NON-DEGENERACY OF THE BUBBLE IN A FRACTIONAL AND SINGULAR 1D LIOUVILLE EQUATION

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Abstract. We prove the non-degeneracy of solutions to a fractional and singular Liouville equation defined on the whole real line in presence of a singular term. We use conformal transformations to rewrite the linearized equation as a Steklov eigenvalue problem posed in a bounded domain, which is defined either by an intersection or a union of two disks. We conclude by proving the simplicity of the corresponding eigenvalue.

1. Introduction

In this work we investigate non-degeneracy properties for solutions to the one-dimensional singular Liouville equation

$$(-\Delta)^{\frac{1}{2}}u = |x|^{\alpha-1}e^u \text{ in } \mathbb{R}, \tag{1.1}$$

with $0 < \alpha < 2$. In order to define the half-Laplacian in (1.1), we require

$$\int_{\mathbb{R}} \frac{|u|}{1+x^2} < +\infty. \tag{1.2}$$

We also assume the integrability condition

$$\int_{\mathbb{R}} |x|^{\alpha-1}e^u < +\infty. \tag{1.3}$$

Under conditions (1.2) and (1.3), weak solutions to (1.1) are completely classified. When $\alpha = 1$, the set of solutions contains only the two-parameter family of solutions

$$\mathfrak{u}_{\mu,\xi}(x) = \ln\left(\frac{2\mu}{|x-\xi|^2+\mu^2}\right), \tag{1.4}$$

with $\mu, \xi \in \mathbb{R}$ and $\mu > 0$. We refer to the work of Da Lio, Martinazzi and Rivière in [12] for the proof. Due to translation and dilation invariance, it is clear that the derivatives

$$z_{0,\mu,\xi}(x) := \partial_\mu \mathfrak{u}_{\mu,\xi}(x) = \frac{1}{\mu}\frac{(x-\xi)^2-\mu^2}{\mu^2+(x-\xi)^2}, \quad z_{1,\mu,\xi}(x) := \partial_\xi \mathfrak{u}_{\mu,\xi}(x) = \frac{2(x-\xi)}{\mu^2+(x-\xi)^2} \tag{1.5}$$

solve the linear problem

$$(-\Delta)^{\frac{1}{2}}z = e^{\mathfrak{u}_{\mu,\xi}}z \quad \text{ in } \mathbb{R}. \tag{1.6}$$

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It is well known (see [13, 39, 11]) that the bubble $\mathfrak{u}_{\mu,\xi}$ is non-degenerate up to the natural invariances of (1.1), i.e. the two functions in (1.5) span the space of all bounded solutions to (1.6). More precisely, if $z \in L^\infty(\mathbb{R})$ is a weak solution to (1.6), then z is a linear combination of $z_{0,\mu,\xi}$ and $z_{1,\mu,\xi}$.

If $\alpha \neq 1$, problem (1.1) is not translation invariant. As we will show in Section 2, it follows from the results obtained by Gálvez, Jiménez and Mira in [26] (see also [43]) that for any $\alpha \in (0,1)\cup(1,2)$, equation (1.1) only has a one-parameter family of solutions given by:

$$u_\rho(x) = \ln\left(\frac{2\alpha\rho\sin\frac{\pi\alpha}{2}}{|x|^{2\alpha}+2\rho|x|^\alpha\cos\frac{\pi\alpha}{2}+\rho^2}\right) \tag{1.7}$$

with $\rho > 0$. We stress that the condition $\alpha \in (0,1)\cup(1,2)$ is necessary, since there exists no solution to (1.1) when $\alpha \geq 2$ (see Proposition 2.5).

In the present work we prove the non-degeneracy of u_ρ. Specifically, for any $\alpha \in (0,1)\cup(1,2)$ and $\rho > 0$, we classify all solutions to the linearized problem

$$(-\Delta)^{\frac{1}{2}}\varphi = |x|^{\alpha-1}e^{u_\rho}\varphi \text{ in } \mathbb{R}, \tag{1.8}$$

in the space of functions satisfying the conditions

$$(-\Delta)^{\frac{1}{4}}\varphi \in L^2(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} |x|^{\alpha-1}e^{u_\rho}\varphi^2 < +\infty. \tag{1.9}$$

We consider the function

$$z_\rho(x) := \partial_\rho u_\rho(x) = \frac{1}{\rho}\frac{|x|^{2\alpha}-\rho^2}{|x|^{2\alpha}+2\rho|x|^\alpha\cos\frac{\pi\alpha}{2}+\rho^2}, \tag{1.10}$$

and give the following result:

Theorem 1.1. *Assume $\alpha \in (0,1)\cup(1,2)$ and $\rho > 0$. Let u_ρ and z_ρ be defined as in (1.7) and (1.10). Let φ be a weak solution to (1.8) such that (1.9) holds. Then there exists $c \in \mathbb{R}$ such that $\varphi = c\, z_\rho$.*

The main idea of the proof consists in proving the equivalence between the *non-local* eigenvalue problem

$$(-\Delta)^{\frac{1}{2}}\varphi = \lambda|x|^{\alpha-1}e^{u_\rho}\varphi \text{ in } \mathbb{R}, \tag{1.11}$$

and the *Steklov* eigenvalue problem

$$\Delta\psi = 0 \text{ in } \Omega_\alpha,\ \partial_\nu\psi = \mu\psi \text{ in } \partial\Omega_\alpha, \tag{1.12}$$

where Ω_α is either the intersection of two disks, when $\alpha \in (0,1)$, or the union of two disks, when $\alpha \in (1,2)$. We will prove that the eigenvalue $\lambda = 1$ of (1.11) corresponds to the eigenvalue $\mu_\alpha = \frac{1}{\sqrt{1+\tau_\alpha^2}}$ of (1.12), being $\tau_\alpha := \frac{1+\cos\alpha\pi}{\sin\alpha\pi}$ and it is always simple. It is worthwhile to point out that μ_α is the second eigenvalue of (1.12) if $\alpha \in (0,1)$, while it is an higher order eigenvalue when $\alpha \in (1,2)$. As a consequence, the Morse index of the bubble u_ρ changes when α crosses the value 1. Indeed, it turns out to be equal to 1 when $\alpha < 1$, while it is greater or equal than 2 when $\alpha > 1$. It would be interesting to compute exactly the Morse index in this second case, which is equivalent to find the order of the eigenvalue μ_α of (1.12).

The proof of Theorem 1.1 is based on harmonic extension techniques (see [8]). Via convolution with the Poisson Kernel, every function satisfying (1.2) can be extended to a harmonic function defined on the upper half-plane $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : y > 0\}$. It is simple to verify that the harmonic extensions of (1.4) and (1.7) are given respectively by

$$\mathcal{U}_{\mu,\xi}(x,y) := \ln\left(\frac{2\alpha\mu}{(x-\xi)^2+(y+\mu)^2}\right)$$

and by

$$U_\rho(x,y) := \ln \frac{2\alpha\rho|\sin\theta_0|}{|z^\alpha - z_0|^2}, \ z = x+iy, \ z_0 = \rho e^{i\theta_0}, \ \theta_0 := \frac{\pi\alpha}{2} + \pi. \tag{1.13}$$

These functions solve the local problem

$$-\Delta U = 0 \text{ in } \mathbb{R}^2_+, \ \partial_\nu U = |x|^{\alpha-1} e^U \text{ on } \partial\mathbb{R}^2_+, \tag{1.14}$$

respectively for $\alpha = 1$ and $\alpha \in (0,1) \cup (1,2)$, where ν is the outward normal to the half-plane $\partial\mathbb{R}^2_+$. Similarly, if φ solves the (1.8)-(1.9), then the harmonic extension Φ of φ satisfies

$$-\Delta\Phi = 0 \text{ in } \mathbb{R}^2_+, \ \partial_\nu \Phi = |x|^{\alpha-1} e^U \Phi \text{ on } \partial\mathbb{R}^2_+, \tag{1.15}$$

as well as

$$\int\limits_{\mathbb{R}^2_+} |\nabla\Phi|^2 + \int\limits_{\mathbb{R}^2_+} |z|^{2(\alpha-1)} e^{2U} \Phi^2 \, dz + \int\limits_{\partial\mathbb{R}^2_+} |x|^{\alpha-1} e^U \Phi^2 < +\infty. \tag{1.16}$$

Theorem 1.2. *Assume $\alpha \in (0,1) \cup (0,2)$ and $\rho > 0$. Let U_ρ be defined as in (1.13). Then U_ρ is non-degenerate. Namely, each solution to the linear problem (1.15) satisfying (1.16) is of the form*

$$\Phi(z) = \mathfrak{c}\frac{\partial U_\rho}{\partial\rho}(z), \ \mathfrak{c} \in \mathbb{R}, \ \textit{with } \frac{\partial U_\rho}{\partial \rho}(z) = \frac{1}{\rho}\frac{|z|^{2\alpha} - |z_0|^2}{|z^\alpha - z_0|^2}.$$

In [13], Dávila, del Pino and Musso studied problem (1.15) with $\alpha = 1$, and proved that it is equivalent to the study of the first nontrivial Steklov eigenspace for the unit disk $\mathscr{D} \subseteq \mathbb{R}^2$ (they use the fact that the half-plane is conformally equivalent to $\mathscr{D}$). In [39, 11] Santra as well as Cozzi and Fernández directly attacked problem (1.8) and, using the stereographic projection of the real line on $\mathbb{S}^1$, they wrote problem (1.8) with $\alpha = 1$ as an eigenvalue problem of the fractional Laplacian on $\mathbb{S}^1$. Neither of the approaches can be followed if $\alpha \neq 1$ because of the presence of the non-autonomous term $|x|^{\alpha-1}$. In the present paper, we find a clever change of variables which allows us to get rid of this term and to reduce the linear problem (1.15) to a classical Steklov eigenvalue problem defined on a Lipschitz continuous bounded domain in the plane. More precisely, we proceed as follows. First, using a conformal change of variables, we rewrite (1.15) on a cone (see (3.2)) so that the boundary condition does not contain the non-autonomous term anymore. Then, using a conformal Möbius map, we rewrite (1.15) as the Steklov eigenvalue problem (see (1.12)) with $\mu = \mu_\alpha$. Finally, we conclude the proof of Theorem 1.2 by showing that μ_α is a simple eigenvalue.

It is interesting to compare Theorem 1.1 with similar results in higher dimension. Equation (1.1) is a one-dimensional analog of the celebrated Liouville equation

$$-\Delta u = |x|^{2(\alpha-1)} e^{2u}, \tag{1.17}$$

which was introduced by Liouville [32] with $\alpha = 1$. Solutions to (1.17) with $|x|^{2(\alpha-1)} \in L^1(\mathbb{R}^2)$ were classified by Chen and Li [10] for $\alpha = 1$, and by Prajapat and Tarantello [36] for a general $\alpha > 0$. Non-degeneracy of solutions was proved by Baraket and Pacard in [7] for $\alpha = 1$, Esposito in [21] for $\alpha \in (0, +\infty) \setminus \mathbb{N}$ and del Pino, Esposito and Musso in [18], for $\alpha \in \mathbb{N} \setminus \{0\}$. We also quote the paper [27], where Gladiali, Grossi and Neves studied the Morse index of the solution of (1.17) showing that it changes and increases whenever α crosses an integer value. In recent years, Liouville equations have been studied also in dimension $n \geq 3$ in connection to problems involving higher order notions of curvature such as prescribed Q-curvature or prescribed fractional curvature problems (see e.g. [28, 29, 19]). In particular in [29], Hyder, Mancini and Martinazzi consider the problem

$$(-\Delta)^{\frac{n}{2}} u = |x|^{n(\alpha-1)} e^{nu} \quad \text{in } \mathbb{R}^n \tag{1.18}$$

with
$$\int_{\mathbb{R}^n} |x|^{n(\alpha-1)} e^{nu}\, dx < +\infty.$$
If $\alpha = 1$, solutions satisfying $u(x) = o(|x|^2)$ as $|x| \to \infty$ are completely classified (see [41, 33] and non-degeneracy has been proved when n is even (see [6, 34]). However, there are also solutions to (1.18) which behave at infinity as a quadratic polynomial (see [31, 9]). The singular case is more difficult to study. Differently from the 1d-case and 2d-case, if $n \geq 3$ and $\alpha \neq 1$, there is no explicit example of solution to (1.18). However, in [29] it is proved that for any $\alpha > 0$, (1.18) has a radially symmetric solution with logarithmic behavior at infinity and infinitely many radially symmetric solutions with polynomial behavior at infinity. To our knowledge no non-degeneracy result has been obtained so far.

We point out that the one-dimensional case that we treat in Theorem 1.1 is the only one in which a restriction on α appears. Moreover, Theorem 1.1 is the first classification result for the linearization of (1.18) with $\alpha \neq 1$ in odd dimension, which makes (1.18) non-local. Non-degeneracy results for non-local problems are extremely delicate to obtain. For sake of completeness we quote some results concerning the non-degeneracy of solutions in the fractional framework. The non-degeneracy of solutions to the non-local critical equation
$$(-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \text{ in } \mathbb{R}^n$$
was studied by Dávila, del Pino and Sire in [15]. In the subcritical regime, i.e $1 < p < \frac{n+2s}{n-2s}$, the non-degeneracy of least energy solution
$$(-\Delta)^s u + u = u^p \text{ in } \mathbb{R}^n,$$
was completely achieved by Frank, Lenzmann and Silvestre in [25], after preliminary works in particular cases discussed by Fall and Valdinoci [22] when s is close to 1 and by Frank and Lenzmann [24] when $n = 1$. The non-degeneracy of minimizers for the fractional Caffarelli-Kohn-Nirenberg inequality, which after multiplication by $|x|^{-\alpha}$ are solutions to
$$(-\Delta)^s u + \tau \frac{u}{|x|^{2s}} = |x|^{-(\beta-\alpha)p} u^{p-1} \text{ in } \mathbb{R}^n$$
with $p = \frac{2n}{n-2s+2(\beta-\alpha)}$, $\tau \geq 0$ and $-2s < \alpha < \frac{n-2s}{2}$ and $\alpha \leq \beta < \alpha + s$, was obtained by Ao, DelaTorre and González in [5] (see also [16]), while the non-degeneracy of minimizers for the fractional Hardy-Sobolev inequality, namely solutions to (i.e. $\tau = 0$ and $\alpha = 0$ in the previous equation)
$$(-\Delta)^s u = |x|^{-\beta p} u^{p-1} \text{ in } \mathbb{R}^n$$
was obtained by Musina and Nazarov in [35] and to the critical fractional Hénon equation
$$(-\Delta)^s u = |x|^{\alpha} u^{\frac{N+2s+2\alpha}{N-2s}} \text{ in } \mathbb{R}^n$$
by Alarcon, Barrios and Quaas in [2].

The non-degeneracy result of Theorem 1.1 plays a role in the description of parameter-depending problems in which concentration phenomena occur and in which (1.1) appears as a limit problem. For example, we refer to [14, 20, 3, 4] for applications of (1.1) and (1.14) to physical models for the description of galvanic corrosion phenomena for simple electrochemical systems (see e.g. [17, 40] and references therein). We believe that 1.1 and 1.2 could be useful in the description of non-simple blow-up phenomena for such models.

Finally, we wish to point out that the argument used to prove our main result, can also be applied to prove that the second eigenvalue of Steklov's problem on the ellipse is simple, as soon as the ellipse is not a circle (see Proposition 4.3). A similar result holds true for the second eigenvalue of the Dirichlet problem [37] and for the Neumann problem [38]. The result for the Steklov problem is widely expected, but since we did not find a suitable reference we decide to write the proof here.

This paper is organized as follows. In Section 2, we introduce the notation and we recall some useful results. In Section 3, we introduce the changes of variable which allow to reduce Theorems 1.1 and 1.2 to the study of a Steklov eigenvalue problem, which is studied in Section 4 concluding the proof.

2. Preliminaries and classification results

Throughout the paper we will denote

$$L_{\frac{1}{2}}(\mathbb{R}) := \left\{ u \in L^1_{loc}(\mathbb{R}) \ : \ \int_{\mathbb{R}} \frac{|u|}{1+x^2} < +\infty \right\}.$$

If $u \in L_{\frac{1}{2}}(\mathbb{R})$, then for any $s \in (0, \frac{1}{2}]$ it is possible to define the fractional Laplacian $(-\Delta)^s$ in the sense of tempered distribution by means of the Fourier Transform:

$$< (-\Delta)^s u, \psi >= \int_{\mathbb{R}} u(\xi)(-\Delta)^s \psi \, d\xi, \qquad \text{where } (-\Delta)^s \psi = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}[\varphi]].$$

In particular, for a function $f \in L^1_{loc}(\mathbb{R})$ we say that $u \in L^1_{loc}(\mathbb{R})$ is a weak solution to $(-\Delta)^{\frac{1}{2}} u = f$ if

$$\int_{\mathbb{R}} u(-\Delta)^{\frac{1}{2}} \psi = \int_{\mathbb{R}} f \psi,$$

for any $\psi \in C_c^\infty(\mathbb{R})$. In particular, if $u \in L_{\frac{1}{2}}(\mathbb{R})$ and (1.3) holds, then we say that u is a weak solution to (1.1) if

$$\int_{\mathbb{R}} u(-\Delta)^{\frac{1}{2}} \psi = \int_{\mathbb{R}} |x|^{\alpha-1} e^u \psi$$

for any $\psi \in C_c^\infty(\mathbb{R})$.

We now state a result concerning regularity of weak solutions. We refer to [29] for the proof.

Lemma 2.1. *Assume $\alpha \in (0, +\infty)$ and let $u \in L_{\frac{1}{2}}(\mathbb{R})$ be a weak solution to (1.1) such that (1.3) holds. Then $u \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^{0,\beta}_{loc}(\mathbb{R})$ for some $\beta \in (0,1)$.*

Condition (1.3) also allows to describe the asymptotic behavior of u as $|x| \to \infty$.

Lemma 2.2. *Assume $\alpha \in (0, +\infty)$ and let $u \in L_{\frac{1}{2}}(\mathbb{R})$ be a weak solution to (1.1) such that (1.3) holds. Then there exist $\beta > \alpha$ and $C > 0$ such that*

$$|u(x) + \beta \ln |x|| \leq C,$$

for all $x \in \mathbb{R}$ with $|x| \geq 1$.

We refer again to [29] for the proof. In fact, following the arguments of [29] one can show that $\beta = 2\alpha$. However, for our purposes here we only need the estimate of $\beta > \alpha$.

To relate (1.1) with (1.14), we let

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$$

denote the Poisson kernel for the half-plane $\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 \ : \ y > 0\}$. For a function $u \in L_{\frac{1}{2}}(\mathbb{R})$, we can define the Poisson extension of u as

$$U(x, y) := (u * P_y)(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y\, u(\xi)}{(x-\xi)^2 + y^2} \, d\xi, \quad (x, y) \in \mathbb{R}^2_+.$$

We recall the following standard properties of Poisson extensions:

Proposition 2.3. *Assume $u \in L_{\frac{1}{2}}(\mathbb{R})$.*

(1) *If $u \in C(a, b)$ for some $a, b \in \mathbb{R}$, $a < b$, then U extends continuously to $(a, b) \times \{0\}$ and $U(x, 0) = u(x)$ for any $x \in (a, b)$.*

(2) *If $u \in C^{1,s}(a,b)$ for some $s \in (0,1)$ and $a,b \in \mathbb{R}$ with $a < b$, then the partial derivatives of U extend continuously to $(a,b)\times\{0\}$ and $\frac{\partial U}{\partial y}(x,0) = -(-\Delta)^{\frac{1}{2}}u(x)$ for any $x \in (a,b)$.*

(3) *If $(-\Delta)^{\frac{1}{4}}u \in L^2(\mathbb{R})$, then $|\nabla U| \in L^2(\mathbb{R}^2_+)$ and $\|\nabla U\|_{L^2(\mathbb{R}^2_+)} = \|(-\Delta)^{\frac{1}{4}}u\|_{L^2(\mathbb{R})}$.*

Lemma 2.4. *Assume that $u \in L_{\frac{1}{2}}(\mathbb{R})$ is a weak solution of (1.1) such that (1.3) holds. Then the harmonic extension $U = u * P_y$ is a solution to (1.14). Moreover the following properties are satisfied.*

(1) *$U \in C^\infty(\mathbb{R}^2_+) \cup C(\overline{\mathbb{R}^2_+})$ and*

$$\int_{\partial\mathbb{R}^2_+} |x|^{\alpha-1}e^U < +\infty \tag{2.1}$$

(2) *Let β be as in Lemma 2.2. Then, there exists $C > 0$ such that*

$$\left|U(x,y) + \beta \ln\sqrt{x^2+y^2}\right| \le C, \qquad \text{in } \mathbb{R}^2_+ \setminus B_2(0,0). \tag{2.2}$$

In particular

$$\int_{\mathbb{R}^2_+} |x|^{2(\alpha-1)}e^{2U} < +\infty. \tag{2.3}$$

Proof. Thanks to Proposition 2.3, we just need to prove (2), which is a consequence of the formula

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{y\ln|\xi|}{(x-\xi)^2+y^2}\,d\xi = \ln\sqrt{x^2+y^2}. \tag{2.4}$$

Indeed, (2.4) gives

$$\left|U(x,y) + \beta\ln\sqrt{x^2+y^2}\right| \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{y|u(\xi)+\beta\ln|\xi||}{(x-\xi)^2+y^2}\,d\xi \le C + \frac{1}{\pi}\int_{-1}^{1}\frac{y|u(\xi)+\beta\ln|\xi||}{(x-\xi)^2+y^2}\,d\xi,$$

where the last inequality follows from Lemma 2.2. Finally, we observe that if $\sqrt{x^2+y^2} \ge 2$, then

$$\sqrt{(x-\xi)^2+y^2} = |(x,y)-(\xi,0)| \ge |(x,y)| - |\xi| \ge \frac{|(x,y)|}{2},$$

for any $\xi \in [-1,1]$, so that

$$\frac{y}{(x-\xi)^2+y^2} \le \frac{4y}{x^2+y^2} \le \frac{4}{\sqrt{x^2+y^2}} \le 2.$$

Then

$$\frac{1}{\pi}\int_{-1}^{1}\frac{y|u(\xi)+\beta\ln|\xi||}{(x-\xi)^2+y^2}\,d\xi \le \frac{2}{\pi}\|u\|_{L^1(-1,1)} + \frac{2}{\pi}\int_{-1}^{1}|\ln|\xi||\,d\xi = \frac{2}{\pi}\|u\|_{L^1(-1,1)} + \frac{4}{\pi}.$$

This proves (2.2). Since $\beta > \alpha$, we get (2.3). □

Proposition 2.5. *Assume $\alpha \in (0,1)\cup(1,2)$. Let $U \in C(\overline{\mathbb{R}^2_+})$ be a solution to (1.14) such that (2.1) and (2.3) hold. Then, there exists $\rho > 0$ such that $U = U_\rho$ where U_ρ is defined as in (1.13). Moreover, if $\alpha \ge 2$, there is no solution to (1.14) which is continuous in $\overline{\mathbb{R}^2_+}$.*

Proof. Taking

$$V(x,y) = 2U(x,y) + 2(\alpha-1)\ln\sqrt{x^2+y^2}$$

we see that V solves

$$\Delta V = 0 \text{ in } \mathbb{R}^2_+,\ \partial_\nu V = 2e^{\frac{V}{2}} \text{ on } \partial\mathbb{R}^2_+ \setminus \{0\}, \tag{2.5}$$

with

$$\int_{\mathbb{R}^2_+} e^{V(z)}dz < +\infty \quad \text{and} \quad \int_{\partial\mathbb{R}^2_+} e^{\frac{V(z)}{2}}dz < +\infty. \tag{2.6}$$

Since U is continuous at 0, we further have

$$\lim_{(x,y)\to(0,0)} V(x,y) - 2(\alpha-1)\ln\sqrt{x^2+y^2} = U(0,0). \tag{2.7}$$

In [26] it is proved that all the solutions to (2.5)-(2.6) can be written in complex variable as

$$V(z) = \ln \frac{4\lambda^2\gamma^2|z|^{2(\gamma-1)}}{|z^\gamma - z_0|^4},\ z_0 = \rho e^{i\theta_0}. \tag{2.8}$$

where $\rho > 0$, $\gamma > 0$ and the parameters $\lambda > 0$ and θ_0 must satisfy

$$\lambda = -\rho\sin\theta_0 \text{ and } \lambda = \rho\sin(\theta_0 - \pi\gamma), \tag{2.9}$$

or

$$V(z) = \ln \frac{\pi^2}{|z|^2|\ln z - z_0|},$$

where $z_0 \in \mathbb{C}$ and $Im(z_0) = \frac{\pi}{2}$. Since (2.7) holds, we must have that (2.8) hold and $\gamma = \alpha$. Furthermore, in order to have V well defined on $\overline{\mathbb{R}^2_+}$, it is necessary that $\alpha = \gamma \in (0,2)$. Since we are also assuming $\alpha \neq 1$, (2.9) yields

$$\theta_0 = \frac{\pi\alpha}{2} + \pi.$$

Then we have proved that U is given by

$$U(x,y) = \frac{1}{2}\ln\left(\frac{4\alpha^2\rho^2\sin^2\theta_0}{|z^\alpha - \rho e^{i\theta_0}|^4}\right) = U_\rho(x,y)$$

for any $(x,y) \in \mathbb{R}^2_+$. □

As a straightforward consequence of Proposition 2.5 and Lemma 2.4 we get the following classification result for (1.1).

Proposition 2.6. *Assume $\alpha \in (0,1)\cup(1,2)$ and let $u \in L_{\frac{1}{2}}(\mathbb{R})$ be a weak solution to (1.1) such that (1.3) holds. Then there exists $\rho > 0$ such that $u = u_\rho$ where u_ρ is defined as in (1.7).*

We now briefly discuss the equivalence of the linarized problems (1.8) and (1.15). Let us fix $\rho > 0$ and $\alpha \in (0,1)\cup(1,2)$. To simplify the notation, in the following we write $u = u_\rho$, without writing explicitly the dependence on ρ. We consider the space

$$\mathcal{H} := \{\varphi \in L^1_{loc}(\mathbb{R}) \ :\ |x|^{\alpha-1}e^u\varphi^2 \in L^1(\mathbb{R}),\ (-\Delta)^{\frac{1}{4}}\varphi \in L^2(\mathbb{R})\}.$$

We observe that the condition $|x|^{\alpha-1}e^u\varphi^2 \in L^1(\mathbb{R})$ implies $\varphi \in L_{\frac{1}{2}}(\mathbb{R})$. Indeed, we have

$$\int_{\mathbb{R}} \frac{|\varphi(x)|}{1+x^2} = \int_{\mathbb{R}} |x|^{\frac{\alpha-1}{2}}e^{\frac{u}{2}}\frac{1}{|x|^{\frac{\alpha-1}{2}}e^{\frac{u}{2}}}\frac{|\varphi(x)|}{1+x^2} \le \left(\int_{\mathbb{R}} |x|^{\alpha-1}e^u\varphi^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} \frac{1}{|x|^{\alpha-1}e^u}\frac{1}{(1+x^2)^2}\right)^{\frac{1}{2}},$$

where, since $|x|^{\alpha-1}e^u \sim \frac{1}{|x|^{\alpha+1}}$ as $|x| \to +\infty$, $|x|^{\alpha-1}e^u \sim |x|^{\alpha-1}$ as $|x| \to 0$, and $\alpha \in (0,2)$,

$$\int_{\mathbb{R}} \frac{1}{|x|^{\alpha-1}e^u}\frac{1}{(1+x^2)^2} \le C\left(\int_{-1}^{1}\frac{1}{|x|^{\alpha-1}} + \int_{|x|\ge1}\frac{1}{|x|^{3-\alpha}}\right) < +\infty.$$

We now show that φ can grow at most logarithmically as $|x| \to +\infty$.

Lemma 2.7. *Assume that $\varphi \in \mathcal{H}$ is a weak solution to (1.8). Then, there exists $C_1, C_2 > 0$ such that $|\varphi(x)| \le C_1 + C_2\ln|x|$ for any $x \in \mathbb{R}$ with $|x| > 1$.*

Proof. We define
$$v(x)=\frac{1}{\pi}\int_{\mathbb{R}}\ln\left(\frac{1+|y|}{|x-y|}\right)|y|^{\alpha-1}e^{u}\varphi(y)dy.$$
Then
$$|v(x)|\le|C|+\frac{1}{\pi}\int_{\mathbb{R}}\ln(1+y)|y|^{\alpha-1}e^{u(y)}\varphi(y)\,dy+\frac{1}{\pi}\int_{\mathbb{R}}|\ln|x-y|||y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy.$$
By Holder's inequality we have
$$\begin{aligned}&\int_{\mathbb{R}}\ln(1+y)|y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy\\&\qquad\le\left(\int_{\mathbb{R}}\ln^2(1+y)|y|^{\alpha-1}e^{u(y)}\,dy\right)^{\frac12}\left(\int_{\mathbb{R}}|y|^{\alpha-1}e^{u(y)}\varphi^2(y)\,dy\right)^{\frac12}.\end{aligned}$$
Moreover,
$$\begin{aligned}&\int_{|x-y|\ge1}\ln|x-y||y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy\\&\quad\le\int_{|x-y|\ge1}\ln|x||y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy+\int_{|x-y|\ge1}\ln\left(1+\frac{|y|}{|x|}\right))|y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy\\&\quad\le\ln|x|\int_{\mathbb{R}}|y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy+\int_{\mathbb{R}}\ln(1+|y|)\,|y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy\le C(1+\ln|x|),\end{aligned}$$
and
$$\begin{aligned}&\int_{|x-y|\le1}|\ln|x-y||\,|y|^{\alpha-1}e^{u(y)}|\varphi(y)|\,dy\\&\quad\le\left(\int_{|x-y|\le1}\ln^2|x-y||y|^{\alpha-1}e^{u(y)}\,dy\right)^{\frac12}\left(\int_{|x-y|\le1}|y|^{\alpha-1}e^{u(y)}\varphi^2(y)\,dy\right)^{\frac12}\\&\quad\le C\left(\int_{|x-y|\le1}\ln^2|x-y|\,dy\right)^{\frac12}\le C.\end{aligned}$$
We can so conclude that there exist $C_1,C_2>0$ such that
$$|v(x)|\le C_1+C_2\ln|x|. \tag{2.10}$$
In particular $v\in L_{\frac12}(\mathbb{R})$. Moreover, v is a weak solution to $(-\Delta)^{\frac12}v=|x|^{\alpha-1}e^{u}\varphi$. Since $h=\varphi-v$ is $\frac12-$harmonic in $\mathbb{R}$ and $h\in L_{\frac12}(\mathbb{R})$, by Liouville's theorem for the fractional Laplacian (see e.g [23, Theorem 4.4]), we find that h is constant. Then the conclusion follows by (2.10). □

Lemma 2.8. *Assume that $\varphi\in\mathcal{H}$ and let Φ be the harmonic extension of φ. Then, there exists $C_1,C_2>0$ such that*
$$|\Phi(x,y)|\le C_1+C_2\ln(x^2+y^2)\quad\forall(x,y)\in\mathbb{R}^2_+\setminus B_1((0,0)).$$

Proof. Indeed, by Lemma 2.7 we know that
$$-C_1-C_2\ln|\xi|\le\phi(\xi)\le C_1+C_2\ln|\xi|,$$
for any $\xi\in\mathbb{R}$ with $|\xi|\ge1$. Then, the conclusion follows by formula (2.4). □

Let us now consider the linearized problem (1.15). We consider the space
$$\mathcal{H}(\mathbb{R}^2_+):=\left\{Z\in L^1_{loc}(\mathbb{R}^2_+)\ :\ |\nabla Z|\in L^2(\mathbb{R}^2_+)\text{ and }|x|^{2(\alpha-1)}e^{U}Z^2\in L^1(\mathbb{R}^2_+)\right\}.$$

We say that a function $Z \in \mathcal{H}(\mathbb{R}^2_+)$ is a weak solution to (1.15) if

$$\int_{\mathbb{R}^2_+} \nabla Z \cdot \nabla \chi = \int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^U Z \chi, \quad \forall \chi \in C_c^\infty(\overline{\mathbb{R}^2_+}). \tag{2.11}$$

Remark 2.9. *A function $Z \in \mathcal{H}(\mathbb{R}^2_+)$ is a weak solution to (1.15) iff*

$$\int_{\mathbb{R}^2_+} \nabla Z \cdot \nabla \chi = \int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^U Z \chi, \quad \forall \chi \in \mathcal{H}(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+). \tag{2.12}$$

Proof. Assume that $\chi \in \mathcal{H}(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$. Let η be a cut-off function such that $\eta \equiv 1$ for $|x| \le 1$, $\eta \in C_c^\infty(B_2(0,0))$, and $0 \le \eta \le 1$. Given $R > 0$, consider the functions $\eta_R(x) = \eta(\frac{x}{R})$ and $\chi_R(x) = \chi(x)\eta_R(x)$. Then, $\chi_R \in H^1(B_{2R}(0,0) \cap \mathbb{R}^2_+)$. A standard density argument (see e.g. [1, Theorem 3.22]) shows that there exists a sequence of functions $\psi_n \in C_c^\infty(\mathbb{R}^2)$ such that $\psi_n \to \chi_R$ in $H^1(B_{2R}(0,0) \cap \mathbb{R}^2_+)$. For any $n \in \mathbb{N}$, we have the identity

$$\int_{\mathbb{R}^2_+} \nabla Z \cdot \nabla \psi_n = \int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^U Z \psi_n.$$

Using that $\psi_n \to \chi_R$ in $H^1(B_{2R}(0,0) \cap \mathbb{R}^2_+)$, we easily get

$$\int_{\mathbb{R}^2_+} \nabla Z \cdot \nabla \chi_R = \int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^U Z \chi_R$$

By dominated convergence, we have that

$$\int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^U Z \chi_R \to \int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^U Z \chi, \quad \text{as } R \to +\infty.$$

Moreover, we have that

$$\int_{\mathbb{R}^2_+} \nabla Z \cdot \nabla \chi_R = \int_{\mathbb{R}^2_+} (\nabla Z \cdot \nabla \chi)\eta_R + (\nabla Z \cdot \nabla \eta_R)\chi,$$

with

$$\int_{\mathbb{R}^2_+} (\nabla Z \cdot \nabla \chi)\eta_R \to \int_{\mathbb{R}^2_+} \nabla Z \cdot \nabla \chi, \quad \text{as } R \to +\infty.$$

xby dominated convergence, and

$$\begin{aligned} \left| \int_{\mathbb{R}^2_+} (\nabla Z \cdot \nabla \eta_R)\chi \right| &\le \|\chi\|_{L^\infty(\mathbb{R}^2_+)} \|\nabla \eta\|_{L^\infty(\mathbb{R}^2)} \frac{1}{R} \int_{\mathbb{R}^2_+ \cap (B_{2R}(0,0) \setminus B_R(0,0))} |\nabla Z| \\ &\le \sqrt{\frac{3}{2}\pi}\, \|\chi\|_{L^\infty(\mathbb{R}^2_+)} \|\nabla \eta\|_{L^\infty(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2_+ \cap (B_{2R}(0,0) \setminus B_R(0,0))} |\nabla Z|^2 \right)^{\frac{1}{2}} \to 0, \end{aligned}$$

as $R \to +\infty$. $\square$

Remark 2.10. *Using the changes of variables given in Section 3, one can actually show that for any $\chi \in \mathcal{H}(\mathbb{R}^2_+)$, $\int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^u \chi^2 < \infty$. Moreover, a function $Z \in \mathcal{H}(\mathbb{R}^2_+)$ is a weak solution to (2.11) iff (2.12) holds for any χ in $\mathcal{H}(\mathbb{R}^2_+)$.*

Proposition 2.11. *Assume that $\varphi \in \mathcal{H}$ and let Φ be the harmonic extension of φ. Then Φ is a weak solution to (1.15), and (1.16) holds.*

Proof. By Proposition 2.3, we know that

$$\int_{\mathbb{R}^2_+} \nabla \Phi \cdot \nabla \chi = \int_{\partial \mathbb{R}^2_+} |x|^{\alpha-1} e^U \Phi \chi, \quad \forall \chi \in C_c^\infty(\overline{\mathbb{R}^2_+} \setminus \{(0,0)\}).$$

We use a cut-off argument to show that (2.11) holds. Fix $\chi \in C_c^\infty(\overline{\mathbb{R}^2_+})$, and let $\eta \in C_c^\infty(B_2(0,0))$ be such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $B_1(0,0)$. For any $\varepsilon > 0$ we denote

$\eta_\varepsilon(x) = (1-\eta(\frac{x}{\varepsilon}))$. If $\chi \in C_c^\infty(\overline{\mathbb{R}^2_+})$, then $\chi_\varepsilon := \chi\eta_\varepsilon \in C_c^\infty(\overline{\mathbb{R}^2_+} \setminus \{0\})$. Then, for any $\varepsilon > 0$, we have

$$\int_{\mathbb{R}^2_+} \nabla\Phi \cdot \nabla\chi_\varepsilon = \int_{\partial\mathbb{R}^2_+} |x|^{\alpha-1} e^U \Phi \chi_\varepsilon\,.$$

Noting that $\nabla\chi_\varepsilon = \eta_\varepsilon \nabla\chi + \chi\nabla\eta_\varepsilon$ and that

$$\begin{aligned}\left|\int_{\mathbb{R}^2_+} (\nabla\Phi\cdot\nabla\eta_\varepsilon)\chi\right| &\leq \|\chi\|_{L^\infty(\mathbb{R}^2_+)}\|\nabla\eta\|_{L^\infty(\mathbb{R}^2)}\frac{1}{\varepsilon}\int_{\mathbb{R}^2_+\cap(B_{2\varepsilon}(0,0)\setminus B_\varepsilon(0,0))}|\nabla\Phi| \\ &\leq \|\chi\|_{L^\infty(\mathbb{R}^2_+)}\|\nabla\eta\|_{L^\infty(\mathbb{R}^2)}\sqrt{\frac{3}{2}\pi}\left(\int_{\mathbb{R}^2_+\cap(B_{2\varepsilon}(0,0)\setminus B_\varepsilon(0,0))}|\nabla\Phi|^2\right)^{\frac12} \to 0,\end{aligned}$$

as $\varepsilon \to 0$, we can use dominated convergence theorem, since $\eta_\varepsilon \to 1$ pointwise on $\overline{\mathbb{R}^2_+} \setminus \{(0,0)\}$, to get

$$\int_{\mathbb{R}^2_+} \nabla\Phi\cdot\nabla\chi = \int_{\partial\mathbb{R}^2_+} |x|^{\alpha-1}e^U\Phi\chi\,.$$

Now (2.12) follows by Remark 2.9. Finally, we observe that (1.16) is a consequence of Lemma 2.8 and Proposition 2.3. □

Remark 2.12. *Proposition 2.11 shows, in particular, that Theorem 1.1 follows by Theorem 1.2.*

3. Proof of Theorem 1.2

In this section, we will transform our problem into an equivalent one via conformal transformations. First, from the upper half-space to a cone, and then to a bounded domain which will be determined by an intersection or union of balls depending on the values of $\alpha \in (0,1)\cup(1,2)$. In the cone we will obtain a linear problem with Neumann-type boundary conditions which, in the bounded domain, will become a Steklov eigenvalue problem. This will allow us to prove the main result of the paper.

For sake of simplicity we rewrite (1.13) as

$$U(z) = \ln\frac{2\alpha|\xi_2|}{|z^\alpha - \xi|^2}, \text{ with } \xi = \xi_1 + i\xi_2,\ \xi_2 < 0,\ \frac{\xi_1}{\xi_2} = \frac{1+\cos\alpha\pi}{\sin\alpha\pi}.$$

3.1. An equivalent problem on a cone.

Let us consider the cone

$$\mathcal{C}_\alpha := \{(r\cos\theta, r\sin\theta)\ :\ r>0,\ \theta\in[0,\pi\alpha)\}\,. \tag{3.1}$$

Let $F_\alpha : \mathbb{R}^2_+ \to \mathcal{C}_\alpha$ be the complex power z^α, which using polar coordinates is written as

$$F_\alpha(x,y) := (r^\alpha\cos\alpha\theta, r^\alpha\sin\alpha\theta),\ x = r\cos\theta,\ y = r\sin\theta.$$

It is known that F_α is a conformal diffeomorphism between $\mathbb{R}^2_+$ and $\mathcal{C}_\alpha$. A straightforward computation shows that a function Φ solves the linear problem (1.15) if and only if the function $\phi = \Phi\circ F_\alpha^{-1}$ solves the linear problem

$$-\Delta\phi = 0 \text{ in } \mathcal{C}_\alpha,\ \partial_\nu\phi = e^{W_\alpha}\phi \text{ on } \partial\mathcal{C}_\alpha, \tag{3.2}$$

where

$$W_\alpha(x,y) := U\left(F_\alpha^{-1}(x,y)\right) - \ln\alpha = \ln\frac{2|\xi_2|}{(x-\xi_1)^2+(y-\xi_2)^2}. \tag{3.3}$$

In fact, for sake of completeness we give a brief proof of the claim. If Φ solves (1.15) for any $\chi \in \mathcal{H}(\mathbb{R}^2_+)\cap L^\infty(\mathbb{R}^2_+)$, we have

$$\int_{\mathbb{R}^2_+}\nabla\Phi\cdot\nabla\chi = \int_{\partial\mathbb{R}^2_+}|x|^{\alpha-1}e^U\Phi\chi.$$

First, we point out that

$$\int\limits_{\mathbb{R}^2_+} \nabla\Phi \cdot \nabla\chi dxdy$$

(setting $x = r\cos\theta,\ y = r\sin\theta$, $\hat{\Phi}(r,\theta) := \Phi(r\cos\theta, r\sin\theta)$ and $\hat{\chi}(r,\theta) := \chi(r\cos\theta, r\sin\theta)$)

$$= \int\limits_0^\infty \int\limits_0^\pi \left(r\partial_r\hat{\Phi}\partial_r\hat{\chi} + \frac{1}{r}\partial_\theta\hat{\Phi}\partial_\theta\hat{\chi} \right) drd\theta$$

(setting $\rho = r^\alpha$, $\gamma = \alpha\theta$, $\tilde{\Phi}(\rho,\gamma) := \hat{\Phi}\left(\rho^{\frac{1}{\alpha}}, \frac{\gamma}{\alpha}\right)$ and $\tilde{\chi}(\rho,\gamma) := \hat{\chi}\left(\rho^{\frac{1}{\alpha}}, \frac{\gamma}{\alpha}\right)$)

$$= \int\limits_0^\infty \int\limits_0^{\alpha\pi} \left(\rho\partial_\rho\tilde{\Phi}\partial_\rho\tilde{\chi} + \frac{1}{\rho}\partial_\gamma\tilde{\Phi}\partial_\gamma\tilde{\chi} \right) d\rho d\gamma$$

(setting $s = \rho\cos\gamma,\ t = \rho\cos\gamma$, $\phi(s,t) := \tilde{\Phi}(\rho,\gamma)$ and $\upsilon(x,y) := \tilde{\chi}(\rho,\gamma)$)

$$= \int\limits_{\mathcal{C}_\alpha} \nabla\phi\nabla\upsilon dsdt$$

Next, since on $\partial\mathbb{R}^2_+$

$$e^{U(x,0)} = \frac{2\alpha|\xi_2|}{(x^\alpha - \xi_1)^2 + |\xi_2|^2}, \qquad \text{if } x \geq 0,$$

and

$$e^{U(x,0)} = \frac{2\alpha|\xi_2|}{(|x|^\alpha\cos\alpha\pi - \xi_1)^2 + (|x|^\alpha\sin\alpha\pi - \xi_2)^2}, \qquad \text{if } x \leq 0,$$

we also have

$$\begin{aligned}
&\int\limits_{\partial\mathbb{R}^2_+} |x|^{\alpha-1}e^U\Phi\chi \\
&= \int\limits_{\{(x,0):x\geq 0\}} \frac{2\alpha|\xi_2||x|^{\alpha-1}}{(|x|^\alpha - \xi_1)^2 + |\xi_2|^2}\Phi\chi + \int\limits_{\{(x,0):x\leq 0\}} \frac{2\alpha|\xi_2||x|^{\alpha-1}}{(|x|^\alpha\cos\alpha\pi - \xi_1)^2 + |\xi_2|^2}\Phi\chi \\
&= \int\limits_0^\infty \frac{2|\xi_2|}{(\sigma - \xi_1)^2 + |\xi_2|^2}\phi\upsilon d\sigma + \int\limits_0^\infty \frac{2|\xi_2|}{(\sigma\cos\alpha\pi - \xi_1)^2 + (\sigma\sin\alpha\pi - \xi_2)^2}\phi\upsilon d\sigma \\
&= \int\limits_{\partial_-\mathcal{C}_\alpha} e^{W_\alpha(s,t)}\phi\upsilon + \int\limits_{\partial_+\mathcal{C}_\alpha} e^{W_\alpha(s,t)}\phi\upsilon = \int\limits_{\partial\mathcal{C}_\alpha} e^{W_\alpha(s,t)}\phi\upsilon,
\end{aligned}$$

because

$$\partial\mathcal{C}_\alpha := \underbrace{\{(\sigma,0)\ :\ \sigma \geq 0\}}_{:=\partial_-\mathcal{C}_\alpha} \cup \underbrace{\{(\sigma\cos\pi\alpha, \sigma\sin\pi\alpha)\ :\ \sigma \geq 0\}}_{:=\partial_+\mathcal{C}_\alpha}. \tag{3.4}$$

Finally, we deduce that for any $\upsilon \in L^\infty(\mathcal{C}_\alpha)$ and thus, such that $\int\limits_{\partial\mathcal{C}_\alpha} e^W\upsilon^2 < +\infty$, if υ satisfies

$$\int\limits_{\mathcal{C}_\alpha} |\nabla\upsilon|^2 < +\infty, \tag{3.5}$$

then,

$$0 = \int\limits_{\mathcal{C}_\alpha} \nabla\phi\nabla\upsilon - \int\limits_{\partial\mathcal{C}_\alpha} e^{W_\alpha(s,0)}\phi\upsilon,$$

that is, ϕ solves (3.2).

3.2. **An equivalent problem on a bounded domain.** Let us consider the map

$$G_\alpha(x,y)=\left(\frac{|\xi|^2-x^2-y^2}{(x-\xi_1)^2+(y-\xi_2)^2},\frac{2(y\xi_1-x\xi_2)}{(x-\xi_1)^2+(y-\xi_2)^2}\right),\ (x,y)\in\mathbb{R}^2\setminus\{\xi\}. \tag{3.6}$$

Lemma 3.1. *The function G_α given by (3.6) satisfies the following properties:*

(1) *G is a conformal diffeomorphism between $\mathbb{R}^2\setminus\{\xi\}$ and $\mathbb{R}^2\setminus\{(-1,0)\}$.*

(2) *The Jacobian of G is given by (see (3.3))*

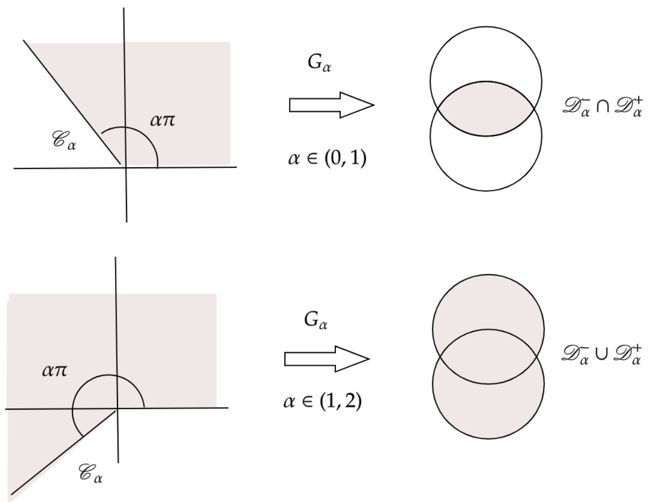
$$JG_\alpha(x,y)=\frac{4|\xi|^2}{((x-\xi_1)^2+(y-\xi_2)^2)^2}=\frac{|\xi|^2}{|\xi_2|^2}e^{2W_\alpha(x,y)}. \tag{3.7}$$

(3) *The image of the cone (3.1), i.e., $G_\alpha(\mathcal{C}_\alpha)$ is*

$$\Omega_\alpha:=\mathscr{D}_\alpha^-\cap\mathscr{D}_\alpha^+ \text{ if } \alpha\in(0,1) \quad\text{or}\quad \Omega_\alpha:=\mathscr{D}_\alpha^-\cup\mathscr{D}_\alpha^+ \text{ if } \alpha\in(1,2),$$

where

$$\mathscr{D}_\alpha^\pm:=\left\{(x,y)\in\mathbb{R}^2\ :\ x^2+(y\pm\tau_\alpha)^2\le 1+\tau_\alpha^2\right\},\ \tau_\alpha:=\frac{1+\cos\alpha\pi}{\sin\alpha\pi}. \tag{3.8}$$



Proof. First of all, we remind that $\frac{\xi_1}{\xi_2}=\tau_\alpha$ and $\frac{|\xi|}{|\xi_2|}=\sqrt{1+\tau_\alpha^2}$.
Now, (1) follows from the complex representation of G_α as

$$G_\alpha(x,y)=g(z)=-\frac{z+\xi}{z-\xi}.$$

Property (2) follows from a direct computation:

$$\mathrm{J}G_\alpha(x,y)=|g'(z)|^2=\frac{4|\xi|^2}{|z-\xi|^4}.$$

To prove (3), first we note that if $\Pi_\alpha:=\{(x,y)\in\mathbb{R}^2\ :\ -x\sin\pi\alpha+y\cos\pi\alpha\le 0\}$, the cone is given by

$$\mathcal{C}_\alpha=\Pi_\alpha\cap\mathbb{R}^2_+,\ \text{if } 0<\alpha<1 \ \text{ and }\ \mathcal{C}_\alpha=\Pi_\alpha\cup\mathbb{R}^2_+,\ \text{if } 1<\alpha<2. \tag{3.9}$$

The claim follows once we prove that

$$G_\alpha(\mathbb{R}^2_+)=\mathscr{D}_\alpha^- \text{ and } G_\alpha(\Pi_\alpha)=\mathscr{D}_\alpha^+. \tag{3.10}$$

Next, we observe that G_α maps the boundary of the half-spaces $\mathbb{R}^2_+$ and Π_α into the boundary of the two disks of radius $\frac{|\xi|}{|\xi_2|}$ centered at the points $\frac{\xi_1}{\xi_2}$ and $-\frac{\xi_1}{\xi_2}$, respectively. Indeed a direct computation shows that

$$\left|G_\alpha(x,0)+\left(0,\frac{\xi_1}{\xi_2}\right)\right|^2=\frac{(x+\xi_1)^2+\xi_2^2}{(x-\xi_1)^2+\xi_2^2}-\frac{4\xi_1 x}{(x-\xi_1)^2+\xi_2^2}+\frac{\xi_1^2}{\xi_2^2}=1+\frac{\xi_1^2}{\xi_2^2}=\frac{|\xi|^2}{\xi_2^2},\ \text{for any } x\in\mathbb{R},$$

and

$$\left|G_\alpha(r\cos\pi\alpha, r\sin\pi\alpha) - \left(0, \frac{\xi_1}{\xi_2}\right)\right|^2 = 1 + \frac{\xi_1^2}{\xi_2^2} = \frac{|\xi|^2}{\xi_2^2}, \text{ for any } r > 0.$$

Finally, we point out that G_α maps the point ξ at ∞. Moreover $\xi \notin \mathbb{R}^2_+$ because $\xi_2 < 0$ and $\xi \notin \Pi_\alpha$ because

$$-\xi_1 \sin\pi\alpha + \xi_2\cos\pi\alpha = \xi_2\left(-\frac{1+\cos\alpha\pi}{\sin\alpha\pi}\sin\pi\alpha + \cos\pi\alpha\right) = -\xi_2 > 0.$$

Therefore, by (3.9) together with the fact that G_α maps the boundary of the half-spaces into the boundary of the disks, we deduce (3.10).

$\square$

Let $\psi(x,y) = \phi\left(G_\alpha^{-1}(x,y)\right)$. Thanks to Lemma 3.1 we see that ϕ solves the linear problem (3.2) if and only if ψ is a solution to the Steklov problem

$$\Delta\psi = 0 \text{ in } \Omega_\alpha,\ \partial_\nu\psi = \frac{1}{\sqrt{1+\tau_\alpha^2}}\psi \text{ in } \partial\Omega_\alpha. \tag{3.11}$$

For the sake of completeness, let us prove the claim. If ϕ solves (3.2), for any $\upsilon \in L^\infty(\mathcal{C}_\alpha)$ satisfying (3.5), it holds

$$0 = \int\limits_{\mathcal{C}_\alpha} \nabla\phi\nabla\upsilon - \int\limits_{\partial\mathcal{C}_\alpha} e^{W_\alpha(s,0)}\phi\upsilon.$$

We set $\Upsilon = \upsilon \circ G_\alpha^{-1}$. On the one hand, via the change of variables $G_\alpha^{-1}(x,y) = (s,t)$ (taking into account that G_α is a conformal map), we have

$$\begin{aligned}\int\limits_{\mathcal{C}_\alpha} \nabla\phi(s,t)\cdot\nabla\upsilon(s,t)dsdt &= \int\limits_{\Omega_\alpha} \mathtt{det}(DG_\alpha^{-1})(x,y)\nabla\phi\left(G_\alpha^{-1}(x,y)\right)\cdot\nabla\upsilon\left(G_\alpha^{-1}(x,y)\right)dxdy \\ &= \int\limits_{\Omega_\alpha} DG_\alpha^{-1}(x,y)\nabla\phi\left(G_\alpha^{-1}(x,y)\right)dG_\alpha^{-1}(x,y)\cdot\nabla\upsilon\left(G_\alpha^{-1}(x,y)\right)dxdy \\ &= \int\limits_{\Omega_\alpha} \nabla\psi(x,y)\cdot\nabla\Upsilon(x,y)dxdy.\end{aligned}$$

On the other hand, we can assert that

$$\begin{aligned}\int\limits_{\mathcal{C}_\alpha} \nabla\phi(s,t)\cdot\nabla\upsilon(s,t)dsdt &= \int\limits_{\partial_-\mathcal{C}_\alpha} e^{W_\alpha(s,t)}\phi\upsilon + \int\limits_{\partial_+\mathcal{C}_\alpha} e^{W_\alpha(s,t)}\phi\upsilon \\ &= \frac{1}{\sqrt{1+\tau_\alpha^2}}\int\limits_{\partial\mathscr{D}_\alpha^-\cap\partial\Omega_\alpha}\psi\Upsilon + \frac{1}{\sqrt{1+\tau_\alpha^2}}\int\limits_{\partial\mathscr{D}_\alpha^+\cap\partial\Omega_\alpha}\psi\Upsilon \\ &= \frac{1}{\sqrt{1+\tau_\alpha^2}}\int\limits_{\partial\Omega_\alpha}\psi\Upsilon,\end{aligned}$$

because of (3.7), (3.8), (3.4) and

$$G_\alpha(\mathcal{C}_\alpha) = \Omega_\alpha,\ G_\alpha(\partial_+\mathcal{C}_\alpha) = \partial\mathscr{D}_\alpha^+\cap\partial\Omega_\alpha \text{ and } G_\alpha(\partial_-\mathcal{C}_\alpha) = \partial\mathscr{D}_\alpha^-\cap\partial\Omega_\alpha.$$

Therefore,

$$0 = \int\limits_{\Omega_\alpha}\nabla\psi\nabla\Upsilon - \frac{1}{\sqrt{1+\tau_\alpha^2}}\int\limits_{\partial\Omega_\alpha}\psi\Upsilon, \tag{3.12}$$

for any $\Upsilon \in H^1(\Omega_\alpha)\cap L^\infty(\Omega_\alpha)$. Since $H^1(\Omega_\alpha)\cap L^\infty(\Omega_\alpha)$ is a dense subspace of $H^1(\Omega_\alpha)$, (3.12) holds for any $\Upsilon \in H^1(\Omega_\alpha)$, namely ψ solves (3.11).

3.3. **Proof of Theorem 1.2: conclusion.** It is immediate to check that $\mu_\alpha := \frac{1}{\sqrt{1+\tau_\alpha^2}}$ is an eigenfunction of the Steklov problem (3.11) and also that

$$\psi(x,y)=x,$$

is an associate eigenfunction.

In Section 4 and, in particular, in Proposition 4.1 (i.e. $\alpha\in(0,1)$) and Proposition 4.2 (i.e. $\alpha\in(1,2)$) we prove that μ_α is simple. Thus, using all the previous arguments, we deduce that all the solutions to (1.15) are a scalar multiple of the function

$$\Phi(x,y)=(\psi\circ G_\alpha\circ F_\alpha)(x,y)=\frac{|\xi|^2-(x^2+y^2)^\alpha}{|(x+iy)^\alpha-\xi_1-i\xi_2|^2},$$

concluding the proof of Theorem 1.2.

4. On the simplicity of the eigenvalue of the Steklov problem

This last section is devoted to the study of the Steklov eigenvalue problem (3.11) and, in particular, to proving the simplicity of a given eigenvalue, which will allow us to conclude the proof of Theorem 1.2, as anticipated in Section 3.3. We also include a more general result showing that the second eigenvalue of the Steklov problem is always simple, when it is posed on an ellipse with different axes. For simplicity of notation, we omit here the underscore index α. Let us consider the disks $\mathscr{D}^\pm$ as defined in (3.8), and denote by Ω either their intersection or their union, i.e.,

$$\Omega:=\mathscr{D}^+\cap\mathscr{D}^- \text{ or } \Omega:=\mathscr{D}^+\cup\mathscr{D}^-,$$

whose boundary is

$$\partial\Omega:=\gamma^+\cup\gamma^-,$$

where

$$\gamma^\pm:=\left\{(x,y)\in\partial\Omega\ :\ x^2+(y\pm\tau)^2=1+\tau^2\right\}.$$

Let us also consider the Steklov eigenvalue problem

$$\Delta u=0 \text{ in } \Omega,\ \partial_\nu u=\mu u \text{ on } \partial\Omega. \tag{4.1}$$

A direct computation shows that $\mu=\frac{1}{\sqrt{1+\tau^2}}$ is an eigenvalue and an associate eigenfunction is $u(x,y)=x$. We will show that μ is simple.
We have to distinguish two cases corresponding to Ω being the union or the intersection of the two disks:

Proposition 4.1. *If $\Omega:=\mathscr{D}^+\cap\mathscr{D}^-$, then $\mu=\frac{1}{\sqrt{1+\tau^2}}$ is the second Steklov eigenvalue of (4.1), and it is simple.*

Proof. First of all, we prove that μ is exactly the second eigenvalue using the lower bound found by Kuttler and Sigillito in [30]. More precisely, we observe that the domain Ω is a piecewise smooth bounded domain with two axes of symmetries whose boundary $\partial\Omega$ can be represented as

$$y=\pm g(x),\ g(x):=\left(\sqrt{1+\tau^2-x^2}-\tau\right),\ -1\le x\le 1,$$

or

$$x=\pm f(y),\ f(y)=\begin{cases}\sqrt{1+\tau^2-(y+\tau)^2} \text{ if } 0\le y\le\sqrt{1+\tau^2}-\tau,\\ \sqrt{1+\tau^2-(y-\tau)^2} \text{ if } -\sqrt{1+\tau^2}+\tau\le y\le 0.\end{cases}$$

Therefore the second eigenvalue Λ_2 must satisfy

$$\frac{1}{\Lambda_2}\le\max\left\{\max_{0\le x\le 1} g(x)\sqrt{1+|g'(x)|^2},\ \max_{0\le y\le\sqrt{1+\tau^2}-\tau} f(y)\sqrt{1+|f'(y)|^2}\right\}=\sqrt{1+\tau^2},$$

because straightforward computations show that

$$\max_{0\le x\le 1} g(x)\sqrt{1+|g'(x)|^2}=\sqrt{1+\tau^2}-\tau,$$

and

$$\max_{0\le y\le \sqrt{1+\tau^2}-\tau} f(y)\sqrt{1+|f'(y)|^2}=\sqrt{1+\tau^2}.$$

Since $\mu=\frac{1}{\sqrt{1+\tau^2}}$, it must be the second eigenvalue.
Finally, we prove that it is simple. Assume φ is an eigenfunction associated with μ. Since μ is the second eigenvalue, we know that the nodal line of φ is an axis of symmetry (see [30]). Without loss of generality, we can assume that φ is positive in the first quadrant. Therefore two cases can be distinguished.

(i) φ is even in x and odd in y and its nodal line is the $x-$axis. Then φ solves the mixed Dirichlet-Neumann-Steklov problem

$$\begin{cases}\Delta\varphi=0 \text{ in } D:=\{(x,y)\in\Omega\ :\ x\ge 0,\ y\ge 0\}\,,\\ \partial_\nu\varphi=\dfrac{1}{\sqrt{1+\tau^2}}\varphi \text{ on } \Gamma_S:=\{(x,y)\in\partial\Omega\ :\ x>0,\ y>0\}\,,\\ \varphi=0 \text{ on } \Gamma_D:=\{0\le x\le 1, y=0\}\,,\\ \partial_\nu\varphi=0 \text{ on } \Gamma_N:=\left\{x=0,\ 0\le y\le\sqrt{1+\tau^2}-\tau\right\}.\end{cases}$$

We test this problem by the function $v(x,y)=y$, and we get

$$0=\int_{\partial D}(v\partial_\nu\varphi-\varphi\partial_\nu v)d\sigma=\frac{1}{\sqrt{1+\tau^2}}\int_{\Gamma_S} y\varphi d\sigma-\frac{1}{\sqrt{1+\tau^2}}\int_{\Gamma_S}(y+\tau)\varphi d\sigma,$$

which leads to contradiction since $\tau\neq 0$ and $\varphi>0$ on Γ_S.

(ii) φ is odd in x and even in y and its nodal line is the $y-$axis. In particular, it satisfies an orthogonality condition with the eigenfunction $u(x,y)=x$, which reads as

$$0=\int_{\partial\Omega}x\varphi(x,y)d\sigma=4\int_{\Gamma_S}x\varphi(x,y)d\sigma.$$

Hence we deduce that φ must change sign on Γ_S and a contradiction arises.

That concludes the proof. □

Proposition 4.2. *If $\Omega:=\mathscr{D}^+\cup\mathscr{D}^-$, then $\mu=\frac{1}{\sqrt{1+\tau^2}}$ is not the second Steklov eigenvalue of* (4.1)*, but it is simple.*

Proof. First of all, we point out that μ is not the second eigenvalue, because by the isoperimetric inequality (see [30, 42])

$$\Lambda_2\le\frac{2\pi}{L}<\frac{1}{\sqrt{\tau^2+1}}=\mu,$$

where L is the length of $\partial\Omega$. The proof carried out in the previous case cannot work in this framework. We will use a different argument.
Let u be an associated eigenfunction to μ. First of all, we point out that since the domain D has two axes of symmetry, every eigenfunction u can be assumed to have one of these symmetries (see [30]):

(i) u is *even-even* if $u(x,y)$ is even in both x and y,
(ii) u is *odd-even* if $u(x,y)$ is odd in x and even in y,
(iii) u is *even-odd* if $u(x,y)$ is even in x and odd in y,
(iv) u is *odd-odd* if $u(x,y)$ is odd in both x and y.

We are going to show that the nodal lines of u must coincide with one of the axes of symmetries. Assume u has a nodal line Γ_D which is not entirely contained in the axes of symmetries. We choose a connected component N of the set $\{(x,y) \in D \,:\, u(x,y) \geq 0\}$ whose boundary is $\partial N := \Gamma_S \cup \Gamma_D$ where Γ_S is the part of the boundary of D where u satisfies the Steklov boundary condition. We also observe that, up to a change of sign, the function u solves the problem

$$\begin{cases} \Delta u = 0 \text{ in } N, \\ \partial_\nu u = \mu u \text{ on } \Gamma_S, \\ u = 0 \text{ in } \Gamma_D, \\ u > 0 \text{ in } N. \end{cases} \tag{4.2}$$

We consider the function $w(x,y) = x$ which solve the Steklov problem in the domain D, i.e.,

$$\Delta w = 0 \text{ in } D, \ \partial_\nu w = \mu w \text{ on } \partial D := \partial\mathscr{D}^+ \cup \partial\mathscr{D}^-.$$

Next, we test problem (4.2) by the function φ, and we get

$$\begin{aligned} 0 &= \int\limits_{\partial N} (u\partial_\nu w - w\partial_\nu u) d\sigma \\ &= \underbrace{\int\limits_{\Gamma_S} (u\partial_\nu w - w\partial_\nu u) d\sigma}_{=0} + \int\limits_{\Gamma_D} (u\partial_\nu w - w\partial_\nu u) d\sigma, \end{aligned}$$

which implies

$$\int\limits_{\Gamma_D} x\partial_\nu u d\sigma = 0. \tag{4.3}$$

Now, Hopf's Lemma ensures that $\partial_\nu u(x,y) < 0$ on Γ_D, because $u > 0$ in N and $u = 0$ on Γ_D. If Γ_D is entirely contained in the set $\{(x,y) \in D \,:\, x \geq 0\}$, by (4.3) and Hopf's Lemma a contradiction immediately arises. As this happens for sure when u is odd-even or odd-odd, since u vanishes on the $y-$axis, we exclude those possibilities. When u is even-odd or even-even instead, it may happen that the nodal line Γ_D crosses the $y-$axes, but in this case using the symmetry (namely the eveness in the x variable), we get

$$\int\limits_{\Gamma_D} x\partial_\nu u d\sigma = \int\limits_{\Gamma_D \cap \{x \geq 0\}} x\partial_\nu u d\sigma + \int\limits_{\Gamma_D \cap \{x \leq 0\}} x\partial_\nu u d\sigma = 2 \int\limits_{\Gamma_D \cap \{x \geq 0\}} x\partial_\nu u d\sigma > 0,$$

and again a contradiction arises by Hopf's Lemma.
Therefore, the nodal line of u must coincide with one of the axes of symmetries. We can assume that u is positive in the first quadrant. We will show that u is a multiple of x.

Let us denote $\Gamma_{\mathtt{x}} := \Omega_\alpha \cap \{(t,0) \,:\, t \in \mathbb{R}, t > 0\}$ and $\Gamma_{\mathtt{y}} := \Omega_\alpha \cap \{(0,t) \,:\, t \in \mathbb{R}, t > 0\}$. We have to distinguish two cases.

(i) if $u = 0$ on $\Gamma_{\mathtt{x}}$ then u has to be even-odd. In fact, if it were odd-odd, it should vanish on both axes, but this is not possible because of the previous discussion. Moreover, if it is even in the $y-$variable, the partial derivative $\partial_y u$ vanishes on $\Gamma_{\mathtt{x}}$ so that $\nabla u = 0$ on $\Gamma_{\mathtt{x}}$ and this is not possible because because of Hopf's Lemma. Finally, we get a contradiction arguing as in (i) of Proposition 4.1.

(ii) if $u = 0$ on $\Gamma_{\mathtt{y}}$ then u has to be odd-even. In fact, if it were odd-odd, it should vanish on both axes, but this is not possible because of the previous discussion. Moreover, if it is even in the $x-$variable, the partial derivative $\partial_x u$ vanishes on $\Gamma_{\mathtt{y}}$ so that $\nabla u = 0$ on $\Gamma_{\mathtt{y}}$ and this is not possible because of Hopf's Lemma. Now, we can prove that u cannot be orthogonal at the eigenfunction x arguing as in (ii) of Proposition 4.1.

That concludes the proof.

□

Finally, we apply the argument used in the proof of Proposition 4.1 to prove the simplicity of the second eigenvalue for the Steklov problem in an ellipse.

Proposition 4.3. *Let* $\Omega := \left\{(x,y) \in \mathbb{R}^2 \ : \ a^2x^2 + b^2y^2 = 1\right\}$. *The second eigenvalue of the Steklov problem is simple if* $a \neq b$.

Proof. Without loss of generality, we can assume $b < a$. Therefore, if μ denotes the second eigenvalue, it satisfies the following lower bound (see [30])

$$\mu \geq \min\{a, b\} = b. \tag{4.4}$$

Now, assume φ is an eigenfunction associated with μ. Since μ is the second eigenvalue we know that the nodal line of φ is an axis of symmetry. Without loss of generality, we can assume that φ is positive in the first quadrant. Therefore two cases can be distinguished.

(i) φ is even in x and odd in y and its nodal line is the $x-$axis. Then φ solves the mixed Dirichlet-Neumann-Steklov problem

$$\begin{cases} \Delta\varphi = 0 \text{ in } D := \{(x,y) \in \Omega \ : \ x \geq 0, \ y \geq 0\}, \\ \partial_\nu \varphi = \mu\varphi \text{ on } \Gamma_S := \{(x,y) \in \partial\Omega \ : \ x > 0, \ y > 0\}, \\ \varphi = 0 \text{ on } \Gamma_D := \{0 \leq x \leq 1/a, y = 0\}, \\ \partial_\nu \varphi = 0 \text{ on } \Gamma_N := \{x = 0, \ 0 \leq y \leq 1/b\}. \end{cases}$$

We test this problem by the function $v(x,y) = y$, and taking into account that

$$\partial_\nu v = \frac{b^2 y}{\sqrt{a^2 - b^2(a^2-b^2)y^2}} \text{ on } \partial\Omega,$$

we get

$$\begin{aligned} 0 &= \int_{\partial D} (v\partial_\nu\varphi - \varphi\partial_\nu v) d\sigma = \int_{\Gamma_S} y\varphi \left(\mu - \frac{b^2}{\sqrt{a^2 - b^2(a^2-b^2)y^2}}\right) d\sigma \\ &\geq \left(\mu - \max_{0 \leq y \leq 1/b} \frac{b^2}{\sqrt{a^2 - b^2(a^2-b^2)y^2}}\right) \int_{\Gamma_S} y\varphi d\sigma \geq 0 \end{aligned}$$

because of (4.4). Finally a contradiction arises since $\varphi \geq 0$ on Γ_S.

(ii) φ is odd in x and even in y and its nodal line is the $y-$axis. Assume by contradiction that there exist two eigenfunctions φ_1 and φ_2 with such properties, then they have to satisfy an orthogonality condition which reads as

$$0 = \int_{\partial\Omega} \varphi_1\varphi_2 d\sigma = 4 \int_{\Gamma_S} \varphi_1\varphi_2 d\sigma,$$

where $\Gamma_S := \{(x,y) \in \partial\Omega \ : \ x > 0, \ y > 0\}$. Hence we deduce that $\varphi_1\varphi_2$ must change sign on Γ_S and a contradiction arises.

That concludes the proof. □

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