i} t^{-s-1} E^{(r_2)}_{q_{r_2}-q_l+i,u} t^{s+1} \end{aligned}$$

$$+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l+i} E^{(r_1)}_{q_{r_1}-q_l+i,q_{r_1}-q_{r_2}+u}t^{-s}E^{(r_2)}_{u,q_{r_2}-q_l+i}t^s$$
$$-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} E^{(r_1)}_{q_{r_1}-q_{r_2}+u,q_{r_1}-q_l+i}t^{-s}E^{(r_2)}_{q_{r_2}-q_l+i,u}t^s$$
$$+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} E^{(r_1)}_{q_{r_1}-q_l+i,q_{r_1}-q_{r_2}+u}t^{-s-1}E^{(r_2)}_{u,q_{r_2}-q_l+i}t^{s+1}$$
$$+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l+i} E^{(r_1)}_{q_{r_1}-q_{r_2}+u,q_{r_1}-q_l+i+1}t^{-s-1}E^{(r_2)}_{q_{r_2}-q_l+i+1,u}t^{s+1}$$
$$-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l+i} E^{(r_1)}_{q_{r_1}-q_l+i+1,q_{r_1}-q_{r_2}+u}t^{-s}E^{(r_2)}_{u,q_{r_2}-q_l+i+1}t^s$$
$$+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} E^{(r_1)}_{q_{r_1}-q_{r_2}+u,q_{r_1}-q_l+i+1}t^{-s}E^{(r_2)}_{q_{r_2}-q_l+i+1,u}t^s$$
$$-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} E^{(r_1)}_{q_{r_1}-q_l+i+1,q_{r_1}-q_{r_2}+u}t^{-s-1}E^{(r_2)}_{u,q_{r_2}-q_l+i+1}t^{s+1}$$

and

$$\begin{aligned}C_i=&\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=1}^{q_r-q_l} E^{(r)}_{u,q_r-q_l+i}t^{-s-1}E^{(r)}_{q_r-q_l+i,u}t^{s+1}\\
&-\hbar\sum_{s\geq 0}\sum_{u=1}^{q_r-q_l} E^{(r)}_{u,q_r-q_l+i+1}t^{-s-1}E^{(r)}_{q_r-q_l+i+1,u}t^{s+1}\\
&+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} E^{(r_1)}_{q_{r_1}-q_{r_2}+u,q_{r_1}-q_l+i}t^{-s-1}E^{(r_2)}_{q_{r_2}-q_l+i,u}t^{s+1}\\
&-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} E^{(r_1)}_{q_{r_1}-q_{r_2}+u,q_{r_1}-q_l+i+1}t^{-s-1}E^{(r_2)}_{q_{r_2}-q_l+i+1,u}t^{s+1}\\
&+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} E^{(r_2)}_{u,q_{r_2}-q_l+i}t^{-s-1}E^{(r_1)}_{q_{r_1}-q_l+i,q_{r_1}-q_{r_2}+u}t^{s+1}\\
&-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} E^{(r_2)}_{u,q_{r_2}-q_l+i+1}t^{-s-1}E^{(r_1)}_{q_{r_1}-q_l+i+1,q_{r_1}-q_{r_2}+u}t^{s+1}.\end{aligned}$$

We note that B_i comes from the coproduct for the affine Yangian and C_i comes from the homomorphism $\Psi^{m.m+n}$.

By the definition of the evaluation map, we obtain

$$\begin{aligned}&\bigotimes_{1\leq s\leq l}\mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon-(q_s-q_{Min})\hbar}(\bigotimes_{1\leq s\leq l}H^{(s)}_{i+q_s-q_l})\\
=&\sum_{r=1}^{l}(\gamma_r-\frac{i}{2}\hbar)(e^{(r)}_{q_1-q_l+i,q_1-q_l+i}-e^{(r)}_{q_1-q_l+i+1,q_1-q_l+i+1})\\
&-\sum_{r=1}^{l}\hbar e^{(r)}_{q_1-q_l+i,q_1-q_l+i}e^{(r)}_{q_1-q_l+i+1,q_1-q_l+i+1}\end{aligned}$$

$$
\begin{aligned}
&+\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=q_1-q_r+1}^{q_1-q_l+i} e^{(r)}_{q_1-q_l+i,u}t^{-s}e^{(r)}_{u,q_1-q_l+i}t^{s}\\
&+\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=q_1-q_l+i+1}^{q_1} e^{(r)}_{q_1-q_l+i,u}t^{-s-1}e^{(r)}_{u,q_1-q_l+i}t^{s+1}\\
&-\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=q_1-q_r+1}^{q_1-q_l+i} e^{(r)}_{q_1-q_l+i+1,u}t^{-s}e^{(r)}_{u,q_1-q_l+i+1}t^{s}\\
&-\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=q_1-q_l+i+1}^{q_1} e^{(r)}_{q_1-q_l+i+1,u}t^{-s-1}e^{(r)}_{u,q_1-q_l+i+1}t^{s+1},
\end{aligned}
\tag{8.6}
$$

$$
\begin{aligned}
&\bigotimes_{1\leq s\leq l}\mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon+(q_s-q_l)\hbar}(B_i)\\
&=-\hbar\sum_{r_1<r_2} e^{(r_1)}_{q_1-q_l+i,q_1-q_l+i}e^{(r_2)}_{q_1-q_l+i+1,q_1-q_l+i+1}-\hbar\sum_{r_1<r_2} e^{(r_1)}_{q_1-q_l+i+1,q_1-q_l+i+1}e^{(r_2)}_{q_1-q_l+i,q_1-q_l+i}\\
&\quad-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i}t^{-s-1}e^{(r_2)}_{q_1-q_{r_2}+i,u}t^{s+1}\\
&\quad+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_l+i,q_1-q_{r_2}+u}t^{-s}e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i}t^{s}\\
&\quad-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i}t^{-s}e^{(r_2)}_{q_1-q_l+i,q_1-q_{r_2}+u}t^{s}\\
&\quad+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_l+i,q_1-q_{r_2}+u}t^{-s-1}e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i}t^{s+1}\\
&\quad+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i+1}t^{-s-1}e^{(r_2)}_{q_1-q_l+i+1,q_1-q_{r_2}+u}t^{s+1}\\
&\quad-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_l+i+1,q_1-q_{r_2}+u}t^{-s}e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i+1}t^{s}\\
&\quad+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i+1}t^{-s}e^{(r_2)}_{q_1-q_l+i+1,q_1-q_{r_2}+u}t^{s}\\
&\quad-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_l+i+1,q_1-q_{r_2}+u}t^{-s-1}e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i+1}t^{s+1},
\end{aligned}
\tag{8.7}
$$

$$
\begin{aligned}
&\bigotimes_{1\leq s\leq l}\mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon-(q_s-q_{Min})\hbar}(C_i)\\
&=\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=1}^{q_r-q_l} e^{(r)}_{q_1-q_r+u,q_1-q_l+i}t^{-s-1}e^{(r)}_{q_1-q_l+i,q_1-q_r+u}t^{s+1}\\
&\quad-\hbar\sum_{s\geq 0}\sum_{u=1}^{q_r-q_l} e^{(r)}_{q_1-q_r+u,q_1-q_l+i+1}t^{-s-1}e^{(r)}_{q_1-q_l+i+1,q_1-q_r+u}t^{s+1}\\
&\quad+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i}t^{-s-1}e^{(r_2)}_{q_1-q_l+i,q_1-q_{r_2}+u}t^{s+1}
\end{aligned}
$$

$$
\begin{aligned}
&-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i+1}t^{-s-1}e^{(r_2)}_{q_1-q_l+i+1,q_1-q_{r_2}+u}t^{s+1}\\
&+\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i}t^{-s-1}e^{(r_1)}_{q_1-q_l+i,q_1-q_{r_2}+u}t^{s+1}\\
&-\hbar\sum_{r_1<r_2}\sum_{s\geq 0}\sum_{u=1}^{q_{r_2}-q_l} e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i+1}t^{-s-1}e^{(r_1)}_{q_1-q_l+i+1,q_1-q_{r_2}+u}t^{s+1}. \qquad (8.8)
\end{aligned}
$$

By the definition of $\widetilde{\mu}$, we have

$$
\begin{aligned}
&\hbar\widetilde{\mu}(W^{(2)}_{q_1-q_l+i,q_1-q_l+i}t)-\hbar\widetilde{\mu}(W^{(2)}_{q_1-q_l+i+1,q_1-q_l+i+1}t)\\
&=\hbar\sum_{r=1}^{n}\gamma_r e^{(r)}_{q_1-q_l+i,q_1-q_l+i}+\hbar\sum_{s\in\mathbb{Z}}\sum_{r_1<r_2}\sum_{u>q_1-q_l} e^{(r_1)}_{u,q_1-q_l+i}t^{-s}e^{(r_2)}_{q_1-q_l+i,u}t^{s}\\
&\quad-\hbar\sum_{s\geq 0}\sum_{r\geq 0}\sum_{1\leq u\leq q_r-q_l} e^{(r)}_{q_1-q_r+u,q_1-q_l+i}t^{-s-1}e^{(r)}_{q_1-q_l+i,q_1-q_r+u}t^{s+1}\\
&\quad-\hbar\sum_{s\geq 0}\sum_{r\geq 0}\sum_{1\leq u\leq q_r-q_l} e^{(r)}_{q_1-q_l+i,q_1-q_r+u}t^{-s}e^{(r)}_{q_1-q_r+u,q_1-q_l+i}t^{s}\\
&\quad-\hbar\sum_{s\in\mathbb{Z}}\sum_{r_1<r_2}\sum_{1\leq u\leq q_{r_2}-q_l} e^{(r_1)}_{q_1-q_l+i,q_1-q_{r_2}+u}t^{s+1}e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i}t^{-s-1}\\
&\quad-\hbar\sum_{r=1}^{n}\gamma_r e^{(r)}_{q_1-q_l+i+1,q_1-q_l+i+1}-\hbar\sum_{s\in\mathbb{Z}}\sum_{r_1<r_2}\sum_{u>q_1-q_l} e^{(r_1)}_{u,q_1-q_l+i+1}t^{-s}e^{(r_2)}_{q_1-q_l+i+1,u}t^{s}\\
&\quad+\hbar\sum_{s\geq 0}\sum_{r\geq 0}\sum_{1\leq u\leq q_r} e^{(r)}_{q_1-q_r+u,q_1-q_l+i+1}t^{-s-1}e^{(r)}_{q_1-q_l+i+1,q_1-q_r+u}t^{s+1}\\
&\quad+\hbar\sum_{s\geq 0}\sum_{r\geq 0}\sum_{1\leq u\leq q_r} e^{(r)}_{q_1-q_l+i+1,q_1-q_r+u}t^{-s}e^{(r)}_{q_1-q_r+u,q_1-q_l+i+1}t^{s}\\
&\quad+\hbar\sum_{s\in\mathbb{Z}}\sum_{r_1<r_2}\sum_{1\leq u\leq q_{r_2}-q_l} e^{(r_1)}_{q_1-q_l+i+1,q_1-q_{r_2}+u}t^{s+1}e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i+1}t^{-s-1}. \qquad (8.9)
\end{aligned}
$$

Then, by a direct computation, the sum of (8.6)-(8.9) is equal to

$$
\bigotimes_{1\leq s\leq l}\mathrm{ev}^{q_s,\gamma_s\hbar}_{\hbar,\varepsilon-(q_s-q_{Min})\hbar}\circ\Delta^l(H_{i,1})+\hbar\widetilde{\mu}(W^{(2)}_{q_1-q_l+i,q_1-q_l+i}t)-\hbar\widetilde{\mu}(W^{(2)}_{q_1-q_l+i+1,q_1-q_l+i+1}t).
$$

For simplifying the notation, we denote the i-th term of the equation $(\cdot)$ by $(\cdot)_i$. We divide the sum into 8 piecies:

$$
(8.6)_1+(8.9)_1+(8.9)_6=-\frac{i}{2}\hbar\widetilde{\mu}(W^{(1)}_{q_1-q_l+i,q_1-q_l+i}-W^{(1)}_{q_1-q_l+i+1,q_1-q_l+i+1}), \qquad (8.10)
$$

$$
(8.6)_2+(8.7)_1+(8.7)_2=-\hbar\widetilde{\mu}(W^{(1)}_{q_1-q_l+i,q_1-q_l+i}W^{(1)}_{q_1-q_l+i+1,q_1-q_l+i+1}), \qquad (8.11)
$$

$$
\begin{aligned}
&(8.6)_3+(8.6)_4+(8.8)_1+(8.9)_3+(8.9)_4\\
&=\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=q_1-q_l+1}^{q_1-q_l+i} e^{(r)}_{q_1-q_l+i,u}t^{-s}e^{(r)}_{u,q_1-q_l+i}t^{s}\\
&\quad+\hbar\sum_{r=1}^{l}\sum_{s\geq 0}\sum_{u=q_1-q_l+i+1}^{q_1} e^{(r)}_{q_1-q_l+i,u}t^{-s-1}e^{(r)}_{u,q_1-q_l+i}t^{s+1}, \qquad (8.12)
\end{aligned}
$$

$$
(8.6)_5+(8.6)_6+(8.8)_2+(8.9)_8+(8.9)_9
$$

$$= -\hbar \sum_{r=1}^{l} \sum_{s\geq 0} \sum_{u=q_1-q_l+1}^{q_1-q_l+i} e^{(r)}_{q_1-q_l+i+1,u} t^{-s} e^{(r)}_{u,q_1-q_l+i+1} t^{s}$$

$$- \hbar \sum_{r=1}^{l} \sum_{s\geq 0} \sum_{u=q_1-q_l+i+1}^{n} e^{(r)}_{q_1-q_l+i+1,u} t^{-s-1} e^{(r)}_{u,q_1-q_l+i+1} t^{s+1}, \tag{8.13}$$

$(8.7)_3 + (8.7)_5 + (8.8)_3 + (8.9)_2$

$$= \hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i} t^{s} e^{(r_2)}_{q_1-q_{r_2}+i,u} t^{-s}$$

$$+ \hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i} t^{s+1} e^{(r_2)}_{q_1-q_l+i,q_1-q_{r_2}+u} t^{-s-1}, \tag{8.14}$$

$(8.7)_4 + (8.7)_6 + (8.8)_5 + (8.9)_5$

$$= \hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_l+i,q_1-q_{r_2}+u} t^{-s} e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i} t^{s}$$

$$+ \hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_l+i,q_1-q_{r_2}+u} t^{-s-1} e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i} t^{s+1}, \tag{8.15}$$

$(8.7)_7 + (8.7)_9 + (8.8)_4 + (8.9)_7$

$$= -\hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i+1} t^{s} e^{(r_2)}_{q_1-q_l+i+1,q_1-q_{r_2}+u} t^{-s}$$

$$- \hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_{r_2}+u,q_1-q_l+i+1} t^{s+1} e^{(r_2)}_{q_1-q_l+i+1,q_1-q_{r_2}+u} t^{-s-1}, \tag{8.16}$$

$(8.7)_8 + (8.7)_{10} + (8.8)_6 + (8.9)_{10}$

$$= -\hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+1}^{q_{r_2}-q_l+i} e^{(r_1)}_{q_1-q_l+i+1,q_1-q_{r_2}+u} t^{-s} e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i+1} t^{s}$$

$$- \hbar \sum_{r_1<r_2} \sum_{s\geq 0} \sum_{u=q_{r_2}-q_l+i+1}^{q_{r_2}} e^{(r_1)}_{q_1-q_l+i+1,q_1-q_{r_2}+u} t^{-s-1} e^{(r_2)}_{q_1-q_{r_2}+u,q_1-q_l+i+1} t^{s+1}. \tag{8.17}$$

By a direct computation, we have

$$\begin{aligned}(8.12) + (8.14) + (8.15) = \widetilde{\mu}(&\hbar \sum_{s\geq 0} \sum_{u=1}^{i} W^{(1)}_{q_1-q_l+i,q_1-q_l+u} t^{-s} W^{(1)}_{q_1-q_l+u,q_1-q_l+i} t^{s} \\ &+ \hbar \sum_{s\geq 0} \sum_{u=i+1}^{n} W^{(1)}_{q_1-q_l+i,q_1-q_l+u} t^{-s-1} W^{(1)}_{q_1-q_l+u,q_1-q_l+i} t^{s+1})\end{aligned}$$

and

$$\begin{aligned}(8.13) + (8.16) + (8.17) = \widetilde{\mu}(&-\hbar \sum_{s\geq 0} \sum_{u=1}^{i} W^{(1)}_{q_1-q_l+i+1,q_1-q_l+u} t^{-s} W^{(1)}_{q_1-q_l+u,q_1-q_l+i+1} t^{s} \\ &- \hbar \sum_{s\geq 0} \sum_{u=i+1}^{n} W^{(1)}_{q_1-q_l+i+1,q_1-q_l+u} t^{-s-1} W^{(1)}_{q_1-q_l+u,q_1-q_l+i+1} t^{s+1}).\end{aligned}$$

Thus, we have proven the relation (8.5). $\square$

In the same way as Φ, we can construct two homomorphisms

$$\Phi_1\colon Y_{\hbar,\varepsilon+(\min(q_1,q_w)-q_{Min})\hbar}(\widehat{\mathfrak{sl}}(\min(q_1,q_w)))\to\mathcal{U}(\mathcal{W}^{k+N_2}(\mathfrak{gl}(N_1),f_1)),$$
$$\Phi_2\colon Y_{\hbar,\varepsilon+(\min(q_{w+1},q_l)-q_{Min})\hbar}(\widehat{\mathfrak{sl}}(\min(q_{w+1},q_l)))\to\mathcal{U}(\mathcal{W}^{k+N_1}(\mathfrak{gl}(N_2),f_2)).$$

For a complex number a, we set a homomorphism called the shift operator of the affine Yangian:

$$\tau_a\colon Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))\to Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$$

determined by $X^{\pm}_{i,0}\mapsto X^{\pm}_{i,0}$ and $H_{i,1}\mapsto H_{i,1}+aH_{i,0}$.

Corollary 8.18. *We obtain the following relations:*

$$(\Phi_1\circ\tau_{-\gamma_{w+1}\hbar}\circ\Psi^{q_l,\min(q_1,q_w)})\otimes\Phi_2\circ\Delta=\Delta_W\circ\Phi \text{ if } q_1\geq q_l,$$
$$.(\Phi_1\circ\tau_{-\gamma_{w+1}\hbar})\otimes(\Phi_2\circ\Psi^{q_1,\min(q_l,q_{w+1})})\circ\Delta=\Delta_W\circ\Phi \text{ if } q_1<q_l.$$

This corollary follows directly from Theorem 8.3 and the relation (7.3).

9 A new interpretation of Φ from the affine coset of a non-rectangular W-algebra $\mathcal{W}^k(\mathfrak{gl}(m+n),(1^{m-n},2^n))$

In this section, we consider the case that $l=2$ and $q_1=m>q_2=n$. We denote the W-algebra of type A associated with $\mathfrak{gl}(m+n)$ and a nilpotent element of type $(1^{m-n},2^n)$ by $\mathcal{W}^k(\mathfrak{gl}(m+n),(1^{m-n},2^n))$. In [24], we construct Φ by a direct computation. Especially, in [24], we compute OPEs of the W-algebra $\mathcal{W}^k(\mathfrak{gl}(m+n),(1^{m-n},2^n))$.

Theorem 9.1 (Theorem 4.5 in [24])**.** *The following equations hold;*

$$(W^{(1)}_{p,q})_{(0)}W^{(1)}_{i,j}=\delta_{q,i}W^{(1)}_{p,j}-\delta_{p,j}W^{(1)}_{i,q},$$
$$(W^{(1)}_{p,q})_{(1)}W^{(1)}_{i,j}=\delta_{q,i}\delta_{p,j}(\alpha_1+\delta(p,i>m-n)\alpha_2)|0\rangle+\delta_{p,q}\delta_{i,j}(1+\delta(p,i>m-n))|0\rangle,$$
$$(W^{(1)}_{p,q})_{(s)}W^{(1)}_{i,j}=0 \text{ for all } s>1.$$

In particular, for all $p,q\leq m-n$, $i,j>m-n$, we obtain

$$[W^{(1)}_{p,q}t^s,W^{(1)}_{i,j}t^u]=\delta_{s+u,0}\delta_{p,q}\delta_{i,j}s.$$

Let us set an inner product on $\mathfrak{gl}(m-n)$ by

$$\kappa(E_{i,j},E_{p,q})=\alpha_1\delta_{i,q}\delta_{j,p}+E_{i,j}E_{p,q}.$$

By Theorem 9.1, we find that there exists an embedding from

$$\iota\colon V^\kappa(\mathfrak{gl}(m-n))\to\mathcal{W}^k(\mathfrak{gl}(m+n),(1^{m-n},2^n))$$

by corresponding $E_{i,j}t^{-1}\in V^\kappa(\mathfrak{gl}(m-n))=U(\mathfrak{gl}(m-n)[t^{-1}]t^{-1})$ to $W^{(1)}_{i,j}$.

Theorem 9.2 (Theorem 4.6 in [24])**.** *The following four equations hold;*

$$\begin{aligned}&(W^{(1)}_{p,q})_{(0)}W^{(2)}_{i,j}\\&=-\delta_{p,j}W^{(2)}_{i,q}+\delta_{i,q}\delta(p>m-n)W^{(2)}_{p,j}-\delta_{i,q}\delta(p\leq m-n)\sum_{w\leq m-n}(W^{(1)}_{w,j})_{(-1)}W^{(1)}_{p,w}\\&\quad-\delta_{i,q}\delta(p\leq m-n)\alpha_2\partial W^{(1)}_{p,j}+\delta(p\leq m-n,q>m-n)(W^{(1)}_{p,j})_{(-1)}W^{(1)}_{i,q},\end{aligned}$$

$$
\begin{aligned}
&(W^{(1)}_{p,q})_{(1)}W^{(2)}_{i,j}\\
&=\delta_{p,j}\alpha_1\delta(q>m-n)W^{(1)}_{i,q}-\delta_{i,q}\delta(p\leq m-n)(\alpha_2+\alpha_1)W^{(1)}_{p,j}\\
&\quad+\delta_{p,q}\delta(j>m-n)W^{(1)}_{i,j}-\delta_{i,j}\delta(q\leq m-n)W^{(1)}_{p,q}+\delta_{p,j}\delta_{q,i}\sum_{w\leq m-n}W^{(1)}_{w,w},
\end{aligned}
$$

$$
\begin{aligned}
&(W^{(1)}_{p,q})_{(2)}W^{(2)}_{i,j}\\
&=-\delta_{q,i}\delta_{p,j}(\alpha_1+\alpha_2)\alpha_1|0\rangle-\delta_{p,q}\delta_{i,j}(\delta(p,q\leq m-n)\alpha_1+2\alpha_2)|0\rangle,
\end{aligned}
$$

$$(W^{(1)}_{p,q})_{(s)}W^{(2)}_{i,j}=0 \textit{ for all } s>2.$$

In particular, for all $p,q\leq m-n,i,j>m-n$, we obtain

$$[W^{(1)}_{p,q}t^s,W^{(2)}_{i,j}t]=s\delta_{p,q}W^{(1)}_{i,j}t^s-\delta_{i,j}sW^{(1)}_{p,q}t^s.$$

For a vertex algebra A and its vertex subalgebra B, we set a coset vertex algebra of the pair (A,B) as follows:

$$C(A,B)=\{v\in A\mid w_{(r)}v=0 \text{ for } w\in B \text{ and } r\geq 0\}.$$

We can consider the coset vertex algebra $C(\mathcal{W}^k(\mathfrak{gl}(m+n),(1^{m-n},2^n)),V^\kappa(\mathfrak{gl}(m-n)))$.

Theorem 9.3. *The image of Φ is contained in the coset $C(\mathcal{W}^k(\mathfrak{gl}(m+n),(1^{m-n},2^n)),V^\kappa(\mathfrak{gl}(m-n)))$.*

Proof. It is enough to show that $\Psi(H_{i,1})$ and $\Psi(X^{\pm}_{j,0})$ are contained in the coset for $1\leq i\leq n-1$ and $0\leq j\leq n-1$. It is trivial that $\Psi(X^{\pm}_{j,0})$ is contained in the coset. We will show the relation

$$[W^{(1)}_{p,q}t^s,\Psi(H_{i,1})]=0$$

for $p,q\leq m-n$ and $s\in\mathbb{Z}$. By Theorem 9.2, we find that

$$
\begin{aligned}
&[W^{(1)}_{p,q}t^s,-\hbar(W^{(2)}_{m-n+i,m-n+i}t-W^{(2)}_{m-n+i+1,m-n+i+1}t)]\\
&=-\hbar(s\delta_{p,q}W^{(1)}_{m-n+i,m-n+i}t^s-s\delta_{p,q}W^{(1)}_{m-n+i+1,m-n+i+1}t^s).
\end{aligned}\tag{9.4}
$$

By Theorem 9.1, we obtain

$$
\begin{aligned}
&[W^{(1)}_{p,q}t^s,\Psi(H_{i,1})-(W^{(2)}_{m-n+i,m-n+i}t-W^{(2)}_{m-n+i+1,m-n+i+1}t)]\\
&=\hbar s\delta_{p,q}W^{(1)}_{m-n+i,m-n+i}t^s-\hbar s\delta_{p,q}W^{(1)}_{m-n+i+1,m-n+i+1}t^s.
\end{aligned}\tag{9.5}
$$

Adding (9.4) and (9.5), we find that $[W^{(1)}_{p,q}t^s,\Psi(H_{i,1})]=0$. □

References

[1] T. Arakawa. Representation theory of W-algebras. *Invent. Math.*, 169(2):219–320, 2007, https://doi.org/10.1007/s00222-007-0046-1.

[2] J. Brundan and A. Kleshchev. Shifted Yangians and finite W-algebras. *Adv. Math.*, 200(1):136–195, 2006, https://doi.org/10.1016/j.aim.2004.11.004.

[3] T. Creutzig, D. E. Diaconescu, and M. Ma. Affine Laumon spaces and iterated $\mathcal{W}$-algebras. *Comm. Math. Phys.*, 402(3):2133–2168, 2023.

[4] T. Creutzig and A. R. Linshaw. Trialities of $\mathcal{W}$-algebras. *Camb. J. Math.*, 10(1):69–194, 2022.

[5] A. De Sole and V. G. Kac. Finite vs affine W-algebras. *Jpn. J. Math.*, 1(1):137–261, 2006, https://doi.org/10.1007/s11537-006-0505-2.

[6] A. De Sole, V. G. Kac, and D. Valeri. A Lax type operator for quantum finite W-algebras. *Selecta Math. (N.S.)*, 24(5):4617–4657, 2018.

[7] V. G. Drinfeld. Hopf algebras and the quantum Yang-Baxter equation. *Dokl. Akad. Nauk SSSR*, 283(5):1060–1064, 1985, https://doi.org/10.1142/9789812798336.0013.

[8] V. G. Drinfeld. A new realization of Yangians and of quantum affine algebras. *Dokl. Akad. Nauk SSSR*, 296(1):13–17, 1987.

[9] M. Finkelberg and A. Tsymbaliuk. Multiplicative slices, relativistic Toda and shifted quantum affine algebras. In *Representations and nilpotent orbits of Lie algebraic systems*, volume 330 of *Progr. Math.*, pages 133–304. Birkhäuser/Springer, Cham, 2019.

[10] I. B. Frenkel and Y. Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.*, 66(1):123–168, 1992, https://doi.org/10.1215/S0012-7094-92-06604-X.

[11] D. Gaiotto and M. Rapčák. Vertex algebras at the corner. *J. High Energy Phys.*, (1):160, front matter+85, 2019.

[12] N. Genra. Screening operators for $\mathcal{W}$-algebras. *Selecta Math. (N.S.)*, 23(3):2157–2202, 2017, https://doi.org/10.1007/s00029-017-0315-9.

[13] N. Genra. Screening operators and parabolic inductions for affine $\mathcal{W}$-algebras. *Adv. Math.*, 369:107179, 62, 2020, https://doi.org/10.1016/j.aim.2020.107179.

[14] N. Guay. Cherednik algebras and Yangians. *Int. Math. Res. Not.*, (57):3551–3593, 2005, https://doi.org/10.1155/IMRN.2005.3551.

[15] N. Guay. Affine Yangians and deformed double current algebras in type A. *Adv. Math.*, 211(2):436–484, https://doi.org/10.1016/j.aim.2006.08.007, 2007.

[16] N. Guay, H. Nakajima, and C. Wendlandt. Coproduct for Yangians of affine Kac-Moody algebras. *Adv. Math.*, 338:865–911, 2018, https://doi.org/10.1016/j.aim.2018.09.013.

[17] V. G. Kac and M. Wakimoto. Quantum reduction and representation theory of superconformal algebras. *Adv. Math.*, 185(2):400–458, 2004, https://doi.org/10.1016/j.aim.2003.12.005.

[18] R. Kodera. Braid group action on affine Yangian. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 15:Paper No. 020, 28, 2019, https://doi.org/10.3842/SIGMA.2019.020.

[19] R. Kodera. On Guay's evaluation map for affine Yangians. *Algebr. Represent. Theory*, 24(1):253–267, 2021, https://doi.org/10.1007/s10468-019-09945-w.

[20] R. Kodera and M. Ueda. Coproduct for affine Yangians and parabolic induction for rectangular W-algebras. *Lett. Math. Phys.*, 112(1):Paper No. 3, 37, 2022.

[21] A. Matsuo, K. Nagatomo, and A. Tsuchiya. Quasi-finite algebras graded by Hamiltonian and vertex operator algebras. *London Math. Soc. Lecture Note Ser.*, 372:282–329, 2010, https://doi.org/10.1017/CBO9780511730054.015.

[22] A. Premet. Special transverse slices and their enveloping algebras. *Adv. Math.*, 170(1):1–55, 2002, https://doi.org/10.1006/aima.2001.2063. With an appendix by Serge Skryabin.

[23] O. Schiffmann and E. Vasserot. Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbf{A}^2$. *Publ. Math. Inst. Hautes Études Sci.*, 118:213–342, 2013, https://doi.org/10.1007/s10240-013-0052-3.

[24] M. Ueda. An Example of Homomorphisms from the Guay's affine Yangians to non-rectangular W-algebras. *Transformation Groups(2023).*

[25] M. Ueda. A homomorphism from the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n))$ to the affine Yangian $Y_{\hbar,\varepsilon}(\widehat{\mathfrak{sl}}(n+1))$. arXiv:2312.09933.

[26] M. Ueda. Notes on a homomorphism from the affine Yangian associated with $\widehat{\mathfrak{sl}}(n)$ to the affine Yangian associated with $\widehat{\mathfrak{sl}}(n)$. arXiv:2402.01870.

[27] M. Ueda. Two homomorphisms from the affine Yangian associated with $\widehat{\mathfrak{sl}}(n)$ to the affine Yangian associated with $\widehat{\mathfrak{sl}}(n)$. arXiv:2404.10923.

[28] M. Ueda. Affine super Yangians and rectangular W-superalgebras. *J. Math. Phys.*, 63(5):Paper No. 051701, 34, 2022.

[29] M. Ueda. Guay's affine Yangians and non-rectangular W-algebras. *Adv. Math.*, 438:Paper No. 109468, 44, 2024.