ON GORENSTEINESS OF ASSOCIATED GRADED RINGS OF FILTRATIONS

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ABSTRACT. Let $(A,\mathfrak{m})$ be a Gorenstein local ring, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. In this paper, we give a criterion for Gorensteinness of the associated graded ring of $\mathcal{F}$ in terms of the Hilbert coefficients of $\mathcal{F}$ in some cases. As a consequence we recover and extend a result proved in [19]. Further, we present ring-theoretic properties of the normal tangent cone of the maximal ideal of $A = S/(f)$ where $S = K[\![x_0, x_1, \dots, x_m]\!]$ is a formal power series ring over an algebraically closed field K, and $f = x_0^a - g(x_1,\dots,x_m)$, where g is a polynomial with $g \in (x_1,\dots,x_m)^b \setminus (x_1,\dots,x_m)^{b+1}$, and a, b, m are integers. We show that the normal tangent cone $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay if A is normal and $a \le b$. Moreover, we give a criterion of the Gorensteinness of $\overline{G}(\mathfrak{m})$.

1. INTRODUCTION

Let $(A,\mathfrak{m})$ be a Noetherian local ring of dimension d with an infinite residue field. An F_1-good filtration, $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ is a descending chain $A = F_0 \supseteq F_1 \supseteq \cdots$ of ideals satisfying $F_iF_j \subseteq F_{i+j}$ for all $i,j \in \mathbb{Z}$, $F_n = A$ for all $n \le 0$, and $F_{n+1} = F_1F_n$ for all $n \gg 0$. In addition, if F_1 is an $\mathfrak{m}$-primary ideal, then $\mathcal{F}$ is called a *Hilbert filtration* (c.f. [1]). We write $G(\mathcal{F}) := \bigoplus_{n\ge 0} F_n/F_{n+1}$ for the associated graded ring of $\mathcal{F}$. Studying the homological properties of $G(\mathcal{F})$, in particular, the Cohen-Macaulay property, has been an active topic of research since long time, see for instance [4, 8, 13].

In this paper we study the Gorenstein property of $G(\mathcal{F})$ with particular attention to the filtration $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ where $\bar{J}$ denotes the integral closure of J. The Gorenstein property of $G(\mathcal{F})$ when $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$ has been studied extensively (eg. [20, 10, 3, 12, 24, 22, 23]).

Let $\mathcal{F}$ be a Hilbert filtration. It is well known that the Hilbert function of $\mathcal{F}$, $H_{\mathcal{F}}(n) := \ell(A/F_n)$ coincides with a polynomial of degree d for large n (see [15]). We denote this polynomial by $P_{\mathcal{F}}(n)$ and write it as

$$P_{\mathcal{F}}(n) = e_0(\mathcal{F})\binom{n+d-1}{d} - e_1(\mathcal{F})\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F}).$$

Here $e_i(\mathcal{F})$ are integers called the *Hilbert coefficients* of $\mathcal{F}$. Recall that Q is said to be a reduction of $\mathcal{F}$ if $Q \subseteq F_1$ and $F_{n+1} = QF_n$ for $n \gg 0$. A reduction Q of I is said to be a minimal reduction of $\mathcal{F}$, if Q is a

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reduction of $\mathcal{F}$ and is minimal with respect to the inclusion. The reduction number of $\mathcal{F}$ with respect to a minimal reduction Q of $\mathcal{F}$,

$$r_Q(\mathcal{F}) := \min\{r \mid F_{n+1} = QF_n \text{ for all } n \geq r\},$$

and the *reduction number* of $\mathcal{F}$ is

$$r(\mathcal{F}) := \min\{r_Q(\mathcal{F}) \mid Q \text{ is a minimal reduction of } \mathcal{F}\}.$$

It is easy to observe that if I is an $\mathfrak{m}$-primary ideal in A, then $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$ is a Hilbert filtration. In this case we write $G(I)$ for $G(\mathcal{F})$, $e_i(I)$ for $e_i(\mathcal{F})$, $r_Q(I)$ for $r_Q(\mathcal{F})$ and $r(I)$ for $r(\mathcal{F})$.

There is another interesting example of a filtration that appears in the study of resolution of singularities, namely $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$.

In [21] Rees showed that if A is an analytically unramified local ring and I is an $\mathfrak{m}$-primary ideal in A, then $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$ is a Hilbert filtration. In this case we write $\overline{G}(I)$ for $G(\mathcal{F})$. We denote $\bar{e}_i(I)$ for $e_i(\mathcal{F})$, $\bar{r}_Q(I)$ for $r_Q(\mathcal{F})$, and $\bar{r}(I)$ for $r(\mathcal{F})$ and call them as the *normal Hilbert coefficients*, *normal reduction number with respect to a minimal reduction Q of $\{\overline{I^n}\}_{n\in\mathbb{Z}}$* and *normal reduction number*, respectively.

It has been observed that $G(I)$ and $\overline{G}(I)$ are rarely Cohen-Macaualy even if A is Cohen-Macaualy. A large attention in the literature is devoted to obtaining criteria for $G(\mathcal{F})$ to be Cohen-Macaualy in terms of Hilbert coefficients and/or reduction number with particular attention to the filtration $\{I^n\}_{n\in\mathbb{Z}}$ and $\{\overline{I^n}\}_{n\in\mathbb{Z}}$. For instance, it is well-known that if $r(\mathcal{F}) \leq 1$ in a Cohen-Macaualy local ring, then $G(\mathcal{F})$ is Cohen-Macaualy (c.f. [15]). Also, it is known that if $e_1(\mathcal{F}) = 0$ in a Cohen-Macaualy local ring, then $G(\mathcal{F})$ is Cohen-Macaualy (c.f. [15]). For these reasons, a huge research is devoted to obtaining a sharp bound on the Hilbert coefficients and reduction number, and studying the behavior of $G(\mathcal{F})$ when these bounds are extremal. This was also a theme studied by several mathematicians, for example, Huneke, Goto, Lipman, Sally among others.

In this paper, we are interested in obtaining the Gorenstein property of $G(\mathcal{F})$ in terms of the Hilbert coefficients. As mentioned before this has also a rich history. Now, we focus on the content of this paper.

In [5] Heinzer, Kim, and Ulrich established a criterion for Gorensteinness of $G(I)$ when A is an Artinian Gorenstein local ring. This has been extended for Hilbert filtrations in [6].

They proved that $G(\mathcal{F})$ is Gorenstein if and only if the h-vector of $G(\mathcal{F})$ is symmetric. In this paper we extend this result for any d-dimensional Gorenstein local ring (Theorem 3.1). We use this result to characterize the Gorensteiness of $G(\mathcal{F})$ in terms of the Hilbert coefficients.

A first result in this direction is obtained when $r(\mathcal{F}) \leq 2$ (Theorems 3.4 and 4.2). We make an additional assumption when $r(\mathcal{F}) = 2$, namely: there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. This condition ensures that $G(\mathcal{F})$ is Cohen-Macaualy (Lemma 3.2), and helps to compute the Hilbert series of $G(\mathcal{F})$. Notice that this condition is not restrictive. For instance, the filtration $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$ and $\mathcal{F} = \{\mathfrak{m}^n\}_{n\in\mathbb{Z}}$ satisfies this criterion. As a consequence of our result, we obtain a criterion for Gorensteiness of $\overline{G}(I)$ (Corollary 4.3). This recovers and extends a recent result by Okuma, Watanabe, and Yoshida in [19].

We next investigate the case $r(\mathcal{F}) > 2$. A difficulty in this case is the computation of the Hilbert series of $G(\mathcal{F})$. In order to compute this we assume that the *relative reduction number is maximal*. We define this term below.

In section 2, we introduce the *relative reduction number* $\operatorname{nr}(\mathcal{F})$ for Hilbert filtration $\mathcal{F}$. This is inspired by the definition of relative reduction number for normal filtrations defined in [18] in dimension two. The *relative reduction number* of $\mathcal{F}$ with respect to a minimal reduction Q of $\mathcal{F}$ is defined as

$$\mathrm{nr}_Q(\mathcal{F}) := \min\{r \in \mathbb{Z}_+ \mid F_{r+1} = QF_r\}.$$

The *relative reduction number* of $\mathcal{F}$ is defined as

$$\mathrm{nr}(\mathcal{F}) := \min\{\mathrm{nr}_Q(\mathcal{F}) \mid Q \text{ is a minimal reduction of } \mathcal{F}\}.$$

Observe that $\mathrm{nr}(\mathcal{F}) \leq r(\mathcal{F})$ and the inequality can be strict in general (see [16, Example 2.8]). If $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$, then $\mathrm{nr}_Q(\mathcal{F}) = r_Q(\mathcal{F})$ for any minimal reduction Q of I and hence $\mathrm{nr}(F) = r(\mathcal{F})$. For $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$, we denote $\mathrm{nr}_Q(\mathcal{F})$ by $\mathrm{nr}_Q(I)$ and $\mathrm{nr}(\mathcal{F})$ by $\mathrm{nr}(I)$ and the latter is called the *relative normal reduction number*. Note that $\mathrm{nr}(I)$ does not refer to the relative reduction number of the filtration $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$.

In Theorem 2.3 we derive an upper bound for $\mathrm{nr}(\mathcal{F})$ in a d-dimensional Cohen-Macaualy local ring when depth $G(\mathcal{F}) \geq d-1$. We prove that

$$\mathrm{nr}(\mathcal{F}) \leq e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1. \tag{1.1}$$

In [16] the inequality (1.1) was proved for $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$ in a two-dimensional normal excellent local domain. We also examine the conditions under which $\mathrm{nr}(\mathcal{F})$ achieves the upper bound, which serves as a crucial technical assumption for the computation of the Hilbert series of $G(\mathcal{F})$. We say $\mathcal{F}$ has *maximal relative reduction number* if $\mathrm{nr}(\mathcal{F}) = e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1$.

We are able to compute the Hilbert series of $G(\mathcal{F})$ when $r(\mathcal{F}) > 2$ and the relative reduction number is maximal in dimensions one and two. Since these computations are slightly different, we separate these cases. We consider the dimension one case in Section 3 and the dimension two case in Section 5. As a consequence we obtain criteria for $G(\mathcal{F})$ to be Gorenstein when $r(\mathcal{F}) > 2$ and $\mathrm{nr}(\mathcal{F})$ is maximal in dimensions one and two in Theorems 3.4 and 5.4, respectively. These criteria stated above enable us to understand the Gorenstein property of $\overline{G}(I)$ even without having the information on the explicit expression of $\overline{I^n}$.

We include some interesting examples of semigroup rings in dimension one in Section 3 and use our criteria to detect Gorensteinness of $\overline{G}(\mathfrak{m})$.

In Section 6 and Section 7 we study Gorensteiness of $\overline{G}(\mathfrak{m})$in hypersurface rings $A = K[\![x_0, x_1, \ldots, x_m]\!]/(f)$. The standard graded ring defined by

$$G(\mathfrak{m}) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

is called the *tangent cone* of $\mathfrak{m}$. It is well-known that for a local ring A, $G(\mathfrak{m})$ is a hypersurface (and thus Gorenstein) with the multiplicity of $G(\mathfrak{m})$ is equal to $\mathrm{ord}(f) = r_Q(\mathfrak{m}) + 1$.

A criterion for Gorensteinness of $\overline{G}(I)$ in terms of $\bar{r}_Q(I)$ and the other geometric properties is given in [19, 17]. In [16, Example 2.8] an explicit example of an $\mathfrak{m}$-primary integrally closed ideal $I \subset A$ is given such that $\bar{r}_Q(I) > \mathrm{nr}_Q(I)$. Such an ideal I gives an example for which $\overline{G}(I)$ is *not* Cohen-Macaulay. However, the ring A is *not* Gorenstein in this example. So it is natural to ask the following:

Question. Let A be a reduced hypersurface. Then
(1) Is $\overline{G}(\mathfrak{m})$ a hypersurface?
(2) Is $\overline{G}(\mathfrak{m})$ Gorenstein?
(3) Is $\overline{G}(\mathfrak{m})$ Cohen-Macaulay?

This question (2) has a negative answer. Surprisingly, there exist many examples of Brieskorn hypersurfaces for which $\overline{G}(\mathfrak{m})$ is NOT Gorenstein (see [19]). However, we have no negative answers to Question (3). So the main aim of Sections 6 and 7 is to prove the Cohen-Macaulayness of $\overline{G}(\mathfrak{m})$ for certain hypersurfaces, and to give a criterion for Gorensteinness for such $\overline{G}(\mathfrak{m})$. Furthermore, we compute $\bar{\mathrm{r}}_Q(\mathfrak{m})$ in this case.

Let K be an algebraically closed field, and let $m \geq 1, 2 \leq a \leq b$ be integers, and put $S = K[\![x_0, x_1, \ldots, x_m]\!]$ be a formal power series ring with $(m+1)$-variables over K. Put $x = x_0$ and $\underline{x} = x_1, \ldots, x_m$. Let g, f be polynomials such that

$$\begin{aligned} g &= g(\underline{x}) \in (\underline{x})^b \setminus (\underline{x})^{b+1}, \\ f &= f(x, \underline{x}) = x^a - g(\underline{x}). \end{aligned}$$

Then a hypersurface $A = S/(f)$ is called "Zariski-type hypersurface" if $a \neq 0$ in K.

The main result of these sections is the following:

Theorem 1.1. *Let $m \geq 1, 2 \leq a \leq b$ be integers. Let $A = S/(x^a - g(\underline{x}))$ be a normal Zariski-type hypersurface as above, and $\mathfrak{m}$ be its unique maximal ideal. Then*

(a) $\bar{\mathrm{r}}_Q(\mathfrak{m}) = \lfloor \frac{(a-1)b}{a} \rfloor$.
(b) $\overline{G} := \overline{G}(\mathfrak{m})$ *is Cohen-Macaulay with* $\dim \overline{G} = m$ *and* $e_0(\overline{G}) = a$.
(c) $\overline{G}$ *is Gorenstein if and only if* $b \equiv 0$ *or* $d \pmod{a}$, *where* $d = \gcd(a, b)$.
(d) $\overline{G}$ *has maximal embedding dimension if and only if either* (i) $a = 2$ *or* (ii) $b \equiv a - 1 \pmod{a}$.

We refer [2] for undefined terms.

2. A Bound on relative reduction number

In this section, we obtain an upper bound for the relative reduction number $\mathrm{nr}(\mathcal{F})$ of $\mathcal{F}$ when the depth of $G(\mathcal{F})$ is almost maximal. In general for an arbitrary filtration $\mathcal{F}$, we have $\mathrm{nr}_Q(\mathcal{F}) \leq r_Q(\mathcal{F})$, and hence $\mathrm{nr}(\mathcal{F}) \leq r(\mathcal{F})$. In [16, Example 2.8] the authors gave an example for which the inequality is strict for the filtration $\mathcal{F} = \{\overline{I^n}\}_{n \in \mathbb{Z}}$. In the following proposition we show that if $G(\mathcal{F})$ is Cohen-Macaualy, then $\mathrm{nr}(\mathcal{F}) = r(\mathcal{F})$ for any Hilbert filtration $\mathcal{F}$.

Proposition 2.1. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ be a Hilbert filtration. If $G(\mathcal{F})$ is Cohen-Macaulay, then $\mathrm{nr}_Q(\mathcal{F}) = r_Q(\mathcal{F})$ for any minimal reduction Q of $\mathcal{F}$. In particular, $\mathrm{nr}(\mathcal{F}) = r(\mathcal{F})$.*

Proof. We always have $\mathrm{nr}_Q(\mathcal{F}) \leq r_Q(\mathcal{F})$. Let Q be a minimal reduction of F_1 such that $\mu(Q) = d$. Since $G(\mathcal{F})$ is Cohen-Macaulay, by [7, Proposition 3.5] $Q \cap F_n = QF_{n-1}$ for all $n \geq 1$. Let $m = \mathrm{nr}_Q(\mathcal{F})$. Then $F_{m+1} = QF_m$. Hence $F_{m+2} \subseteq F_{m+1} \subseteq Q$. Therefore

$$F_{m+2} = F_{m+2} \cap Q = QF_{m+1}.$$

Continuing this successively we get $F_{k+1} = QF_k$ for all $k \geq m$. This implies that $r_Q(\mathcal{F}) \leq m = \mathrm{nr}_Q(\mathcal{F})$. Thus $r_Q(\mathcal{F}) = \mathrm{nr}_Q(\mathcal{F})$. $\square$

In general, the reduction number $r_Q(\mathcal{F})$ of $\mathcal{F}$ and the relative reduction number $\mathrm{nr}_Q(\mathcal{F})$ of $\mathcal{F}$ are dependent on the minimal reduction Q. It is well-known that if $\operatorname{depth} G(\mathcal{F}) \geq d-1$, then $r_Q(\mathcal{F})$ is independent of the minimal reduction Q. In the following proposition we show that $\mathrm{nr}_Q(\mathcal{F})$ is also independent of Q if depth $G(\mathcal{F}) \geq d-1$. Recall that if $f : \mathbb{Z} \to \mathbb{N}$ is a numerical function, then $\Delta(f) : \mathbb{Z} \to \mathbb{N}$ is a numerical function defined by

$$\Delta(f)(n) := f(n) - f(n-1).$$

Proposition 2.2. *Let (A, m) be a d-dimensional Cohen-Macaulay local ring, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. If* depth $G(\mathcal{F}) \geq d-1$, *then* $\mathrm{nr}_Q(\mathcal{F})$ *is independent of any minimal reduction Q.*

Proof. Using [15, Theorem 3.6] for all $n \in \mathbb{Z}$, we have

$$\Delta^d(P_{\mathcal{F}}(n) - H_{\mathcal{F}}(n)) = \ell(F_n/QF_{n-1}).$$

Since the expression on the left-hand side is independent of Q, we get $\ell(F_n/QF_{n-1})$ is independent of Q. Hence it is easy to verify that $\mathrm{nr}_Q(\mathcal{F})$ is independent of Q. □

If $(A, \mathfrak{m})$ is a Cohen-Macaualy local ring of dimension two and $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$, then depth $G(\mathcal{F}) \geq 1$ by [15, Proposition 3.25]. Hence by Proposition 2.2 $\mathrm{nr}_Q(I)$ is independent of Q and thus $\mathrm{nr}(I) = \mathrm{nr}_Q(I)$. This is a reason that the authors use the notation $nr(I)$ (without suffix) for the relative normal reduction number in [16].

In the next theorem we extend a result by Okuma, Rossi, Watanabe, and Yoshida. In [16, Theorem 2.7] they showed that if $(A, \mathfrak{m})$ is an excellent two-dimensional normal local domain containing an algebraically closed field $A/\mathfrak{m}$, then for an $\mathfrak{m}$-primary ideal I

$$\mathrm{nr}(I) \leq \bar{e}_1(I) - \bar{e}_0(I) + \ell(A/\bar{I}) + 1.$$

They also provided a criterion for the equality to hold. In the following theorem we prove an analogue bound for any Hilbert filtration in a d-dimensional Cohen-Macaualy local ring, provided $G(\mathcal{F})$ has almost maximal depth.

Theorem 2.3. *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. If* depth $G(\mathcal{F}) \geq d-1$, *then*

$$\mathrm{nr}(\mathcal{F}) \leq e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1.$$

Moreover, the equality holds if and only if $\ell(F_n/QF_{n-1}) = 1$ for all $2 \leq n \leq \mathrm{nr}(\mathcal{F})$ for any minimal reduction Q of $\mathcal{F}$ and $\mathrm{nr}(\mathcal{F}) = r(\mathcal{F})$. Also, $e_i(\mathcal{F}) = \binom{\mathrm{nr}(\mathcal{F})}{i}$ for $2 \leq i \leq d$.

Proof. Using Proposition [7, Proposition 4.6], for any minimal reduction Q of $\mathcal{F}$ we have,

$$\begin{aligned} e_1(\mathcal{F}) &= \sum_{n\geq 1}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) \\ &= \ell(F_1/Q) + \sum_{n\geq 2}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) \\ &= e_0(\mathcal{F}) - \ell(A/F_1) + \sum_{n\geq 2}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) \quad (\text{as } e_0(\mathcal{F}) = \ell(A/Q)) \end{aligned} \tag{2.1}$$

We observe that $\ell(F_n/QF_{n-1}) \geq 1$ for $2 \leq n \leq \mathrm{nr}(\mathcal{F})$. Therefore by (2.1)

$$\mathrm{nr}(\mathcal{F}) - 1 \leq \sum_{n\geq 2}^{r(\mathcal{F})} \ell(F_n/QF_{n-1}) = e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1), \tag{2.2}$$

and hence $\operatorname{nr}(\mathcal{F}) \leq e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1$.

Suppose $\operatorname{nr}(\mathcal{F}) = e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) + 1$. Then by (2.2) $\ell(F_n/QF_{n-1}) = 0$ for $n \geq \operatorname{nr}_Q(\mathcal{F}) + 1$ and $\ell(F_n/QF_{n-1}) = 1$ for $2 \leq n \leq \operatorname{nr}_Q(\mathcal{F})$. Therefore $\operatorname{nr}_Q(\mathcal{F}) = r_Q(\mathcal{F})$, which yields $\operatorname{nr}(\mathcal{F}) = r(\mathcal{F})$. Again using [7, Proposition 4.6] we have, for $i \geq 2$,

$$e_i(\mathcal{F}) = \sum_{n \geq i}^{\operatorname{nr}(\mathcal{F})} \binom{n-1}{i-1} = \binom{\operatorname{nr}(\mathcal{F})}{i}.$$

□

Using Theorem 2.3 for the filtration $\mathcal{F} = \{\overline{I^n}\}_{n \in \mathbb{Z}}$ in dimension two, we recover and extend the result [16, Theorem 2.7] by Okuma, Rossi, Watanabe and Yoshida.

Corollary 2.4 ((cf. [16, Theorem 2.7])). *Let $(A, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay analytically unramified local ring, and I an $\mathfrak{m}$-primary ideal. Let Q a minimal reduction of I. Then*

$$\operatorname{nr}(I) \leq \bar{e}_1(I) - e_0(I) + \ell(A/\bar{I}) + 1.$$

Moreover, the equality holds if and only if $\ell(\overline{I^n}/Q\overline{I^{n-1}}) = 1$ for all $2 \leq n \leq \operatorname{nr}(I)$ and $\operatorname{nr}(I) = \bar{r}(I)$. Also, $\bar{e}_2(I) = \binom{\operatorname{nr}(I)}{2}$.

Proof. Since depth $\overline{G}(I) \geq 1$ by [15, Proposition 3.25], the result follows from Theorem 2.3. □

3. GORENSTEINNESS OF $G(\mathcal{F})$ IN DIMENSION ONE

In this section, we discuss the Gorenstein property of $G(\mathcal{F})$ of a Hilbert filtration $\mathcal{F}$ in a one-dimensional Cohen-Macaulay local ring.

In [6] Heinzer, Kim and Ulrich gave criteria for $G(\mathcal{F})$ to be Gorenstein for an Artinian Gorenstein local ring. In the following theorem, we extend this result for any d-dimensional Gorenstein local ring. This plays a very important role in determining the Gorenstein property of the associated graded ring in terms of the Hilbert coefficients of $\mathcal{F}$.

Theorem 3.1. *Let $(A, \mathfrak{m})$ be a d-dimensional Gorenstein local ring, $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. Let*

$$HS_{G(\mathcal{F})}(z) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d},$$

where $h_s \neq 0$ be the Hilbert series of $G(\mathcal{F})$. Suppose $G(\mathcal{F})$ is Cohen-Macaulay. Then $G(\mathcal{F})$ is Gorenstein if and only if $h_{s-i} = h_i$ for all $i = 0, 1, \ldots, s$.

Proof. Let Q be a minimal reduction of $\mathcal{F}$. By [9, Theorem 8.6.3] there exists a generating set $a_1, \ldots, a_d$ of Q such that $a_1, \ldots, a_d$ is a superficial sequence. Since $G(\mathcal{F})$ is Cohen-Macaualy, $a_1^*, \ldots a_d^*$ is a $G(\mathcal{F})$-regular sequence by [7, Lemma 2.1], where $a_i^* = a_i + F_2$ for $i = 1, 2, \ldots, d$. Therefore $G(\mathcal{F})/(a_1^*, \ldots, a_d^*) \cong G(\mathcal{F}/Q)$. Since $a_1^*, \ldots, a_d^*$ is a regular sequence in $G(\mathcal{F})$,

$$HS_{G(\mathcal{F})/(a_1^*, \ldots, a_d^*)}(t) = h_0 + h_1 t + \cdots + h_s t^s.$$

Hence

$$HS_{G(\mathcal{F}/Q)}(t) = h_0 + h_1 t + \cdots + h_s t^s.$$

Therefore by [6, Theorem 4.2] $G(\mathcal{F}/Q)$ is Gorenstein if and only if $h_{s-i} = h_i$ for $i = 0, 1, \ldots, s$. Now the result follows because $a_1^*, \ldots, a_d^*$ is a regular sequence in $G(\mathcal{F})$ and $G(\mathcal{F})/(a_1^*, \ldots, a_d^*) \cong G(\mathcal{F}/Q)$. □

Our goal is to use Theorem 3.1 to obtain criteria for $G(\mathcal{F})$ to be Gorenstein in terms of the Hilbert coefficients. For this, first of all we should know whether $G(\mathcal{F})$ is Cohen-Macaualy. There is a huge research in the literature on obtaining criteria for $G(\mathcal{F})$ to be Cohen-Macaualy in terms of numerical invariants such as Hilbert coefficients and reduction number. Below we present a well-known criteria for $G(\mathcal{F})$ to be Cohen-Macaualy in terms of reduction number. We remark that the condition in (b) is not restrictive. For instance, the filtration $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$ and $\mathcal{F} = \{\mathfrak{m}^n\}_{n\in\mathbb{Z}}$ satisfies this criterion.

Lemma 3.2. *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. Then $G(\mathcal{F})$ is Cohen-Macaulay in each of the following cases*:

(a) $r(\mathcal{F}) = 1$;

(b) there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$.

Proof. The proof follows by [7, Proposition 3.5]. □

Now in order to provide conditions for $G(\mathcal{F})$ to be Gorenstein, we need to compute the Hilbert series of $G(\mathcal{F})$. This computation is slightly different for the one-dimensional case. In this section, we compute the Hilbert series of $G(\mathcal{F})$ in dimension one, and using this, we obtain criteria for $G(\mathcal{F})$ to be Gorenstein in terms of Hilbert coefficients of $\mathcal{F}$. We consider $d \geq 2$ in the next section.

The computation of the Hilbert series of $G(\mathcal{F})$ is not an easy task in general. In the following proposition we compute the Hilbert series of $G(\mathcal{F})$ in the case $\mathcal{F}$ satisfies the conditions (a), (b) of Lemma 3.2, and under certain additional assumptions when $r(\mathcal{F}) \geq 3$. For this, we recall the following definition: For a Hilbert filtration $\mathcal{F}$, the *postulation number* of $\mathcal{F}$ is given by

$$n(\mathcal{F}) := \sup\{n \in \mathbb{Z} | H_{\mathcal{F}}(n) \neq P_{\mathcal{F}}(n)\}.$$

If $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$, then $n(\mathcal{F})$ is denoted by $\overline{n}(I)$. We also recall the result [15, corollary 3.8] that will be used frequently in the remainder of the paper. Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field and $\mathcal{F} = \{F_n\}_{n\mathbb{Z}}$ a Hilbert filtration of ideals. Suppose depth $G(\mathcal{F}) \geq d - 1$. Then

$$r(\mathcal{F}) = n(\mathcal{F}) + d. \tag{3.1}$$

Proposition 3.3. *Let $(A, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration.*

(a) *If $r(\mathcal{F}) = 1$, then $HS_{G(\mathcal{F})}(t) = \dfrac{(e_0(\mathcal{F}) - e_1(\mathcal{F})) + e_1(\mathcal{F})t}{(1-t)}$.*

(b) *Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. Then*

$$HS_{G(\mathcal{F})}(t) = \frac{1}{(1-t)}\left(\ell(A/F_1) + ((\ell(F_1/F_2) - \ell(A/F_1)))t + (e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1))t^2\right).$$

(c) *Suppose $r(\mathcal{F}) \geq 3$ and $\mathrm{nr}(\mathcal{F})$ is maximal. Then*

$$HS_{G(\mathcal{F})}(t) = \frac{1}{(1-t)}\left(\ell(A/F_1) - t(e_0(\mathcal{F}) - \ell(A/F_1) - 1) + t^{r(\mathcal{F})}\right).$$

Proof. (a) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaulay. Hence using Equation (3.1) we get $n(\mathcal{F}) = 1 - 1 = 0$. Hence for all $n \geq 1$, $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$. Taking $n = 1$, we obtain

$$\ell(A/F_1) = e_0(\mathcal{F}) - e_1(\mathcal{F}). \tag{3.2}$$

Also,

$$\ell(F_n/F_{n+1}) = \ell(A/F_{n+1}) - \ell(A/F_n) = e_0(\mathcal{F}) \text{ for all } n \geq 1. \tag{3.3}$$

Therefore

$$\begin{aligned}
HS_{G(\mathcal{F})}(t) &= \sum_{n\in\mathbb{Z}} \ell(F_n/F_{n+1})t^n \\
&= \ell(A/F_1) + \sum_{n\geq 1} \ell(F_n/F_{n+1})t^n \\
&= \ell(A/F_1) - e_0(\mathcal{F}) + \sum_{n\geq 0} e_0(\mathcal{F})t^n && \text{(by Equation (3.3))} \\
&= -e_1(\mathcal{F}) + \frac{e_0(\mathcal{F})}{(1-t)} && \text{(by Equation (3.2))} \\
&= \frac{(e_0(\mathcal{F}) - e_1(\mathcal{F})) + e_1(\mathcal{F})t}{(1-t)}.
\end{aligned}$$

(b) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaualy. Hence using Equation (3.1) $n(\mathcal{F}) = 2 - 1 = 1$. Hence for all $n \geq 2$, $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$. Taking $n = 2$, we obtain

$$\ell(A/F_2) = 2e_0(\mathcal{F}) - e_1(\mathcal{F}). \tag{3.4}$$

The Hilbert series of $G(\mathcal{F})$ is given by

$$\begin{aligned}
HS_{G(\mathcal{F})}(t) &= \sum_{n\in\mathbb{Z}} \ell(F_n/F_{n+1})t^n \\
&= \ell(A/F_1) + \ell(F_1/F_2)t + \sum_{n\geq 2} \ell(F_n/F_{n+1})t^n \\
&= \ell(A/F_1) - e_0(\mathcal{F}) + (\ell(F_1/F_2) - e_0(\mathcal{F}))t + \sum_{n\geq 0} e_0(\mathcal{F})t^n \\
&= \frac{1}{(1-t)}\left((\ell(A/F_1) - e_0(\mathcal{F}))(1-t) + (\ell(F_1/F_2) - e_0(\mathcal{F}))t(1-t) + e_0(\mathcal{F})\right) \\
&= \frac{1}{(1-t)}\left(\ell(A/F_1) + (\ell(F_1/F_2) - \ell(A/F_1))t + (e_0(\mathcal{F}) - \ell(F_1/F_2))t^2\right) \\
&= \frac{1}{(1-t)}\left(\ell(A/F_1) + (\ell(F_1/F_2) - \ell(A/F_1))t + (e_0(\mathcal{F}) - \ell(A/F_2) + \ell(A/F_1))t^2\right) \\
&= \frac{1}{(1-t)}\left(\ell(A/F_1) + (\ell(F_1/F_2) - \ell(A/F_1))t + (e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1))t^2\right),
\end{aligned}$$

where the last equality follows from Equation (3.4).

(c) Let $Q = (x)$ a minimal reduction of $\mathcal{F}$. By Theorem 2.3 we have $r(\mathcal{F}) = \mathrm{nr}(\mathcal{F})$ and $\ell(F_n/QF_{n-1}) = 1$ for all $2 \leq n \leq r(\mathcal{F})$. Let $r = r(\mathcal{F})$. Observe that for $1 \leq n \leq r-1$,

$$\begin{aligned}
\ell(F_n/F_{n+1}) &= \ell(A/F_{n+1}) - \ell(A/F_n) \\
&= \ell(A/xF_n) - \ell(F_{n+1}/xF_n) - \ell(A/F_n) \\
&= \ell(A/(x)) + \ell((x)/xF_n) - \ell(F_{n+1}/xF_n) - \ell(A/F_n) \\
&= \ell(A/(x)) + \ell(A/F_n) - \ell(F_{n+1}/xF_n) - \ell(A/F_n) && (as (x)/xF_n \cong A/F_n) \\
&= e_0(\mathcal{F}) - 1 && \text{(since } e_0(\mathcal{F}) = A/(x)). \qquad (3.5)
\end{aligned}$$

Therefore

$$\begin{aligned}
HS_{G(\mathcal{F})}(t) &= \ell(A/F_1) + \sum_{n=1}^{r-1} \ell(F_n/F_{n+1})t^n + \sum_{n\geq r} e_0(\mathcal{F})t^n \\
&= \ell(A/F_1) - e_0(\mathcal{F}) + \sum_{n=1}^{r-1} \left(\ell(F_n/F_{n+1}) - e_0(\mathcal{F})\right) t^n + \frac{e_0(\mathcal{F})}{(1-t)} \\
&= \frac{1}{(1-t)}\left(\ell(A/F_1) - t(\ell(A/F_1) - e_0(\mathcal{F})) - (1-t)\sum_{n=1}^{r-1} t^n\right) \quad \text{(by equation (3.5))} \\
&= \frac{1}{(1-t)}\left(\ell(A/F_1) + t(e_0(\mathcal{F}) - \ell(A/F_1)) - t(1-t^{r-1})\right) \\
&= \frac{1}{(1-t)}\left(\ell(A/F_1) + t(e_0(\mathcal{F}) - \ell(A/F_1) - 1) + t^r\right).
\end{aligned}$$

□

Using Proposition 3.3 we provide criteria for $G(\mathcal{F})$ to be Gorenstein in terms of the Hilbert coefficients.

Theorem 3.4. *Let $(A, \mathfrak{m})$ be a one-dimensional Gorenstein local ring and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration.*

(a) *Let $r(\mathcal{F}) = 1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = 2e_1(\mathcal{F})$.*
(b) *Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = e_1(\mathcal{F})$.*
(c) *Suppose $r(\mathcal{F}) \geq 3$, $\mathrm{nr}(\mathcal{F})$ is maximal, and $G(\mathcal{F})$ is Cohen-Macaulay. Then $G(\mathcal{F})$ is Gorenstein if and only if $F_1 = \mathfrak{m}$ and $e_0(\mathcal{F}) = 2$.*

Proof. (a) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaualy. By Proposition 3.3, we have

$$HS_{G(\mathcal{F})}(t) = \frac{e_0(\mathcal{F}) - e_1(\mathcal{F}) + e_1(\mathcal{F})t}{(1-t)}.$$

Note that $e_1(\mathcal{F}) \neq 0$. Using Theorem 3.1, we obtain $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = 2e_1(\mathcal{F})$.
(b) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaualy. Using Proposition 3.3, we have

$$HS_{G(\mathcal{F})}(t) = \frac{1}{(1-t)}\left(\ell(A/F_1) + (\ell(F_1/F_2) - \ell(A/F_1))t + (e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1))t^2\right).$$

By Theorem 3.1, we obtain $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = e_1(\mathcal{F})$.
(c) By Proposition 3.3(c),

$$HS_{G(\mathcal{F})}(t) = \frac{\ell(A/F_1) + (e_0(\mathcal{F}) - \ell(A/F_1) - 1)t + t^{r(\mathcal{F})}}{1-t}.$$

Using Theorem 3.1 we get $G(\mathcal{F})$ is Gorenstein if and only if $F_1 = \mathfrak{m}$ and $e_0(\mathcal{F}) = 2$. □

Following is an easy consequence of Theorem 3.4.

Corollary 3.5. *Let $(A, \mathfrak{m})$ be a one-dimensional analytically unramified Gorenstein local ring, I an $\mathfrak{m}$-primary ideal in A.*

(a) *Let $\bar{r}(I) = 1$. Then $\overline{G}(I)$ is Gorenstein if and only if $e_0(I) = 2\bar{e}_1(I)$.*
(b) *Let $\bar{r}(I) = 2$. Then $\overline{G}(I)$ is Gorenstein if and only if $e_0(I) = \bar{e}_1(I)$.*
(c) *Suppose $\bar{r}(I) = r \geq 3$ and $\mathrm{nr}(I)$ is maximal. Then $\overline{G}(I)$ is Gorenstein if and only if $\bar{I} = \mathfrak{m}$ and $e_0(I) = 2$.*

Note that if $(A, \mathfrak{m})$ is a positive dimensional Cohen-Macaulay local ring and I is an $\mathfrak{m}$-primary ideal, then depth $\overline{G}(I) \geq 1$ by [15, Proposition 3.25]. In particular, if dim $A = 1$, then $\overline{G}(I)$ is Cohen-Macaulay.

Next, we give examples of one-dimensional semigroup rings and apply our results.

Definition 3.6. Let $m_1 < m_2 < \cdots < m_n$ be positive integers with $\gcd(m_1, m_2, \ldots, m_n) = 1$, and S be the semigroup generated by $m_1, m_2, \ldots, m_n$. Then the Fröbenius number $F(S)$ of S is defined as the smallest integer k such that $n \in S$ for all $n > k$.

Lemma 3.7. *Let $A = K[\![t^{m_1}, t^{m_2}, \ldots, t^{m_n}]\!]$ be a semigroup ring of the semigroup generated by $m_1 < m_2 < \cdots < m_n$ with* $\gcd(m_1, m_2, \ldots, m_n) = 1$. *Then* $\bar{r}(\mathfrak{m}) = \left\lceil \dfrac{F(S)}{m_1} \right\rceil$.

Proof. We note that $Q = (t^{m_1})$ is a minimal reduction of $\mathfrak{m} = (t^{m_1}, t^{m_2}, \ldots, t^{m_n})$, the maximal ideal of A. By [6, Example 7.3] we have $\overline{\mathfrak{m}^i} = \{\alpha \in A \mid \operatorname{ord}(\alpha) \geq m_1 i\}$ where $\operatorname{ord}(\alpha)$ is the degree of the initial form of α. Indeed, the natural inclusion map $A \to K[\![t]\!]$ is an integral extension. We have $\bar{I} = \overline{IK[\![t]\!]} \cap A$ for any ideal I in A. Since $K[\![t]\!]$ is a PID, $(t^{m_1})K[\![t]\!]$ is integrally closed. As $Q = (t^{m_1})$ is a minimal reduction of $\mathfrak{m}$, $\overline{Q^i} = \overline{\mathfrak{m}^i}$. Let k be the smallest integer such that $m_1 k > F(S)$.

Consider $\overline{\mathfrak{m}^k} = (t^{m_1 k}, t^{m_1 k + 1}, \cdots)$. Hence

$$Q\overline{\mathfrak{m}^k} = (t^{m_1(k+1)}, t^{m_1(k+1)+1} \ldots) = \overline{\mathfrak{m}^{k+1}}.$$

Therefore $\bar{r}(\mathfrak{m}) \leq k$.

By the definition of the Fröbenius number $m_1(k-1) \neq F(S)$. Hence, $m_1(k-1) < F(S)$. Hence $F(S) + m_1 > m_1 k$. Therefore $t^{F(S)+m_1} \in \overline{\mathfrak{m}^k}$. Suppose $t^{F(S)+m_1} \in Q\overline{\mathfrak{m}^{k-1}}$. Then $t^{F(S)+m_1} = t^{m_1} b$ where $b \in A$. Since t^{m_1} is a regular element in $k[\![t]\!]$, we obtain $t^{F(S)} = b \in A$. This is a contradiction. Hence $t^{F(S)+m_1} \in \overline{m^k} \setminus Q\overline{\mathfrak{m}^{k-1}}$. Therefore $\bar{r}(\mathfrak{m}) = k$. $\square$

Example 3.8. Let $A = K[\![t^3, t^4]\!]$ be a semigroup ring generated by t^3 and t^4. The Fröbenius number $F(S)$ of the semigroup $S = \mathbb{N}3 + \mathbb{N}4$ is 5. Since S is symmetric, A is Gorenstein. Also, $\overline{\mathfrak{m}^i} = \{\alpha \in A \mid \operatorname{ord}(\alpha) \geq 3i\}$. Note that $Q = (t^3)$ is a minimal reduction of $\mathfrak{m}$. Hence by Lemma 3.7 $\bar{r}(\mathfrak{m}) = 2$. Since depth $\overline{G}(m) \geq 1$, $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay. Using equation 3.1, $\bar{n}(\mathfrak{m}) = 2 - 1 = 1$. Hence $\overline{P}_{\mathfrak{m}}(n) = \overline{H}_{\mathfrak{m}}(n)$ for all $n \geq 2$. Therefore to calculate $e_0(\mathfrak{m})$ and $\overline{e_1}(m)$ we solve the following two equations:

$$2e_0(\mathfrak{m}) - \overline{e_1}(\mathfrak{m}) = \ell(A/\overline{\mathfrak{m}^2})$$
$$3e_0(\mathfrak{m}) - \overline{e_1}(\mathfrak{m}) = \ell(A/\overline{\mathfrak{m}^3}).$$

Since $\ell(A/\overline{\mathfrak{m}^2}) = 3, \ell(A/\overline{\mathfrak{m}^3}) = 6$, we obtain $e_0(\mathfrak{m}) = 3$ and $\overline{e_1}(\mathfrak{m}) = 3$. Since $e_0(\mathfrak{m}) = \overline{e_1}(\mathfrak{m})$, by Theorem 3.4 $\overline{G}(\mathfrak{m})$ is Gorenstein.

Example 3.9. Let $A = K[\![t^2, t^m]\!]$ be a semigroup ring generated by t^2 and t^m where m is an odd integer ≥ 3. The Fröbenius number $F(S)$ of the semigroup $S = \mathbb{N}2 + \mathbb{N}m$ is $m - 2$. Since S is symmetric, A is Gorenstein. Also, $\overline{\mathfrak{m}^i} = \{\alpha \in A \mid \operatorname{ord}(\alpha) \geq 2i\}$. Note that $Q = (t^2)$ is a minimal reduction of $\mathfrak{m}$. Hence by Lemma 3.7 $\bar{r}(\mathfrak{m}) = (m-1)/2$. Therefore $\overline{P}_{\mathfrak{m}}(n) = \overline{H}_{\mathfrak{m}}(n)$ for all $n \geq (m-1)/2$.

Also,

$$\ell(A/\overline{\mathfrak{m}^{(m-1)/2}}) = (m-3)/2 + 1 = (m-1)/2,$$
$$\ell(A/\overline{\mathfrak{m}^{(m+1)/2}}) = (m-3)/2 + 3 = (m+3)/2.$$

Hence

$$e_0(\mathfrak{m})((m-1)/2) - \overline{e_1}(\mathfrak{m}) = (m-1)/2,$$
$$e_0(\mathfrak{m})((m+1)/2) - \overline{e_1}(\mathfrak{m}) = (m+3)/2.$$

Therefore $e_0(\mathfrak{m}) = 2$ and $\overline{e_1}(\mathfrak{m}) = (m-1)/2$. Hence $\bar{r}(\mathfrak{m}) = \overline{e_1}(\mathfrak{m}) - e_0(\mathfrak{m}) + \ell(A/\overline{m}) + 1$. Therefore by Corollary 3.5 $\overline{G}(\mathfrak{m})$ is Gorenstein.

Example 3.10. Let $A = K[\![t^4, t^5, t^6]\!]$ be a semigroup ring generated by 4, 5 and 6. The Fröbenius number of the semigroup $F(S) = 7$. Since S is symmetric, A is Gorenstein. Let $Q = (t^4)$. Then Q is a minimal reduction of $\mathfrak{m}$. By Lemma 3.7 we get $\bar{r}(\mathfrak{m}) = 2$. Using similar arguments as above we obtain $e_0(\mathfrak{m}) = 4$ and $\overline{e_1}(\mathfrak{m}) = 4$. Hence by Corollary 3.5 $\overline{G}(\mathfrak{m})$ is Gorenstein.

The following example is from [6, Example 7.3]. They prove that $\overline{G}(\mathfrak{m})$ is not Gorenstein. We also prove the same by using our techniques.

Example 3.11. Let $A = K[\![t^4, t^6, t^7]\!]$ be a semigroup ring generated by 4, 6 and 7. The Fröbenius number of the semigroup $F(S) = 9$ and the semigroup is symmetric. Hence A is a one-dimensional Gorenstein local ring. Let $\mathfrak{m} = (t^4, t^6, t^7)$. Then $\overline{\mathfrak{m}^i} = \{\alpha \in A \mid \operatorname{ord}(\alpha) \geq 4i\}$, and $Q = (t^4)$ is a minimal reduction of $\mathfrak{m}$ with reduction number $\bar{r}(\mathfrak{m}) = 3$ by Lemma 3.7. Since the depth $\overline{G}(\mathfrak{m}) \geq 1$, $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay. By equation 3.1 $\bar{n}(\mathfrak{m}) = 3 - 1 = 2$. Hence $\overline{P}_{\mathfrak{m}}(n) = \overline{H}_{\mathfrak{m}}(n)$ for all $n \geq 3$. Therefore to calculate $e_0(\mathfrak{m})$ and $\overline{e_1}(m)$ we solve the following two equations:

$$3e_0(\mathfrak{m}) - \overline{e_1}(\mathfrak{m}) = \ell(A/\overline{\mathfrak{m}^3})$$
$$4e_0(\mathfrak{m}) - \overline{e_1}(\mathfrak{m}) = \ell(A/\overline{\mathfrak{m}^4}).$$

Since $\ell(A/\overline{\mathfrak{m}^3}) = 7$ and $\ell(A/\overline{\mathfrak{m}^4}) = 11$, we get $e_0(\mathfrak{m}) = 4$ and $\overline{e_1}(\mathfrak{m}) = 5$. Since $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay, by Theorem 2.3 $\operatorname{nr}(\mathfrak{m}) = \bar{r}(\mathfrak{m})$. We have

$$\overline{e_1}(\mathfrak{m}) - e_0(\mathfrak{m}) + \ell(A/\mathfrak{m}) + 1 = 5 - 4 + 1 + 1 = 3 = \operatorname{nr}(\mathfrak{m}).$$

Since $e_0(\mathfrak{m}) = 4$, by Theorem 3.5 $\overline{G}(\mathfrak{m})$ is not Gorenstein.

4. Gorensteinness of $G(\mathcal{F})$ in dimension $d \geq 2$ and $r(\mathcal{F}) \leq 2$

Let $(A, \mathfrak{m})$ be a d-dimensional Gorenstein local ring where $d \geq 2$ and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration. In this section, we determine criteria for $G(\mathcal{F})$ to be Gorenstein in terms of the Hilbert coefficients in arbitrary dimension $d \geq 2$ when $r(\mathcal{F}) \leq 2$. A reason for the assumption $r(\mathcal{F}) \leq 2$ is that we are able to compute the Hilbert series of $G(\mathcal{F})$ in this case, and hence use Theorem 3.1. As a consequence we recover the criterion proved in [19, Theorem 2.1].

Proposition 4.1. *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring where $d \geq 2$ and $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ a Hilbert filtration.*

(a) *If $r(\mathcal{F}) = 1$, then $HS_{G(\mathcal{F})}(t) = \dfrac{(e_0(\mathcal{F}) - e_1(\mathcal{F})) + e_1(\mathcal{F})t}{(1-t)}$.*

(b) *Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. Then*

$$HS_{G(\mathcal{F})}(t) = \frac{1}{(1-t)^d}\left([e_0(\mathcal{F}) - e_1(\mathcal{F}) + e_2(\mathcal{F})] + (e_1(\mathcal{F}) - 2e_2(\mathcal{F}))t + e_2(\mathcal{F})t^2\right).$$

Proof. (a) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaualy. Hence using Equation (3.1) we obtain that $n(\mathcal{F}) = 1-d \leq -1$. Therefore $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$ for all $n \geq 0$. Since $G(\mathcal{F})$ is Cohen-Macaulay and $r(\mathcal{F}) = 1$, by [7, Proposition 4.6] $e_i(\mathcal{F}) = 0$ for all $i \geq 2$. Therefore for all $n \geq 0$

$$\ell(F_n/F_{n+1}) = \ell(A/F_{n+1}) - \ell(A/F_n) = e_0(\mathcal{F})\binom{n+d-1}{d-1} - e_1(\mathcal{F})\binom{n+d-2}{d-2} \tag{4.1}$$

Hence

$$\begin{aligned} HS_{G(\mathcal{F})}(t) &= \sum_{n\in\mathbb{Z}} \ell(F_n/F_{n+1})t^n \\ &= \sum_{n\geq 0}\left[e_0(\mathcal{F})\binom{n+d-1}{d-1} - e_1(\mathcal{F})\binom{n+d-2}{d-2}\right]t^n \qquad \text{(by Equation (4.1))} \\ &= \frac{e_0(\mathcal{F})}{(1-t)^d} - \frac{e_1(\mathcal{F})}{(1-t)^{d-1}} \\ &= \frac{[e_0(\mathcal{F}) - e_1(\mathcal{F})] + e_1(\mathcal{F})t}{(1-t)^d}. \end{aligned}$$

(b) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaualy.

<u>Case 1:</u> <u>$d = 2$.</u> Then $n(\mathcal{F}) = 0$. Therefore $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$ for all $n \geq 1$. In particular, when $n = 1$ we have

$$\ell(A/F_1) = e_0(\mathcal{F}) - e_1(\mathcal{F}) + e_2(\mathcal{F}). \tag{4.2}$$

The Hilbert series of $G(\mathcal{F})$ is given by

$$\begin{aligned} HS_{G(\mathcal{F})}(t) &= \sum_{n\in\mathbb{Z}} \ell(F_n/F_{n+1})t^n \\ &= \ell(A/F_1) - e_0(\mathcal{F}) + e_1(\mathcal{F}) + \sum_{n\geq 0}(e_0(\mathcal{F})(n+1) - e_1(\mathcal{F}))t^n \\ &= e_2(\mathcal{F}) + \frac{e_0(\mathcal{F})}{(1-t)^2} - \frac{e_1(\mathcal{F})}{(1-t)} \\ &= \frac{1}{(1-t)^2}\left(e_2(\mathcal{F})(1-t)^2 + (e_0(\mathcal{F}) - e_1(\mathcal{F})) + e_1(\mathcal{F})t\right) \\ &= \frac{1}{(1-t)^2}\left((e_0(\mathcal{F}) - e_1(\mathcal{F}) + e_2(\mathcal{F})) + (e_1(\mathcal{F}) - 2e_2(\mathcal{F}))t + e_2(\mathcal{F})t^2\right). \end{aligned}$$

<u>Case 2</u>: <u>$d \geq 3$.</u> Then $n(\mathcal{F}) \leq -1$. Hence $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$ for all $n \geq 0$, and

$$P_{\mathcal{F}}(n) = e_0(\mathcal{F})\binom{n+d-1}{d} - e_1(\mathcal{F})\binom{n+d-2}{d-1} + e_2(\mathcal{F})\binom{n+d-3}{d-2}.$$

The Hilbert series of $G(\mathcal{F})$ is given by

$$\begin{aligned}
HS_{G(\mathcal{F})}(t) &= \sum_{n\in\mathbb{Z}} \ell(F_n/F_{n+1})t^n \\
&= \sum_{n\geq 0}\left[e_0(\mathcal{F})\binom{n+d-1}{d-1} - e_1(\mathcal{F})\binom{n+d-2}{d-2} + e_2(\mathcal{F})\binom{n+d-3}{d-3}\right]t^n \\
&= \frac{e_0(\mathcal{F})}{(1-t)^d} - \frac{e_1(\mathcal{F})}{(1-t)^{d-1}} + \frac{e_2(\mathcal{F})}{(1-t)^{d-2}} \\
&= \frac{1}{(1-t)^d}\left(e_0(\mathcal{F}) - e_1(\mathcal{F})(1-t) + e_2(\mathcal{F})(1-t)^2\right) \\
&= \frac{1}{(1-t)^d}\left((e_0(\mathcal{F}) - e_1(\mathcal{F}) + e_2(\mathcal{F})) + (e_1(\mathcal{F}) - 2e_2(\mathcal{F}))t + e_2(\mathcal{F})t^2\right).
\end{aligned}$$

□

Theorem 4.2. *Let $(A,\mathfrak{m})$ be a d-dimensional Gorenstein local ring of dimension $d\geq 2$. Let $\mathcal{F}=\{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration.*

(a) *Let $r(\mathcal{F})=1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F})=2e_1(\mathcal{F})$.*
(b) *Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F})=2$ and $F_2\cap Q=QF_1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F})=e_1(\mathcal{F})$.*

Proof. In both the cases, by Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaulay.
(a) By Proposition 4.1,

$$HS_{G(\mathcal{F})}(t) = \frac{[e_0(\mathcal{F})-e_1(\mathcal{F})]+e_1(\mathcal{F})t}{(1-t)^d}.$$

By Theorem 3.1, $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F})-e_1(\mathcal{F})=e_1(\mathcal{F})$. Hence the result follows.
(b) Using Proposition 4.1,

$$HS_{G(\mathcal{F})}(t) = \frac{1}{(1-t)^d}\left(e_0(\mathcal{F})-e_1(\mathcal{F})+e_2(\mathcal{F})+(e_1(\mathcal{F})-2e_2(\mathcal{F}))t+e_2(\mathcal{F})t^2\right).$$

By Theorem 3.1, $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F})-e_1(\mathcal{F})+e_2(\mathcal{F}) = e_2(\mathcal{F})$. Hence the result follows. □

Now we recover the result in [19, Theorem 2.1]. Moreover, we extend it for any d-dimensional analytically unramified Gorenstein local ring.

Corollary 4.3. *Let $(A,\mathfrak{m})$ be a d-dimensional analytically unramified Gorenstein local ring of dimension $d\geq 2$, and I an $\mathfrak{m}$-primary ideal in A.*

(a) *If $\overline{r}(I)=1$, then $\overline{G}(I)$ is Gorenstein if and only if $e_0(I)=2\bar{e}_1(I)$.*
(b) *If $\overline{r}(I)=2$, then $\overline{G}(I)$ is Gorenstein if and only if $e_0(I)=\bar{e}_1(I)$.*

Proof. (a) If $\overline{r}(I)=1$, then the result follows by Theorem 4.2.
(b) Let $\overline{r}(I)=2$. By [11, Theorem 1], we obtain $Q\cap\overline{I^2}=Q\overline{I}$. By Theorem 4.2, the result follows. □

Below we provide criteria for $G(\mathcal{F})$ to be Gorenstein under the additional assumption that $\operatorname{nr}(\mathcal{F})$ is maximal.

Corollary 4.4. *Let $(A,\mathfrak{m})$ be a d-dimensional Gorenstein local ring of dimension $d\geq 2$, and $\mathcal{F}=\{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. Suppose $\operatorname{nr}(\mathcal{F})$ is maximal.*

(a) *If $r(\mathcal{F}) = 1$, then $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = 2\,\ell(A/F_1)$.*
(b) *Suppose there exists a minimal reduction Q of $\mathcal{F}$ such that $r_Q(\mathcal{F}) = 2$ and $F_2 \cap Q = QF_1$. Then $G(\mathcal{F})$ is Gorenstein if and only if $F_1 = \mathfrak{m}$.*

Proof. (a) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaulay. Hence by Theorem 2.3, $\operatorname{nr}(\mathcal{F}) = r(\mathcal{F})$. Therefore by assumption

$$e_1(\mathcal{F}) = e_0(\mathcal{F}) - \ell(A/F_1). \tag{4.3}$$

By Theorem 4.2, $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = 2e_1(\mathcal{F})$. Substituting $e_1(\mathcal{F})$ from equation 4.3, we obtain the result.
(b) By Lemma 3.2 $G(\mathcal{F})$ is Cohen-Macaulay. Hence by Theorem 2.3, $\operatorname{nr}(\mathcal{F}) = r(\mathcal{F})$. Therefore by assumption

$$e_1(\mathcal{F}) - e_0(\mathcal{F}) + \ell(A/F_1) = 1. \tag{4.4}$$

By Theorem 4.2, $G(\mathcal{F})$ is Gorenstein if and only if $e_0(\mathcal{F}) = e_1(\mathcal{F})$. Hence by equation 4.4, we obtain the result. □

Applying the above result for the filtration $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ we obtain the following result:

Corollary 4.5. *Let $(A,\mathfrak{m})$ be a d-dimensional analytically unramified Gorenstein local ring, and I an $\mathfrak{m}$-primary ideal in A. Suppose $\operatorname{nr}(I)$ is maximal.*
(a) *If $\bar{r}(I) = 1$, then $\overline{G}(I)$ is Gorenstein if and only if $\bar{e}_0(I) = 2\,\ell(A/\bar{I})$.*
(a) *If $\bar{r}(I) = 2$, then $\overline{G}(I)$ is Gorenstein if and only if $\bar{I} = \mathfrak{m}$.*

Proof. Follows from Corollary 4.4. □

5. Gorensteinness of $G(\mathcal{F})$ in dimension two

In this section we present results on Gorensteinness of $G(\mathcal{F})$ in two-dimensional Gorenstein local rings. If $r(\mathcal{F}) \leq 2$, the Gorensteinness of $G(\mathcal{F})$ is characterized in Section 4. In this section we consider the case $r(\mathcal{F}) \geq 3$. In order to use Theorem 3.1 we need to compute the Hilbert series of $G(\mathcal{F})$. This is not easy when $r(\mathcal{F}) \geq 3$ even in dimension two. In this section we compute the Hilbert series of $G(\mathcal{F})$ in dimension two under the assumption that the relative reduction number is maximal. As a consequence we characterize the Gorensteinness of $G(\mathcal{F})$ in terms of the Hilbert coefficients of $\mathcal{F}$.

Let $(A,\mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. Let $\operatorname{depth} G(\mathcal{F}) \geq 1$. Then we can find $a, b \in F_1 \setminus \mathfrak{m}F_1$ such that $Q = (a,b)$ a minimal reduction of F_1 where a^* is regular in $G(\mathcal{F})$(see [15, Proposition A.2.4] and [7, Lemma 2.1]).

We need the following lemmas in order to compute the Hilbert series of $G(\mathcal{F})$.

Lemma 5.1. *Let $(A,\mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. Let $\operatorname{depth} G(\mathcal{F}) \geq 1$, and $Q = (a,b)$ a minimal reduction of F_1 where a^* is regular in $G(\mathcal{F})$. Then for all $n \geq 1$,*

$$\ell(Q/QF_n) = 2\ell(A/F_n) - \ell(A/F_{n-1}).$$

Proof. Consider the A-module homomorphism

$$f : A/F_n \oplus A/F_n \to Q/QF_n$$

defined by $f(\bar{x},\bar{y}) = \overline{ax+by}$ for $x,y \in A$ where $\bar{x},\bar{y}$ denote the image of $x,y \in A/F_n$ and $\overline{ax+by}$ is the image of $ax+by \in Q/QF_n$. Then $\ker(f) = \{(\bar{r},\bar{s}) \mid ra+sb \in QF_n\}$. Let $(\bar{r},\bar{s}) \in \ker(f)$, then $ra+sb = ap+bq$ for some $p,q \in F_n$. Hence

$$a(r-p)+b(s-q) = 0. \tag{5.1}$$

Since a,b is a regular sequence in A, we get $r-p = bt$ for some $t \in A$. Substituting in equation (5.1) we obtain $s-q = -at$, hence $(\bar{r},\bar{s}) = t(\bar{b},-\bar{a})$. Therefore $\ker(f) = (\bar{b},-\bar{a})A$.

Consider $\varphi : A \to (\bar{b},-\bar{a})A$ an A-module homomorphism given by

$$\varphi(r) = (r\bar{b},-r\bar{a}).$$

Then φ is surjective and $\ker(\varphi) = (F_n : a) \cap (F_n : b)$. Note that $F_{n-1} = (F_n : a)$, since a^* is a nonzerodivisor in $G(\mathcal{F})$. Indeed, if $x \in (F_n : a)$, then $xa \in F_n \cap (a) = aF_{n-1}$ by [7, Proposition 3.5]. Moreover, since a is a nonzerodivisor in A, $x \in F_{n-1}$. Clearly $F_{n-1} \subseteq (F_n : b)$. Thus $\ker(\varphi) = F_{n-1}$. Hence we have $(\bar{b},-\bar{a})A \cong A/F_{n-1}$. Therefore

$$0 \longrightarrow A/F_{n-1} \longrightarrow A/F_n \oplus A/F_n \longrightarrow Q/QF_n \longrightarrow 0$$

is an exact sequence and the result follows. □

Lemma 5.2. *Let $(A,\mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. Let* $\operatorname{depth} G(\mathcal{F}) \geq 1$. *Suppose* $\operatorname{nr}(\mathcal{F})$ *is maximal. Then for all* $1 \leq n \leq \operatorname{nr}(\mathcal{F})-1$,

$$f(n) := \ell(F_n/F_{n+1}) = n(e_0(\mathcal{F})-1)+\ell(A/F_1).$$

Proof. We first claim that $\ell(F_n/F_{n+1}) = e_0(\mathcal{F})-1+\ell(F_{n-1}/F_n)$ for all $1 \leq n \leq \operatorname{nr}(\mathcal{F})-1$. Let $Q = (a,b)$ be a minimal reduction of $\mathcal{F}$ such that a^* is a nonzerodivisor on $G(\mathcal{F})$. Since

$$QF_n \subseteq F_{n+1} \subseteq F_n \subseteq A \quad \text{and} \quad QF_n \subseteq Q \subseteq A,$$

we have

$$\begin{aligned}
\ell(F_n/F_{n+1}) &= \ell(A/QF_n) - \ell(A/F_n) - \ell(F_{n+1}/QF_n)\\
&= e_0(\mathcal{F}) + \ell(Q/QF_n) - \ell(A/F_n) - \ell(F_{n+1}/QF_n)\\
&= e_0(\mathcal{F}) + 2\ell(A/F_n) - \ell(A/F_{n-1}) - \ell(A/F_n) - \ell(F_{n+1}/QF_n)\\
&= e_0(\mathcal{F}) + \ell(A/F_n) - \ell(A/F_{n-1}) - \ell(F_{n+1}/QF_n)\\
&= e_0(\mathcal{F}) + \ell(F_{n-1}/F_n) - \ell(F_{n+1}/QF_n)
\end{aligned}$$

Using Theorem 2.3 we have $\ell(F_{n+1}/QF_n) = 1$ for all $1 \leq n \leq \operatorname{nr}(\mathcal{F})-1$. Hence the claim follows. Therefore, for $1 \leq n \leq \operatorname{nr}(\mathcal{F})-1$, we have

$$\begin{aligned}
f(n) &= e_0(\mathcal{F}) - 1 + f(n-1)\\
f(n-1) &= e_0(\mathcal{F}) - 1 + f(n-2)\\
&\vdots\\
f(1) &= e_0(\mathcal{F}) - 1 + \ell(A/F_1).
\end{aligned}$$

Adding all the equalities above, we obtain $f(n) = n(e_0(\mathcal{F})-1)+\ell(A/F_1)$. □

Now we compute the Hilbert series of $G(\mathcal{F})$ when $\operatorname{nr}(\mathcal{F})$ is maximal.

Proposition 5.3. *Let $(A,\mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring, and $\mathcal{F}=\{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. Let* $\operatorname{depth} G(\mathcal{F})\geq 1$*, and* $\operatorname{nr}(\mathcal{F})=r\geq 3$*. Suppose* $\operatorname{nr}(\mathcal{F})$ *is maximal. Then*

$$HS_{G(\mathcal{F})}(t)=\frac{\ell(A/F_1)+(e_1(\mathcal{F})-r)t+t^r}{(1-t)^2}$$

Proof. Since $\operatorname{nr}(\mathcal{F})=e_1(\mathcal{F})-e_0(\mathcal{F})+\ell(A/F_1)+1$, by Theorem 2.3 we get, $\operatorname{nr}(\mathcal{F})=r(\mathcal{F})$. Let $r(\mathcal{F})=r$. By equation 3.1, $n(\mathcal{F})=r-2$, $P_{\mathcal{F}}(n)=H_{\mathcal{F}}(n)$ for all $n\geq r-1$. Therefore, for all $n\geq r-1$,

$$\ell(F_n/F_{n+1})=P_{\mathcal{F}}(n+1)-P_{\mathcal{F}}(n).$$

Therefore

$$\begin{aligned}
HS_{G(\mathcal{F})}(t) &= \sum_{n\in\mathbb{Z}}\ell(F_n/F_{n+1})t^n\\
&= \sum_{n=0}^{r-2}\ell(F_n/F_{n+1})t^n+\sum_{n\geq r-1}\ell(F_n/F_{n+1})t^n\\
&= \sum_{n=0}^{r-2}\left(n(e_0(\mathcal{F})-1)+\ell(A/F_1)-e_0(\mathcal{F})(n+1)+e_1(\mathcal{F})\right)t^n\\
&\quad+\sum_{n\geq 0}(e_0(\mathcal{F})(n+1)-e_1(\mathcal{F}))t^n \qquad \text{[by Lemma 5.2]}\\
&= \sum_{n=0}^{r-2}(-n+\ell(A/F_1)-e_0(\mathcal{F})+e_1(\mathcal{F}))t^n+\frac{e_0(\mathcal{F})}{(1-t)^2}-\frac{e_1(\mathcal{F})}{(1-t)}\\
&= \frac{1}{(1-t)^2}\left(\sum_{n=1}^{r-2}(-nt^n)(1-t)^2+\sum_{n=0}^{r-2}(\ell(A/F_1)-e_0(\mathcal{F})+e_1(\mathcal{F}))t^n(1-t)^2\right)\\
&\quad+\frac{1}{(1-t)^2}\left(e_0(\mathcal{F})-e_1(\mathcal{F})(1-t)\right)\\
&= \frac{1}{(1-t)^2}\left(\sum_{n=1}^{r-2}(-nt^n)(1-t)^2+(\ell(A/F_1)-e_0(\mathcal{F})+e_1(\mathcal{F}))(1-t^{r-1})(1-t)\right)\\
&\quad+\frac{1}{(1-t)^2}\left(-e_1(\mathcal{F})(1-t)+e_0(\mathcal{F})\right)\\
&= \frac{1}{(1-t)^2}\left(\sum_{n=1}^{r-2}(-nt^n)(1-t)^2+(r-1)(1-t^{r-1})(1-t)-e_1(\mathcal{F})(1-t)+e_0(\mathcal{F})\right).
\end{aligned}$$

Now for further simplification consider the h-vector,

$$\begin{aligned}
h(t) &= \sum_{n=1}^{r-2}(-nt^n)(1-t)^2+(r-1)(1-t^{r-1})(1-t)+e_0(\mathcal{F})-e_1(\mathcal{F})(1-t)\\
&= \sum_{n=1}^{r-2}(-nt^n)+2\sum_{n=1}^{r-2}nt^{n+1}-\sum_{n=1}^{r-2}nt^{n+2}+(r-1)(1-t^{r-1})(1-t)+e_0(\mathcal{F})-e_1(\mathcal{F})(1-t)\\
&= -t+\sum_{n=2}^{r-2}-nt^n+2\sum_{n=2}^{r-2}(n-1)t^n+2(r-2)t^{r-1}+\sum_{n=2}^{r-2}-(n-2)t^n-(r-3)t^{r-1}-(r-2)t^r\\
&\quad +(r-1)(1-t^{r-1}-t+t^r)+e_0(\mathcal{F})-e_1(\mathcal{F})(1-t)\\
&= (e_0(\mathcal{F})-e_1(\mathcal{F})+r-1)+(-1-r+1+e_1(\mathcal{F}))t+(2r-4-r+3-r+1)t^{r-1}\\
&\quad +(-r+2+r-1)t^r\\
&= \ell(A/F_1)+(e_1(\mathcal{F})-r)t+t^r.
\end{aligned}$$

□

As a consequence, we obtain:

Theorem 5.4. *Let $(A,\mathfrak{m})$ be a two-dimensional Gorenstein local ring, $\mathcal{F}=\{F_n\}_{n\in\mathbb{Z}}$ a Hilbert filtration. Let $G(\mathcal{F})$ be Cohen-Macaulay, and $\operatorname{nr}(\mathcal{F})=r\geq 3$. Suppose $\operatorname{nr}(\mathcal{F})$ is maximal. Then $G(\mathcal{F})$ is Gorenstein if and only if $F_1=\mathfrak{m}$ and $e_0(\mathfrak{m})=2$.*

Proof. Since $\operatorname{nr}(\mathcal{F})=e_1(\mathcal{F})-e_0(\mathcal{F})+\ell(A/F_1)+1$, by Theorem 2.3 $\operatorname{nr}(\mathcal{F})=r(\mathcal{F})$. By Theorem 3.1 and Theorem 5.3, we get $G(\mathcal{F})$ is Gorenstein if and only if $\ell(A/F_1)=1$ and $e_1(\mathcal{F})=r(\mathcal{F})$ if and only if $F_1=\mathfrak{m}$ and $e_0(\mathcal{F})=2$. □

Applying Theorem 5.4 for the filtration $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ we get the following result.

Corollary 5.5. *Let $(A,\mathfrak{m})$ be a two-dimensional analytically unramified Gorenstein local ring, and I an $\mathfrak{m}$-primary ideal in A. Let $\operatorname{nr}(I)\geq 3$. Suppose $\operatorname{nr}(I)$ is maximal, and $\overline{G}(I)$ is Cohen-Macaulay. Then $\overline{G}(I)$ is Gorenstein if and only if $\bar{I}=\mathfrak{m}$ and $e_0(\mathfrak{m})=2$.*

Proof. Follows from Theorem 5.4. □

6. Gorensteineness of $\overline{G}(\mathfrak{m})$ in Zariski-type hypersurface rings

Consider the following setup in the rest of the paper. Let K be an algebraically closed field, and let $m\geq 1$, $2\leq a\leq b$ be integers, and put $S=K[[x_0,x_1,\ldots,x_m]]$ be a formal power series ring with $(m+1)$-variables over K. Then S is a complete regular local ring with the unique maximal ideal $\mathfrak{m}_S=(x_0,x_1,\ldots,x_m)$. Put $x=x_0$ and $\underline{x}=x_1,\ldots,x_m$. Let g, f be polynomials such that

$$\begin{aligned}
g &= g(\underline{x})\in(\underline{x})^b\setminus(\underline{x})^{b+1},\\
f &= f(x,\underline{x})=x^a-g(\underline{x}).
\end{aligned}$$

Then a hypersurface $A=S/(f)$ is called "*Zariski-type hypersurface*" if $a\neq 0$ in K. Let $Q=(\underline{x})A$. Note that Q is a minimal reduction of $\mathfrak{m}=\mathfrak{m}_SA$. Indeed, $\mathfrak{m}^a=Q\mathfrak{m}^{a-1}$. In this section, we compute $\bar{\mathfrak{r}}(\mathfrak{m})$ and prove the Cohen-Macaulayness of $\overline{G}(\mathfrak{m})$. The proof is a slight generalization of the proof in [18]. We fix the following notation for the rest of the paper:

$$d = \gcd(a,b), \quad a' = \frac{a}{d}, \quad b' = \frac{b}{d} \quad \text{and} \quad n_k = \lfloor \frac{kb}{a} \rfloor \quad \text{for } k = 1,2,\ldots,a-1.$$

For all $n \geq 1$, we define the following two ideals:

$$\begin{aligned} J_n &:= Q^n + xQ^{n-n_1} + x^2Q^{n-n_2} + \cdots + x^{a-1}Q^{n-n_{a-1}}, \\ I_n &:= \left(x^k x_1^{i_1} \cdots x_m^{i_m} \,\middle|\, kb' + (i_1 + \cdots + i_m)a' \geq n, k, i_1, \ldots, i_m \in \mathbb{Z}_{\geq 0} \right) A. \end{aligned}$$

Proposition 6.1. *$J_n = I_{na'}$ for every $n \geq 1$.*

Proof. First we prove $J_n \subset I_{na'}$. It is enough to show $x^kQ^{n-n_k} \subset I_{na'}$ for every $k = 1,2,\ldots,a-1$. For each k with $1 \leq k \leq a-1$, if $i_1 + \cdots + i_m \geq n - n_k$, then

$$kb' + (i_1 + \cdots + i_m)a' \geq kb' + (n - n_k)a' \geq kb' + \big(n - \frac{kb'}{a'}\big)a' = na',$$

as required. Next we prove the converse. If $x^k x_1^{i_1} \cdots x_m^{i_m} \in I_{na'}$, then $kb' + (i_1 + \cdots + i_m)a' \geq na'$. Thus

$$n_k = \lfloor \frac{kb'}{a'} \rfloor \geq n - (i_1 + \cdots + i_m), \text{ that is, } i_1 + \cdots + i_m \geq n - n_k.$$

Hence $x^k x_1^{i_1} \cdots x_m^{i_m} \in x^kQ^{n-n_k}$. This yields $I_{na'} \subset J_n$. $\square$

Proposition 6.2. *Under the notation above,*

(a) *For every $k, n \geq 1$, $x^k \in \overline{Q^n}$ if and only if $n \leq n_k$.*
(b) *$Q^n \subset J_n \subset \overline{Q^n}$, that is, $\overline{J_n} = \overline{Q^n}$ for each $n \geq 1$.*

Proof. (a) We first show that if $n \leq n_k$ then $x^k \in \overline{Q^n}$. By definition of n_k, we get

$$(x^k)^a = x^{ka} \in (Q^b)^k = Q^{kb} \subset Q^{an_k} = (Q^{n_k})^a.$$

Hence $x^k \in \overline{Q^{n_k}} \subset \overline{Q^n}$.

Suppose $x^k \in \overline{Q^n}$. We will show $n \leq n_k$. By assumption, there exists a nonzero divisor $c \in A$ such that $c(x^k)^\ell \subset Q^{n\ell}$ for sufficiently large ℓ. By Artin-Rees' lemma, we can take an integer $\ell_0 \geq 1$ such that if $\ell \gg 0$, then $Q^\ell \cap cA = c(Q^\ell : c) \subset cQ^{\ell-\ell_0}$. Thus $Q^\ell : c \subset Q^{\ell-\ell_0}$ because c is a nonzero divisor.

Now suppose that $n \geq n_k + 1$. By the choice of n_k, we have

$$\frac{kb}{a} + \frac{1}{a} \leq n_k + 1 \leq n.$$

Then $kb + 1 \leq a(n_k + 1) \leq an$. If $\ell \geq \ell_0 + 1$, then

$$g^{k\ell} = x^{ak\ell} \in Q^{an\ell} : c \subset Q^{an\ell-\ell_0} \subset Q^{(kb+1)\ell-\ell_0} \subset Q^{kb\ell+1}.$$

Moreover, since $g \in Q^b \setminus Q^{b+1}$, if necessary, by replacing $\underline{x} = x_1, x_2, \ldots, x_m$ with $\underline{y} = y_1, \ldots, y_m$, the other minimal generators of Q, we may assume that $g \equiv x_1^b \pmod{(x_2, \ldots, x_m)A}$. Hence $x_1^{bk\ell} \in (x_1^{kb\ell+1}, x_2, \ldots, x_n)$. This contradicts the fact that $x_1, x_2, \ldots, x_m$ forms a regular sequence on A.

(b) It is enough to show $x^k x_1^{i_1} \cdots x_m^{i_m} \in \overline{Q^n}$ if and only if $i_1 + \cdots + i_m \geq n - n_k$. Since $\operatorname{depth} \overline{G}(Q) \geq 1$, we have $\overline{Q^n} : x_j = \overline{Q^{n-1}}$ for every n, j. So one can easily see that

$$x^k x_1^{i_1} \cdots x_m^{i_m} \in \overline{Q^n} \Longleftrightarrow x^k \in \overline{Q^{n-(i_1+\cdots+i_m)}}.$$

By (1), the last condition means that $i_1 + \cdots + i_m \geq n - n_k$. Hence we get the required inclusions. $\square$

Set $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$, which is a filtration of ideals in A introduced above and, $I_n = A$ for every integer $n \le 0$. We put

$$\mathcal{R}'(\mathcal{I}) = \bigoplus_{n\in\mathbb{Z}} I_n\, t^n \subset A[t,t^{-1}], \qquad G(\mathcal{I}) = \bigoplus_{n\ge 0} I_n/I_{n+1} \cong \mathcal{R}'(\mathcal{I})/t^{-1}\mathcal{R}'(\mathcal{I}).$$

In the following, we set $\underline{x}t^k = x_1t^k, \ldots, x_mt^k$ and $\underline{Y} = Y_1, \ldots, Y_m$.

Lemma 6.3. *Write $g = g_b + g_{b+1} + \cdots + g_c$, where g_i denotes a homogeneous polynomial of degree $i = b, b+1, \ldots, c$. Then $\mathcal{R}'(\mathcal{I}) \cong A[xt^{b'}, \underline{x}t^{a'}, t^{-1}]$ and $G(\mathcal{I}) \cong K[X, \underline{Y}]/(X^a - g_b(\underline{Y}))$.*

Proof. We have

$$\begin{aligned}
\mathcal{R}'(\mathcal{I}) &= \bigoplus_{n\in\mathbb{Z}} I_n\, t^n \\
&= \sum_{k,i_1,\ldots,i_m\in\mathbb{Z}_{\ge 0}} x^k x_1^{i_1}\cdots x_m^{i_m}\, t^{kb'+(i_1+\cdots+i_m)a'} + \sum_{n\ge 0} A\, t^{-n} \\
&= \sum_{k,i_1,\ldots,i_m\in\mathbb{Z}_{\ge 0}} (xt^{b'})^k (x_1t^{a'})^{i_1}\cdots(x_mt^{a'})^{i_m} + \sum_{n\ge 0} A\, t^{-n} \\
&= A[xt^{b'}, x_1t^{a'}, \ldots, x_mt^{a'}, t^{-1}].
\end{aligned}$$

Put $X = xt^{b'}, Y_i = x_it^{a'}$ for $i = 1, 2, \ldots, m$ and $U = t^{-1}$. Then

$$\begin{aligned}
X^a &= x^a t^{ab'} = x^a t^{a'b} \\
&= g_b(x_1,\ldots,x_m)t^{a'b} + g_{b+1}(x_1,\ldots,x_m)t^{a'b} + \cdots + g_c(x_1,\ldots,x_m)t^{a'b} \\
&= g_b(Y_1,Y_2,\ldots,Y_m) + g_{b+1}(Y_1,Y_2,\ldots,Y_m)t^{-a'} + \cdots + g_c(Y_1,Y_2,\ldots,Y_m)t^{-(c-b)a'} \\
&= g_b(Y_1,Y_2,\ldots,Y_m) + g_{b+1}(Y_1,Y_2,\ldots,Y_m)U^{a'} + \cdots + g_c(Y_1,Y_2,\ldots,Y_m)U^{(c-b)a'}.
\end{aligned}$$

Hence

$$\begin{aligned}
G(\mathcal{I}) &= \mathcal{R}'(\mathcal{I})/t^{-1}\mathcal{R}'(\mathcal{I}) \\
&= K[X, Y_1, Y_2, \ldots, Y_m]/(X^a - g_b(Y_1,\ldots,Y_m)).
\end{aligned}$$

□

The normality of A is needed in the following proposition.

Proposition 6.4. *Suppose that A is a Zariski-type normal hypersurface. Under the notation as above,*

(a) *$G(\mathcal{I})$ is reduced.*
(b) *$\overline{I_n} = I_n$ for every $n \ge 1$.*
(c) *$\overline{J_n} = J_n$ for every $n \ge 1$.*

Proof. (a) By Lemma 6.3, since $G(\mathcal{I}) \cong K[X, \underline{Y}]/(X^a - g_b(\underline{Y}))$ is a hypersurface, it satisfies the Serre condition (S_1).

Put $f = X^a - g_b(\underline{Y})$. Then

$$\operatorname{ht}\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y_1}, \ldots, \frac{\partial f}{\partial Y_m}, f\right) \ge \operatorname{ht}(X, g_b(\underline{Y})) \ge 1.$$

Hence $G(\mathcal{I})$ satisfies (R_0) and it is reduced.

(b) It follows from [14, Lemma 3.1].

(c) Since $J_n = I_{n\alpha'}$ for every $n \geq 1$ by Proposition 6.1, J_n is also integrally closed by (b). $\square$

The following theorem generalizes [18, Theorem 3.1].

Theorem 6.5. *Under the notation as above,*

(a) $\overline{\mathfrak{m}^n} = \overline{Q^n} = Q^n + xQ^{n-n_1} + x^2Q^{n-n_2} + \cdots + x^{a-1}Q^{n-n_{a-1}}$.

(b) $\bar{\mathrm{r}}_Q(\mathfrak{m}) = \mathrm{nr}_Q(\mathfrak{m}) = n_{a-1} = \lfloor \dfrac{(a-1)b}{a} \rfloor$.

(c) $\overline{G}(\mathfrak{m})$ *is Cohen-Macaulay.*

Proof. (a) It follows from Proposition 6.2(b) and Proposition 6.4(c).

(b) It immediately follows from (a).

(c) By Proposition 6.4, $G(\mathcal{I})$ is a hypersurface. Therefore it is Cohen-Macaulay. Hence $\mathcal{R}'(\mathcal{I})$ is Cohen-Macaulay.

On the other hand, $\overline{\mathcal{R}'}(\mathfrak{m}) = \mathcal{R}'(\{J_n\}) = \mathcal{R}'(\{I_{na'}\})$ by Propositions 6.1 and 6.4(c). Then $\overline{\mathcal{R}'}(\mathfrak{m})$ is also Cohen-Macaulay since $\overline{\mathcal{R}'}(\mathfrak{m})$ is isomorphic to a direct summand of $\mathcal{R}'(\mathcal{I})$. Therefore $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay. $\square$

Next, we give a criterion for $\overline{G}(\mathfrak{m})$ to be Gorenstein in the case A is a Zariski-type hypersurface. First, we need the following criterion from [17]. We put $L_n = Q + \overline{\mathfrak{m}^n}$ and $\ell_n = \ell_A(A/L_n)$ for every $n \geq 1$.

Lemma 6.6. *Put $r = \bar{\mathrm{r}}_Q(\mathfrak{m})$. Then the following conditions are equivalent*:

(a) $\overline{G}(\mathfrak{m})$ *is Gorenstein.*

(b) $Q \colon L_n = L_{r+1-n}$ *for every* $n = 1, 2, \ldots, r$.

(c) $\ell_n + \ell_{r+1-n} = a$ *for every* $n = 1, 2, \ldots, \lceil \dfrac{r}{2} \rceil$.

As an application of Proposition 6.4, we obtain the following proposition.

Proposition 6.7. *Under the same notation as above, we have*

$$\ell_n = \min\{k \in \mathbb{Z}_+ \mid n \leq n_k\}.$$

Proof. Suppose $n > n_{k-1}$ and $n \leq n_k$. Then

$$\begin{aligned} L_n &= Q + \overline{\mathfrak{m}^n} \\ &= Q + (Q^n + xQ^{n-n_1} + \cdots + x^{k-1}Q^{n-n_{k-1}} + x^kQ^{n-n_k} + \cdots) \\ &= Q + (x^k). \end{aligned}$$

Hence $\ell_n = \ell_A(K[x]/(x^k)) = k$. $\square$

The following theorem is a main result in this section. Recall $d = \gcd(a, b)$.

Theorem 6.8. $\overline{G}(\mathfrak{m})$ *is Gorenstein if and only if* $b \equiv 0, d \pmod{a}$. *Moreover,*

(a) *If* $b \equiv 0 \pmod{a}$, *then* $\overline{G}(\mathfrak{m})$ *is a reduced, hypersurface.*

(b) *If* $b \equiv 1 \pmod{a}$, *then* $\overline{G}(\mathfrak{m})$ *is a non-reduced, hypersurface.*

(c) *If* $b \equiv d \pmod{a}$ *and* $d > 1$, *then* $\overline{G}(\mathfrak{m})$ *is a complete intersection but not a hypersurface.*

In what follows, we prove the above theorem in several steps.

Proposition 6.9. *If a divides b, then $\overline{G}(\mathfrak{m})$ is a reduced hypersurface, and thus Gorenstein. In fact,*

$$\overline{G}(\mathfrak{m}) \cong K[X, \underline{Y}]/(X^a - g_b(\underline{Y})).$$

Proof. One can easily see that

$$\begin{aligned} \overline{\mathcal{R}}'(\mathfrak{m}) &= K[xt, \underline{x}t, \{x^k t^{n_k}\}_{1 \le k \le a-1}, t^{-1}] \\ &= K[xt^{n_1}, \underline{x}t,\, t^{-1}] \\ &\cong K[X, \underline{Y}, U]/(X^a - g_b(\underline{Y}) - \cdots - g_c(\underline{Y})U^{c-b}), \end{aligned}$$

where $X = xt^{n_1}$, $Y_i = x_i t$ $(i = 1, 2, \ldots, m)$, and $U = t^{-1}$.

Moreover, it immediately follows from the fact that $\overline{G}(\mathfrak{m}) \cong \overline{\mathcal{R}'}(\mathfrak{m})/t^{-1}\overline{\mathcal{R}'}(\mathfrak{m})$. □

The case $1 \le d < a$. First, we consider the case of $d = 1$ and $b \equiv 1 \pmod{a}$.

Proposition 6.10. *If $d = 1$ and $b = n_1 a + 1$, then $\overline{G}(\mathfrak{m})$ is a non-reduced hypersurface, and thus Gorenstein. In fact,*

$$\overline{G}(\mathfrak{m}) \cong K[X, \underline{Y}]/(X^a) \cong G(\mathfrak{m}).$$

Proof. For every k with $1 \le k \le a - 1$, we have

$$n_k = \lfloor \frac{kb}{a} \rfloor = \lfloor \frac{k(n_1 a + 1)}{a} \rfloor = kn_1 + \lfloor \frac{k}{a} \rfloor = kn_1.$$

Hence

$$\begin{aligned} \overline{\mathcal{R}'}(\mathfrak{m}) &= K[xt, \underline{x}t, \{x^k t^{n_k}\}_{1 \le k \le a-1}, t^{-1}] \\ &= K[xt, \underline{x}t, xt^{n_1}, t^{-1}] \\ &\cong K[X, \underline{Y}, U]/(X^a - g_b(\underline{Y})U - \cdots - g_c(\underline{Y})U^{c-b+1}). \end{aligned}$$

Indeed, put $X = xt^{n_1}, Y_i = x_i t$ for $i = 1, 2, \ldots, m$ and $U = t^{-1}$. Then $xt = (xt^{n_1})(t^{-1})^{n_1 - 1}$ and

$$\begin{aligned} X^a = (xt^{n_1})^a &= (g_b + g_{b+1} + \cdots + g_c)t^{an_1} \\ &= (g_b + g_{b+1} + \cdots + g_c)t^{b-1} \\ &= (g_b t^b)U + g_{b+1}t^{b+1}U^2 + \cdots + g_c t^c U^{c-b+1} \\ &= g_b(\underline{Y})U + \cdots + g_c(\underline{Y})U^{c-b+1}. \end{aligned}$$

The other assertion follows from this. □

Next, we consider the case $1 < d < a'$, $b' \equiv 1 \pmod{a'}$.

Proposition 6.11. *If $1 < d < a'$ and $b' = n_1 a' + 1$, then $\overline{G}(\mathfrak{m})$ is a complete intersection, and thus Gorenstein. In fact,*

$$\overline{G}(m) \cong K[X, \underline{Y}, Z]/(X^{a'}, Z^d - g_b(\underline{Y})).$$

Proof. Note that $b' \equiv 1 \pmod{a'}$. One can easily see that $n_k = kn_1$ for $1 \le k \le a' - 1$, and $n_{a'} = b'$. For any integer p $(1 \le p \le d - 1)$, if $pa' + 1 \le k \le pa' + a' - 1$, then

$$n_k = \lfloor \frac{k(n_1 a' + 1)}{a'} \rfloor = kn_1 + \lfloor \frac{k}{a'} \rfloor = kn_1 + p = pb' + (k - pa')n_1.$$

Hence $(k, n_k) = p(a', b') + (k - pa')(1, n_1)$. Similarly, if $k = pa'$ $(2 \le p \le d-1)$, then $(k, n_k) = p(a', b')$. Therefore

$$\begin{aligned} \overline{\mathcal{R}'}(\mathfrak{m}) &= K[xt, x_1t, \ldots, x_mt, xt^{n_1}, x^{a'}t^{b'}, t^{-1}] \\ &\cong K[X, \underline{Y}, Z, U]/(X^{a'} - ZU, Z^d - g_b(\underline{Y}) - \cdots - g_c(\underline{Y})U^{c-b}), \end{aligned}$$

where $X = xt^{n_1}$, $Y_i = x_it$, $Z = x^{a'}t^{b'}$, and $U = t^{-1}$. Indeed,

$$X^{a'} = x^{a'}t^{n_1a'} = (x^{a'}t^{b'})(t^{-1})^{b'-n_1a'} = (x^{a'}t^{b'})(t^{-1}) = ZU.$$

and

$$\begin{aligned} Z^d &= (x^{a'}t^{b'})^d \\ &= x^a t^b \\ &= g_b(x_1, \ldots, x_m)t^b + \cdots + g_c(x_1, \ldots, x_m)t^c \cdot t^{b-c} \\ &= g_b(\underline{Y}) + \cdots + g_c(\underline{Y})U^{c-b}. \end{aligned}$$

The other assertion follows from this. $\square$

In the rest of this section, we consider the case of $1 < d < a$ and $b' \equiv \delta \pmod{a'}$ for some δ $(2 \le \delta \le a' - 1)$. Note that

$$\bar{\mathrm{r}}(\mathfrak{m}) = n_{a-1} = \lfloor \frac{(a-1)b'}{a'} \rfloor = (a-1)n_1 + d\delta - 1.$$

In order to prove Theorem 6.8, it suffices to prove the following proposition.

Proposition 6.12. *In the above notation, if we put* $k_1 = \lceil \frac{a'}{\delta} \rceil$ *and* $t_1 = (k_1 - 1)n_1 + 1$, *then*

$$\ell_{t_1} + \ell_{r+1-t_1} = a + 1 (\neq a).$$

In particular, $\overline{G}(\mathfrak{m})$ *is* not *Gorenstein.*

Proof. We first prove the following claim.

Claim 1. $\lfloor \frac{k\delta}{a'} \rfloor = 0$, $\lceil \frac{k\delta}{a'} \rceil = 1$ for each k $(1 \le k \le k_1 - 1)$.

By definition, $k \le k_1 - 1 < \frac{a'}{\delta}$. Thus $0 < \frac{k\delta}{a'} < 1$. Hence $\lfloor \frac{k\delta}{a'} \rfloor = 0$ and $\lceil \frac{k\delta}{a'} \rceil = 1$.

Claim 2. $\lfloor \frac{k_1\delta}{a'} \rfloor = 1$, $\lceil \frac{k_1\delta}{a'} \rceil = 2$.

By definition, $k_1 - 1 < \frac{a'}{\delta} \le k_1$. Since $(a', \delta) = 1$, $\frac{a'}{\delta}$ is not an integer. Hence $\frac{a'}{\delta} < k_1$, that is, $\frac{k_1\delta}{a'} > 1$. If $\frac{k_1\delta}{a'} \ge 2$, then one has

$$\frac{2a'}{\delta} - 1 \le k_1 - 1 < \frac{a'}{\delta}.$$

This yields $a' < \delta$, which is a contradiction. Hence $1 < \frac{k_1\delta}{a'} < 2$. The claim follows from here.

Claim 3. $n_k = kn_1$ for every k $(1 \le k \le k_1 - 1)$ and $n_{k_1} = k_1n_1 + 1$.

Indeed, for every k $(1 \le k \le k_1 - 1)$, Claim 1 implies

$$n_k = \lfloor \frac{kb'}{a'} \rfloor = \lfloor \frac{k(n_1a' + \delta)}{a'} \rfloor = kn_1 + \lfloor \frac{k\delta}{a'} \rfloor = kn_1.$$

On the other hand, Claim 2 implies

$$n_{k_1} = \lfloor \frac{k_1b'}{a'} \rfloor = \lfloor \frac{k_1(n_1a' + \delta)}{a'} \rfloor = k_1n_1 + \lfloor \frac{k_1\delta}{a'} \rfloor = k_1n_1 + 1.$$

Claim 4. $n_{a-k} = (a-k)n_1 + d\delta - 1$ for each k $(1 \le k \le k_1 - 1)$ and $n_{a-k_1} = (a-k_1)n_1 + d\delta - 2$.

For every k $(1 \le k \le k_1 - 1)$, Claim 1 implies

$$n_{a-k} = = \lfloor \frac{(a-k)(n_1a' + \delta)}{a'} \rfloor \ = (a-k)n_1 + d\delta - \lceil \frac{k\delta}{a'} \rceil = (a-k)n_1 + d\delta - 1.$$

On the other hand, Claim 2 implies

$$n_{a-k_1} = = \lfloor \frac{(a-k_1)(n_1a' + \delta)}{a'} \rfloor \ = (a-k_1)n_1 + d\delta - \lceil \frac{k_1\delta}{a'} \rceil = (a-k_1)n_1 + d\delta - 2.$$

Claim 5. $\ell_{t_1} + \ell_{r+1-t_1} = a + 1$.

By Claim 3, $n_{k_1-1} + 1 = t_1 \le n_{k_1}$. Hence $\ell_{t_1} = k_1$ by Proposition 6.7. By Claim 4, we have

$$\begin{aligned} n_{a-k_1} + 1 &= (a-k_1)n_1 + d\delta - 1 = r + 1 - t_1 \\ n_{a-k_1+1} &= (a-k_1+1)n_1 + d\delta - 1 > r + 1 - t_1. \end{aligned}$$

Hence $\ell_{r+1-t_1} = a - k_1 + 1$ and thus $\ell_{t_1} + \ell_{r+1-t_1} = (a - k_1 + 1) + k_1 = a + 1$. $\square$

Example 6.13. Let $m \ge 1$ be an integer, and let $A = \mathbb{C}[x_0, x_1, \ldots, x_m]/(x_0^3 + x_1^5 + \cdots + x_m^5)$. Put $\mathfrak{m} = (x_0, x_1, \ldots, x_m)$ and $x_0 = x$. Then

(a) A is a rational singularity if and only if $m \ge 4$.
(b) $\overline{\mathfrak{m}^2} = \mathfrak{m}^2$, $\overline{\mathfrak{m}^3} = \mathfrak{m}^3 + (x^2)$, and $\overline{\mathfrak{m}^4} = (x_1, x_2, \ldots, x_m)\overline{\mathfrak{m}^3}$.
(c) $\overline{\mathcal{R}'}(\mathfrak{m}) = K[xt, x_1t, x_2t, \ldots, x_mt, x^2t^3, t^{-1}]$.
(d) $\overline{G}(\mathfrak{m})$ is a Cohen-Macaulay ring of $e_0(\overline{G}) = 3$ and CM type 2.

If we put $X = xt$, $Y_i = x_it$ $(i = 1, 2, \ldots .m)$ and $Z = x^2t^3$, and $U = t^{-1}$, then

$$\overline{\mathcal{R}'}(\mathfrak{m}) \cong K[X, \underline{Y}, Z, U]/(X^2 - ZU, XZ + \underline{Y^5}U, Z^2 + X\underline{Y^5}).$$

In particular,

$$\overline{G}(\mathfrak{m}) \cong \overline{\mathcal{R}'}(\mathfrak{m})/t^{-1}\overline{\mathcal{R}'}(\mathfrak{m}) \cong K[X, \underline{Y}, Z]/(X^2, XZ, Z^2 + X\underline{Y^5}),$$

where $\underline{Y^5} = Y_1^5 + \cdots + Y_m^5$.

Example 6.14. Let $A = K[\![t^a, t^b]\!]$ be a semigroup ring generated by the positive integers $a \le b$ such that $\gcd(a, b) = 1$. We know that $A \cong k[\![x, y]\!]/(x^a - y^b)$ is a hypersurace. By Theorem 6.5 $\bar{r}_Q(\mathfrak{m}) = \lfloor \frac{(a-1)b}{a} \rfloor$. By Theorem 6.8 $\overline{G}(\mathfrak{m})$ is Gorenstein if and only if $b \equiv 1 \pmod{a}$.

The examples 3.8 and 3.9 can also be verified using Theorem 6.8.

7. Normal Tangent cones of maximal embedding dimension

Let $\mathfrak{M}_{\overline{G}}$ denote the unique graded maximal ideal of $\overline{G} = \overline{G}(\mathfrak{m})$. Then $\operatorname{embdim}(\overline{G})$, the *embedding dimension* of $\overline{G}$, is the number of minimal system of generators of $\mathfrak{M}_{\overline{G}}$. Since $\overline{G}$ is Cohen-Macaulay,

$$\operatorname{embdim}(\overline{G}) \le e_0(\overline{G}) + \dim \overline{G} - 1 = a + m - 1 \tag{7.1}$$

If equality holds in Eq.(7.1), then $\overline{G}$ is said to have *maximal embedding dimension* (in the sense of Sally). Note that $\mathfrak{M}_{\overline{G}}$ is generated by $\{x^k t^{n_k}\}_{k=1,2,\ldots,a-1}$, and $x_1 t, x_2 t, \ldots, x_m t$.

We can characterize those $\overline{G}$ in terms of a and b.

Proposition 7.1. *Let $A = S/(x^a - g(\underline{x}))$ be a Zariski-type hypersurface with $b = \operatorname{ord}(g) \ge a$. Then $\overline{G}(\mathfrak{m})$ has maximal embedding dimension if and only if one of the following cases occurs*:

(a) $a = 2$.
(b) $a \ge 3$, $\gcd(a, b) = 1$ *and* $b \equiv a - 1 \pmod{a}$.

When this is the case, $\overline{G}(\mathfrak{m})$ is Gorenstein if and only if $a = 2$.

Proof. In the case of $a = 2$, since $\overline{G}$ is a hypersurface of $e_0(\overline{G}) = 2$, it is a Gorenstein ring having maximal embedding dimension. So we may assume that $a \ge 3$.

If part: We may compute n_k as follows:

$$n_k = \lfloor \frac{k(n_1 a + a - 1)}{a} \rfloor = kn_1 + k + \lfloor \frac{-k}{a} \rfloor = kn_1 + k - \lceil \frac{k}{a} \rceil = kn_1 + k - 1$$

for every $k = 1, 2, \ldots, a-1$. Then a sequence

$$xt^{n_1},\ x^2 t^{2n_1+1},\ x^3 t^{3n_1+2},\ \ldots,\ x^{a-1} t^{(a-1)n_1+a-2},\ x_1 t, \ldots, x_m t$$

forms a minimal system of generators of $\mathfrak{M}_{\overline{G}}$, we have

$$\operatorname{embdim}(\overline{G}) = a + m - 1 = e_0(\overline{G}) + \dim \overline{G} - 1.$$

Only if part: In order to complete the proof, we prove the following two lemmata.

Lemma 7.2. *If $d > 1$ and $a \ge 3$, then $\overline{G}$ does not have maximal embedding dimension.*

Proof. We note that $\mathfrak{M}_{\overline{G}} = (\{x^k t^{n_k}\}_{k=1,2,\ldots,a-1}, x_1 t, \ldots, x_m t)$. For every s $(1 \le s \le d)$, δ $(1 \le \delta \le a' - 1)$, we have

$$n_{sa'+\delta} = \lfloor \frac{(sa' + \delta)b}{a} \rfloor = \lfloor \frac{(sa' + \delta)b'}{a'} \rfloor = sb' + \lfloor \frac{\delta b'}{a'} \rfloor = sn_{a'} + n_\delta,$$

and hence

$$x^{sa'+\delta} t^{n_{sa'+\delta}} = x^{sa'+\delta} t^{sn_{a'} + n_\delta} = (x^{a'} t^{n_{a'}})^s \cdot (x^\delta t^{n_\delta}) \in \mathfrak{M}_{\overline{G}}^2.$$

Moreover, if $s \ge 2$, then

$$x^{sa'} t^{n_{sa'}} = x^{sa'} t^{sn_{a'}} = (x^{a'} t^{n_{a'}})^s \in \mathfrak{M}_{\overline{G}}^2.$$

Hence $\mathfrak{M}_{\overline{G}} = (\{x^k t^{n_k}\}_{k=1,2,\ldots,a'}, x_1 t, \ldots, x_m t)$. Since $a \ge 3$ and $d \ge 2$, we obtain that

$$\operatorname{embdim}(\overline{G}) \le a' + m \le \frac{a}{2} + m < (a-1) + m = a + m - 1,$$

as required. □

Lemma 7.3. *If $d = 1$ and $b \equiv \delta \pmod{a}$ with $1 \le \delta \le a - 2$, then $\overline{G}$ does not have maximal embedding dimension.*

Proof. For each k $(1 \leq k \leq a-1)$, if we put $m_k = n_k - kn_1$, we have

$$m_k = n_k - kn_1 = \lfloor \frac{kb}{a} \rfloor - kn_1 = \lfloor \frac{k(n_1 a + \delta)}{a} \rfloor - kn_1 = \lfloor \frac{k\delta}{a} \rfloor.$$

Since $1 \leq \delta \leq a-2$ and $a \geq 3$, we get

$$m_{a-1} = \lfloor \frac{(a-1)\delta}{a} \rfloor \leq \lfloor \frac{(a-1)(a-2)}{a} \rfloor = a - 3 + \lfloor \frac{2}{a} \rfloor = a-3.$$

Then since $0 \leq m_1 \leq m_2 \leq \cdots \leq m_{a-1} \leq a-3$, there exists an integer i with $2 \leq i \leq a-1$ such that $m_{i-1} = m_i$ by Pigeonhole principle. Since $n_1 + n_{i-1} = n_1 + \{(i-1)n_1 + m_{i-1}\} = in_1 + m_i = n_i$, we get

$$x^i t^{n_i} = x^i t^{n_1 + n_{i-1}} = (xt^{n_1})(x^{i-1}t^{n_{i-1}}) \in \mathfrak{M}_{\overline{G}}^2.$$

This implies $\operatorname{embdim}(\overline{G}) \leq a + m - 2 < e_0(\overline{G}) + \dim \overline{G} - 1$, that is, $\overline{G}$ does not have maximal embedding dimension. $\square$

By the lemmata as above, we complete the proof of Proposition 7.1. $\square$

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