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Non-trivial r-wise agreeing families

Peter Frankl* and Andrey Kupavskii†

Abstract

A family of sets is r-wise agreeing if for any r sets from the family there is an element x that is either contained in all or contained in none of the r sets. The study of such families is motivated by questions in discrete optimization. In this paper, we determine the size of the largest non-trivial r-wise agreeing family. This can be seen as a generalization of the classical Brace-Daykin theorem.

Let $[n] = \{1, 2, \ldots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. Let $\mathcal{F} \subset 2^{[n]}$ be a family. We say that sets $F_1, \ldots, F_r \subset [n]$ *agree on a coordinate* $x \in [n]$ if either $x \in \cap_{i\in[t]} F_i$ or $x \notin \cup_{i\in[r]} F_i$. We call a family $\mathcal{F}$ *r-wise t-agreeing* if any r sets from $\mathcal{F}$ agree on at least t coordinates. For $t = 1$ we call such families *r-wise agreeing* for shorthand. Additionally, we call $\mathcal{F}$ *non-trivial* if $\cap_{A\in\mathcal{F}} A = \emptyset$ and $\cup_{A\in\mathcal{F}} A = [n]$ (that is, all sets from $\mathcal{F}$ do not agree on a coordinate).

r-wise agreeing families appear in the context of packing and covering problems in combinatorial optimization [1, 2]. In particular, Abdi et al. [1] proposed a conjecture that states that non-trivial r-wise agreeing families cannot be cube-ideal for sufficiently large r (cf. [1] for the definition of cube-idealness). While discussing possible strategies to attack this conjecture with Ahmad Abdi, the following question was raised:

How big can a non-trivial r-wise agreeing family be?

The goal of this note is to answer this question. We prove the following theorem.

Theorem 1. *Let $n > r \geq 2$ and $t \leq 2^r - r - 1$. Suppose that $\mathcal{F} \subset 2^{[n]}$ is non-trivial r-wise t-agreeing. Then $|\mathcal{F}| \leq (r + t + 1)2^{n-r-t}$.*

Let us mention that in the case $r = 2, t = 1$ the bound is exactly 2^{n-1}. The proof in this case is easy: it is immediate to see that $\mathcal{F}$ contains at most 1 set out of each pair of complementary sets $A, [n] \setminus A$. The same argument shows that any r-wise agreeing family has size at most 2^{n-1}. At the same time, the family of all sets not containing 1 provides an example of an r-wise agreeing family of size 2^{n-1}. If we drop the non-triviality assumption then the family with some fixed t coordinates provides a lower bound of 2^{n-t} for the size of the largest r-wise t-agreeing family. Theorem 1 shows that non-triviality forces the family to be considerably smaller.

It is natural to draw parallel with union[1] families. A family $\mathcal{F} \subset 2^{[n]}$ is *r-wise t-union* if $|A_1 \cup \ldots \cup A_r| \leq n - t$ for any $A_1, \ldots, A_r \in \mathcal{F}$. It is *non-trivial* if $\cup_{A\in\mathcal{F}} A = [n]$. Note that r-wise t-union families are r-wise agreeing, but not vice versa.

*Rényi Institute, Budapest, Hungary; Email: peter.frankl@gmail.com

†Moscow Institute of Physics and Technology, St. Petersburg State University, Russia; Email: kupavskii@ya.ru

[1]A notion dual to intersecting families, which is more convenient to work with here.

Theorem 2 (Brace–Daykin–Frankl theorem [3, 4, 5])**.** *Let $n > r \geq 2$ and $t \leq 2^r - r - 1$. Suppose that $\mathcal{F} \subset 2^{[n]}$ is non-trivial r-wise t-union. Then*

$$|\mathcal{F}| \leq (r + t + 1)2^{n-r-t}. \tag{1}$$

The following example shows that the bound (1) is tight:

$$\mathcal{B} = \{B \in 2^{[n]} : |B \cap [r + t]| \leq 1\}. \tag{2}$$

Note that $\cup_{b \in \mathcal{B}} B = [n], \cap_{b \in \mathcal{B}} B = \emptyset$. In general, it is not difficult to see that inclusion-maximal non-trivial r-wise t-union families are non-trivial r-wise t-agreeing. That is, Theorem 1 is a strengthening of Theorem 2.

1 Proof of Theorem 1

Let us introduce a convenient definition.

Definition 1. For a collection of sets $A_1, \ldots, A_r$ let $W(A_1, \ldots, A_r)$ denote the set $(A_1 \cup \ldots \cup A_r) \setminus (A_1 \cap \ldots \cap A_r)$. For integers $n > \ell \geq 2$ and $r \geq 2$ let

$$w(n, \ell, r) = \max\{|\mathcal{A}| : \mathcal{A} \subset 2^{[n]}, |W(A_1, \ldots, A_r)| \leq \ell \text{ for all } A_1, \ldots, A_r \in \mathcal{A}\}.$$

Let $w^*(n, \ell, r)$ denote the maximum taken over all non-trivial (in the agreeing sense) families $\mathcal{A} \subset 2^{[n]}$.

In this terminology, Theorem 1 states that $w^*(n, n - t, r) = (r + t + 1)2^{n-t-r}$ for $t \leq 2^r - r - 1$. Kleitman [7] determined $w(n, \ell, 2)$ for all ℓ. Except for the trivial case $\ell = 1$, the extremal construction is non-trivial whence $w^*(n, \ell, 2) = w(n, \ell, 2)$. For the proof he introduced an operation on families of sets, called *squashing* (cf. [6]).

For $\mathcal{F} \subset 2^{[n]}$ and $i \in [n]$ define $\mathcal{F}(i) = \{F \setminus \{i\} : i \in F, F \in \mathcal{F}\}$ and $\mathcal{F}(\bar{i}) = \{F : i \notin F, F \in \mathcal{F}\}$. Note that $|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|$ and that $\mathcal{F}$ is uniquely determined by the two families $\mathcal{F}(i), \mathcal{F}(\bar{i}) \subset 2^{[n]\setminus\{i\}}$.

For $\mathcal{F} \subset 2^{[n]}$ and $i \in [n]$ the squashed family $S_i(\mathcal{F})$ is the (unique) family $\mathcal{G} \subset 2^{[n]}$ determined by $\mathcal{G}(i) = \mathcal{F}(i) \cap \mathcal{F}(\bar{i})$ and $\mathcal{G}(\bar{i}) = \mathcal{F}(i) \cup \mathcal{F}(\bar{i})$. Note that $|\mathcal{G}(i)| + |\mathcal{G}(\bar{i})| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|$, implying $|\mathcal{G}| = |\mathcal{F}|$. The following statement is quintessential for the proofs.

Lemma 1. *Let $\mathcal{F} \subset 2^{[n]}$ and $i \in [n]$. Put $\mathcal{G} = S_i(\mathcal{F})$. Then for arbitrary r,*

$$\max\{|W(F_1, \ldots, F_r)| : F_1, \ldots, F_r \in \mathcal{F}\} \geq \max\{|W(G_1, \ldots, G_r)| : G_1, \ldots, G_r \in \mathcal{G}\}.$$

Proof. Let $G_1, \ldots, G_r \in \mathcal{G}$. Then there exist $F_1, \ldots, F_r \in \mathcal{F}$ so that $G_j \setminus \{i\} = F_j \setminus \{i\}$ for $i \in [r]$. Hence $W(F_1, \ldots, F_r) \subset W(G_1, \ldots, G_r)$ is automatically satisfied unless $i \notin W(F_1, \ldots, F_r)$ and $i \in W(G_1, \ldots, G_r)$. The latter implies $i \in G_1 \cup \ldots \cup G_r$ and $i \notin G_1 \cap \ldots \cap G_r$. By symmetry, we may assume $i \in G_1, i \notin G_2$.

Since $\mathcal{G}(i) = \mathcal{F}(i) \cap \mathcal{F}(\bar{i})$, both G_1 and $G_1 \setminus \{i\}$ must be members of $\mathcal{F}$. Hence no matter whether $i \in F_2$ or $i \notin F_2$, we may choose $F_1 \in \mathcal{F}$ so that $i \in (F_1 \cup F_2) \setminus (F_1 \cap F_2)$ whence $i \in W(F_1, \ldots, F_r)$. Consequently, $W(F_1, \ldots, F_r) \supset W(G_1 \ldots, G_r)$. This proves the lemma. ∎

The problem with squashing is that it might destroy non-triviality. E.g., for $\mathcal{F}_{even} = \{F \subset [n] : |F| \text{ is even }\}$, which is an extremal constriction for $w^*(n, n-1, 2)$ if n is odd, $S_i(\mathcal{F}) = 2^{[n]\setminus\{i\}}$ for all $i \in [n]$. The following lemma permits us to circumvent this difficulty.

Lemma 2. *Let $\mathcal{F} \subset 2^{[n]}$ be non-trivial r-wise t-agreeing family. For $j \in [n]$, consider $\mathcal{F}_j := \{F \setminus \{j\} : F \in \mathcal{F}$, thought of as a subfamily of $2^{[n]\setminus\{j\}}$. If $t \geq 2$ then $\mathcal{F}_j$ is non-trivial r-wise $(t-1)$-agreeing. If $t = 1$ then $\mathcal{F}_j$ is non-trivial $(r-1)$-wise agreeing.*

Proof. It should be clear that $\cap_{A\in\mathcal{F}_j} A \subset \cap_{A\in\mathcal{F}} A = \emptyset$ and $\cup_{A\in\mathcal{F}_j} A = \cup_{A\in\mathcal{F}} A \setminus \{j\} = [n] \setminus \{j\}$, and thus $\mathcal{F}_j$ is non-trivial. By the definition of $\mathcal{F}_j$, for any ℓ and $A_1, \dots, A_\ell \in \mathcal{F}_j$ there are $B_1, \dots, B_\ell \in \mathcal{F}$ such that $B_i \in \{A_i, A_i \cup \{j\}\}$, and thus

$$W(B_1, \dots B_\ell) \setminus \{j\} = W(A_1, \dots A_\ell).$$

If $t \geq 2$, then, using the above for $\ell = r$, we get $|W(A_1, \dots A_r)| \geq |W(B_1, \dots B_r)| - 1 \geq t - 1$. If $t = 1$ then, using the above for $\ell = r - 1$, we get $W(B_1, \dots B_{r-1}) \setminus \{j\} = W(A_1, \dots A_{r-1})$. Since $\mathcal{F}$ is non-trivial, we can find a set B_r such that $B_1, \dots, B_r$ do not agree on j, and thus, using that $\mathcal{F}$ is r-wise agreeing, we get $\emptyset \neq W(B_1, \dots, B_r) \subset W(B_1, \dots B_{r-1}) \setminus \{j\} = W(A_1, \dots A_{r-1})$, which proves that $\mathcal{F}_j$ is $(r-1)$-wise agreeing. $\square$

Proof of Theorem 1. The proof is by induction on $t + r$, subject to the constraint $t \leq 2^r - r - 1$. The case $r = 2$ serves as the base case (note that here only $t = 1$ is allowed).

Take the largest non-trivial r-wise t-agreeing family $\mathcal{F} \subset 2^{[n]}$ and sequentially apply the squashing operations to $\mathcal{F}$ for $j = 1, 2, \dots, n$. There are two possible outcomes of this procedure. The first outcome is that the family (by the abuse of notation, also $\mathcal{F}$) always stays non-trivial. The resulting family is down-closed: for any set $A \in \mathcal{F}$ and $B \subset A$, we have $B \in \mathcal{F}$. Then, whenever $x \in A_1 \cap \ldots \cap A_r$, $A_i \in \mathcal{F}$, we also have $A_1 \setminus \{x\} \in \mathcal{F}$, and the set of agreeing coordinates for $A_1 \setminus \{x\}, A_2, \dots, A_r$ does not include x. This implies that, in order to guarantee the r-wise t-agreeing property, we must have $|A_1 \cup \ldots \cup A_r| \leq n - t$. In other words, $\mathcal{F}$ is a non-trivial r-wise t-union family, and we may apply Theorem 2 to $\mathcal{F}$ and get the desired bound $|\mathcal{F}| \leq (r + t + 1)2^{n-r-t}$.

The second outcome is that at a certain stage we lose non-triviality: while $\mathcal{F}$ is non-trivial, $S_j(\mathcal{F})$ is trivial. This means that no set in $S_j(\mathcal{F})$ contains j, and thus $S_j(\mathcal{F})$ coincides with $\mathcal{F}_j$ (as defined in Lemma 2), in particular, $|S_j(\mathcal{F})| = |\mathcal{F}_j|$. By Lemma 2, $\mathcal{F}_j$ is non-trivial r-wise $(t-1)$-agreeing for $t \geq 2$, and non-trivial $(r-1)$-wise agreeing for $t = 1$. In any case, we may apply the induction hypothesis to $\mathcal{F}_j$ and get

$$|\mathcal{F}| = |S_j(\mathcal{F})| = |\mathcal{F}_j| \leq (r + t)2^{(n-1)-r-t+1} < (r + t + 1)2^{n-r-t},$$

which proves the required bound. $\square$

Working a bit harder, one can determine the families for which equality in Theorem 1 for $r \geq 3$ is attained. Suppose that $\mathcal{F}$ is r-wise t-agreeing for $t < 2^r - r - 1$. We want to show that there is a set $A \in \binom{[n]}{r+t}$ and $R \subset A$ such that

$$\mathcal{F} = \{F \Delta R : F \subset [n], |F \cap A| \leq 1\}. \tag{3}$$

To do so, we analyze the squashing procedure. If, while running the procedure, we lose non-triviality, then the size of the family is smaller than the extremal value, and thus we must arrive at a non-trivial r-wise t-union family at the end of the procedure. The first author showed [5] that the extremal family in Theorem 2 for $t < 2^r - r - 1$ is unique and, up to permuting the coordinates, is of the form (2). Thus, $\mathcal{F}$ is also of the form (3) at the end of the procedure. In order to complete the proof, we need to show that, provided $S_j(\mathcal{F})$ is of the form (3), $\mathcal{F}$ itself must be of the form (3). In order to simplify the exposition, let us assume that $j = 1$ and $A = [r+t]$. Next, replacing $\mathcal{F}$ with $\mathcal{F}\Delta R = \{F\Delta R : F \in \mathcal{F}\}$ preserves the property of being non-trivial r-wise t-intersecting and transforms a family of the form (3) into a family of the same form. Thus, we may replace $S_1(\mathcal{F})$ with $S_1(\mathcal{F})\Delta R$ for a suitably chosen R, and w.l.o.g. assume that $S_1(\mathcal{F}) = \{F \subset [n] : |F \cap [r+t]| \le 1\}$.

By definition of squashing, we must have $\mathcal{F}(1) \cap \mathcal{F}(\bar{1}) = \{F \subset [2,n] : F \cap [2, r+t] = \emptyset\}$ and $\mathcal{F}(1) \cup \mathcal{F}(\bar{1}) = \{F \subset [2,n] : |F \cap [2, r+t]| \le 1\}$. Note that

$$\mathcal{F}(1)\Delta\mathcal{F}(\bar{1}) = \{F \subset [2,n] : |A \cap [2, r+t]| = 1\}.$$

Saying that $\mathcal{F}$ is not of the form (3) is the same as saying that both $\mathcal{F}(1) \setminus \mathcal{F}(\bar{1})$ and $\mathcal{F}(\bar{1}) \setminus \mathcal{F}(1)$ are non-empty. Arguing indirectly, let us assume that. Further, assume w.l.o.g. that $|\mathcal{F}(1) \setminus \mathcal{F}(\bar{1})| \le |\mathcal{F}(\bar{1}) \setminus \mathcal{F}(1)|$ and take $A_1 \in \mathcal{F}(1) \setminus \mathcal{F}(\bar{1})$. Assume that $A_1 \cap [2, r+t] = \{i_1\}$. Take $A_2 \in \mathcal{F}(\bar{1}) \setminus \mathcal{F}(1)$ such that $A_2 \cap [2, r+t] = \{i_2\}$, $i_2 \ne i_1$. This is possible since

$$|\mathcal{F}(\bar{1})\setminus\mathcal{F}(1)| \ge \frac{1}{2}|\mathcal{F}(1)\Delta\mathcal{F}(\bar{1})| = \frac{1}{2}(r+t-1)2^{n-r-t} > 2^{n-r-t} = \Big|\{F \subset [2,n] : F \cap [2, r+t] = \{i_1\}\}\Big|.$$

Next, fix some distinct $i_3, \ldots, i_r \in [2, r+t] \setminus \{i_1, i_2\}$ and for each i_s, $s \in [3, r]$, take a set $A_s \in \mathcal{F}(\bar{1}) \cup \mathcal{F}(1)$ such that $A_s \cap [2, r+t] = \{i_s\}$ and $A_s \cap [r+t+1, n] = [r+t+1, n] \setminus A_1$. For each $s \in [r]$ let A'_i be the set in $\mathcal{F}$ that corresponds to A_i. Note that $\cup_{i\in[r]} A'_i \cap [2,n] = \cup_{i\in[r]} A_i \cap [2,n] = \{i_1, \ldots, i_r\} \cup [r+t+1, n]$ and that $i \in A'_1 \cup A'_2 \subset \cup_{i\in[r]} A'_i$. Thus, $|\cup_{i\in[r]} A'_i| = n - t + 1$. Similarly, $\cap_{i\in[r]} A'_i \cap [2,n] = \cap_{i\in[r]} A_i \cap [2,n] = \emptyset$ and $i \notin A'_1 \cap A'_2 \supset \cap_{i\in[r]} A'_i$. Thus, $\cap_{i\in[r]} A'_i = \emptyset$. We conclude that $|W(A'_1, \ldots, A'_r)| = n - t + 1$, contradicting the fact that $\mathcal{F}$ is r-wise t-agreeing.

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